

# On the regularity criteria for liquid crystal flows involving the gradient of one velocity component

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**Abstract** In this paper, we show a regularity criteria for three dimensional nematic liquid crystal flows. More precisely, we prove that the strong solution  $(u, d)$  can be extended beyond  $T$ , provided  $\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3))$ ,  $\frac{2}{p} + \frac{3}{q} \leq \frac{19}{12} + \frac{1}{2q}(\frac{30}{19} < q \leq 3)$  or  $\frac{2}{p} + \frac{3}{q} \leq \frac{3}{2} + \frac{3}{4q}(q > 3)$  with some conditions about the orientation field  $\nabla_h d \in L^\alpha(0, T; L^\beta(\mathbb{R}^3))$ ,  $\frac{2}{\alpha} + \frac{3}{\beta} \leq \frac{3}{4} + \frac{1}{2\beta}(\beta > \frac{10}{3})$ .

**Keywords:** Liquid crystal flows; Regularity criteria; Velocity component; Orientation field component.

**MSC:** 35B65; 35Q35; 76A15.

## 1 Introduction

We consider the following equations:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla P = \nu \Delta u - \lambda \nabla \cdot (\nabla d \otimes \nabla d), & \text{in } \mathbb{R}^3 \times (0, T), \\ d_t + (u \cdot \nabla)d = \gamma(\Delta d - f(d)), & \text{in } \mathbb{R}^3 \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0, T), \end{cases} \quad (1.1)$$

with the initial conditions

$$u(x, 0) = u_0(x), \quad d(x, 0) = d_0(x), \quad \operatorname{div} u_0 = 0, \quad x \in \mathbb{R}^3, \quad (1.2)$$

where  $u$  is the velocity field,  $P$  is the scalar pressure,  $d$  represents the macroscopic molecular orientation field of the liquid crystal materials,  $\nabla \cdot$  denotes the divergence operator, the  $(i, j)$ -th entry of  $\nabla d \otimes \nabla d$  is given by  $\nabla_{x_i} d \cdot \nabla_{x_j} d$  ( $1 \leq i, j \leq 3$ ),  $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d$  and  $\nu, \lambda, \gamma, \eta$  are positive constants. For simplicity, we suppose that they are all one.

In the 1960s, the hydrodynamic theory of liquid crystals was established by Ericksen and Leslie, see [6, 7]. The above system is a simplified version of the Ericksen-Leslie equations for liquid crystal flows. When the orientation field  $d$  equals to a constant, the above equations become the incompressible Navier-Stokes equations. Many regularity results on the solutions

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to the three-dimensional Navier-Stokes equations have been extensively studied, see e.g., [3, 9, 10, 12, 13, 14, 15]. Especially, Zhou and Pokorný in [10] have proved the regularity of the strong solution under the assumption  $\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3))$ ,  $\frac{2}{p} + \frac{3}{q} \leq \frac{19}{12} + \frac{1}{2q}$ ,  $q \in (\frac{30}{19}, 3]$ , or  $\frac{2}{p} + \frac{3}{q} \leq \frac{3}{2} + \frac{3}{4q}$ ,  $q \in (3, \infty]$ . Similarly, there are many interesting results on the regularity criteria for liquid crystals equations, see [1, 2, 4, 5, 8, 11, 16, 17, 18, 20]. Recently, Zhao and Li [20] proved that the solution to liquid crystals equations can be regular in terms of one velocity component and two components of the nematic liquid crystal orientation field, provided  $u_3, \nabla_h d \in L^p(0, T; L^q(\mathbb{R}^3))$ ,  $\frac{2}{p} + \frac{3}{q} \leq \frac{3}{4} + \frac{1}{2q}$  ( $q > \frac{10}{3}$ ). Moreover, Zhao and Li [21] gave the regularity criteria for the system (1.1) involving one derivative components of the pressure and components of gradient of the orientation field, provided

$$\partial_{x_3} P \in L^s(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{s} + \frac{3}{q} < \frac{29}{10}, \quad \frac{30}{23} < q \leq \frac{10}{3},$$

$$\partial_{x_3} d \in L^\gamma(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\gamma} + \frac{3}{\beta} \leq \frac{19}{20}, \quad \frac{60}{13} < \beta \leq \frac{20}{3}$$

and

$$\nabla_h d \in L^{\gamma_1}(0, T; L^{\beta_1}(\mathbb{R}^3)), \quad \frac{2}{\gamma_1} + \frac{3}{\beta_1} \leq \frac{3}{4} + \frac{1}{2\beta_1}, \quad \frac{60}{13} < \beta_1 \leq \frac{20}{3}.$$

Motivated by their ideas, in this paper we show the regularity of the strong solution to 3D liquid crystals equations in terms of gradients of one velocity component.

**Theorem 1.1.** *Let  $u_0 \in H^1(\mathbb{R}^3)$ ,  $d_0 \in H^2(\mathbb{R}^3)$ , and assume that  $(u, d)$  be a strong solution to the liquid crystals equations (1.1)-(1.2) on  $[0, T)$  for some  $0 < T < \infty$ . If  $(u, d)$  satisfies*

$$\nabla u_3 \in L^p(0, T; L^q(\mathbb{R}^3)) \tag{1.3}$$

and

$$\nabla_h d \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \tag{1.4}$$

where  $p, q, \alpha, \beta$  satisfy

$$\begin{cases} \frac{2}{p} + \frac{3}{q} \leq \frac{19}{12} + \frac{1}{2q}, & q \in (\frac{30}{19}, 3], \\ \frac{2}{\alpha} + \frac{3}{\beta} \leq \frac{3}{4} + \frac{1}{2\beta}, & \beta \in (\frac{10}{3}, \infty], \end{cases} \tag{1.5}$$

or

$$\begin{cases} \frac{2}{p} + \frac{3}{q} \leq \frac{3}{2} + \frac{3}{4q}, & q \in (3, \infty], \\ \frac{2}{\alpha} + \frac{3}{\beta} \leq \frac{3}{4} + \frac{1}{2\beta}, & \beta \in (\frac{10}{3}, \infty]. \end{cases} \tag{1.6}$$

Then  $(u, d)$  can be extended beyond  $T$ .

## 2 Preliminaries and Proof of the main result

Denote

$$\begin{aligned}\nabla_h u &= \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right), \nabla_h d = \left( \frac{\partial d}{\partial x_1}, \frac{\partial d}{\partial x_2} \right), \Delta_h u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \Delta_h d = \frac{\partial^2 d}{\partial x_1^2} + \frac{\partial^2 d}{\partial x_2^2}, \\ \partial_i u &= \frac{\partial u}{\partial x_i}; \partial_i d = \frac{\partial d}{\partial x_i} (i = 1, 2, 3), \quad \int_{\mathbb{R}^3} \psi dx = \int \psi dx, \quad \psi \text{ is some function.}\end{aligned}$$

**Lemma 2.1.** (see [11]) *Let  $u_0 \in H^1(\mathbb{R}^3)$ ,  $d_0 \in H^2(\mathbb{R}^3)$ , then there exists  $(u, d)$  be the weak solution to the liquid crystal (1.1)-(1.2) on  $[0, T)$  for some  $0 < T < \infty$ , satisfying the following:*

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} + \|d\|_{L^\infty(0,T;H^1)} + \|d\|_{L^2(0,T;H^2)} \leq C. \quad (2.1)$$

Recall the following Sobolev and Ladyzhenskaya inequality in  $\mathbb{R}^3$  (see [19])

$$\|\phi\|_p \leq C_p \|\phi\|_2^{\frac{6-p}{2p}} \|\partial_1 \phi\|_2^{\frac{p-2}{2p}} \|\partial_2 \phi\|_2^{\frac{p-2}{2p}} \|\partial_3 \phi\|_2^{\frac{p-2}{2p}} \leq C_p \|\phi\|_2^{\frac{6-p}{2p}} \|\phi\|_{H^1}^{\frac{3(p-2)}{2p}}, \quad (2.2)$$

for each  $\phi \in H^1(\mathbb{R}^3)$ , here  $C_p$  is a some positive constant,  $p \in [2, 6]$ .

Multiply (1.1)<sub>2</sub> by  $|d|^4 d$ , and integrate over  $\mathbb{R}^3$ , then we get

$$\frac{1}{6} \frac{d}{dt} \int |d|^6 dx + \int (5|d|^4 |\nabla d|^2 + |d|^8) dx = \int |d|^6 dx,$$

which follows

$$\|d(\cdot, t)\|_{L^\infty(0,T;L^6)} \leq C \|d_0\|_6 \leq C \|d_0\|_{H^1}. \quad (2.3)$$

Next, we shall give the proof of Theorem 1.1.

**Proof of Theorem 1.1:** Suppose  $[0, T^*)$  be the maximal time interval for the existence of the local smooth solution, if  $T^* \geq T$ , the conclusion is obviously valid; but for  $T^* < T$ , we shall show

$$\limsup_{t \rightarrow T^*} (\|\nabla u(\cdot, t)\|_2 + \|\Delta d(\cdot, t)\|_2) \leq C, \quad (2.4)$$

which however contradicts with the definition of  $T^*$ .

Take  $\nabla_h$  on both sides of (1.1)<sub>1</sub> and multiply by  $\nabla_h u$ , it then follows from (1.1)<sub>3</sub> that

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_2^2 + \|\nabla_h \nabla u\|_2^2 + \int \nabla_h(u \cdot \nabla u) \cdot \nabla_h u dx = - \int \nabla_h[\nabla \cdot (\nabla d \otimes \nabla d)] \cdot \nabla_h u dx. \quad (2.5)$$

Take  $\Delta$  on both sides of (1.1)<sub>2</sub>, multiply by  $\Delta_h d$ , we can still get

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h \nabla d\|_2^2 + \int \Delta(u \cdot \nabla d) \cdot \Delta_h d dx = - \|\nabla_h \Delta d\|_2^2 - \int \Delta f(d) \cdot \Delta_h d dx. \quad (2.6)$$

It then follows from (2.5) and (2.6) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2 \\
&= - \int \nabla_h(u \cdot \nabla u) \cdot \nabla_h u dx - \int \nabla_h[\nabla \cdot (\nabla d \otimes \nabla d)] \cdot \nabla_h u dx \\
&\quad - \int \Delta(u \cdot \nabla d) \cdot \Delta_h d dx - \int \Delta f(d) \cdot \Delta_h d dx \\
&= L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{2.7}$$

Next, we will estimate each term of (2.7) step by step, for the term  $L_1$  (see [10]), by using (1.1)<sub>3</sub>, we have

$$\begin{aligned}
L_1 &= - \int \nabla_h(u \cdot \nabla u) \cdot \nabla_h u dx = \int (u \cdot \nabla u) \cdot \Delta_h u dx \\
&= \frac{1}{2} \sum_{i,j=1}^2 \int \partial_3 u_3 \partial_j u_i \partial_j u_i dx - \int \partial_3 u_3 \partial_1 u_1 \partial_2 u_2 dx + \int \partial_3 u_3 \partial_2 u_1 \partial_1 u_2 dx \\
&\quad - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3 dx + \sum_{j=1}^2 \int u_3 \partial_3 u_j \Delta_h u_j dx \\
&= \frac{1}{2} \sum_{i,j=1}^2 \int \partial_3 u_3 \partial_j u_i \partial_j u_i dx - \int \partial_3 u_3 \partial_1 u_1 \partial_2 u_2 dx + \int \partial_3 u_3 \partial_2 u_1 \partial_1 u_2 dx \\
&\quad - \sum_{i=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3 dx - \sum_{j,k=1}^2 \left( \int \partial_k u_3 \partial_3 u_j \partial_k u_j dx - \frac{1}{2} \int \partial_3 u_3 \partial_k u_j \partial_k u_j dx \right).
\end{aligned} \tag{2.8}$$

Then we obtain

$$L_1 \leq C \int |\nabla u_3| |\nabla_h u|^2 dx + C \int |\nabla u_3| |\nabla u| |\nabla_h u| dx. \tag{2.9}$$

By using (1.1)<sub>3</sub> again, we have

$$\begin{aligned}
L_2 &= - \int \nabla_h[\nabla \cdot (\nabla d \otimes \nabla d)] \cdot \nabla_h u dx \\
&= - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k (\partial_j \partial_i d \partial_j d + \partial_i d \Delta d) \partial_k u_i dx \\
&= - \int \sum_{k=1}^2 \sum_{i,j=1}^3 (\partial_k \partial_i d \partial_j \partial_j d + \partial_i d \partial_k \partial_j \partial_j d) \partial_k u_i dx \\
&= - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k \partial_i d \partial_j \partial_j d \partial_k u_i dx - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_i d \partial_k \partial_j \partial_j d \partial_k u_i dx \\
&= L_{21} + L_{22},
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
L_{21} &= \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k d \partial_j \partial_i d \partial_k u_i dx \\
&= \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_k d \partial_j \partial_i d \partial_k u_i dx + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k d \partial_j \partial_j \partial_3 d \partial_k u_3 dx \\
&= \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_k d \partial_j \partial_j \partial_i d \partial_k u_i dx - \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_j \partial_k d \partial_j \partial_3 d \partial_k u_3 dx \\
&\quad - \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k d \partial_j \partial_3 d \partial_k \partial_j u_3 dx \\
&= \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_k d \partial_j \partial_j \partial_i d \partial_k u_i dx - \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_j \partial_k d \partial_j \partial_3 d \partial_k u_3 dx \\
&\quad + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k \partial_k d \partial_j \partial_3 d \partial_j u_3 dx + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k d \partial_j \partial_k \partial_3 d \partial_j u_3 dx,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
L_{22} &= - \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_i d \partial_k \partial_j \partial_j d \partial_k u_i dx - \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_3 d \partial_k \partial_j \partial_j d \partial_k u_3 dx \\
&= - \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_i d \partial_k \partial_j \partial_j d \partial_k u_i dx + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_j \partial_3 d \partial_k \partial_j d \partial_k u_3 dx \\
&\quad + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_3 d \partial_k \partial_j d \partial_j \partial_k u_3 dx \\
&= L_{221} + L_{222} + L_{223}.
\end{aligned} \tag{2.12}$$

Combining (2.10)-(2.12), we have

$$L_2 \leq C \int |\nabla_h d| |\nabla_h \nabla^2 d| |\nabla u| dx + C \int |\nabla u_3| |\nabla_h \nabla d| |\nabla^2 d| dx + L_{223}. \tag{2.13}$$

Using (1.1)<sub>3</sub>, we obtain

$$\begin{aligned}
L_3 &= - \int \Delta(u \cdot \nabla d) \cdot \Delta_h d dx = \int \nabla(u \cdot \nabla d) \cdot \Delta_h \nabla d dx \\
&= \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx + \int \sum_{k=1}^2 \sum_{i,j=1}^3 u_j \partial_i \partial_j d \partial_i \partial_k \partial_k d dx \\
&= \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_j \partial_i \partial_j d \partial_i \partial_k d dx \\
&\quad - \int \sum_{k=1}^2 \sum_{i,j=1}^3 u_j \partial_k \partial_i \partial_j d \partial_i \partial_k d dx \\
&= \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx + \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k \partial_j u_j \partial_i d \partial_i \partial_k d dx \\
&\quad + \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_j \partial_i d \partial_j \partial_i \partial_k d dx \\
&= \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx + \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_j \partial_i d \partial_j \partial_i \partial_k d dx \\
&= L_{31} + L_{32},
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
L_{31} &= \int \sum_{k=1}^2 \sum_{i=1}^3 \sum_{j=1}^2 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx + \int \sum_{k=1}^2 \sum_{i=1}^3 \partial_i u_3 \partial_3 d \partial_i \partial_k \partial_k d dx \\
&= \int \sum_{k=1}^2 \sum_{i=1}^3 \sum_{j=1}^2 \partial_i u_j \partial_j d \partial_i \partial_k \partial_k d dx - \int \sum_{k=1}^2 \sum_{i=1}^3 \partial_k \partial_i u_3 \partial_3 d \partial_i \partial_k d dx \\
&\quad - \int \sum_{k=1}^2 \sum_{i=1}^3 \partial_i u_3 \partial_3 \partial_k d \partial_i \partial_k d dx \\
&= L_{311} + L_{312} + L_{313},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
L_{32} &= \int \sum_{k=1}^2 \sum_{j=1}^3 \sum_{i=1}^2 \partial_k u_j \partial_i d \partial_j \partial_i \partial_k d dx + \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k u_j \partial_3 d \partial_j \partial_3 \partial_k d dx, \\
&= L_{321} + L_{322},
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
L_{322} &= - \int \sum_{k=1}^2 \sum_{j=1}^3 \partial_k u_j \partial_j \partial_3 d \partial_3 \partial_k d dx \\
&= - \int \sum_{k=1}^2 \partial_k u_3 \partial_3 \partial_3 d \partial_3 \partial_k d dx - \int \sum_{k=1}^2 \sum_{j=1}^2 \partial_k u_j \partial_j \partial_3 d \partial_3 \partial_k d dx \\
&= - \int \sum_{k=1}^2 \partial_k u_3 \partial_3 \partial_3 d \partial_3 \partial_k d dx + \int \sum_{k=1}^2 \sum_{j=1}^2 \partial_3 \partial_k u_j \partial_j d \partial_3 \partial_k d dx \\
&\quad + \int \sum_{k=1}^2 \sum_{j=1}^2 \partial_k u_j \partial_j d \partial_3 \partial_3 \partial_k d dx.
\end{aligned} \tag{2.17}$$

Thus, combining (2.14)-(2.17), we have

$$\begin{aligned}
L_3 &\leq C \int |\nabla u| |\nabla_h d| |\nabla_h \nabla^2 d| dx + C \int |\nabla u_3| |\nabla^2 d| |\nabla_h \nabla d| dx \\
&\quad + C \int |\nabla_h d| |\nabla_h \nabla u| |\nabla_h \nabla d| dx + L_{312}.
\end{aligned} \tag{2.18}$$

Noting that  $L_{223} + L_{312} = 0$ .

For the term  $L_4$ , we have

$$\begin{aligned}
L_4 &= - \int \Delta f(d) \cdot \Delta_h d dx = - \int (\Delta(|d|^2 d) - \Delta d) \cdot \Delta_h d dx \\
&= \int \nabla_h(|d|^2 d) \nabla_h \Delta d dx + \int |\nabla_h \nabla d|^2 dx \\
&= \int \sum_{k=1}^2 \partial_k(|d|^2 d) \partial_k \Delta d dx + \int |\nabla_h \nabla d|^2 dx \\
&= \int \sum_{k=1}^2 |d|^2 \partial_k d \partial_k \Delta d dx + 2 \int \sum_{k=1}^2 (d \cdot \partial_k d) (d \cdot \partial_k \Delta d) dx + \int |\nabla_h \nabla d|^2 dx.
\end{aligned} \tag{2.19}$$

Then using (2.2) and (2.3), we can obtain

$$\begin{aligned}
L_4 &\leq C \|d\|_6^2 \|\nabla_h d\|_6 \|\nabla_h \Delta d\|_2 + \|\nabla_h \nabla d\|_2^2 \\
&\leq C \|d_0\|_{H^1}^2 \|\nabla_h \nabla d\|_2 \|\nabla_h \Delta d\|_2 + \|\nabla_h \nabla d\|_2^2 \\
&\leq \frac{1}{8} \|\nabla_h \Delta d\|_2^2 + C \|\nabla_h \nabla d\|_2^2.
\end{aligned} \tag{2.20}$$

Combining (2.8)-(2.20), then (2.7) becomes

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2 \\
& \leq C \int |\nabla u_3| |\nabla u| |\nabla_h u| dx + C \int |\nabla u_3| |\nabla_h u|^2 dx \\
& \quad + C \int |\nabla_h d| |\nabla_h \nabla^2 d| |\nabla u| dx + C \int |\nabla u_3| |\nabla_h \nabla d| |\nabla^2 d| dx \\
& \quad + C \int |\nabla_h d| |\nabla_h \nabla u| |\nabla_h \nabla d| dx + \frac{1}{8} \|\nabla_h \Delta d\|_2^2 + C \|\nabla_h \nabla d\|_2^2 \\
& = M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7.
\end{aligned} \tag{2.21}$$

Next, we will estimate terms from  $M_1 - M_7$  step by step. By using Hölder's inequality, the Sobolev and Ladyzhenskaya inequality(2.2) and Young's inequality, we get

$$M_2 \leq \|\nabla u_3\|_q \|\nabla_h u\|_{\frac{2q}{q-1}}^2 \leq C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla_h u\|_2^2 + \frac{1}{8} \|\nabla_h \nabla u\|_2^2, \tag{2.22}$$

$$\begin{aligned}
M_3 & \leq C \|\nabla_h d\|_p \|\nabla_h \nabla^2 d\|_2 \|\nabla u\|_2^{1-\frac{3}{p}} \|\nabla u\|_6^{\frac{3}{p}} \\
& \leq C \|\nabla_h d\|_p \|\nabla_h \nabla^2 d\|_2 \|\nabla u\|_2^{1-\frac{3}{p}} \|\nabla_h \nabla u\|_2^{\frac{2}{p}} \|\nabla^2 u\|_2^{\frac{1}{p}} \\
& \leq \frac{1}{8} \|\nabla_h \nabla^2 d\|_2^2 + \frac{1}{8} \|\nabla_h \nabla u\|_2^2 + C \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}},
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
M_5 & \leq C \|\nabla_h d\|_p \|\nabla_h \nabla u\|_2 \|\nabla_h \nabla d\|_2^{1-\frac{3}{p}} \|\nabla_h \nabla d\|_6^{\frac{3}{p}} \\
& \leq C \|\nabla_h d\|_p \|\nabla_h \nabla u\|_2 \|\nabla_h \nabla d\|_2^{1-\frac{3}{p}} \|\nabla_h \nabla^2 d\|_2^{\frac{3}{p}} \\
& \leq \frac{1}{8} \|\nabla_h \nabla u\|_2^2 + \frac{1}{8} \|\nabla_h \nabla^2 d\|_2^2 + C \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2.
\end{aligned} \tag{2.24}$$

Next, we will estimate terms  $M_1$  and  $M_4$  by two cases.

- If  $q \in (3, \infty]$ , then

$$\begin{aligned}
M_1 & \leq C \|\nabla u_3\|_q \|\nabla u\|_2 \|\nabla_h u\|_{\frac{2q}{q-2}}^{\frac{2q}{q-2}} \\
& \leq \|\nabla u_3\|_q \|\nabla u\|_2 \|\nabla_h u\|_2^{1-\frac{3}{q}} \|\nabla_h u\|_6^{\frac{3}{q}} \\
& \leq \frac{1}{8} \|\nabla_h \nabla u\|_2^2 + C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla u\|_2^2,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
M_4 & \leq C \|\nabla u_3\|_q \|\nabla^2 d\|_2 \|\nabla_h \nabla d\|_{\frac{2q}{q-2}}^{\frac{2q}{q-2}} \\
& \leq C \|\nabla u_3\|_q \|\nabla^2 d\|_2 \|\nabla_h \nabla d\|_6^{\frac{3}{q}} \|\nabla_h \nabla d\|_2^{1-\frac{3}{q}} \\
& \leq \frac{1}{8} \|\nabla_h \nabla^2 d\|_2^2 + C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla^2 d\|_2^2.
\end{aligned} \tag{2.26}$$



• If  $q \in [\frac{3}{2}, 3]$ , then

$$\begin{aligned}
M_1 &\leq C \|\nabla u_3\|_q \|\nabla_h u\|_6 \|\nabla u\|_2^{\frac{6q}{5q-6}} \\
&\leq C \|\nabla u_3\|_q \|\nabla_h \nabla u\|_2 \|\nabla u\|_2^{\frac{2q-3}{q}} \|\nabla u\|_6^{\frac{3-q}{q}} \\
&\leq C \|\nabla u_3\|_q \|\nabla_h \nabla u\|_2^{1+\frac{6-2q}{3q}} \|\nabla u\|_2^{\frac{2q-3}{q}} \|\nabla^2 u\|_2^{\frac{3-q}{3q}} \\
&\leq \frac{1}{8} \|\nabla_h \nabla u\|_2^2 + C \|\nabla u_3\|_q^{\frac{6q}{5q-6}} \|\nabla u\|_2^{\frac{12q-18}{5q-6}} \|\nabla^2 u\|_2^{\frac{6-2q}{5q-6}},
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
M_4 &\leq C \|\nabla u_3\|_q \|\nabla^2 d\|_2^{\frac{6q}{5q-6}} \|\nabla_h \nabla d\|_6 \\
&\leq C \|\nabla u_3\|_q \|\nabla_h \nabla^2 d\|_2 \|\nabla^2 d\|_2^{\frac{2q-3}{q}} \|\nabla^2 d\|_6^{\frac{3-q}{q}} \\
&\leq \frac{1}{8} \|\nabla_h \nabla^2 d\|_2^2 + C \|\nabla u_3\|_q^{\frac{6q}{5q-6}} \|\nabla^2 d\|_2^{\frac{12q-18}{5q-6}} \|\nabla \Delta d\|_2^{\frac{6-2q}{5q-6}}.
\end{aligned} \tag{2.28}$$

By combining (2.22)-(2.26), then (2.21) becomes

$$\begin{aligned}
&\frac{d}{dt} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2 \\
&\leq C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} (\|\nabla_h u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) + C \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} \\
&\quad + C \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2.
\end{aligned} \tag{2.29}$$

By integrating in time, then we have

$$\begin{aligned}
&\sup_{0 \leq \tau \leq t} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \int_0^t (\|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2) d\tau \\
&\leq \|\nabla_h u_0\|_2^2 + \|\nabla_h \nabla d_0\|_2^2 + C \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} (\|\nabla_h u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \\
&\quad + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau.
\end{aligned} \tag{2.30}$$

Besides, by combing (2.22)-(2.24) and (2.27)-(2.28), then (2.21) becomes

$$\begin{aligned}
&\frac{d}{dt} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2 \\
&\leq C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla_h u\|_2^2 + C \|\nabla u_3\|_q^{\frac{6q}{5q-6}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{12q-18}{5q-6}} (\|\nabla^2 u\|_2 + \|\nabla \Delta d\|_2)^{\frac{6-2q}{5q-6}} \\
&\quad + C \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} + C \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2.
\end{aligned} \tag{2.31}$$

By integrating in time, then we have

$$\begin{aligned}
& \sup_{0 \leq \tau \leq t} (\|\nabla_h u\|_2^2 + \|\nabla_h \nabla d\|_2^2) + \int_0^t (\|\nabla_h \nabla u\|_2^2 + \|\nabla_h \Delta d\|_2^2) d\tau \\
& \leq \|\nabla_h u_0\|_2^2 + \|\nabla_h \nabla d_0\|_2^2 + C \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla_h u\|_2^2 d\tau \\
& \quad + C \int_0^t \|\nabla u_3\|_q^{\frac{6q}{5q-6}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{12q-18}{5q-6}} (\|\nabla^2 u\|_2 + \|\nabla \Delta d\|_2)^{\frac{6-2q}{5q-6}} d\tau \\
& \quad + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau.
\end{aligned} \tag{2.32}$$

Taking  $\Delta$  on both sides of (1.1)<sub>2</sub> and multiplying it by  $\Delta d$ , then multiplying (1.1)<sub>1</sub> by  $-\Delta u$ , by adding them, it follows from (1.1)<sub>3</sub> that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + \|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2 \\
& = \int (u \cdot \nabla) u \cdot \Delta u dx - 2 \sum_{i=1}^3 \int \nabla u_i \partial_i \nabla d \cdot \Delta d dx - \int \Delta f(d) \cdot \Delta d dx \\
& = F_1 + F_2 + F_3.
\end{aligned} \tag{2.33}$$

Then we will compute  $F_1, F_2$  and  $F_3$  step by step. By using (1.1)<sub>3</sub>,

$$\begin{aligned}
F_1 &= \int (u \cdot \nabla) u \cdot \Delta u dx = \int \sum_{k=1}^2 \sum_{i,j=1}^3 u_i \partial_i u_j \partial_k \partial_k u_j dx + \int \sum_{i,j=1}^3 u_i \partial_i u_j \partial_3 \partial_3 u_j dx \\
&= - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_i \partial_i u_j \partial_k u_j dx + \int \sum_{i=1}^2 \sum_{j=1}^3 u_i \partial_i u_j \partial_3 \partial_3 u_j dx + \int \sum_{j=1}^3 u_3 \partial_3 u_j \partial_3 \partial_3 u_j dx \\
&= - \int \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_i \partial_i u_j \partial_k u_j dx - \int \sum_{i=1}^2 \sum_{j=1}^3 \partial_3 u_i \partial_i u_j \partial_3 u_j dx \\
&\quad + \int \sum_{j=1}^3 (\partial_1 u_1 + \partial_2 u_2) \partial_3 u_j \partial_3 u_j dx \\
&\leq C \int |\nabla_h u| |\nabla u|^2 dx \\
&\leq C \|\nabla_h u\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\nabla u\|_6^{\frac{3}{2}} \\
&\leq C \|\nabla_h u\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{\frac{1}{2}},
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
F_2 &= -2 \sum_{i=1}^3 \sum_{j=1}^3 \int \partial_j u_i \partial_i \partial_j d \Delta d dx \\
&= -2 \sum_{i=1}^2 \sum_{j=1}^3 \int \partial_j u_i \partial_i \partial_j d \Delta d dx - 2 \sum_{j=1}^3 \int \partial_j u_3 \partial_3 \partial_j d \Delta d dx \\
&= F_{21} + F_{22},
\end{aligned} \tag{2.35}$$

$$F_{21} = 2 \sum_{i=1}^2 \sum_{j=1}^3 \int \partial_j \partial_j u_i \partial_i d \Delta d dx + 2 \sum_{i=1}^2 \sum_{j=1}^3 \int \partial_j u_i \partial_i d \partial_j \Delta d dx = F_{211} + F_{212}, \tag{2.36}$$

$$\begin{aligned}
F_{211} &\leq C \int |\nabla_h d| |\nabla^2 u| |\nabla^2 d| dx \\
&\leq C \|\nabla_h d\|_p \|\nabla^2 u\|_2 \|\nabla^2 d\|_2^{1-\frac{3}{p}} \|\nabla^2 d\|_6^{\frac{3}{p}} \\
&\leq C \|\nabla_h d\|_p \|\nabla^2 u\|_2 \|\nabla^3 d\|_2^{\frac{3}{p}} \|\nabla^2 d\|_2^{1-\frac{3}{p}} \\
&\leq \frac{1}{4} \|\nabla^2 u\|_2^2 + \frac{1}{8} \|\nabla \Delta d\|_2^2 + C \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla^2 d\|_2^2,
\end{aligned}$$

$$\begin{aligned}
F_{212} &\leq C \int |\nabla_h d| |\nabla u| |\nabla \Delta d| dx \\
&\leq C \|\nabla_h d\|_p \|\nabla \Delta d\|_2 \|\nabla u\|_2^{1-\frac{3}{p}} \|\nabla u\|_6^{\frac{3}{p}} \\
&\leq C \|\nabla_h d\|_p \|\nabla \Delta d\|_2 \|\nabla u\|_2^{1-\frac{3}{p}} \|\nabla^2 u\|_2^{\frac{3}{p}} \\
&\leq \frac{1}{4} \|\nabla^2 u\|_2^2 + \frac{1}{8} \|\nabla \Delta d\|_2^2 + C \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla u\|_2^2.
\end{aligned}$$

We estimate the term of  $F_{22}$  by two cases.

- If  $q \in (3, \infty]$ , then

$$\begin{aligned}
F_{22} &\leq C \|\nabla u_3\|_q \|\nabla^2 d\|_2 \|\nabla^2 d\|_2^{\frac{2q}{q-2}} \\
&\leq \|\nabla u_3\|_q \|\nabla^2 d\|_2 \|\nabla^2 d\|_2^{1-\frac{3}{q}} \|\nabla^2 d\|_6^{\frac{3}{q}} \\
&\leq \frac{1}{8} \|\nabla \Delta d\|_2^2 + C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla^2 d\|_2^2.
\end{aligned} \tag{2.37}$$

• If  $q \in [\frac{3}{2}, 3]$ , then

$$\begin{aligned}
F_{22} &\leq C \|\nabla u_3\|_q \|\nabla^2 d\|_6 \|\nabla^2 d\|_6^{\frac{6q}{5q-6}} \\
&\leq C \|\nabla u_3\|_q \|\nabla \Delta d\|_2 \|\nabla^2 d\|_2^{\frac{2q-3}{q}} \|\nabla^2 d\|_6^{\frac{3-q}{q}} \\
&\leq C \|\nabla u_3\|_q \|\nabla \Delta d\|_2^{\frac{3}{q}} \|\nabla^2 d\|_2^{\frac{2q-3}{q}} \\
&\leq \frac{1}{8} \|\nabla \Delta d\|_2^2 + C \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla^2 d\|_2^2.
\end{aligned} \tag{2.38}$$

For the term  $F_3$ , similar to the estimate of  $L_4$  in (2.20),

$$\begin{aligned}
F_3 &= - \int \Delta f(d) \cdot \Delta d dx \\
&= \int \nabla(|d|^2 d) \nabla \Delta d dx + \|\Delta d\|_2^2 \\
&= \int |d|^2 \nabla d \nabla \Delta d dx + 2 \int (d \cdot \nabla d)(d \cdot \nabla \Delta d) dx + \|\Delta d\|_2^2 \\
&\leq \frac{1}{8} \|\nabla \Delta d\|_2^2 + C \|\Delta d\|_2^2.
\end{aligned} \tag{2.39}$$

Combining (2.34)-(2.37) and (2.39), (2.33) becomes

$$\begin{aligned}
&\frac{d}{dt} (\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + \|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2 \\
&\leq C \|\nabla_h u\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{\frac{1}{2}} + C(1 + \|\nabla u_3\|_q^{\frac{2q}{2q-3}} + \|\nabla_h d\|_p^{\frac{2p}{p-3}}) (\|\Delta d\|_2^2 + \|\nabla u\|_2^2).
\end{aligned} \tag{2.40}$$

Then by integrating in time, we obtain

$$\begin{aligned}
&\|\nabla u\|_2^2 + \|\Delta d\|_2^2 + \int_0^t (\|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2) d\tau \\
&\leq \|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2 + C \int_0^t \|\nabla_h u\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{\frac{1}{2}} d\tau \\
&\quad + C \int_0^t (1 + \|\nabla u_3\|_q^{\frac{2q}{2q-3}} + \|\nabla_h d\|_p^{\frac{2p}{p-3}}) (\|\Delta d\|_2^2 + \|\nabla u\|_2^2) d\tau \\
&\leq \|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2 + CH(t) + C \int_0^t (1 + \|\nabla u_3\|_q^{\frac{2q}{2q-3}} + \|\nabla_h d\|_p^{\frac{2p}{p-3}}) (\|\Delta d\|_2^2 + \|\nabla u\|_2^2) d\tau,
\end{aligned} \tag{2.41}$$

where

$$H(t) = \int_0^t \|\nabla_h u\|_2 \|\nabla u\|_2^{\frac{1}{2}} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{\frac{1}{2}} d\tau.$$

Combining (2.30), we have

$$\begin{aligned}
H(t) &\leq \sup_{0 \leq \tau \leq t} \|\nabla_h u\|_2 \left( \int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{4}} \left( \int_0^t \|\nabla_h \nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq C \left( \sup_{0 \leq \tau \leq t} \|\nabla_h u\|_2^2 + \int_0^t \|\nabla_h \nabla u\|_2^2 d\tau \right) \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq C \left( \|\nabla_h u_0\|_2^2 + \|\nabla_h \nabla d_0\|_2^2 + C \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} (\|\nabla_h u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \right. \\
&\quad \left. + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau \right. \\
&\quad \left. + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau \right) \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq C \left( \|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}} \right) + C \left[ \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \right]^{\frac{4}{3}} \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau \right]^{\frac{4}{3}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\quad + \frac{1}{4} \int_0^t \|\Delta u\|_2^2 d\tau \\
&\leq C \left( \|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}} \right) + \frac{1}{4} \int_0^t \|\Delta u\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla u\|_2^2 d\tau \right]^{\frac{4(p-3)}{3(p-2)}} \left[ \int_0^t \|\Delta u\|_2^2 d\tau \right]^{\frac{4}{3(p-2)}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla^2 d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\leq C \left( \|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}} \right) + \frac{1}{2} \int_0^t \|\Delta u\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla u\|_2^2 d\tau \right]^{\frac{4(p-3)}{3(p-10)}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla^2 d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\leq C \left( \|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}} \right) + \frac{1}{2} \int_0^t \|\Delta u\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \\
&\quad + C \int_0^t \|\nabla_h d\|_p^{\frac{8p}{3p-10}} \|\nabla u\|_2^2 d\tau + C \int_0^t \|\nabla_h d\|_p^{\frac{8p}{3(p-3)}} \|\nabla^2 d\|_2^2 d\tau.
\end{aligned} \tag{2.42}$$

Then combining (2.41) and (2.42), we have

$$\begin{aligned}
&\|\nabla u\|_2^2 + \|\Delta d\|_2^2 + \int_0^t (\|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2) d\tau \\
&\leq C \int_0^t (\|\nabla u_3\|_q^{\frac{8q}{6q-9}} + \|\nabla_h d\|_p^{\frac{8p}{3p-10}}) (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau + C.
\end{aligned} \tag{2.43}$$

By Gronwall's inequality, (1.3), (1.4) and (1.6), then the solution can be extended beyond  $T$ .

On the other hand, combining (2.32), we have

$$\begin{aligned}
H(t) &\leq \sup_{0 \leq \tau \leq t} \|\nabla_h u\|_2 \left( \int_0^\tau \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{4}} \left( \int_0^\tau \|\nabla_h \nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^\tau \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq C \left( \sup_{0 \leq \tau \leq t} \|\nabla_h u\|_2^2 + \int_0^t \|\nabla_h \nabla u\|_2^2 d\tau \right) \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq \{ \|\nabla_h u_0\|_2^2 + \|\nabla_h \nabla d_0\|_2^2 + C \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla_h u\|_2^2 d\tau \\
&\quad + C \int_0^t \|\nabla u_3\|_q^{\frac{6q}{5q-6}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{12q-18}{5q-6}} (\|\nabla^2 u\|_2 + \|\nabla \Delta d\|_2)^{\frac{6-2q}{5q-6}} d\tau \\
&\quad + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau + C \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau \} \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{1}{4}} \\
&\leq C (\|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}}) + C \left[ \int_0^t \|\nabla u_3\|_q^{\frac{2q}{2q-3}} \|\nabla u\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\quad + C \left[ \int_0^t \|\nabla u_3\|_q^{\frac{6q}{5q-6}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{12q-18}{5q-6}} (\|\nabla^2 u\|_2 + \|\nabla \Delta d\|_2)^{\frac{6-2q}{5q-6}} d\tau \right]^{\frac{4}{3}} \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-2}} \|\nabla u\|_2^{\frac{2(p-3)}{p-2}} \|\nabla^2 u\|_2^{\frac{2}{p-2}} d\tau \right]^{\frac{4}{3}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla_h \nabla d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\quad + \frac{1}{4} \int_0^t \|\Delta u\|_2^2 d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq C(\|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}}) + \frac{1}{4} \int_0^t \|\Delta u\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} \|\nabla u\|_2^2 d\tau \\
&\quad + C \left[ \int_0^t \|\nabla u_3\|_q^{\frac{12q}{12q-18}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \right]^{\frac{2(12q-18)}{3(5q-6)}} \left[ \int_0^t (\|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2) d\tau \right]^{\frac{2(6-2q)}{3(5q-6)}} \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla u\|_2^2 d\tau \right]^{\frac{4(p-3)}{3(p-2)}} \left( \int_0^t \|\Delta u\|_2^2 d\tau \right)^{\frac{4}{3(p-2)}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla^2 d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\leq C(\|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}}) + \frac{1}{2} \int_0^t \|\Delta u\|_2^2 d\tau + \frac{1}{4} \int_0^t \|\nabla \Delta d\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} \|\nabla u\|_2^2 d\tau \\
&\quad + C \left[ \int_0^t \|\nabla u_3\|_q^{\frac{12q}{12q-18}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau \right]^{\frac{2(12q-18)}{19q-30}} + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla u\|_2^2 d\tau \right]^{\frac{4(p-3)}{3p-10}} \\
&\quad + C \left[ \int_0^t \|\nabla_h d\|_p^{\frac{2p}{p-3}} \|\nabla^2 d\|_2^2 d\tau \right]^{\frac{4}{3}} \\
&\leq C(\|\nabla_h u_0\|_2^{\frac{8}{3}} + \|\nabla_h \nabla d_0\|_2^{\frac{8}{3}}) + \frac{1}{2} \int_0^t \|\Delta u\|_2^2 d\tau + \frac{1}{4} \int_0^t \|\nabla \Delta d\|_2^2 d\tau + C \int_0^t \|\nabla u_3\|_q^{\frac{8q}{6q-9}} \|\nabla u\|_2^2 d\tau \\
&\quad + C \int_0^t \|\nabla u_3\|_q^{\frac{24q}{19q-30}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau + C \int_0^t \|\nabla_h d\|_p^{\frac{8p}{3p-10}} \|\nabla u\|_2^2 d\tau \\
&\quad + C \int_0^t \|\nabla_h d\|_p^{\frac{8p}{3(p-3)}} \|\nabla^2 d\|_2^2 d\tau.
\end{aligned} \tag{2.44}$$

Then combining (2.41) and (2.44), we have

$$\begin{aligned}
&\|\nabla u\|_2^2 + \|\Delta d\|_2^2 + \int_0^t (\|\Delta u\|_2^2 + \|\nabla \Delta d\|_2^2) d\tau \\
&\leq C \int_0^t (\|\nabla u_3\|_q^{\frac{24q}{19q-30}} + \|\nabla_h d\|_p^{\frac{8p}{3p-10}}) (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) d\tau + C.
\end{aligned} \tag{2.45}$$

By Gronwall's inequality, (1.3), (1.4) and (1.5), then the solution can be extended beyond  $T$ .

## References

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