Dynamic behaviors of abundant solutions for the Lakshmanan–Porsezian–Daniel equation in an optical fiber

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Abstract

The integrable Lakshmanan–Porsezian–Daniel (LPD) equation originating in nonlinear fiber is studied in this work via the Riemann–Hilbert (RH) approach. Firstly we perform the spectral analysis of the Lax pair along with LPD equation, from which a RH problem is formulated. Afterwards, using the symmetry relations of the potential matrix, the formula of N-soliton solutions can be obtained by solving the special RH problem with reflectionless under the conditions of irregularity. In particular, the localized structures and dynamic behaviors of the breathers and solitons corresponding to the real part, imaginary part and modulus of the resulting solution $r(x, t)$ are shown graphically and discussed in detail. One of the innovations in the paper is that the higher-order linear and nonlinear term β has important impact on the velocity, phase, period, and wavewidth of wave dynamics. The other is that collisions of the high-order breathers and soliton solutions are elastic interaction which imply they remain bounded all the time.

Keywords:
Riemann–Hilbert approach; spectral analysis; Lakshmanan–Porsezian–Daniel equation; breathers; solitons.

1. Introduction

Communication system depended on optical solitons [1, 2] have attracted a considerable interest of mathematical physicist due to its ultrafast response to time. It is known nonlinear pulses propagation in monomode fiber is modeled by the classical nonlinear Schrödinger (NLS) equation [3]

$$iq_t = \alpha q_{xx} + \beta |q|^2 q, \tag{1}$$

in which only the group velocity dispersion α and the self-phase modulation effect β are considered. Later on, the higher-order nonlinear Schrödinger (HNLS) equation that was more accurate for describing the propagation of femtosecond optical pulse is investigated [4].

In order to understand the ultrashort pulse propagation in optical fibers profoundly, we should contemplate the effects of the higher-order dispersion, the higher-order nonlinearity, the self-steepening as well as the stimulated Raman scattering. The following LPD equation we would study in this paper is as follows [5–7]

$$ir_t - \alpha(\frac{1}{2} r_{xx} + |r|^2 r) - \beta(r_{xxxx} + 8|r|^2 r_{xx} + 2r^2 r^*_{xx} + 4r|r_x|^2 + 6r_x^2 r^* + 6|r|^4 r) = 0. \tag{2}$$

Eq.(2) possess the second-order dispersion, the fourth-order dispersion, the cubic and quintic nonlinearities by adding higher order nonlinear terms to the NLS equation. A long-distance, high-speed optical fiber transmission system can be governed by the LPD equation due to its ultrashort optical pulse propagation.

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Here $r(x,t)$ is the complex amplitude of the pulse envelope as well as β represents strength of higher-order linear and nonlinear effects. In the mean time, the subscripts of $r(x,t)$ denote the partial derivatives in regard to the corresponding variables, while the asterisk stands for the complex conjugate. In optical fibers, x and t are equivalent to the retarded time coordinate and propagation distance respectively.

As a member of the NLS integrable hierarchy, a lot of interesting studies have been done about Eq. (2). Lakshmanan, Porsezian, and Daniel presented the LPD equation firstly from a classical 1-dimensional isotropic biquadratic Heisenberg spin chain [5] with the help of geometric method and discussed the integrable characteristics in some literatures [8, 9]. The breathers and rogue wave solutions of Eq. (2) are constructed and discussed in detail by means of Darboux transformation [10]. Conservation laws, soliton solutions and modulational instability are analyzed in the presence of small perturbation [11]. Furthermore, the authors constructed the NLS equation hierarchy beyond the LPD equation and obtained plane wave solutions, Akhmediev breathers, Kuznetsov–Ma breathers, periodic solutions, and rogue wave solutions for this infinite-order hierarchy in certain particular cases [12]. In addition, nonlocal LPD equations [6, 7] are also explored and dynamical behaviors of the obtained solutions are illustrated in the form of some visualized graphs.

The matrix RH problem was proposed by Fokas in 1997 [13] which can be used to figure out the initial-boundary value problem [14], the long-time asymptotic behaviors [15], the associated random matrix and orthogonal polynomials problems. Besides address the important mathematical problems above, RH approach [16–23] has been successfully applied to formulate multiple-soliton solutions to many nonlinear evolution equations (NLEEs), such as the coupled derivative NLS equation [24], the coupled NLS equation with higher-order effects [25], the multicomponent AKNS integrable hierarchies [26], the generalized Sasa–Satsuma equation [27], the Kundu–Eckhaus equation [28] and so on.

Inspired by the work [29], we focus on constructing multi-soliton solutions to Eq. (2) under the zero boundary condition at infinity by the RH method. The paper is organized as follows. In Sec. 2, a RH problem is set up and solved in the reflectionless cases from the Lax pair related to Eq. (2). In Secs. 3 and 4, we construct general expression of the N-soliton solutions of Eq. (2) and examine the spacial structures and collision dynamics behaviors of 1-order, 2-order and 3-order breathers and soliton solutions in detail as an example. Sec. 5 is devoted to conclusions and discussions.

2. The Riemann–Hilbert problem

In the following, a RH problem is constructed for the LPD equation (2) through the scattering and inverse scattering transforms. To this end, we give spectral analysis of the Lax pair which is in the following form [7]:

$$
\begin{aligned}
\Phi_x = U\Phi =& (i\lambda\Lambda + iP)\Phi,\\
\Phi_t = V\Phi =& (-8i\beta\Lambda\lambda^4 - 8i\beta P\lambda^3 + \Lambda(4i\beta(P^2 + iP_x) + i\alpha)\lambda^2\\
&+ (2\beta(P_xP - PP_x + 2iP^3 + iP_{xx}) + i\alpha P)\lambda\\
&+ \Lambda(\frac{\alpha}{2}(P_x - iP^2) - i\beta(3P^4 + P_{xx}P - PP_{xx} - P_x^2 + iP_{xxx} + 6iP^2P_x)))\Phi,
\end{aligned}
\tag{3}
$$

where $\Phi = \Phi(x,t)$ is an eigenfunction vector. The column vector Φ and these matrices Λ and P are given by:

$$
\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix}. \tag{4}
$$

The zero curvature equation $U_t - V_x + [U,V] = 0$ generates the LPD equation (2). Here $[U,V] = UV - VU$ denotes a commutator and $\lambda \in \mathbb{C}$ is a spectral parameter. Then the Lax pair (3) is converted into this equivalent form in the case $\alpha = 2$

$$
\begin{aligned}
\Phi_x =& i\lambda\Lambda\Phi + U_1\Phi,\\
\Phi_t =& -8i\beta\lambda^4\Lambda\Phi + U_2\Phi,
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
U_1 =& iP,\\
U_2 =& -8i\beta P\lambda^3 + 2i\Lambda(2\beta(P^2 + iP_x) + 1)\lambda^2 + (2\beta(P_xP - PP_x + 2iP^3 + iP_{xx}) + 2iP)\lambda\\
&+ \Lambda(P_x - iP^2 - i\beta(3P^4 + P_{xx}P - PP_{xx} - P_x^2 + iP_{xxx} + 6iP^2P_x)).
\end{aligned}
$$

Assume that the potential function r in the Lax pair (3) decays to zero sufficiently fast as $x \to \pm\infty$. From Lax pair (5), it is easy to know that $\Phi \propto e^{i\lambda\Lambda x - 8i\beta\lambda^4\Lambda t}$, so we propose the following transformation

$$
\Phi = Je^{i(\lambda x - 8\lambda^4\beta t)\Lambda}. \tag{6}
$$

On the basis of transformation (6), the Lax pair (5) is then transformed into a desired form

$$
\begin{aligned}
J_x - i\lambda[\Lambda, J] &= U_1 J,\\
J_t + 8i\beta\lambda^4[\Lambda, J] &= U_2 J.
\end{aligned} \tag{7}
$$

Firstly, we construct two matrix jost solutions through direct scattering process on the x-part of the Lax pair (7),

$$
J_- = ([J_-]_1, [J_-]_2), \quad J_+ = ([J_+]_1, [J_+]_2). \tag{8}
$$

They meet the asymptotic conditions

$$
J_- \to I,\ x \to -\infty, \quad J_+ \to I,\ x \to +\infty,
$$

where each $[J_\pm]_l$ $(l = 1, 2)$ denotes the l-th column of the matrices $J_\pm$ respectively, I is a 2×2 identity matrix, and $J_\pm$ are unique solutions of the Volterra integral equations

$$
\begin{aligned}
J_-(x, \lambda) =& I + \int_{-\infty}^{x} e^{i\lambda(x-\xi)\Lambda} U_1(\xi; \lambda) J_-(\xi; \lambda) e^{-i\lambda(x-\xi)\Lambda} d\xi,\\
J_+(x, \lambda) =& I - \int_{x}^{+\infty} e^{i\lambda(x-\xi)\Lambda} U_1(\xi; \lambda) J_+(\xi; \lambda) e^{-i\lambda(x-\xi)\Lambda} d\xi.
\end{aligned} \tag{9}
$$

After a series of analysis, it can be seen that $[J_-]_1, [J_+]_2$ are analytic for $\lambda \in \mathbb{C}^+$ and continuous for $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, whereas $[J_+]_1, [J_-]_2$ are analytic for $\lambda \in \mathbb{C}^-$ and continuous for $\lambda \in \mathbb{C}^- \cup \mathbb{R}$, where

$$
\mathbb{C}^+ = \{\lambda | \arg\lambda \in (\pi, 2\pi)\}, \quad \mathbb{C}^- = \{\lambda | \arg\lambda \in (0, \pi)\}.
$$

Subsequently we investigate the properties of $J_\pm$. Resorting to the Abel's identity and $\mathrm{tr}U_1 = \mathrm{tr}U_2 = 0$ that the determinants of $J_\pm$ are independent of the variable x . Evaluating $\det J_-$ at $x = -\infty$ and $\det J_+$ at $x = +\infty$, we find $\det J_\pm = 1$ for $\lambda \in \mathbb{R}$. In addition, J_-E and J_+E are both fundamental matrix solutions of the original spectral problem of the first formula in Eq. (3), where $E = e^{i\lambda\Lambda x}$, they must be associated by a scattering matrix $S(\lambda) = (s_{ij})_{2\times 2}$

$$
J_-E = J_+E \cdot S(\lambda), \quad \lambda \in \mathbb{R}. \tag{10}
$$

$\mathrm{Det}S(\lambda) = 1$ is obtained directly from Eq. (10). Furthermore, we know s_{11} and s_{22} can be analytic extension to $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively according to the property of J_-.

In what follows, we shall formulate a RH problem by utilizing the analytic properties of the Jost solutions $J_\pm$. Two matrix functions related to the solutions (8) are reconstructed so that one is analytic in $\mathbb{C}^+$ and the other in $\mathbb{C}^-$. The first analytic function of λ in $\mathbb{C}^+$ is defined as the form

$$
P_1(x, \lambda) = ([J_-]_1, [J_+]_2)(x, \lambda). \tag{11}
$$

And then, one can expand P_1 into the asymptotic series at very large-λ

$$
P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\lambda} + O(\frac{1}{\lambda^2}), \quad \lambda \to \infty. \tag{12}
$$

After substituting expansion (12) into the spectral problem of the first formula in (7) and equating terms with same power of λ, we have

$$\begin{aligned} O(\lambda): &\quad -i[\Lambda, P_1^{(0)}] = 0, \\ O(1): &\quad P_{1x}^{(0)} - i[\Lambda, P_1^{(1)}] = U_1 P_1^{(0)}, \end{aligned} \tag{13}$$

which yields $P_1^{(0)} = I$, namely $P_1 \to I$ as $\lambda \in \mathbb{C}^+ \to \infty$.

An analytic counterpart of P_1 in $\mathbb{C}^-$ is still need to be constructed for the RH problem. Noting that the adjoint scattering equation [30] of the first formula in (7) is of the form

$$K_x = i\lambda[\Lambda, K] - KU_1. \tag{14}$$

Actually, we only need to consider the inverse matrices of $J_\pm$,

$$J_-^{-1} = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \end{pmatrix}, \quad J_+^{-1} = \begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}, \tag{15}$$

which obey the boundary conditions $J_\pm^{-1} \to I$ as $x \to \pm\infty$. Then through calculations we know that $J_\pm^{-1}$ are the solutions of adjoint equation (14). From (10) it is easy to see that

$$E^{-1}J_-^{-1} = R(\lambda) \cdot E^{-1}J_+^{-1}, \tag{16}$$

with $R(\lambda) = (r_{ij})_{2\times 2}$ as the inverse matrix of $S(\lambda)$. Consequently, the matrix function P_2 which is analytic in $\mathbb{C}^-$ can be defined as

$$P_2(x,\lambda) = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}(x,\lambda). \tag{17}$$

Analogously to the analysis of P_1, the very large-λ asymptotic behavior of P_2 comes out to be $P_2 \to I$ as $\lambda \in \mathbb{C}^- \to \infty$.

Inserting Jost solutions (8) into (10) gives rise to

$$([J_-]_1, [J_-]_2) = ([J_+]_1, [J_+]_2) \times \begin{pmatrix} s_{11} & s_{12}e^{2i\lambda x} \\ s_{21}e^{-2i\lambda x} & s_{22} \end{pmatrix}, \tag{18}$$

from which we derive

$$[J_-]_1 = s_{11}[J_+]_1 + s_{21}e^{-2i\lambda x}[J_+]_2. \tag{19}$$

Hence, P_1 can be eventually rewritten as the form

$$P_1 = ([J_-]_1, [J_+]_2) = ([J_+]_1, [J_+]_2)\begin{pmatrix} s_{11} & 0 \\ s_{21}e^{-2i\lambda x} & 1 \end{pmatrix}. \tag{20}$$

On the other hand, substituting (15) into (16), we acquire

$$\begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}e^{2i\lambda x} \\ r_{21}e^{-2i\lambda x} & r_{22} \end{pmatrix}\begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}, \tag{21}$$

from which we can denote $[J_-^{-1}]^1$ as

$$[J_-^{-1}]^1 = r_{11}[J_+^{-1}]^1 + r_{12}e^{2i\lambda x}[J_+^{-1}]^2. \tag{22}$$

Then P_2 is represented as

$$P_2 = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}e^{2i\lambda x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix}. \tag{23}$$

Up to now, the resulting functions P_1 and P_2 are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. After denoting that the limit of P_1 is P^+ when $\lambda \in \mathbb{C}^+ \to \mathbb{R}$ and the limit of P_2 is P^- when $\lambda \in \mathbb{C}^- \to \mathbb{R}$, based on which a RH problem would be set up as follows

$$P^-(x,\lambda)P^+(x,\lambda) = \begin{pmatrix} 1 & r_{12}e^{2i\lambda x} \\ s_{21}e^{-2i\lambda x} & 1 \end{pmatrix}, \tag{24}$$

whose canonical normalization conditions [31] are

$$\begin{aligned}
&P_1(x,\lambda) \to I, \quad \lambda \in \mathbb{C}^+ \to \infty,\\
&P_2(x,\lambda) \to I, \quad \lambda \in \mathbb{C}^- \to \infty,
\end{aligned} \tag{25}$$

and $r_{11}s_{11} + r_{12}s_{21} = 1$.

3. General formula of the N-soliton solutions

In this section, we construct the N-soliton solutions to the LPD equation (2) based on the RH problem. For this purpose, we suppose the RH problem is irregular which means both $\det P_1$ and $\det P_2$ own zeros in their analytic domains. Recalling the definitions of P_1 and P_2, we have

$$\begin{aligned}
&\det P_1 = s_{11}(\lambda), \quad \lambda \in \mathbb{C}^+,\\
&\det P_2 = r_{11}(\lambda), \quad \lambda \in \mathbb{C}^-,
\end{aligned} \tag{26}$$

which immediately enables us to know that $\det P_1$ and $\det P_2$ possess the same zeros as s_{11} and r_{11} respectively.

Utilizing the resulting analysis above, we now discuss the characteristics of zeros. We note the potential matrix U_1 has the symmetry relation

$$U_1^\dagger = -U_1,$$

here † stands for Hermitian of a matrix. On basis of this relation, we deduce

$$J_\pm^\dagger(\lambda^*) = J_\pm^{-1}(\lambda). \tag{27}$$

Introducing two matrices H_1=diag(1; 0) and H_2=diag(0; 1) allows the expressions (20) and (23) to be reformulated as

$$P_1 = J_- H_1 + J_+ H_2, \quad P_2 = H_1 J_-^{-1} + H_2 J_+^{-1}. \tag{28}$$

Performing the Hermitian of the first formula in (28) and making use of the relation (27), we have

$$P_1^\dagger(\lambda^*) = P_2(\lambda), \quad S^\dagger(\lambda^*) = R(\lambda), \tag{29}$$

for $\lambda \in \mathbb{C}^-$. By means of the second equation of (29), we further have $s_{11}^*(\lambda^*) = r_{11}(\lambda)$. It implies that each zero λ_k of s_{11} results in each zero λ_k^* of r_{11} correspondingly. Therefore, our assumption in the general case is that, $\det P_1$ has N simple zeros $\{\lambda_j\}_1^N$ in $\mathbb{C}^+$ and $\det P_2$ has N simple zeros$\{\hat{\lambda}_j\}_1^N$ in $\mathbb{C}^-$, where $\hat{\lambda}_j = \lambda_j^*$, $1 \le j \le N$. Each of $\ker P_1(\lambda_j)$ and $\ker P_2(\hat{\lambda}_j)$ contain only a single basis column vector v_j and vector $\hat{v}_j$ respectively,

$$P_1(\lambda_j)v_j = 0, \quad \hat{v}_j P_2(\hat{\lambda}_j) = 0. \tag{30}$$

Taking the Hermitian of the first formula in (30) and using (29) as well as comparing with the second formula in (30), we find that the eigenvectors fulfill the relation

$$\hat{v}_j = v_j^\dagger, \quad 1 \le j \le N. \tag{31}$$

Differentiating the first formula in (30) about x and t respectively and taking advantage of the Lax pair (7), we have the following relationships

$$\begin{aligned}
&P_1(\lambda_j)(\frac{\partial v_j}{\partial x} - i\lambda_j \Lambda x) = 0,\\
&P_1(\lambda_j)(\frac{\partial v_j}{\partial t} + 8i\beta\lambda_j^4 \Lambda t) = 0,
\end{aligned} \tag{32}$$

which results in

$$v_j = \mathrm{e}^{i\lambda_j \Lambda x - 8i\beta\lambda_j^4 \Lambda t} v_{j,0}, \quad 1 \le j \le N. \tag{33}$$

Here $\nu_{j,0}$ is independent of the variables x and t. By means of the relation (31), we thus derive

$$\hat{\nu}_j = \nu_{j,0}^{\dagger}\mathrm{e}^{-i\lambda_j^*\Lambda x+8i\beta\lambda_j^{*4}\Lambda t}, \quad 1 \leq j \leq N. \tag{34}$$

In order to obtain soliton solutions for the LPD equation (2), we choose the jump matrix as the 2×2 identity matrix. This can be achieved by supposing the vanishing coefficient $s_{21} = r_{12} = 0$, which corresponds to the reflectionless case. Consequently, the expression of the unique solution for this particular RH problem can be described

$$\begin{aligned} P_1(\lambda) &= I - \sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\nu_k\hat{\nu}_j(M^{-1})_{kj}}{\lambda-\hat{\lambda}_j}, \\ P_2(\lambda) &= I + \sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\nu_k\hat{\nu}_j(M^{-1})_{kj}}{\lambda-\lambda_k}, \end{aligned} \tag{35}$$

in which M is a $N\times N$ matrix defined by

$$M = (M_{kj})_{N\times N} = (\frac{\hat{\nu}_k\nu_j}{\lambda_j-\hat{\lambda}_k})_{N\times N}, \quad 1 \leq k, j \leq N,$$

and $(M^{-1})_{kj}$ represents the (k, j)-entry of the invere matrix M.

In the following, we are going to retrieve the potential function r. Expanding $P_1(\lambda)$ at large-λ as

$$P_1 = I + \frac{P_1^{(1)}}{\lambda} + O(\frac{1}{\lambda^2}), \quad \lambda \to \infty, \tag{36}$$

and carrying it into the first formula in (7) generates the reconstructed potential function [32]

$$r = -2(p_1^{(1)})_{12}, \tag{37}$$

where $(p_1^{(1)})_{12}$ is the $(1, 2)$-entry of matrix $P_1^{(1)}$. From expression of the first formula in (35), the $P_1^{(1)}$ can be evidently obtained

$$P_1^{(1)} = -\sum_{k=1}^{N}\sum_{j=1}^{N}\nu_k\hat{\nu}_j(M^{-1})_{kj}. \tag{38}$$

For the purpose of obtaining explicit multi-soliton solutions to the LPD equation (2), we set the assumption $\nu_{j,0} = (\alpha_j, \beta_j)^T$ and $\theta_j = i\lambda_j x - 8i\beta\lambda_j^4 t$, $\mathrm{Im}\lambda_j < 0$, $1 \leq j \leq N$, here $\alpha_j, \beta_j \in \mathbb{C}$ are elements of $\nu_{j,0}$. On the basis of above results, the formula of N-soliton solutions to the LPD equation (2) can be ultimately obtained as follows:

$$r(x,t) = 2\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\beta_j^*\mathrm{e}^{\theta_k-\theta_j^*}(M^{-1})_{kj}, \tag{39}$$

where

$$M_{kj} = \frac{\alpha_k^*\alpha_j\mathrm{e}^{\theta_k^*+\theta_j} + \beta_k^*\beta_j\mathrm{e}^{-\theta_k^*-\theta_j}}{\lambda_j-\lambda_k^*}, \quad 1 \leq k, j \leq N.$$

Following the same argument as that in Ref. [29], the solutions (39) can be rewritten in a more elegant form

$$r(x,t) = -2\frac{\det F}{\det M}, \tag{40}$$

where F is a $(N + 1)\times(N + 1)$ matrix given by

$$F = \begin{pmatrix} 0 & \beta_1\mathrm{e}^{-\theta_1} & \dots & \beta_N\mathrm{e}^{-\theta_N} \\ \alpha_1^*\mathrm{e}^{\theta_1^*} & M_{11} & \dots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N^*\mathrm{e}^{\theta_N^*} & M_{N1} & \dots & M_{N1} \end{pmatrix}$$

4. Dynamic behaviors of explicit breathers and soliton solutions

In the following, we focus on obtaining explicit breathers and soliton solutions and further examine their dynamic behaviors by various graphs.

In the case of $N = 1$, 1-soliton solution is constructed from N-soliton solution (39) as the following form

$$r_1(x,t) = 2\alpha_1\beta_1^* e^{\theta_1-\theta_1^*} \frac{\lambda_1 - \lambda_1^*}{|\alpha_1|^2 e^{\theta_1^*+\theta_1} + |\beta_1|^2 e^{-\theta_1^*-\theta_1}}, \tag{41}$$

where $\theta_1 = i\lambda_1 x - 8i\beta\lambda_1^4 t$. In addition, by choosing $\alpha_1 = 1$ and letting $\lambda_1 = \xi_1 + i\eta_1$ as well as $|\beta_1|^2 = e^{2\tau_1}$, the solution (41) is further converted into the brief form:

$$r_1(x,t) = 2i\beta_1^*\eta_1 e^{2iY} e^{-\tau_1}\operatorname{sech}(2X - \tau_1), \tag{42}$$

here

$$\begin{aligned} X =& -\eta_1 x + 32\beta\xi_1\eta_1(\xi_1^2 - \eta_1^2)t, \\ Y =& \xi_1 x - 8\beta(\xi_1^4 - 6\xi_1^2\eta_1^2 + \eta_1^4)t. \end{aligned}$$

The solution (42) is demonstrating as the form of a hyperbolic secant function with maximum amplitude $H = 2|\beta_1^*\eta_1|e^{-\tau_1}$ and velocity $V = 32\beta\xi_1(\xi_1^2 - \eta_1^2)$. When the parameters are properly selected, the real and imaginary parts of the solution (42) exhibit periodicity and local oscillation behaviors and the modulus of the solution degenerate to exact 1-soliton soluiton which can be shown in Fig 1. As can be seen from Fig. 1 (d), (e) and (f), the breather solutions propagate periodically along the straight line $l: \ -\eta_1 x + 32\beta\xi_1\eta_1(\xi_1^2 - \eta_1^2)t = 0$ with period $T_l = \frac{\pi}{\sqrt{\xi_1^2 + 64\beta^2(\xi_1^4 - 6\xi_1^2\eta_1^2 + \eta_1^4)^2}}$ from the positive to the negative direction corresponding to x-axis. On the contrary, 1-soliton solution propagates from the left to the right hand side along x-axis.

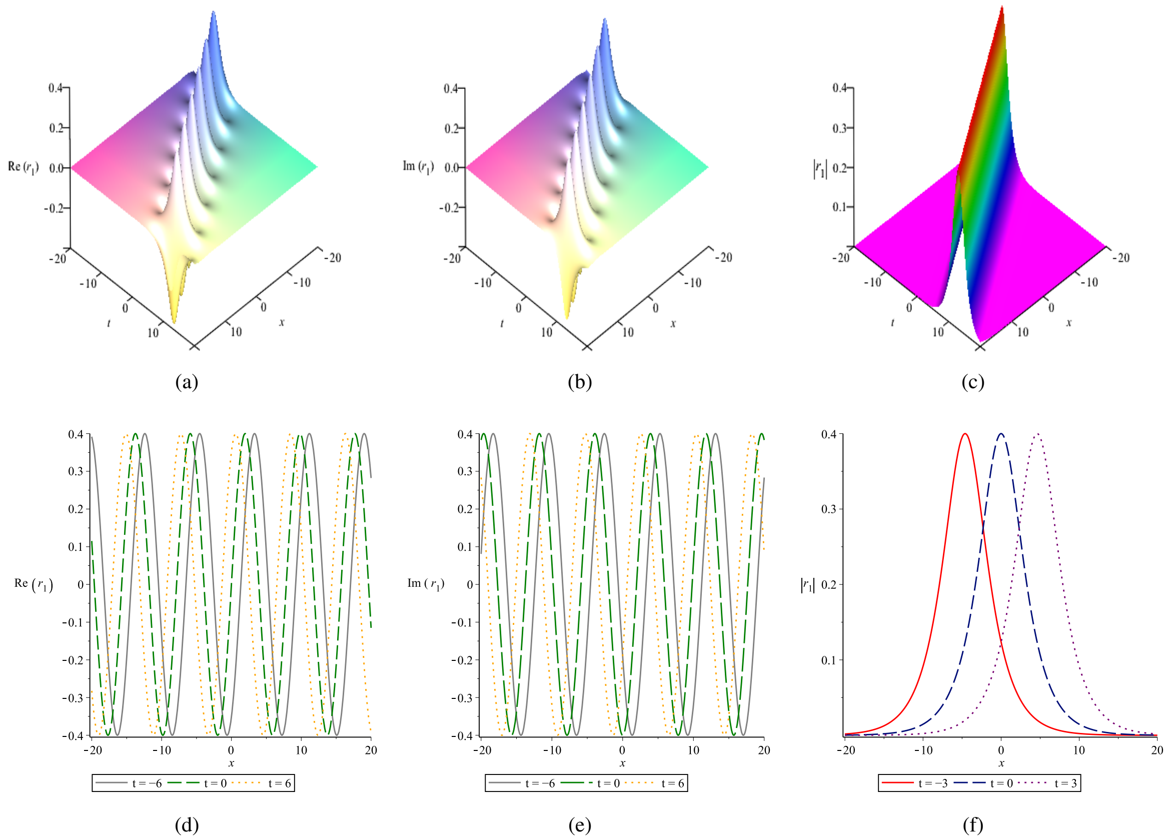

Figure 1: Two types of solutions of $r_1(x,t)$ in (42) with the parameters chosen as $\alpha_1 = \beta_1 = 1,\ \xi_1 = 2/5,\ \eta_1 = -1/5,\ \beta = 1,\ \tau_1 = 0$; (a), (b): 1-order breathers corresponding to real and imaginary part of $r_1(x,t)$; (c): The spatial structure of $|\ r_1(x,t)\ |$; (d), (e) and (f) are the propagation patterns of the solutions along the x-axis at different times in Fig. 1 (a), (b) and (c) respectively.

In the case of $N = 2$, the 2-soliton solution for Eq. (2) is generated as

$$r_2(x,t) = \frac{2}{M_{11}M_{22} - M_{12}M_{21}}(\alpha_1\beta_1^* M_{22} e^{\theta_1-\theta_1^*} - \alpha_1\beta_2^* M_{12} e^{\theta_1-\theta_2^*} - \alpha_2\beta_1^* M_{21} e^{\theta_2-\theta_1^*} + \alpha_2\beta_2^* M_{11} e^{\theta_2-\theta_2^*}), \tag{43}$$

where

$$M_{11} = \frac{|\alpha_1|^2 e^{\theta_1^*+\theta_1} + |\beta_1|^2 e^{-\theta_1^*-\theta_1}}{\lambda_1 - \lambda_1^*}, \quad M_{12} = \frac{\alpha_1^*\alpha_2 e^{\theta_1^*+\theta_2} + \beta_1^*\beta_2 e^{-\theta_1^*-\theta_2}}{\lambda_2 - \lambda_1^*},$$
$$M_{21} = \frac{\alpha_2^*\alpha_1 e^{\theta_2^*+\theta_1} + \beta_2^*\beta_1 e^{-\theta_2^*-\theta_1}}{\lambda_1 - \lambda_2^*}, \quad M_{22} = \frac{|\alpha_2|^2 e^{\theta_2^*+\theta_2} + |\beta_2|^2 e^{-\theta_2^*-\theta_2}}{\lambda_2 - \lambda_2^*},$$

and

$$\theta_1 = i\lambda_1 x - 8i\beta\lambda_1^4 t,\ \theta_2 = i\lambda_2 x - 8i\beta\lambda_2^4 t,\ \lambda_1 = \xi_1 + i\eta_1,\ \lambda_2 = \xi_2 + i\eta_2.$$

After assuming that $\alpha_1 = \alpha_2 = 1,\ \beta_1 = \beta_2$ and $|\beta_1|^2 = e^{2\tau_1}$, the 2-soliton solution (43) becomes

$$r_2(x,t) = \frac{2\beta_1^*}{M_{11}M_{22} - M_{12}M_{21}}(e^{\theta_1-\theta_1^*}M_{22} - e^{\theta_1-\theta_2^*}M_{12} - e^{\theta_2-\theta_1^*}M_{21} + e^{\theta_2-\theta_2^*}M_{11}), \tag{44}$$

where

$$M_{11} = -\frac{ie^{\tau_1}}{\eta_1}\cosh(\theta_1^* + \theta_1 - \tau_1), \quad M_{12} = \frac{2e^{\tau_1}}{(\xi_2 - \xi_1) + i(\eta_1 + \eta_2)}\cosh(\theta_1^* + \theta_2 - \tau_1),$$
$$M_{22} = -\frac{ie^{\tau_1}}{\eta_2}\cosh(\theta_2^* + \theta_2 - \tau_1), \quad M_{21} = \frac{2e^{\tau_1}}{(\xi_1 - \xi_2) + i(\eta_1 + \eta_2)}\cosh(\theta_2^* + \theta_1 - \tau_1).$$

The spatial structures and dynamical behaviors of solution (44) are presented in Fig. 2 by choosing a set of appropriate parameters. Intersectant pattern breather solutions of the real and imaginary parts of $r_2(x,t)$ can be seen clearly. One of the breathers propagates along the straight line $l_1 : -\eta_1 x + 32\beta\xi_1\eta_1(\xi_1^2 - \eta_1^2)t = 0$ with constant velocity $V_1 = 32\beta\xi_1(\xi_1^2 - \eta_1^2)$ and period $T_{l_1} = \frac{\pi}{\sqrt{\xi_1^2+64\beta^2(\xi_1^4-6\xi_1^2\eta_1^2+\eta_1^4)^2}}$. Meanwhile the other propagates along the straight line $l_2 : -\eta_2 x+32\beta\xi_2\eta_2(\xi_2^2-\eta_2^2)t = 0$ with constant velocity $V_2 = 32\beta\xi_2(\xi_2^2-\eta_2^2)$ and period $T_{l_2} = \frac{\pi}{\sqrt{\xi_2^2+64\beta^2(\xi_2^4-6\xi_2^2\eta_2^2+\eta_2^4)^2}}$. But the modulus of $r_2(x,t)$ degenerate into 2-soliton solutions which interact with each other at origin of the coordinates and reaches the maximum amplitude 1. The general 2-soliton describes an elastic collision among two fundamental solitons and is bounded for all the time, as observed in many physical phenomena.

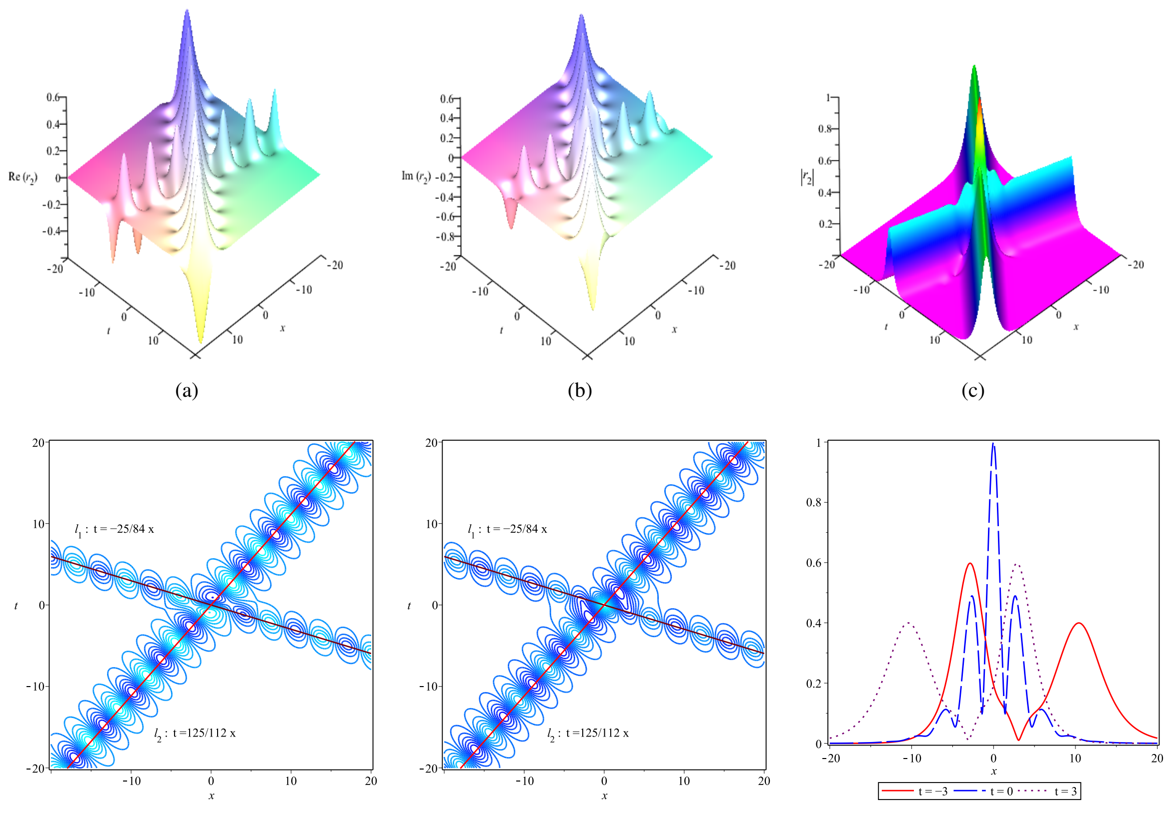

Figure 2: Two types of intersectant pattern solutions of $r_2(x,t)$ in (44) with the parameters chosen as $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1,\ \xi_1 = -1/2,\ \eta_1 = -1/5,\ \xi_2 = 2/5,\ \eta_2 = -3/10,\ \beta = 1,\ \tau_1 = 0$; (a), (b): Breathers of real and imaginary part of $r_2(x,t)$; (c): 2-soliton solutions corresponding to $|\ r_2(x,t)\ |$; (d), (e): Contour plots and propagation orbit of Fig. 2(a) and (b); (f): Sectional view of 2-soliton along the x-axis at different times in Fig. 2(c).

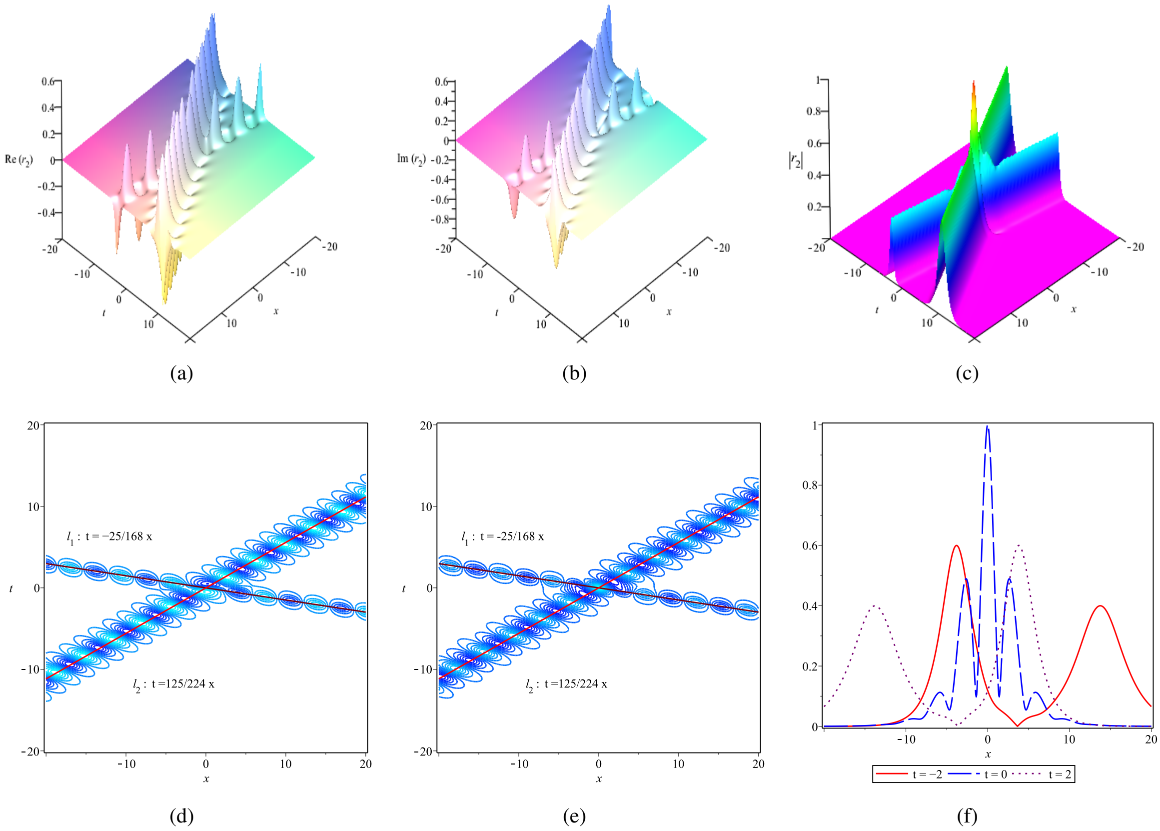

Figure 3: The parameters are the same as Fig. 2 only β is changed to 2.

Subsequently, the influence of higher-order linear and nonlinear effect r on the dynamics would be analyzed. Comparing Fig. 3 with Fig. 2, with the increasing of β from 1 to 2 under other parameters unchanged, the periods of the two second-order breathers decrease, the velocities increase and the phases change, but the amplitudes keep unchanged. Furthermore, for the 2-order breathers and 2-soliton solutions, the angle between two single one also decrease from $arctan(13300/6283)$ to $arctan(19950/28849)$.

When $N = 3$, analogously to the procedures above, the formula of $r(x,t)$ is presented

$$\begin{aligned} r_3(x,t) =& \frac{2}{\det(M)}\Big[\alpha_1\beta_1^* e^{\theta_1-\theta_1^*}(M^{-1})_{11} + \alpha_1\beta_2^* e^{\theta_1-\theta_2^*}(M^{-1})_{12} + \alpha_1\beta_3^* e^{\theta_1-\theta_3^*}(M^{-1})_{13} \\ &+ \alpha_2\beta_1^* e^{\theta_2-\theta_1^*}(M^{-1})_{21} + \alpha_2\beta_2^* e^{\theta_2-\theta_2^*}(M^{-1})_{22} + \alpha_2\beta_3^* e^{\theta_2-\theta_3^*}(M^{-1})_{23} \\ &+ \alpha_3\beta_1^* e^{\theta_3-\theta_1^*}(M^{-1})_{31} + \alpha_3\beta_2^* e^{\theta_3-\theta_2^*}(M^{-1})_{32} + \alpha_3\beta_3^* e^{\theta_3-\theta_3^*}(M^{-1})_{33}\Big], \end{aligned} \tag{45}$$

here $M = (M_{kj})_{3\times3}$ is a 3×3 matrix function. We omit the explicit expressions of elements M_{kj} and $(M^{-1})_{kj}$ $(1 \le k, j \le 3)$ due to the limited space. When the appropriate parameters are selected, 3-order breather solutions corresponding to the real and imaginary parts as well as 3-soliton solutions for modulus of the $r_3(x,t)$ are similarly obtained and their dynamic behaviors are vividly shown in Fig. 4. Obviously distinct to the case in Fig. 2, they do not propagate along the original trajectories after the collisions between them.

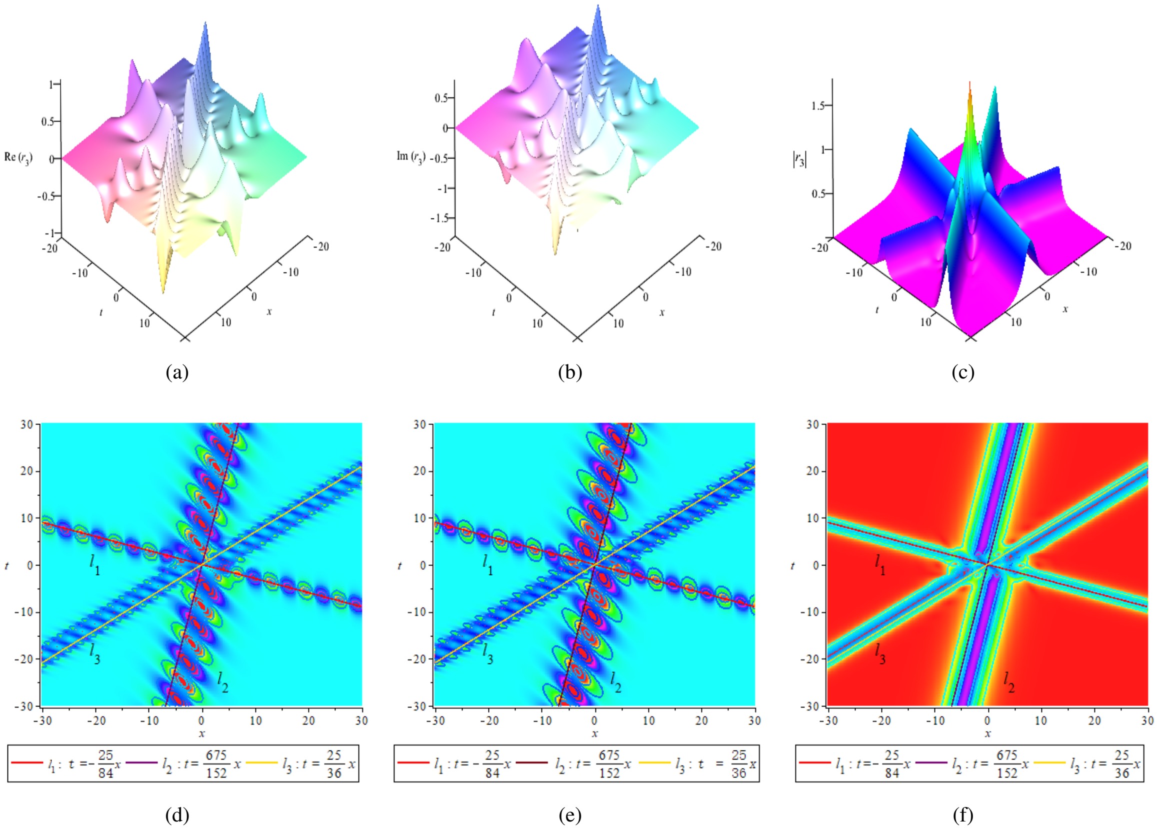

Figure 4: Abundant graphs of $r_3(x,t)$ in (45) with choices of parameters as $\alpha_j = \beta_j = 1,\ 1 \le j \le 3,\ \xi_1 = -1/2,\ \eta_1 = -1/5,\ \xi_2 = 1/3,\ \eta_2 = -3/10,\ \xi_3 = 1/2,\ \eta_3 = -2/5,\ \beta = 1,\ \tau_1 = 0$; (a), (b): 3-order breather solutions and propagation orbits of real part and imaginary part of $r_3(x,t)$; (c): 3-soliton solutions corresponding to $|\ r_3(x,t)\ |$; (d), (e) and (f) are corresponding hybrid graphs containing contours, density and propagation orbits.

5. Conclusions

In this investigation, we have studied the LPD equation in an optical fiber associated with a 2×2 Lax pair via the RH approach. Depending on the spectral analysis of the Lax pair, then a matrix RH problem about the real spectral parameter λ is formulated. By solving the particular Riemann–Hilbert problems with vanishing scattering coefficients, which correspond to the reflectionless cases, the general N-soliton solutions are obtained for Eq. (2).

Furthermore, as a result, 1-, 2- and 3-order breather solutions corresponding to the real and imaginary parts of the general N-soliton solutions $r(x,t)$ can be acquired and the dynamic characteristics are analyzed graphically in resulting graphs. When we take the modulus of the solutions, the breather solutions degenerate into soliton solutions immediately and their dynamic behaviors and perspective views are vividly shown in different types of graphs. The results derived are novel and may provide a feasible way in applications in the communication of nonlinear optical fiber.

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