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A frictional contact problem for piezoelectric materials with internal state variables and normal damped response

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We consider a dynamic frictional contact problem between two electro-elasto-viscoplastic bodies with internal state variables and damage. The contact is frictional, modelled with a normal damped condition involving adhesion effect of contact surfaces. We present a variational formulation for the model and state an existence and uniqueness result of the weak solution. The proof is based on arguments of time dependent variational inequalities, differential equations and fixed point.

KEYWORDS:
Electro-elasto-viscoplastic materials, internal state variable, dynamic process, frictional contact, normal damped response, adhesion, damage, weak solution

1 | INTRODUCTION

The Mathematical Theory of Contact Mechanics is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, different contact conditions and various geometries. The aim of this theory is to predict reliably the evolution of contact processes and provide a rigorous mathematical background for the construction of models for contact phenomena. Contact problems with additional effects are of the special importance in applications. In particular, the effects due to the damage may lead to decrease the carrying capacity of the bodies in contact. The effective functioning and safety of a mechanical system may be deteriorated by this decrease as the material undergoes damage. The damage is an extremely important topic in engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [10] from the virtual power principle. The new idea of [11] was the introduction of the *damage function* $\xi = \xi(x, t)$, which is the ratio between the elastic modulus of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y be the Young modulus of the original material and E_{eff} be the current modulus, then the damage function is defined by $\xi = E_{eff}/E_Y$. Clearly, it follows from this definition that the damage function ξ is restricted to have values between zero and one. When $\xi = 1$, there is no damage in the material, when $\xi = 0$, the material is completely damaged, when $0 < \xi < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [6,9,21,22]. In this paper, we investigate the dynamic process of frictional adhesive contact with normal damped response between two electro-elasto-viscoplastic deformable bodies.

The model we suppose the following constitutive law

$$\sigma^m = \mathcal{A}^m \varepsilon(u^m) + \mathcal{B}^m \varepsilon(\dot{u}^m) - (\mathcal{E}^m)^* E(\varphi^m) + \int_0^t \mathcal{Q}^m \Big(\sigma^m(s) - \mathcal{B}^m \varepsilon(\dot{u}^m(s)) + (\mathcal{E}^m)^* E(\varphi^m(s)), \varepsilon(u^m(s)), \xi^m(s), k^m(s) \Big)\, ds, \tag{1}$$

$$\dot{k}^m = \phi^m\big(\sigma^m - \mathcal{B}^m\varepsilon(\dot{u}^m) + (\mathcal{E}^m)^* E(\varphi^m), \varepsilon(u^m), k^m\big), \tag{2}$$

$$D^m = \mathcal{E}^m \varepsilon(u^m) + \beta^m E(\varphi^m), \tag{3}$$

where σ^m denotes the stress tensor, u^m represents the displacement field, $\varepsilon(u^m)$ is the linearized strain tensor, k^m denotes an internal state variable, ξ^m is the damage field and D^m is the electric displacement field. Here $\mathcal{A}^m$ and $\mathcal{B}^m$ are nonlinear operators describing the purely elastic and the viscous properties of the material, respectively. $\mathcal{Q}^m$ is a nonlinear constitutive function describing the plastic behaviour of the material. We also consider that the plastic function $\mathcal{Q}^m$ depends on the internal state variable k^m. ϕ^m is also a nonlinear constitutive function which depends on k^m. There is a variety of choices for the internal state variables, for reference in the field see[5,7]. Some commonly used internal state variables are the plastic strain and a number of tensor variables that take into account the spatial display of dislocations and the work-hardening of the material. $E(\varphi^m)$ is the electric field that satisfies $E(\varphi^m) = -\nabla\varphi^m$, where φ^m is the electric potential. Also, $\mathcal{E}^m$ represents the third order piezoelectric tensor, $(\mathcal{E}^m)^*$ is its transposition and β^m denotes the electric permittivity tensor.

It follows from (1) that at each time moment, the stress tensor σ^m is split into three parts: $\sigma^m = \sigma^m_V + \sigma^m_E + \sigma^m_R$, where $\sigma^m_V = \mathcal{B}^m\varepsilon(\dot{u}^m)$ represents the purely viscous part of the stress, $\sigma^m_E = -(\mathcal{E}^m)^* E(\varphi^m)$ represents the electric part of the stress and σ^m_R is the elasto-viscoplastic part of the stress which satisfies

$$\sigma^m_R = \mathcal{A}^m \varepsilon(u^m) + \int_0^t \mathcal{Q}^m\big(\sigma^m_R(s), \varepsilon(u^m(s)), \xi^m(s), k^m(s)\big)\, ds. \tag{4}$$

Note also that when $\mathcal{Q}^m = 0$ the constitutive law (1) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation,

$$\sigma^m = \mathcal{B}^m\varepsilon(\dot{u}^m) + \mathcal{A}^m\varepsilon(u^m) - (\mathcal{E}^m)^* E(\varphi^m). \tag{5}$$

Dynamic contact problems with Kelvin-Voigt materials of the form (5) can be found in[2,15]. Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. General models with adhesion can be found in[8]. Results on the mathematical analysis of various adhesive contact problems can be found in[3,4,12,16,20,21]. In all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by ς, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following[13], the bonding field satisfies the restriction $0 \le \varsigma \le 1$, when $\varsigma = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\varsigma = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \varsigma < 1$ the adhesion is partial and only a fraction ς of the bonds is active. We turn now to describe the contact conditions. We assume that the normal stress σ_ν satisfies a general normal damped response condition with adhesion

$$\sigma_\nu = -p_\nu(\dot{u}^1_\nu + \dot{u}^2_\nu) + \gamma_\nu \varsigma^2 R_\nu(u^1_\nu + u^2_\nu), \tag{6}$$

where p_ν is a prescribed function and γ_ν is a given adhesion coefficient. Equality (6) states a general dependence of the normal stress on the normal velocity. A commonly used example of the normal damped function p_ν, is

$$p_\nu(r) = \alpha r, \tag{7}$$

with $\alpha \ge 0$. This type of behavior was considered in[19] modeling the motion of a deformable bodies on sand or a granular materials. We may also consider the case (see[18])

$$p_\nu(r) = \alpha r_+ + p_0, \tag{8}$$

where $\alpha \ge 0$ and $p_0 > 0$. Here α is the damping resistance coefficient, assumed positive, $r_+ = \max(0, r)$ and p_0 is the oil pressure, which is given and nonnegative. .

The associated friction law is chosen as

$$\sigma_\tau = -p_\tau(\dot{u}^1_\tau - \dot{u}^2_\tau) + q_\tau(\varsigma) R_\tau(u^1_\tau - u^2_\tau), \tag{9}$$

where p_τ is a prescribed vector-valued function, q_τ is a given positive function and $\boldsymbol{\sigma}_\tau$ represents the tangential force on contact boundary. As an example we may consider the function

$$p_\tau(\mathbf{r}) = \mu|\mathbf{r}|^{\beta-1}\mathbf{r}, \tag{10}$$

where μ represents the coefficient of friction, assumed positive, and $0 < \beta \leq 1$. This is the case when the contact surface is lubricated with a thin layer of a non-Newtonian fluid, see e.g.[22].

The paper is organized as follows. In Section 2 we describe the mathematical models for the frictional contact problem between two electro-elasto-viscoplastic bodies with internal state variables and damage. The contact is modelled with normal damped and adhesion. In Section 3 We introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model, and state an existence and uniqueness result stated in Theorem 1. We prove in Section 4 the existence and uniqueness of the solution, where it is carried out in several steps and is based on arguments of time-dependent parabolic variational inequalities, differential equations and fixed point.

2 | PROBLEM STATEMENT

We consider two electro-elastic-viscoplastics bodies whose material particles occupy bounded domains Ω^1 and Ω^2 of $\mathbb{R}^d$ ($d \leq 3$ in applications). We put a superscript m to indicate that the quantity is related to the domain Ω^m. In the following, the superscript m ranges between 1 and 2. For each domain Ω^m, the boundary Γ^m is assumed to be Lipschitz continuous. We use the notation $x = (x_i)$ for a typical point in Ω^m and we denote by $\nu^m = (\nu_i^m)$ the outward unit normal at Γ^m. For each domain Ω^m, the boundary Γ^m is partitioned into three disjoint measurable parts Γ_1^m, Γ_2^m and Γ_3^m, on one hand, and on two measurable parts Γ_a^m and Γ_b^m, on the other hand, such that $meas\Gamma_1^m > 0$, $meas\Gamma_a^m > 0$. We are interested in the quasistatic process of evolution of the bodies on the time interval $[0,T]$, with $T > 0$. The Ω^m body is submitted to $\boldsymbol{f}_0^m$ forces and volume electric charges of density q_0^m. The bodies are assumed to be clamped on $\Gamma_1^m \times [0,T]$. The surface tractions $\boldsymbol{f}_2^m$ act on $\Gamma_2^m \times [0,T]$. The two bodies are in contact along the common part $\Gamma_3^1 = \Gamma_3^2$, which will be denoted Γ_3 below. The bodies are in adhesive contact with normal damped response condition. We also assume that the electrical potential vanishes on $\Gamma_a^m \times [0,T]$ and a surface electric charge of density q_2^m is prescribed on $\Gamma_b^m \times [0,T]$. Then, the classical formulation of the mechanical frictional contact problem with damped, internal state variable, adhesion and damage between two electro-elastic-viscoplastics bodies is the following.

Problem P. For $m = 1, 2$, find a displacement field $\boldsymbol{u}^m : \Omega^m \times [0,T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^m : \Omega^m \times [0,T] \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^m : \Omega^m \times [0,T] \longrightarrow \mathbb{R}$, a damage field $\xi^m : \Omega^m \times [0,T] \longrightarrow \mathbb{R}$, a bonding field $\varsigma : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}$, an internal state variable field $\boldsymbol{k}^m : \Omega^m \times [0,T] \longrightarrow \mathbb{R}^\ell$ and a electric displacement field $\boldsymbol{D}^m : \Omega^m \times [0,T] \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}&\boldsymbol{\sigma}^m = \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) + \mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}^m) + (\mathcal{E}^m)^*\nabla\varphi^m + \\ &\int_0^t \mathcal{Q}^m\Big(\boldsymbol{\sigma}^m(s) - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m(s)) - (\mathcal{E}^m)^*\nabla\varphi^m(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m(s)), \xi^m(s), \boldsymbol{k}^m(s)\Big)\,ds\end{aligned} \quad \text{in } \Omega^m \times (0,T), \tag{11}$$

$$\dot{\boldsymbol{k}}^m = \phi^m\big(\boldsymbol{\sigma}^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) - (\mathcal{E}^m)^*\nabla\varphi^m, \boldsymbol{\varepsilon}(\boldsymbol{u}^m), \boldsymbol{k}^m\big) \quad \text{in } \Omega^m \times (0,T), \tag{12}$$

$$\boldsymbol{D}^m = \mathcal{E}^m\varepsilon(\boldsymbol{u}^m) - \boldsymbol{\beta}^m\nabla\varphi^m \quad \text{in } \Omega^m \times (0,T), \tag{13}$$

$$\dot{\xi}^m - \kappa^m\Delta\xi^m + \partial\psi_{K^m}(\xi^m) \ni \Psi^m\big(\boldsymbol{\sigma}^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) - (\mathcal{E}^m)^*\nabla\varphi^m, \boldsymbol{\varepsilon}(\boldsymbol{u}^m), \xi^m\big) \quad \text{in } \Omega^m \times (0,T), \tag{14}$$

$$\operatorname{Div}\boldsymbol{\sigma}^m + \boldsymbol{f}_0^m = \rho^m\ddot{\boldsymbol{u}}^m \quad \text{in } \Omega^m \times (0,T), \tag{15}$$

$$\operatorname{div}\boldsymbol{D}^m - q_0^m = 0 \quad \text{in } \Omega^m \times (0,T), \tag{16}$$

$$\boldsymbol{u}^m = 0 \quad \text{on } \Gamma_1^m \times (0,T), \tag{17}$$

$$\boldsymbol{\sigma}^m\boldsymbol{\nu}^m = \boldsymbol{f}_2^m \quad \text{on } \Gamma_2^m \times (0,T), \tag{18}$$

$$\begin{cases}\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu(\dot{u}_\nu^1 + \dot{u}_\nu^2) + \gamma_\nu\varsigma^2 R_\nu(u_\nu^1 + u_\nu^2)\end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{19}$$

$$\begin{cases}\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = -p_\tau(\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2) + q_\tau(\varsigma)\boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)\end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{20}$$

$$\dot{\varsigma} = H_{ad}(\varsigma, R_\nu(u_\nu^1 + u_\nu^2), \boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)) \quad \text{on } \Gamma_3 \times (0,T), \tag{21}$$

$$\frac{\partial \xi^m}{\partial \nu^m} = 0 \quad \text{on } \Gamma^m \times (0,T), \tag{22}$$

$$\varphi^m = 0 \quad \text{on } \Gamma_a^m \times (0,T), \tag{23}$$

$$\boldsymbol{D}^m.\boldsymbol{\nu}^m = q_2^m \quad \text{on } \Gamma_b^m \times (0,T), \tag{24}$$

$$\boldsymbol{u}^m(0) = \boldsymbol{u}_0^m,\ \dot{\boldsymbol{u}}^m(0) = \boldsymbol{v}_0^m,\ \xi^m(0) = \xi_0^m,\ \boldsymbol{k}^m(0) = \boldsymbol{k}_0^m \quad \text{in } \Omega^m, \tag{25}$$

$$\varsigma(0) = \varsigma_0 \quad \text{on } \Gamma_3. \tag{26}$$

First, (11), (12) and (13) represent the electro-elastic-viscoplastic constitutive law with internal state variable and damage. Inclusion (14) describes the evolution of the damage field, where K^m denotes the set of admissible damage functions defined by

$$K^m = \{\zeta \in H^1(\Omega^m);\ 0 \le \zeta \le 1,\ \text{a.e. in } \Omega^m\}, \tag{27}$$

κ^m is a positive coefficient, $\partial\psi_{K^m}$ represents the subdifferential of the indicator function of the set K^m and Ψ^m is a given constitutive function which describes the sources of the damage in the system. Equations (15) and (16) are the equations of motion written for the stress field and of balance written for the electric displacement field, respectively, ρ^m denotes the mass density. Next, (17) and (18) are the displacement and traction boundary condition, respectively. Condition (19) represents the normal damped response with adhesion, in this condition the interpenetrability between two bodies is allowed, that is $\varepsilon u_\nu^1 + u_\nu^2 \varepsilon$ can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \varsigma^2 R_\nu(u_\nu^1 + u_\nu^2)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu \varsigma^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (20) represents the adhesive contact condition on the tangential plane, where $\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (21) represents the ordinary differential equation which describes the evolution of the bonding field, where H_{ad} is the adhesion evolution rate function and it was already used in[4,3], see also[21] for more details. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (21), $\dot{\varsigma} \le 0$. Boundary condition (22) describes a homogeneous Neumann boundary condition where $\frac{\partial \xi^m}{\partial \nu^m}$ is the normal derivative of ξ^m. (23) and (24) represent the electric boundary conditions. (25) represents the initial displacement field, the initial velocity , the initial internal state variable and the initial damage. Finally, (26) represents the initial condition in which ς_0 is the given initial bonding field.

3 | VARIATIONAL FORMULATIONS AND MAIN RESULT

In this section we list the assumptions on the data, derive a variational formulation for the contact problem (11)–(26) and state our main existence and uniqueness result, Theorem 1. To this end we need to introduce notation and preliminary material.

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$, we recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}.\boldsymbol{v} = u_i.v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v}.\boldsymbol{v})^{\frac{1}{2}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}.\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below the indices i and j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable x, for example $u_{i,j} = \partial u_i / \partial x_j$.

Everywhere below we use the classical notation for L^p and Sobolev spaces associated to Ω^m and Γ^m. Moreover, we use the notation

$$H^m = L^2(\Omega^m)^d = \{\boldsymbol{v}^m = (v_i^m)_{1\le i\le d};\ \ v_i^m \in L^2(\Omega^m)\},$$
$$\mathcal{H}^m = \{\boldsymbol{\tau}^m = (\tau_{ij}^m)_{1\le i,j\le d};\ \ \tau_{ij}^m = \tau_{ji}^m \in L^2(\Omega^m)\},$$
$$H_1^m = \{\boldsymbol{v}^m = (v_i^m)_{1\le i\le d};\ \ v_i^m \in H^1(\Omega^m)\},$$
$$\mathcal{H}_1^m = \{\boldsymbol{\tau}^m = (\tau_{ij}^m)_{1\le i,j\le d};\ \ \tau_{ij}^m \in \mathcal{H}^m;\ \operatorname{div}\boldsymbol{\tau}^m \in H^m\},$$
$$Y^m = \left\{\boldsymbol{\alpha}^m = (\alpha_i^m)_{1\le i\le \ell};\ \ \alpha_i^m \in L^2(\Omega^m)\right\}.$$

The spaces H^m, $\mathcal{H}^m$, H_1^m, $\mathcal{H}_1^m$ and Y^m are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^m, \boldsymbol{v}^m)_{H^m} = \int_{\Omega^m} \boldsymbol{u}_i^m.\boldsymbol{v}_i^m dx,$$
$$(\boldsymbol{\sigma}^m, \boldsymbol{\tau}^m)_{\mathcal{H}^m} = \int_{\Omega^m} \boldsymbol{\sigma}_{ij}^m.\boldsymbol{\tau}_{ij}^m dx,$$
$$(\boldsymbol{u}^m, \boldsymbol{v}^m)_{H_1^m} = \int_{\Omega^m} \boldsymbol{u}_i^m.\boldsymbol{v}_i^m dx + \int_{\Omega^m} \boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}^m).\boldsymbol{\varepsilon}_{ij}(\boldsymbol{v}^m) dx,$$
$$(\boldsymbol{\sigma}^m, \boldsymbol{\tau}^m)_{\mathcal{H}_1^m} = \int_{\Omega^m} \boldsymbol{\sigma}_{ij}^m.\boldsymbol{\tau}_{ij}^m dx + \int_{\Omega^m} \operatorname{Div}\boldsymbol{\sigma}^m.\operatorname{Div}\boldsymbol{\tau}^m dx,$$
$$(\boldsymbol{\alpha}^m, \boldsymbol{\beta}^m)_{Y^m} = \int_{\Omega^m} \boldsymbol{\alpha}_i^m.\boldsymbol{\beta}_i^m dx,$$

and the associated norms $\|.\|_{H^m}$, $\|.\|_{H_1^m}$, $\|.\|_{\mathcal{H}^m}$, $\|.\|_{\mathcal{H}_1^m}$ and $\|.\|_{Y^m}$, respectively. Let $H_{\Gamma^m} = H^{\frac{1}{2}}(\Gamma^m)^d$ and $\gamma^m : H^1(\Omega^m)^d \to H_{\Gamma^m}$ be the trace map. For every element $\boldsymbol{v}^m \in H^1(\Omega^m)^d$, we also use the notation $\boldsymbol{v}^m$ to denote the trace $\gamma^m\boldsymbol{v}^m$ of $\boldsymbol{v}^m$ on Γ^m and we denote by v_ν^m and $\boldsymbol{v}_\tau^m$ the normal and the tangential components of $\boldsymbol{v}^m$ on Γ^m given by

$$v_\nu^m = \boldsymbol{v}^m.\nu^m, \quad \boldsymbol{v}_\tau^m = \boldsymbol{v}^m - v_\nu^m\boldsymbol{\nu}^m.$$

We note that v_ν^m is a scalar, whereas $\boldsymbol{v}_\tau^m$ is a tangent vector to Γ^m. In particular, in what follows, v_ν^m and $\boldsymbol{v}_\tau^m$ will represent the normal and tangential displacement. Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma}^m : \Omega^m \to \mathbb{S}^d$ we define its normal and tangential components

$$\sigma_\nu^m = (\boldsymbol{\sigma}^m\boldsymbol{\nu}^m).\boldsymbol{\nu}^m, \quad \boldsymbol{\sigma}_\tau^m = \boldsymbol{\sigma}^m\boldsymbol{\nu}^m - \sigma_\nu^m\boldsymbol{\nu}^m.$$

and we recall that the following Green's formula holds:

$$(\boldsymbol{\sigma}^m, \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + (\operatorname{Div}\boldsymbol{\sigma}^m, \boldsymbol{v}^m)_{H^m} = \int_{\Gamma^m} \boldsymbol{\sigma}^m\boldsymbol{\nu}^m.\boldsymbol{v}^m da \quad \forall \boldsymbol{v}^m \in H_1^m.$$

Let us now consider the bonding field the set

$$\mathcal{Z} = \left\{\theta \in L^\infty\big(0,T; L^2(\Gamma_3)\big);\ 0 \le \theta(t) \le 1\ \ \forall t \in [0,T],\ \text{a.e. on}\ \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of H_1^m defined by

$$V^m = \left\{\boldsymbol{v}^m \in H_1^m;\ \boldsymbol{v}^m = 0 \text{ on } \Gamma_1^m\right\}.$$

Since $meas\Gamma_1^m > 0$, the following Korn's inequality holds :

$$\|\varepsilon(\boldsymbol{v}^m)\|_{\mathcal{H}^m} \geq c_K \|\boldsymbol{v}^m\|_{H_1^m} \quad \forall \boldsymbol{v}^m \in V^m, \tag{28}$$

where the constant c_K denotes a positive constant which may depends only on Ω^m, Γ_1^m (see [17]). Over the space V^m we consider the inner product given by

$$(\boldsymbol{u}^m, \boldsymbol{v}^m)_{V^m} = (\varepsilon(\boldsymbol{u}^m), \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m}, \quad \forall \boldsymbol{u}^m, \boldsymbol{v}^m \in V^m, \tag{29}$$

and let $\|.\|_{V^m}$ be the associated norm. It follows from Korn's inequality (28) that the norms $\|.\|_{H_1^m}$ and $\|.\|_{V^m}$ are equivalent on V^m. Then $(V^m, \|.\|_{V^m})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (29), there exists a constant $c_0 > 0$, depending only on Ω^m, Γ_1^m and Γ_3 such that

$$\|\boldsymbol{v}^m\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}^m\|_{V^m} \quad \forall \boldsymbol{v}^m \in V^m. \tag{30}$$

We also introduce the spaces

$$I_0^m = L^2(\Omega^m),\ I_1^m = H^1(\Omega^m),\ W^m = \left\{\psi^m \in I_1^m;\ \psi^m = 0 \text{ on } \Gamma_a^m\right\},$$
$$\mathcal{W}^m = \left\{\boldsymbol{D}^m = (D_i^m);\ D_i^m \in L^2(\Omega^m),\ \operatorname{div} \boldsymbol{D}^m \in L^2(\Omega^m)\right\}.$$

Since $meas\Gamma_a^m > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi^m\|_{L^2(\Omega^m)^d} \geq c_F \|\psi^m\|_{H^1(\Omega^m)} \quad \forall \psi^m \in W^m, \tag{31}$$

where $c_F > 0$ is a constant which depends only on Ω^m, Γ_a^m.
Over the space W^m, we consider the inner product given by

$$(\varphi^m, \psi^m)_{W^m} = \int_{\Omega^m} \nabla\varphi^m.\nabla\psi^m dx$$

and let $\|.\|_{W^m}$ be the associated norm. It follows from (31) that $\|.\|_{H^1(\Omega^m)}$ and $\|.\|_{W^m}$ are equivalent norms on W^m and therefore $(W^m, \|.\|_{W^m})$ is areal Hilbert space. On the space $\mathcal{W}^m$, we use the inner product

$$(\boldsymbol{D}^m, \boldsymbol{\Psi}^m)_{\mathcal{W}^m} = \int_{\Omega^m} \boldsymbol{D}^m.\boldsymbol{\Psi}^m dx + \int_{\Omega^m} \operatorname{div} \boldsymbol{D}^m. \operatorname{div} \boldsymbol{\Psi}^m dx,$$

where $\operatorname{div} \boldsymbol{D}^m = (\boldsymbol{D}_{i,i}^m)$, and the associated norm $\|.\|_{\mathcal{W}^m}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2,\ \ H = H^1 \times H^2,\ \ H_1 = H_1^1 \times H_1^2,\ \ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,$$
$$\mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2,\ \ Y = Y^1 \times Y^2,\ \ I_0 = I_0^1 \times I_0^2,\ \ I_1 = I_1^1 \times I_1^2,$$
$$W = W^1 \times W^2,\ \ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, Y, I_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_Y$, $(.,.)_{I_1}$, $(.,.)_W$, and $(.,.)_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_Y$, $\|.\|_{I_1}$, $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

We will use a modified inner product on H, given by

$$(\!(\boldsymbol{u}, \boldsymbol{v})\!)_H = \sum_{m=1}^{2} (\rho^m \boldsymbol{u}^m, \boldsymbol{v}^m)_{H^m}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H,$$

that is, it is weighted with ρ^m, and we let $|\!|\!|.|\!|\!|_H$ be the associated norm, i.e.,

$$|\!|\!|\boldsymbol{v}|\!|\!|_H = (\!(\boldsymbol{v}, \boldsymbol{v})\!)_H^{\frac{1}{2}}, \quad \forall v \in H.$$

It follows from assumption (43) that $|\!|\!|.|\!|\!|_H$ and $\|.\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V}, \|.\|_{\boldsymbol{V}})$ into $(H, |\!|\!|.|\!|\!|_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual, we can write the Gelfand triple $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$. Using the notation $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ to represent the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$ we have

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = (\!(\boldsymbol{u}, \boldsymbol{v})\!)_H, \quad \forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}.$$

In the study of the mechanical problem (11)–(26), we assume that.

The *viscosity operator* $\mathcal{B}^m : \Omega^m \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{B}^m} > 0 \text{ such that} \\ \quad |\mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_1) - \mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_2)| \le L_{\mathcal{B}^m} |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2|, \quad \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) There exists } m_{\mathcal{B}^m} > 0 \text{ such that} \\ \quad (\mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_1) - \mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_2)) \cdot (\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2) \ge n_{\mathcal{B}^m} |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2|^2, \ \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}) \text{ is Lebesgue measurable on } \Omega^m, \quad \text{for any } \boldsymbol{\epsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{\epsilon} \mapsto \mathcal{B}^m(\boldsymbol{x}, \boldsymbol{\epsilon}) \text{ is continuous on } \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega^m. \end{cases} \tag{32}$$

The *elasticity operator* $\mathcal{A}^m : \Omega^m \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^m} > 0 \text{ such that} \\ \quad |\mathcal{A}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_1) - \mathcal{A}^m(\boldsymbol{x}, \boldsymbol{\epsilon}_2)| \le L_{\mathcal{A}^m} |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| \quad \forall \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^m(\boldsymbol{x}, \boldsymbol{\epsilon}) \text{ is Lebesgue measurable on } \Omega^m, \quad \text{for any } \boldsymbol{\epsilon} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^m(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}^m. \end{cases} \tag{33}$$

The *viscoplasticity operator* $\mathcal{Q}^m : \Omega^m \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{Q}^m} > 0 \text{ such that} \\ \quad |\mathcal{Q}^m(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\epsilon}_1, d_1) - \mathcal{Q}^m(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_2, d_2)| \le L_{\mathcal{Q}^m}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| + |d_1 - d_2|\big), \\ \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^m(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\epsilon}, d) \text{ is Lebesgue measurable in } \Omega^m, \text{ for any } \boldsymbol{\eta}, \boldsymbol{\epsilon} \in \mathbb{S}^d, \ d \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^m(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \text{ belongs to } \mathcal{H}^{\mathrm{m}}. \end{cases} \tag{34}$$

The *piezoelectric tensor* $\mathcal{E}^m : \Omega^m \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^m(\boldsymbol{x}, \eta) = (e^m_{ijk}(\boldsymbol{x})\eta_{jk}), \quad \forall \eta = (\eta_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) } e^m_{ijk} = e^m_{ikj} \in L^\infty(\Omega^m), \ 1 \le i, j, k \le d. \end{cases} \tag{35}$$

The *function* $\phi^m : \Omega^m \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\phi^m} > 0 \text{ such that} \\ \quad |\phi^m(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\epsilon}_1, k_1) - \phi^m(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_2, k_2)| \le L_{\phi^m}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| + |d_1 - d_2|\big), \\ \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d, \text{ and } k_1, k_2 \in \mathbb{R}^\ell, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \phi^m(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\epsilon}, k) \text{ is Lebesgue measurable in } \Omega^m, \text{ for any } \boldsymbol{\eta}, \boldsymbol{\epsilon} \in \mathbb{S}^d, k \in \mathbb{R}^\ell. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \phi^m(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \text{ belongs to } \mathrm{L}^2(\Omega^{\mathrm{m}})^\ell. \end{cases} \tag{36}$$

The *adhesion rate function* $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{ad} > 0 \text{ such that :} \\ \quad |H_{ad}(\boldsymbol{x}, \varsigma_1, r_1, \mathbf{d}_1) - H_{ad}(\boldsymbol{x}, \varsigma_2, r_2, \mathbf{d}_2)| \le L_\nu\big(|\varsigma_1 - \varsigma_2| + |r_1 - r_2| + |\mathbf{d}_1 - \mathbf{d}_2|\big), \\ \quad \forall \varsigma_1, \varsigma_2, r_1, r_2 \in \mathbb{R}, \ \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x}, \varsigma, r, \mathbf{d}) \text{ is measurable on } \Gamma_3, \quad \text{for any } \varsigma, r \in \mathbb{R}, \ \mathbf{d} \in \mathbb{R}^{d-1}, \\ \text{(c) The mapping } (\varsigma, r, \mathbf{d}) \mapsto H_{ad}(\boldsymbol{x}, \varsigma, r, \mathbf{d}) \text{ is continuous on } \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(d) } H_{ad}(\boldsymbol{x}, 0, r, \mathbf{d}) = 0, \forall r \in \mathbb{R}, \ \mathbf{d} \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(e) } H_{ad}(\boldsymbol{x}, \varsigma, r, \mathbf{d}) \ge 0, \quad \forall \varsigma \le 0, r \in \mathbb{R}, \ \mathbf{d} \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ and} \\ \quad H_{ad}(\boldsymbol{x}, \varsigma, r, \mathbf{d}) \le 0, \quad \forall \varsigma \ge 1, r \in \mathbb{R}, \ \mathbf{d} \in \mathbb{R}^{d-1}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{37}$$

The *damage source function* $\Psi^m : \Omega^m \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\Psi^m} > 0 \text{ such that} \\ \quad |\Psi^m(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\epsilon}_1, \alpha_1) - \Psi^m(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_2, \alpha_2)| \le L_{\Psi^m}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2| + |\alpha_1 - \alpha_2|\big), \\ \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2 \in \mathbb{S}^d \text{ and } \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Psi^m(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\epsilon}, \alpha) \text{ is Lebesgue measurable on } \Omega^m, \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\epsilon} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Psi^m(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \text{ belongs to } L^2(\Omega^m). \end{cases} \tag{38}$$

The *electric permittivity operator* $\boldsymbol{\beta}^m = (\beta^m_{ij}) : \Omega^m \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\begin{cases} \text{(a) } \boldsymbol{\beta}^m(\boldsymbol{x}, \mathbf{d}) = (\beta^m_{ij}(\boldsymbol{x}) d_j) \quad \forall \mathbf{d} = (d_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \\ \text{(b) } \beta^m_{ij} = \beta^m_{ji} \in L^\infty(\Omega^m), \quad 1 \le i, j \le d. \\ \text{(c) There exists } m_{\beta^m} > 0 \text{ such that } \boldsymbol{\beta}^m \mathbf{d}.\mathbf{d} \ge m_{\mathcal{B}^m} |\mathbf{d}|^2, \ \forall \mathbf{d} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^m. \end{cases} \tag{39}$$

The *normal contact functions* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) There exists a constant } C_1^\nu, C_2^\nu > 0 \text{ such that} \\ \qquad |p_\nu(\boldsymbol{x}, r)| \leq C_1^\nu |r| + C_2^\nu, \quad \forall r \in \mathbb{R} \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0, \quad \forall r_1, r_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R} \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is continuous on } \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{40}$$

The *tangential contact functions* $p_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) There exists a constant } C_1^\tau, C_2^\tau \text{ such that} \\ \qquad |p_\tau(\boldsymbol{x}, \boldsymbol{d})| \leq C_1^\tau |\mathbf{d}| + C_2^\tau, \qquad \forall \mathbf{d} \in \mathbb{R}^d \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(b) } (p_\tau(\boldsymbol{x}, \mathbf{d}_1) - p_\tau(\boldsymbol{x}, \mathbf{d}_2))(\mathbf{d}_1 - \mathbf{d}_2) \geq 0, \quad \forall \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^d \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, \mathbf{d}) \text{ is measurable on } \Gamma_3 \text{ for any } \mathbf{d} \in \mathbb{R}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, \mathbf{d}) \text{ is continuous on } \mathbb{R}^d \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{41}$$

The *The tangential contact functions* $q_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\tau > 0 \text{ such that } |q_\tau(\boldsymbol{x}, d_1) - q_\tau(\boldsymbol{x}, d_2)| \leq L_\tau |d_1 - d_2|, \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } \exists\, M_\tau > 0 \text{ such that } |q_\tau(\boldsymbol{x}, d)| \leq M_\tau \ \forall d \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto q_\tau(\boldsymbol{x}, d) \text{ is measurable on } \Gamma_3, \ \forall d \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto q_\tau(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{42}$$

We suppose that the mass density satisfies

$$\rho^m \in L^\infty(\Omega^m) \text{ and } \exists \rho_0 > 0 \text{ such that } \rho^m(x) \geq \rho_0 \text{ a.e. } x \in \Omega^m, \ m = 1, 2. \tag{43}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} &\mathbf{f}_0^m \in L^2(0, T; L^2(\Omega^m)^d), \quad \mathbf{f}_2^m \in L^2(0, T; L^2(\Gamma_2^m)^d), \\ &q_0^m \in C(0, T; L^2(\Omega^m)), \quad q_2^m \in C(0, T; L^2(\Gamma_b^m)). \end{aligned} \tag{44}$$

The adhesion coefficient γ_ν satisfy the condition

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \gamma_\nu \geq 0, \text{ a.e. on } \Gamma_3. \tag{45}$$

The microcrack diffusion coefficient verifies

$$\kappa^m > 0, \tag{46}$$

and, finally, the initial data satisfy

$$\begin{aligned} \boldsymbol{u}_0^m \in \boldsymbol{V}^m, \quad \boldsymbol{v}_0^m \in H^m, \ \boldsymbol{k}_0^m \in Y^m, \ \xi_0^m \in K^m, m = 1, 2, \\ \varsigma_0 \in L^2(\Gamma_3), \quad 0 \leq \varsigma_0 \leq 1, \text{ a.e. on } \Gamma_3. \end{aligned} \tag{47}$$

We define four mappings $\mathbf{f} : [0, T] \to \boldsymbol{V}'$, $q : [0, T] \to \boldsymbol{W}$, $a : I_1 \times I_1 \to \mathbb{R}$, $j_{dm} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{ad} : L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ respectively, by

$$(\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} = \sum_{m=1}^{2} \int_{\Omega^m} \mathbf{f}_0^m(t) \cdot \boldsymbol{v}^m \, dx + \sum_{m=1}^{2} \int_{\Gamma_2^m} \mathbf{f}_2^m(t) \cdot \boldsymbol{v}^m \, da \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{48}$$

$$(q(t), \zeta)_W = \sum_{m=1}^{2} \int_{\Omega^m} q_0^m(t) \zeta^m \, dx - \sum_{m=1}^{2} \int_{\Gamma_b^m} q_2^m(t) \zeta^m \, da \quad \forall \zeta \in W, \tag{49}$$

$$a(\xi, \zeta) = \sum_{m=1}^{2} \kappa^m \int_{\Omega^m} \nabla \xi^m . \nabla \zeta^m dx, \tag{50}$$

$$j_{dm}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u^1_\nu + u^2_\nu)v_\nu da + \int_{\Gamma_3} p_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau).\boldsymbol{v}_\tau da, \tag{51}$$

$$j_{ad}(\varsigma,\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu\varsigma^2 R_\nu(u^1_\nu + u^2_\nu)v_\nu + q_\tau(\varsigma)\boldsymbol{R}_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau).\boldsymbol{v}_\tau \Big)\, da. \tag{52}$$

We note that the definitions of $\mathbf{f}$ and q are based on the Riesz representation theorem, moreover, it follows from assumptions (40), (41) and (42), that the integrals in (51) and (52) are well-defined and we note that conditions (44) imply

$$\mathbf{f} \in L^2(0,T;\boldsymbol{V}'), \quad q \in C(0,T;W). \tag{53}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of Problem **P.**

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1,\boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential field $\varphi = (\varphi^1,\varphi^2) : [0,T] \to W$, a damage field $\xi = (\xi^1,\xi^2) : [0,T] \to I_1$, a bonding field $\varsigma : [0,T] \to L^\infty(\Gamma_3)$, and an internal state variable field $\boldsymbol{k} = (\boldsymbol{k}^1,\boldsymbol{k}^2) : [0,T] \longrightarrow Y$ such that

$$\begin{aligned}&\boldsymbol{\sigma}^m = \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) + \mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}^m) + (\mathcal{E}^m)^*\nabla\varphi^m + \\ &\int_0^t \mathcal{Q}^m\Big(\boldsymbol{\sigma}^m(s) - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m(s)) - (\mathcal{E}^m)^*\nabla\varphi^m(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m(s)), \xi^m(s), \boldsymbol{k}^m(s)\Big)\, ds \quad \text{in } \Omega^m \times (0,T)\end{aligned} \tag{54}$$

$$\dot{\boldsymbol{k}}^m = \phi^m\big(\boldsymbol{\sigma}^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) - (\mathcal{E}^m)^*\nabla\varphi^m(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m), \boldsymbol{k}^m\big) \quad \text{in } \Omega^m \times (0,T), \tag{55}$$

$$\begin{aligned}&(\ddot{u},v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{m=1}^{2}(\boldsymbol{\sigma}^m,\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + j_{dm}(\dot{\boldsymbol{u}}(t),\boldsymbol{v}) + j_{ad}(\varsigma(t),\boldsymbol{u}(t),\boldsymbol{v}) = (\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}, \\ &\qquad \forall\boldsymbol{v} \in \boldsymbol{V},\ \text{a.e. } t \in (0,T),\end{aligned} \tag{56}$$

$$\begin{aligned}&\xi(t) \in K, \quad \sum_{m=1}^{2}(\dot{\xi}^m(t), \zeta^m - \xi^m(t))_{L^2(\Omega^m)} + a(\xi(t), \zeta - \xi(t)) \geq \\ &\sum_{m=1}^{2}\Big(\Psi^m\big(\boldsymbol{\sigma}^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m) - (\mathcal{E}^m)^*\nabla\varphi^m, \boldsymbol{\varepsilon}(\boldsymbol{u}^m), \zeta^m\big),\ \zeta^m - \xi^m(t)\Big)_{L^2(\Omega^m)}, \quad \forall\zeta \in K,\ \text{ a.e. } t \in (0,T),\end{aligned} \tag{57}$$

$$\sum_{m=1}^{2}(\boldsymbol{\beta}^m\nabla\varphi^m(t), \nabla\Psi^m)_{H^m} - \sum_{m=1}^{2}(\mathcal{E}^m\varepsilon(\boldsymbol{u}^m(t)), \nabla\Psi^m)_{H^m} = (q(t),\phi)_W, \quad \forall\Psi \in W,\ \text{ a.e. } t \in (0,T), \tag{58}$$

$$\dot{\varsigma} = H_{ad}(\varsigma, R_\nu(u^1_\nu + u^2_\nu), \boldsymbol{R}_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau)) \quad \text{a.e. } t \in (0,T), \tag{59}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0, \quad \zeta(0) = \zeta_0, \quad \beta(0) = \beta_0, \tag{60}$$

where $K = K^1 \times K^2$.
We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field and a bonding field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 1. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0 \leq \varsigma \leq 1$. Indeed, equation (59) guarantees that $\varsigma(x,t) \leq \varsigma_0(x)$ and, therefore, assumption (47) shows that $\varsigma(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\varsigma(x,t_0) = 0$ at time t_0, then it follows from (59) that $\dot{\varsigma}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\varsigma(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \varsigma(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

Now, we propose our existence and uniqueness result

Theorem 1 (Existence and uniqueness)**.** Assume that (32)–(47) hold. Then there exists a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, K, \varphi, \xi, \varsigma\}$ to Problem **PV**, Moreover, the solution satisfies

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}) \cap C^1(0,T;H), \tag{61}$$

$$\ddot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'), \tag{62}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \tag{63}$$

$$(\mathrm{Div}\,\boldsymbol{\sigma}^1, \mathrm{Div}\,\boldsymbol{\sigma}^2) \in L^2(0,T;\boldsymbol{V}'), \tag{64}$$

$$\varphi \in C(0,T;W), \tag{65}$$

$$\xi \in H^1(0,T;I_0) \cap L^2(0,T;I_1), \tag{66}$$

$$\varsigma \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{67}$$

$$\boldsymbol{k} \in W^{1,2}(0,T;Y). \tag{68}$$

The functions $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \xi, \varsigma, \boldsymbol{k}, \boldsymbol{D}\}$ which satisfy (54)–(60) and (13) are called weak solution of the piezoelectric contact Problem **P**. We conclude by Theorem 1 that, under the assumptions (32)– (47), the mechanical problem (11)–(26) has a unique weak solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \xi, \varsigma, \boldsymbol{k}, \boldsymbol{D}\}$. To precise the regularity of the weak solution, we note that the constitutive relation (13), the assumptions (35) and (39), and the regularities (61)–(65) show that $\boldsymbol{D} \in C(0,T;H)$; moreover, using (58) and notation (49), we obtain

$$div\boldsymbol{D}^m(t) = q_0^m(t) \quad \forall t \in [0,T],\ m = 1,2.$$

It follows now from the regularities (44) that div $\boldsymbol{D}^m \in C(0,T;H^m)$, $m = 1,2$, which shows that

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{69}$$

We conclude that the weak solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \xi, \varsigma, \boldsymbol{k}, \boldsymbol{D}\}$ of the electro-elastic-viscoplastic contact problem has the regularity (61)–(69).

4 | PROOF OF THEOREM

The proof of Theorem 1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments, similar to those used in [1,4]. To this end, we assume in what follows that (32)–(47) hold, and we consider that $\boldsymbol{C}$ is a generic positive constant which depends on Ω^m, Γ_1^m, Γ_2^m, Γ_3, Γ_a^m, Γ_b^m, p_ν, p_τ, q_τ, $\mathcal{A}^m$, $\mathcal{B}^m$, $\mathcal{Q}^m$, $\not{f}^m$, $\mathcal{E}^m$, ϕ^m, H_{ad}, γ_ν, ρ^m and T with $m = 1,2$. But does not depend on t nor of the rest of input data, and whose value may change from place to place.

Let a $\pi \in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\pi^u$. Find $\boldsymbol{u}_\pi = (\boldsymbol{u}_\pi^1, \boldsymbol{u}_\pi^2) : [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}(\ddot{u}_\pi(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{m=1}^{2}(\mathcal{B}^m\varepsilon(\dot{\boldsymbol{u}}_\pi^m(t)),\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + j_{dm}(\dot{\boldsymbol{u}}_\pi(t), \boldsymbol{v})\\ +(\pi(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in (0,T),\end{aligned} \tag{70}$$

$$\boldsymbol{u}_\pi^m(0) = \boldsymbol{u}_0^m,\ \ \dot{\boldsymbol{u}}_\pi^m(0) = \boldsymbol{v}_0^m \quad \text{in } \Omega^m. \tag{71}$$

To solve Problem $\mathbf{PV}_\pi^u$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. In the sequel, $\mathbb{V}$ and $\mathbb{H}$ denote real Hilbert spaces such that $\mathbb{V}$ is dense in $\mathbb{H}$ and the inclusion map is continuous, $\mathbb{H}$ is identified with its dual and with a subspace of the dual $\mathbb{V}'$ of $\mathbb{V}$, i.e., $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|.\|_{\mathbb{V}}$, $\|.\|_{\mathbb{V}'}$ and $(.,.)_{\mathbb{V}'\times\mathbb{V}}$ represent the norms on $\mathbb{V}$ and on $\mathbb{V}'$ and the duality pairing between $\mathbb{V}'$ them, respectively. The following abstract result may be found in [21] p.48.

Theorem 2. Let $\mathbb{V}$, $\mathbb{H}$ be as above, and let $A : \mathbb{V} \to \mathbb{V}'$ be a hemicontinuous and monotone operator which satisfies

$$\exists w > 0, \lambda \in \mathbb{R},\ \text{such that,}\quad (Av, v)_{\mathbb{V}'\times\mathbb{V}} \geq w\|v\|_{\mathbb{V}}^2 + \lambda \quad \forall v \in \mathbb{V}, \tag{72}$$

$$\exists C > 0,\ \text{such that}\quad \|Av\|_{\mathbb{V}'} \leq C(\|v\|_{\mathbb{V}} + 1), \quad \forall v \in \mathbb{V}. \tag{73}$$

Then, given $\boldsymbol{v}_0 \in \mathbb{H}$ and $f \in L^2(0,T;\mathbb{V}')$, there exists a unique function $\boldsymbol{v}$ which satisfies

$$\boldsymbol{v} \in L^2(0,T;\mathbb{V}) \cap C(0,T;\mathbb{H}), \quad \dot{\boldsymbol{v}} \in L^2(0,T;\mathbb{V}'),$$
$$\dot{\boldsymbol{v}}(t) + A\boldsymbol{v}(t) = f(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{v}(0) = \boldsymbol{v}_0.$$

The first step in the proof of Theorem 1 concerns the variational Problem PV^u_π. We have the following result.

Lemma 1. There exists a unique solution to Problem $\mathbf{PV}^u_\pi$ and it has its regularity expressed in (61)-(62).

Proof. Using Riesz representation theorem we define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{m=1}^{2}(\mathcal{B}^m\varepsilon(\boldsymbol{u}^m),\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + j_{dm}(\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}. \tag{74}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V}$, using (51) and (74) we find

$$\begin{aligned}(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} &= \sum_{m=1}^{2}(\mathcal{B}^m\varepsilon(\boldsymbol{u}_1^m) - \mathcal{B}^m\varepsilon(\boldsymbol{u}_2^m),\ \varepsilon(\boldsymbol{u}_1^m - \boldsymbol{u}_2^m))_{\mathcal{H}^m}\\
&+ \int_{\Gamma_3}(p_\nu(u_{1\nu}^1 + u_{1\nu}^2) - p_\nu(u_{2\nu}^1 + u_{2\nu}^2))(u_{1\nu} - u_{2\nu})da\\
&+ \int_{\Gamma_3}(p_\tau(\boldsymbol{u}_{1\tau}^1 - \boldsymbol{u}_{1\tau}^2) - p_\tau(\boldsymbol{u}_{2\tau}^1 - \boldsymbol{u}_{2\tau}^2)).(\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})da\end{aligned}$$

and keeping in mind (32), (40) and (41), we obtain

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} \geq \min(n_{\mathcal{B}^1}, n_{\mathcal{B}^2})\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}}^2, \ \ \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V}. \tag{75}$$

On the other hand, by (51) and (74) we obtain

$$\begin{aligned}(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} &= \sum_{m=1}^{2}(\mathcal{B}^m\varepsilon(\boldsymbol{u}_1^m) - \mathcal{B}^m\varepsilon(\boldsymbol{u}_2^m),\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m}\\
&+ \int_{\Gamma_3}(p_\nu(u_{1\nu}^1 + u_{1\nu}^2) - p_\nu(u_{2\nu}^1 + u_{2\nu}^2))v_\nu da\\
&+ \int_{\Gamma_3}(p_\tau(\boldsymbol{u}_{1\tau}^1 - \boldsymbol{u}_{1\tau}^2) - p_\tau(\boldsymbol{u}_{2\tau}^1 - \boldsymbol{u}_{2\tau}^2)).\boldsymbol{v}_\tau da, \ \ \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in \boldsymbol{V},\end{aligned}$$

and by (30) and (32), we deduce that

$$\begin{aligned}\|A\boldsymbol{u}_1 - A\boldsymbol{u}_2\|_{\boldsymbol{V}'} \leq{}& \max(L_{\mathcal{B}^1}, L_{\mathcal{B}^2})|\boldsymbol{u}_1 - \boldsymbol{u}_2|_{\boldsymbol{V}} + C_0|p_\nu(u_{1\nu}^1 + u_{1\nu}^2) - p_\nu(u_{2\nu}^1 + u_{2\nu}^2)|_{L^2(\Gamma_3)}\\
&+C_1|p_\tau(\boldsymbol{u}_{1\tau}^1 - \boldsymbol{u}_{1\tau}^2) - p_\tau(\boldsymbol{u}_{2\tau}^1 - \boldsymbol{u}_{2\tau}^2)|_{L^2(\Gamma_3)^d} \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V},\end{aligned}$$

and keeping in mind the Krasnoselski Theorem (see[14] p.60), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. From (75) we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a monotone operator. Now, by (32), (29) and (74), we find where the positive constant $n = \min\{n_{\mathcal{B}^1}, n_{\mathcal{B}^2}\}$. Choosing $\boldsymbol{u}_2 = 0_{\boldsymbol{V}}$ in (75) we obtain

$$\begin{aligned}(A\boldsymbol{u}_1, \boldsymbol{u}_1)_{\boldsymbol{V}'\times\boldsymbol{V}} &\geq n\|\boldsymbol{u}_1\|_{\boldsymbol{V}}^2 - \|A0_{\boldsymbol{V}}\|_{\boldsymbol{V}'}^2\|\boldsymbol{u}_1\|_{\boldsymbol{V}}\\
&\geq \frac{1}{2}n\|\boldsymbol{u}_1\|_{\boldsymbol{V}}^2 - \frac{1}{2n}\|A0_{\boldsymbol{V}}\|_{\boldsymbol{V}'}^2, \quad \forall \boldsymbol{u}_1 \in \boldsymbol{V},\end{aligned}$$

which implies that A satisfies condition (72) with $\omega = \frac{n}{2}$ and $\lambda = -\frac{1}{2n}\|A0_{\boldsymbol{V}}\|_{\boldsymbol{V}'}^2$. Moreover, by (32) and (74) we find

$$\|A\boldsymbol{u}_1\|_{\boldsymbol{V}'} \leq C^1\|u_1\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u}_1 \in \boldsymbol{V}$$

where $C^1 = \max\{C^1_{\mathcal{B}^1}, C^1_{\mathcal{B}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{B}^1}, C^2_{\mathcal{B}^2}\}$. This inequality and (29) imply that A satisfies condition (73). Finally, we recall that by (44) and (48) we have $\mathbf{f} - \pi \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.

It follows now from Theorem 2 that there exists a unique function $\boldsymbol{v}_\pi$ which satisfies

$$\boldsymbol{v}_\pi \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \;\; \dot{\boldsymbol{v}}_\pi \in L^2(0,T;\boldsymbol{V}'), \tag{76}$$

$$\dot{\boldsymbol{v}}_\pi(t) + A\boldsymbol{v}_\pi(t) + \pi(t) = \mathbf{f}(t), \;\; a.e.\; t \in [0,T] \tag{77}$$

$$\boldsymbol{v}_\pi(0) = \boldsymbol{v}_0. \tag{78}$$

Let $\boldsymbol{u}_\pi : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\pi(t) = \int_0^t \boldsymbol{v}_\pi(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{79}$$

It follows from (74) and (76)–(79) that $\boldsymbol{u}_\pi$ is a unique solution of the problem $\mathbf{PV}^u_\pi$, and it satisfies the regularity expressed in (61)-(62). □

In the second step, we use the displacement field $\boldsymbol{u}_\pi$ obtained in Lemma 1 and we consider the following variational problem.

Problem $\mathbf{PV}^\varphi_\pi$. Find $\varphi_\pi : [0,T] \to W$ such that

$$\sum_{m=1}^{2}(\boldsymbol{\beta}^m \nabla\varphi^m_\pi(t), \nabla\Psi^m)_{H^m} - \sum_{m=1}^{2}(\mathcal{E}^m\varepsilon(\boldsymbol{u}^m_\pi(t)), \nabla\Psi^m)_{H^m} = (q(t),\Psi)_W, \;\; \forall\Psi \in W, \; a.e.\; t \in (0,T). \tag{80}$$

We have the following result.

Lemma 2. Problem $\mathbf{PV}^\varphi_\pi$ has a unique solution φ_π which satisfies the regularity (65).

Proof. We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = \sum_{m=1}^{2}(\mathcal{B}^m\nabla\varphi^m, \nabla\phi^m)_{H^m} \quad \forall\varphi,\phi \in W. \tag{81}$$

We use (31), (39) and (81) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on W, moreover using (49) and the Riesz representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\pi(t),\phi)_W = (q(t),\phi)_W + \sum_{m=1}^{2}(\mathcal{E}^m\varepsilon(\boldsymbol{u}^m_\pi(t)), \nabla\phi^m)_{H^m} \quad \forall\phi \in W.$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\pi(t) \in W$ such that

$$b(\varphi_\pi(t),\phi) = (q_\pi(t),\phi)_W \quad \forall\phi \in W. \tag{82}$$

We conclude that $\varphi_\pi(t)$ is a solution of Problem $\mathbf{PV}^\varphi_\pi$. Let $t_1, t_2 \in [0,T]$, it follows from (80) that

$$\|\varphi_\pi(t_1) - \varphi_\pi(t_2)\|_W \le C\big(\|\boldsymbol{u}_\pi(t_1) - \boldsymbol{u}_\pi(t_2)\|_V + \|q(t_1) - q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\pi$ and q imply that $\varphi_\pi \in C(0,T;W)$. □

In the third step, we use the displacement field $\boldsymbol{u}_\pi$ obtained in Lemma1 and we consider the following initial-value problem.

Problem $\mathbf{PV}^\varsigma_\pi$. Find $\varsigma_\pi : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\varsigma}_\pi(t) = H_{ad}\big(\varsigma_\pi(t), R_\nu(u^1_{\pi\nu}(t) + u^2_{\pi\nu}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}^1_{\pi\tau}(t) - \boldsymbol{u}^2_{\pi\tau}(t))\big), \tag{83}$$

$$\varsigma_\pi(0) = \varsigma_0. \tag{84}$$

We have the following result.

Lemma 3. There exists a unique solution $\varsigma_\pi \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem $\mathbf{PV}^\varsigma_\pi$.

Proof. Let the mapping $H_\pi : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_\pi(t,\varsigma) = H_{ad}\big(\varsigma, R_\nu(u^1_{\pi\nu}(t) + u^2_{\pi\nu}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}^1_{\pi\tau}(t) - \boldsymbol{u}^2_{\pi\tau}(t))\big),$$

for all $t \in [0,T]$ and $\varsigma \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that H_π is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\varsigma \in L^2(\Gamma_3)$, the mapping $t \to H_\pi(t,\varsigma)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy–Lipschitz theorem given in[21] p. 48, we deduce that there exists a unique function $\varsigma_{\pi\alpha} \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (83)-(84). Also, the arguments used in Remark 1 show that $0 \leq \varsigma_{\pi\alpha}(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. This completes the proof. □

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In the fourth step, we let $\lambda \in L^2(0.T; I_0)$ be given and consider the following variational problem.

Problem $\mathbf{PV}^{\xi}_{\lambda}$. *Find* $\xi_\lambda = (\xi^1_\lambda, \xi^2_\lambda) : [0,T] \to I_0$ such that

$$\begin{aligned} \xi_\lambda(t) \in K,\ \sum_{m=1}^{2}(\dot{\xi}^m_\lambda(t), \zeta^m - \xi^m_\lambda(t))_{L^2(\Omega^m)} + a(\xi_\lambda(t), \zeta - \xi_\lambda(t)) \geq \\ \sum_{m=1}^{2}\left(\lambda^m(t), \zeta^m - \xi^m_\lambda(t)\right)_{L^2(\Omega^m)},\ \ \forall \zeta \in K,\ \text{a.e. } t \in (0,T). \end{aligned} \tag{85}$$

The following abstract result for parabolic variational inequalities (see, e.g.,[21] p.47)

Theorem 3. Let $\mathbb{E} \subset \mathbb{F} = \mathbb{F}' \subset \mathbb{E}'$ be a Gelfand triple. Let G be a nonempty, closed, and convex set of $\mathbb{E}$. Assume that $a(.,.) : \mathbb{E}\times\mathbb{E} \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,

$$a(\zeta,\zeta) + c_0\|\zeta\|^2_{\mathbb{F}} \geq \alpha\|\zeta\|^2_{\mathbb{E}} \quad \forall \zeta \in \mathbb{E}.$$

Then, for every $\xi_0 \in G$ and $f \in L^2(0,T;\mathbb{F})$, there exists a unique function $\xi \in H^1(0,T;\mathbb{F}) \cap L^2(0,T;\mathbb{E})$ such that $\xi(0) = \xi_0$, $\xi(t) \in G\ \ \forall t \in [0,T]$, and

$$(\dot{\xi}(t), \zeta - \xi(t))_{\mathbb{E}'\times\mathbb{E}} + a(\xi(t), \zeta - \xi(t)) \geq (f(t), \zeta - \xi(t))_F\ \ \forall \zeta \in G \text{ a.e. } t \in (0,T).$$

We prove next the unique solvability of Problem $\mathbf{PV}^{\xi}_{\lambda}$.

Lemma 4. There exists a unique solution ξ_λ of Problem $\mathbf{PV}^{\xi}_{\lambda}$ and it satisfies

$$\xi_\lambda \in H^1(0,T;I_0) \cap L^2(0,T;I_1).$$

Proof. The inclusion mapping of $(I_1, \|.\|_{I_1})$ into $(I_0, \|.\|_{I_0})$ is continuous and its range is dense. We denote by I_1' the dual space of I_1 and, identifying the dual of I_0 with itself, we can write the Gelfand triple

$$I_1 \subset I_0 = I_0' \subset I_1'.$$

We use the notation $(.,.)_{I_1'\times I_1}$ to represent the duality pairing between I_1' and I_1. We have

$$(\zeta,\xi)_{I_1'\times I_1} = (\zeta,\xi)_{I_0} \quad \forall \zeta \in I_0,\ \xi \in I_1,$$

and we note that K is a closed convex set in I_1. Then, using (46), (50) and the fact that $\xi_0 \in K$ in (47), it is easy to see that Lemma 4 is a straight consequence of Theorem 3. □

In the fifth step, we let $\theta \in L^2(0,T;Y)$ be given, and define $\boldsymbol{k}_\theta \in W^{1,2}(0,T;Y)$ by

$$\boldsymbol{k}_\theta(t) = \boldsymbol{k}_0 + \int_0^t \theta(s)ds. \tag{86}$$

We use the displacement field $\boldsymbol{u}_\pi$ obtained in Lemma 1, φ_π obtained in Lemma 2, ξ_λ obtained in Lemma 4 and $\boldsymbol{k}_\theta$ defined in (86) to construct the following Cauchy problem.

Problem $\mathbf{PV}^{\sigma}_{\pi\lambda\theta}$. *Find* $\boldsymbol{\sigma}_{\pi\lambda\theta} = (\boldsymbol{\sigma}^1_{\pi\lambda\theta}, \boldsymbol{\sigma}^2_{\pi\lambda\theta}) : [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}^m_{\pi\lambda\theta}(t) = \mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}^m_\pi(t)) + \int_0^t \mathcal{Q}^m(\boldsymbol{\sigma}^m_{\pi\lambda\theta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m_\pi(s)), \xi^m_\lambda(s), \boldsymbol{k}^m_\theta(s))\,ds, \tag{87}$$

for all $m = 1,2$, and $t \in [0,T]$.

In the study of Problem $\mathbf{PV}^{\sigma}_{\pi\lambda\theta}$ we have the following result.

Lemma 5. There exists a unique solution of Problem $\mathbf{PV}^{\sigma}_{\pi\lambda\theta}$ and it satisfies $\boldsymbol{\sigma}_{\pi\lambda\theta} \in L^2(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_i$, $\boldsymbol{u}_i$ and ξ_i represent the solutions of problems $\mathbf{PV}^{\sigma}_{\pi_i\lambda_i\theta_i}$, $\mathbf{PV}^{\boldsymbol{u}}_{\pi_i}$ and $\mathbf{PV}^{\xi}_{\lambda_i}$ respectively, and $\boldsymbol{k}_i$ is defined in (86) for $(\boldsymbol{\pi}_i, \lambda_i, \theta_i) \in L^2(0,T;\boldsymbol{V}'\times I_0 \times Y)$, $i = 1,2$, then there exists $c > 0$ such that

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_{\boldsymbol{V}}\, ds + \int_0^t \|\xi_1(s) - \xi_2(s)\|^2_{I_0}\, ds + \int_0^t \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_{Y}\, ds\Big) \quad \forall t \in [0,T]. \tag{88}$$

Proof. Let $\Pi_{\pi\lambda\theta} = (\Pi^1_{\pi\lambda\theta}, \Pi^2_{\pi\lambda\theta}) : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Pi^m_{\pi\lambda\theta}\boldsymbol{\sigma}(t) = \mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}^m_\pi(t)) + \int_0^t \mathcal{Q}^m\big(\boldsymbol{\sigma}^m(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m_\pi(s)), \xi^m_\lambda(s), \boldsymbol{k}^m_\theta(s)\big)\, ds, \tag{89}$$

for all $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in L^2(0,T;\mathcal{H})$, $t \in [0,T]$, and $m = 1,2$.
For $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in L^2(0,T;\mathcal{H})$ we use (34) and (89) to obtain

$$\|\Pi_{\pi\lambda\theta}\,\boldsymbol{\sigma}(t) - \Pi_{\pi\lambda\theta}\,\boldsymbol{\tau}(t)\|_{\mathcal{H}} \le \max(L_{\mathcal{Q}^1}, L_{\mathcal{Q}^2}) \int_0^t \|\boldsymbol{\sigma}(s) - \boldsymbol{\tau}(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for n large enough, a power $\Pi^n_{\pi\lambda\theta}$ of the operator $\Pi_{\pi\lambda\theta}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\pi\lambda\theta} \in L^2(0,T;\mathcal{H})$ such that $\Pi_{\pi\lambda\theta}\boldsymbol{\sigma}_{\pi\lambda\theta} = \boldsymbol{\sigma}_{\pi\lambda\theta}$. Moreover, $\boldsymbol{\sigma}_{\pi\lambda\theta}$ is the unique solution of Problem $\mathbf{PV}^{\sigma}_{\pi\lambda\theta}$. Using (86), the regularity of $\boldsymbol{u}_\pi$, the regularity of $\boldsymbol{k}_\theta$ and the properties of the operators $\mathcal{A}^m$ and $\mathcal{Q}^m$, it follows that $\boldsymbol{\sigma}_{\pi\lambda\theta} \in W^{1,2}(0;T;\mathcal{H})$.

Consider now $(\boldsymbol{\pi}_1, \lambda_1, \theta_1)$, $(\boldsymbol{\pi}_2, \lambda_2, \theta_1)$, $\in L^2(0,T;\boldsymbol{V}' \times I_0 \times Y)$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\pi_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\pi_i\lambda_i\theta_i} = \boldsymbol{\sigma}_i$, $\xi_{\lambda_i} = \xi_i$. and $\boldsymbol{k}_{\theta_i} = \boldsymbol{k}_i$. We have, for $m = 1,2$,

$$\boldsymbol{\sigma}^m_i(t) = \mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}^m_i(t)) + \int_0^t \mathcal{Q}^m\big(\boldsymbol{\sigma}^m_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^m_i(s)), \xi^m_i(s), \boldsymbol{k}^m_i(s)\big)\, ds,$$

and, using the properties (33) and (34) of $\mathcal{G}^m$, and $\mathcal{A}^m$ we find

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\mathrm{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_{V}\, ds + \int_0^t \|\xi_1(s) - \xi_2(s)\|^2_{I_0}\, ds + \int_0^t \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_{Y}\, ds\Big) \quad \forall t \in [0,T].$$

Using now a Gronwall argument in the previous inequality we deduce (88), which concludes the proof. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{A}^m$, the operator $\mathcal{E}^m$, the operator $\mathcal{Q}^m$, the operator $\oint^m$ and the function Ψ^m for $t \in [0,T]$, we consider the element

$$\Pi(\pi,\lambda,\theta)(t) = \big(\Pi^1(\pi,\lambda,\theta)(t),\ \Pi^2(\pi,\lambda,\theta)(t),\ \Pi^3(\pi,\lambda,\theta)(t)\big) \in \boldsymbol{V}' \times I_0 \times \boldsymbol{Y}, \tag{90}$$

defined by the equations

$$(\Pi^1(\pi,\lambda,\theta)(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = j_{ad}(\varsigma_\pi(t),\boldsymbol{u}_\pi(t),\boldsymbol{v}) + \sum_{m=1}^{2}\left(\mathcal{A}^m\varepsilon(\boldsymbol{u}_\pi^m(t)),\ \varepsilon(\boldsymbol{v}^m)\right)_{\mathcal{H}^m}$$

$$+\sum_{m=1}^{2}\left((\mathcal{E}^m)^*\nabla\varphi_\pi^m,\ \varepsilon(\boldsymbol{v}^m)\right)_{\mathcal{H}^m} + \sum_{m=1}^{2}\Big(\int_0^t \mathcal{Q}^m\left(\boldsymbol{\sigma}^m_{\pi\lambda\theta},\varepsilon(\boldsymbol{u}_\pi^m(s)),\xi_\lambda(s),\boldsymbol{k}_\theta^m(s)\right)ds,\ \varepsilon(\boldsymbol{v}^m)\Big)_{\mathcal{H}^m},\ \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{91}$$

$$\Pi^2(\pi,\lambda,\theta)(t) = \Big(\phi^1\left(\boldsymbol{\sigma}^1_{\pi\lambda\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\pi^1(t)),\boldsymbol{k}_\theta^1(t)\right),\ \phi^2\left(\boldsymbol{\sigma}^2_{\pi\lambda\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\pi^2(t)),\boldsymbol{k}_\theta^2(t)\right)\Big), \tag{92}$$

$$\Pi^3(\pi,\lambda,\theta)(t) = \Big(\Psi^1\left(\boldsymbol{\sigma}^1_{\pi\lambda\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\pi^1(t)),\xi_\lambda^1(t)\right),\ \Psi^2\left(\boldsymbol{\sigma}^2_{\pi\lambda\theta}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\pi^2(t)),\xi_\lambda^2(t)\right)\Big). \tag{93}$$

Here, for every $(\pi,\lambda,\theta)\in L^2(0,T;\boldsymbol{V}'\times I_0\times Y)$, $\boldsymbol{u}_\pi$, φ_π, ς_π ξ_λ, and $\boldsymbol{\sigma}_{\pi\lambda\theta}$ represent the displacement field, the stress field, the the potential electric field and bonding field obtained in Lemmas 1, 2, 3, 4 , 5 respectively, and $\boldsymbol{k}_\theta$ is the internal state variable given by (86). We have the following result.

Lemma 6. The operator Π has a unique fixed point $(\pi^*,\lambda^*,\theta^*)\in L^2(0,T;\boldsymbol{V}'\times I_0\times Y)$.

Proof. We show that, for a positive integer n, the mapping Π^n is a contraction on $L^2(0,T;\boldsymbol{V}'\times I_0\times Y)$. To this end, we suppose that $(\pi_1,\lambda_1,\theta_1)$ and $(\pi_2,\lambda_2,\theta_2)$ are two functions in $L^2(0,T;\boldsymbol{V}'\times I_0\times Y)$ and denote $\boldsymbol{u}_{\pi_i}=\boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\pi_i}=\boldsymbol{v}_i$, $\varphi_{\pi_i}=\varphi_i$, $\varsigma_{\pi_i}=\varsigma_i$, $\xi_{\lambda_i}=\xi_i$, $\boldsymbol{\sigma}_{\pi_i\lambda_i\theta_i}=\boldsymbol{\sigma}_i$ and $\boldsymbol{k}_{\theta_i}=\boldsymbol{k}_i$ for $i=1,2$. We use (52) and (91) we have

$$\begin{aligned}\|\Pi^1(\pi_1,\lambda_1,\theta_1)(t)-\Pi^1(\pi_2,\lambda_2,\theta_2)(t)\|^2_{\boldsymbol{V}'} &\le C\|R_\nu(u^1_{1\nu}+u^2_{1\nu})-R_\nu(u^1_{2\nu}+u^2_{2\nu})\|^2_{L^2(\Gamma_3)}\\ &+C\|\boldsymbol{R}_\tau(\boldsymbol{u}^1_{1\tau}-\boldsymbol{u}^2_{1\tau})-\boldsymbol{R}_\tau(\boldsymbol{u}^1_{2\tau}-\boldsymbol{u}^2_{2\tau})\|^2_{L^2(\Gamma_3)}\\ &+C\|q_\tau(\varsigma_1(t))\boldsymbol{R}_\tau(\boldsymbol{u}^1_{1\tau}-\boldsymbol{u}^2_{1\tau})-q_\tau(\varsigma_2)\boldsymbol{R}_\tau(\boldsymbol{u}^1_{2\tau}-\boldsymbol{u}^2_{2\tau})\|^2_{L^2(\Gamma_3)}\\ &+\sum_{m=1}^{2}\|\mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}_1^m(t))-\mathcal{A}^m\boldsymbol{\varepsilon}(\boldsymbol{u}_2^m(t))\|^2_{\mathcal{H}^m}\\ &+\sum_{m=1}^{2}\|(\mathcal{E}^m)^*\nabla\varphi_1^m(t)-(\mathcal{E}^m)^*\nabla\varphi_2^m(t)\|^2_{\mathcal{H}^m}+C\|\varsigma_1-\varsigma_2\|^2_{L^2(\Gamma_3)}\end{aligned}$$

$$+\sum_{m=1}^{2}\int_0^t\left(\left\|\mathcal{Q}^m\left(\boldsymbol{\sigma}_1^m(s),\varepsilon(\boldsymbol{u}_1^m(s)),\xi_1^m(s),\boldsymbol{k}_1^m(s)\right)-\ \mathcal{Q}^m\left(\boldsymbol{\sigma}_2^m(s),\varepsilon(\boldsymbol{u}_2^m(s)),\xi_2^m(s),\boldsymbol{k}_2^m(s)\right)\right\|^2_{\mathcal{H}^m}\right)ds.$$

Therefore, from (33), (34), (35)–(42) and the definition of R_ν, $\boldsymbol{R}_\tau$, we obtain

$$\|\Pi^1(\pi_1,\lambda_1,\theta_1)(t)-\Pi^1(\pi_2,\lambda_2,\theta_2)(t)\|^2_{\boldsymbol{V}'}\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds$$

$$+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s))\|^2_{\mathcal{H}}\,ds+\int_0^t\|\xi_1(s)-\xi_2(s))\|^2_{I_0}\,ds+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{Y}\,ds$$

$$+C\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Gamma_3)}\Big).$$

We use estimate (88) to obtain

$$\|\Pi^1(\pi_1,\lambda_1,\theta_1)(t)-\Pi^1(\pi_2,\lambda_2,\theta_2)(t)\|^2_{\boldsymbol{V}'}\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\int_0^t\|\xi_1(s)-\xi_2(s))\|^2_{I_0}\,ds$$

$$+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{Y}\,ds+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Gamma_3)}\Big).$$

Recall that above $u^m_{\pi\nu}$ and $\boldsymbol{u}^m_{\pi\tau}$ denote the normal and the tangential component of the function $\boldsymbol{u}^m_{\pi}$ respectively. By similar arguments, from (36), (86) and (92) it follows that

$$\|\Pi^2(\pi_1,\lambda_1,\theta_1)(t)-\Pi^2(\pi_2,\lambda_2,\theta_2)(t)\|^2_Y \le C\Big(\int_0^t \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s))\|^2_{\mathcal{H}}\,ds+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\|\boldsymbol{k}_1(t)-\boldsymbol{k}_2(t))\|^2_Y\Big).$$

$$\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\|\boldsymbol{k}_1(t)-\boldsymbol{k}_2(t))\|^2_Y+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_{\boldsymbol{V}}ds+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_Y ds\Big).$$

On the other hand by (38), (88) and (93), we get

$$\|\Pi^3(\pi_1,\lambda_1,\theta_1)(t)-\Pi^3(\pi_2,\lambda_2,\theta_2)(t)\|^2_{I_0}\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\|\varphi_1(t)-\varphi_2(t)\|^2_W$$
$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\xi_1(t)-\xi_2(t))\|^2_{I_0}+\int_0^t\|\xi_1(s)-\xi_2(s))\|^2_{I_0}\,ds\Big).$$

Also, since

$$\boldsymbol{u}^m_i(t)=\int_0^t\boldsymbol{v}^m_i(s)ds+\boldsymbol{u}^m_0,\quad t\in[0,T],\ \ m=1,2,$$

we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}\le\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}\,ds$$

which implies

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds\le c\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds. \tag{94}$$

Therefore

$$\begin{aligned}\|\Pi(\pi_1,\lambda_1,\theta_1)(t)-\Pi(\pi_2,\lambda_2,\theta_2)(t)\|^2_{\boldsymbol{V}'\times I_0\times Y}&\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds\\&+\|\xi_1(t)-\xi_2(t))\|^2_{I_0}+\int_0^t\|\xi_1(s)-\xi_2(s))\|^2_{I_0}\,ds+\|\boldsymbol{k}_1(t)-\boldsymbol{k}_2(t))\|^2_Y+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_Y ds\\&+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{95}$$

Moreover, from (70) we obtain

$$\begin{aligned}&(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{m=1}^2(\mathcal{B}^m\varepsilon(\boldsymbol{v}^m_1)-\mathcal{B}^m\varepsilon(\boldsymbol{v}^m_2),\varepsilon(\boldsymbol{v}^m_1-\boldsymbol{v}^m_2))_{\mathcal{H}^m}\\&+j_{dm}(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j_{dm}(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)+(\pi_1-\pi_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}=0.\end{aligned} \tag{96}$$

We use (40), (41) and (51) to deduce that

$$j_{dm}(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j_{dm}(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)\ge 0. \tag{97}$$

It follows from (96) and (97) that

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{m=1}^2(\mathcal{B}^m\varepsilon(\boldsymbol{v}^m_1)-\mathcal{B}^m\varepsilon(\boldsymbol{v}^m_2),\varepsilon(\boldsymbol{v}^m_1-\boldsymbol{v}^m_2))_{\mathcal{H}^m}\le-(\pi_1-\pi_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}. \tag{98}$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$ and condition (32) to find

$$\min(n_{\mathcal{B}^1}, n_{\mathcal{B}^2}) \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2 \, ds \leq - \int_0^t (\pi_1(s) - \pi_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))_{\boldsymbol{V}' \times \boldsymbol{V}} \, ds,$$

for all $t \in [0, T]$. Then, using the inequality $2ab \leq \frac{a^2}{\alpha} + \alpha b^2$, we obtain

$$\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_{\boldsymbol{V}}^2 \, ds \leq C \int_0^t \|\pi_1(s) - \pi_2(s)\|_{\boldsymbol{V}'}^2 \, ds \quad \forall t \in [0, T]. \tag{99}$$

On the other hand, from the Cauchy problem (83)–(84) we can write

$$\varsigma_i(t) = \varsigma_0 - \int_0^t H_{ad}\big(\varsigma_i(s), R_\nu(u_{i\nu}^1(s) + u_{i\nu}^2(s)), \boldsymbol{R}_\tau(\boldsymbol{u}_{i\tau}^1(s) - \boldsymbol{u}_{i\tau}^2(s))\big) ds$$

and then

$$\Big\|\varsigma_1(t) - \varsigma_2(t)\Big\|_{L^2(\Gamma_3)} \leq C \int_0^t \Big\|\varsigma_1(s) - \varsigma_2(s)\Big\|_{L^2(\Gamma_3)} ds + C \int_0^t \Big\|R_\nu(u_{1\nu}^1 + u_{1\nu}^2) - R_\nu(u_{2\nu}^1 + u_{2\nu}^2)\Big\|_{L^2(\Gamma_3)} ds$$

$$+ C \int_0^t \Big\|\boldsymbol{R}_\tau(\boldsymbol{u}_{1\tau}^1 - \boldsymbol{u}_{1\tau}^1) - \boldsymbol{R}_\tau(\boldsymbol{u}_{2\tau}^1 - \boldsymbol{u}_{2\tau}^2)\Big\|_{L^2(\Gamma_3)} ds.$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\varsigma_1 = \varsigma_1 - \varsigma_2 + \varsigma_2$, we get

$$\Big\|\varsigma_1(t) - \varsigma_2(t)\Big\|_{L^2(\Gamma_3)} \leq C\Big(\int_0^t \|\varsigma_1(s) - \varsigma_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \Big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\Big\|_{L^2(\Gamma_3)^d} ds\Big).$$

Next, we apply Gronwall's inequality to deduce

$$\|\varsigma_1(t) - \varsigma_2(t)\|_{L^2(\Gamma_3)} \leq C \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d} ds$$

and from the relation (30) we obtain

$$\|\varsigma_1(t) - \varsigma_2(t)\|_{L^2(\Gamma_3)}^2 \leq C \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{\boldsymbol{V}}^2 ds. \tag{100}$$

Furthermore, from 86 we have

$$\|\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)\|_W^2 \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|_Y^2 ds. \tag{101}$$

We use now (80), (31), (35) and (39) to find

$$\|\varphi_1(t) - \varphi_2(t)\|_W^2 \leq C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2. \tag{102}$$

We substitute (94), (100) and (102) in (95) to obtain

$$\|\Pi(\pi_1, \lambda_1, \theta_1)(t) - \Pi(\pi_2, \lambda_1, \theta_2)(t)\|_{\boldsymbol{V}' \times I_0 \times Y}^2$$

$$\leq C\Big(\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2 \, ds + \|\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t))\|_Y^2 + \int_0^t \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s))\|_Y^2 ds$$

$$+ \|\xi_1(t) - \xi_2(t))\|_{I_0}^2 + \int_0^t \|\xi_1(s) - \xi_2(s))\|_{I_0}^2 \, ds\Big). \tag{103}$$

On the other hand, from (85) we deduce that

$$(\dot{\xi}_1 - \dot{\xi}_2, \xi_1 - \xi_2)_{I_0} + a(\xi_1 - \xi_2, \xi_1 - \xi_2) \le \big(\lambda_1 - \lambda_2, \xi_1 - \xi_2\big)_{I_0}, \text{ a.e. } t \in (0, T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\xi_1(0) = \xi_2(0) = \xi_0$ and inequality $a(\xi_1 - \xi_2, \xi_1 - \xi_2) \ge 0$, to find

$$\frac{1}{2}\|\xi_1(t) - \xi_2(t)\|^2_{I_0} \le \int_0^t \big(\lambda_1(s) - \lambda_2(s), \xi_1(s) - \xi_2(s)\big)_{I_0}\, ds,$$

which implies that

$$\|\xi_1(t) - \xi_2(t)\|^2_{I_0} \le \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{I_0}\, ds + \int_0^t \|\xi_1(s) - \xi_2(s)\|^2_{I_0}\, ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\xi_1(t) - \xi_2(t)\|^2_{I_0} \le C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{I_0}\, ds \quad \forall t \in [0, T]. \tag{104}$$

We substitute (99), (101) and (104) in (103) to obtain

$$\|\Pi(\pi_1, \lambda_1, \theta_1)(t) - \Pi(\pi_2, \lambda_2, \theta_2)(t)\|^2_{\boldsymbol{V}' \times I_0 \times Y} \le C \int_0^t \|(\pi_1, \lambda_1, \theta_1)(s) - (\pi_2, \lambda_2, \theta_2)(s)\|^2_{\boldsymbol{V}' \times I_0 \times Y}\, ds.$$

Reiterating this inequality n times we obtain

$$\|\Pi^n(\pi_1, \lambda_1, \theta_1) - \Pi^n(\pi_2, \lambda_2, \theta_2)\|^2_{L^2(0,T;\boldsymbol{V}' \times I_0 \times Y)} \le \frac{C^n T^n}{n!}\|(\pi_1, \lambda_1, \theta_1) - (\pi_2, \lambda_2, \theta_2)\|^2_{L^2(0,T;\boldsymbol{V}' \times I_0 \times Y)}.$$

Thus, for n sufficiently large, Π^n is a contraction on the Banach space $L^2(0, T; \boldsymbol{V}' \times I_0 \times Y)$, and so Π has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. Let $(\pi^*, \lambda^*, \theta^*) \in L^2(0, T; \boldsymbol{V}' \times I_0 \times Y)$ be the fixed point of Π defined by (90)–(93) and denote by

$$\boldsymbol{u}_* = \boldsymbol{u}_{\pi^*}, \quad \varphi_* = \varphi_{\pi^*}, \quad \xi_* = \xi_{\lambda^*}, \quad \boldsymbol{k}_* = \boldsymbol{k}_{\theta^*}, \quad \varsigma_* = \varsigma_{\pi^*}. \tag{105}$$

Let by $\boldsymbol{\sigma}_* = (\boldsymbol{\sigma}^1_*, \boldsymbol{\sigma}^2_*) : [0, T] \to \mathcal{H}$ the functions defined by

$$\boldsymbol{\sigma}^m_* = \mathcal{B}^m \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^m_*) + (\mathcal{E}^m)^* \nabla \varphi^m_* + \boldsymbol{\sigma}^m_{\pi^* \lambda^* \theta^*}, \quad m = 1, 2. \tag{106}$$

We prove that the $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \xi_*, \boldsymbol{k}_*, \varsigma_*\}$ satisfies (54)–(60) and the regularites (61)–(68). Indeed, we write (70) for $\pi = \pi^*$ and use (105) to find

$$\begin{aligned}(\ddot{u}_*(t), v)_{\boldsymbol{V}' \times \boldsymbol{V}} &+ \sum_{m=1}^{2} (\mathcal{B}^m \varepsilon(\dot{\boldsymbol{u}}^m_*(t)),\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + j_{dm}(\dot{\boldsymbol{u}}_*(t), \boldsymbol{v}) \\ &+(\pi^*(t), v)_{\boldsymbol{V}' \times \boldsymbol{V}} = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in [0, T].\end{aligned} \tag{107}$$

We use equalities $\Pi^1(\pi^*, \lambda^*, \theta^*) = \pi^*$, $\Pi^2(\pi^*, \lambda^*, \theta^*) = \theta^*$ and $\Pi^3(\pi^*, \lambda^*, \theta^*) = \lambda^*$ it follows that

$$\begin{aligned}(\pi_*(t), v)_{\boldsymbol{V}' \times \boldsymbol{V}} = j_{ad}(\varsigma_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) &+ \sum_{m=1}^{2} \big(\mathcal{A}^m \varepsilon(\boldsymbol{u}^m_*(t)),\ \varepsilon(\boldsymbol{v}^m)\big)_{\mathcal{H}^m} + \sum_{m=1}^{2} \big((\mathcal{E}^m)^* \nabla \varphi^m_*(t),\ \varepsilon(\boldsymbol{v}^m)\big)_{\mathcal{H}^m} \\ + \sum_{m=1}^{2} \Big(\int_0^t \mathcal{Q}^m\big(\boldsymbol{\sigma}^m_{\pi^* \lambda^* \theta^*}(s), \varepsilon(\boldsymbol{u}^m_*(s)), \xi_*(s), \boldsymbol{k}^m_*(s)\big)\, ds,\ \varepsilon(\boldsymbol{v}^m)\Big)_{\mathcal{H}^m}, &\ \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{108}$$

$$\theta^m_*(t) = \phi^m\big(\boldsymbol{\sigma}^m_{\pi^* \lambda^* \theta^*}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^m_*(t)), \boldsymbol{k}^m_*(t)\big),\ m = 1, 2, \tag{109}$$

$$\lambda^m_*(t) = \Psi^m\big(\boldsymbol{\sigma}^m_{\pi^* \lambda^* \theta^*}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^m_*(t)), \xi^m_*(t)\big),\ m = 1, 2. \tag{110}$$

We now substitute (108) in (107) to obtain

$$
\begin{aligned}
&(\ddot{\boldsymbol{u}}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{m=1}^{2}(\mathcal{B}^m\varepsilon(\dot{\boldsymbol{u}}_*^m(t)),\ \varepsilon(\boldsymbol{v}^m))_{\mathcal{H}^m} + j_{dm}(\dot{\boldsymbol{u}}_*(t), \boldsymbol{v})\\
&+\sum_{m=1}^{2}\left(\mathcal{A}^m\varepsilon(\boldsymbol{u}_*^m(t)),\ \varepsilon(\boldsymbol{v}^m)\right)_{\mathcal{H}^m} + \sum_{m=1}^{2}\left((\mathcal{E}^m)^*\nabla\varphi_*^m,\ \varepsilon(\boldsymbol{v}^m)\right)_{\mathcal{H}^m}\\
&+\sum_{m=1}^{2}\Big(\int_0^t \mathcal{Q}^m\left(\boldsymbol{\sigma}^m_{\pi^*\lambda^*\theta^*}(s), \varepsilon(\boldsymbol{u}_*^m(s)), \xi_*^m(s), \boldsymbol{k}_*^m(s)\right) ds,\ \varepsilon(\boldsymbol{v}^m)\Big)_{\mathcal{H}^m}\\
&+j_{ad}(\varsigma_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) = (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}},\ \ \forall \boldsymbol{v}\in\boldsymbol{V}.
\end{aligned}
\tag{111}
$$

From (106), (109) and (86) we see that (55) is satisfied.

$$
\dot{\boldsymbol{k}}_*^m(t) = \phi^m\left(\boldsymbol{\sigma}^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^m) - (\mathcal{E}^m)^*\nabla\varphi_*^m(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_*^m), \boldsymbol{k}_*^m\right) \quad m = 1, 2,
\tag{112}
$$

and we substitute (106), (110) in (85) to have

$$
\begin{aligned}
&\xi_*(t)\in K, \qquad \sum_{m=1}^{2}(\dot{\xi}_*^m(t), \zeta^m - \xi_*^m(t))_{L^2(\Omega^m)} + a(\xi_*(t), \zeta - \xi_*(t)) \geq\\
&\sum_{m=1}^{2}\Big(\Psi^{\ell}\left(\boldsymbol{\sigma}_*^m - \mathcal{B}^m\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^m) - (\mathcal{E}^m)^*\nabla\varphi_*^m, \boldsymbol{\varepsilon}(\boldsymbol{u}_*^m), \xi_*^m\right), \zeta^m - \xi_*^m(t)\Big)_{L^2(\Omega^m)},\quad \forall\zeta\in K,\ \text{a.e. } t\in[0,T].
\end{aligned}
\tag{113}
$$

We write now (80) for $\pi = \pi^*$ and use (105) to see that

$$
\sum_{m=1}^{2}(\boldsymbol{\beta}^m\nabla\varphi_*^m(t), \nabla\Psi^m)_{H^m} - \sum_{m=1}^{2}(\mathcal{E}^m\varepsilon(\boldsymbol{u}_*^m(t)), \nabla\Psi^m)_{H^m} = (q(t), \Psi)_W,\ \forall\Psi\in W,\ \ \text{a.e. } t\in[0,T].
\tag{114}
$$

Additionally, we use $\boldsymbol{u}_{\pi^*}$ in (83) and (105) to find

$$
\dot{\varsigma}_*(t) = H_{ad}\big(\varsigma_*(t), R_\nu(\boldsymbol{u}^1_{*\nu}(t) + \boldsymbol{u}^2_{*\nu}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}^1_{*\tau}(t) - \boldsymbol{u}^2_{*\tau}(t))\big),\ \text{a.e. } t\in[0,T].
\tag{115}
$$

The relations (105)–(106), (111)–(115) allow us to conclude now that $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \xi_*, \boldsymbol{k}_*, \varsigma_*\}$ satisfies (54)–(59). Next, (60) and the regularity (61), (65), (66) and (67) follow from Lemmas 1, 2, 4 and 3. Since $\boldsymbol{u}_*$ and φ_* satisfy (61) and (67), it follows from lemma 5 and (106) that

$$
\sigma_* \in L^2(0, T; \mathcal{H}).
\tag{116}
$$

We choose $v = (v^1, v^2)$ with $v^m = \omega^m \in D(\Omega^m)^d$ and $v^{3-m} = 0$ in (111) and by (105) and (48):

$$
\rho^m\ddot{\boldsymbol{u}}_*^m = \operatorname{Div}\boldsymbol{\sigma}_*^m + \boldsymbol{f}_0^m,\ \text{a.e. } t\in[0,T],\ \ m = 1, 2.
$$

Also, by (43), (44), (61) and (116) we have:

$$
(\operatorname{Div}\boldsymbol{\sigma}_*^1, \operatorname{Div}\boldsymbol{\sigma}_*^2) \in L^2(0, T; \boldsymbol{V}').
$$

Finally we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \zeta_*, \boldsymbol{k}_*, \varsigma_*\}$ of the piezoelectric contact Problem **PV** has the regularity (61)–(68), which concludes the existence part of Theorem 1. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Π defined by (90)-(93) and the unique solvability of the Problems PV^u_π, PV^φ_π, PV^ς_π, PV^ξ_λ and $\text{PV}^\sigma_{\pi\lambda\theta}$.

CONCLUSION

We presented a model for the dynamic process of frictional contact between two electro-elasto-viscoplastic bodies with internal state variables and damage. The contact was modeled with the normal damped and adhesion, and the adhesion be written by the differential equation of the form (21). The difficulty of solving this type of problem lies not only in the coupling of elasto-viscoplastic, electrical and internal state variables, but also in the nonlinearity of the boundary conditions modeling this type of physical phenomena (contact and friction conditions), which gives us a variational inequalities and type of nonlinear, parabolic variational equalities. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities, parabolic equalities and fixed point theorem.

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