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ARTICLE TYPE

The incomplete exponential ${}_pR_q(\alpha, \beta; z)$ function with some applications †

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Abstract

In this paper, we are motivated to establish a generalization of ${}_pR_q(\alpha, \beta; z)$ function in terms of incomplete exponential functions and obtain some properties of the incomplete exponential ${}_pR_q(\alpha, \beta; z)$ function. Further we give generating relations for incomplete exponential ${}_pR_q(\alpha, \beta; z)$ function. Some applications related to ground water pumping (Hydrology) modelling and probability theory have also been discussed .

KEYWORDS:
Incomplete gamma function; Incomplete exponential function; Generalized Hypergeometric function; Generating function.

1 | INTRODUCTION AND PRELIMINARIES

The theory of incomplete gamma functions was studied by Tricomi[23] in 1950 . These functions are very essential special functions and that are used in number of problems in mathematical physics, astrophysics, applied statistics and engineering. These functions are also useful in the study of various transform such as Fourier transform and Laplace transforms and probability theory.

Srivastava et al.[20] defined the familiar incomplete gamma functions $\gamma(\omega, x)$ and $\Gamma(\omega, x)$ as,

$$\gamma(\omega, x) := \int_0^x u^{\omega-1} e^{-u} du, \qquad (\Re(\omega) > 0; x \geq 0)\,, \tag{1}$$

and

$$\Gamma(\omega, x) := \int_x^\infty u^{\omega-1} e^{-u} du, \qquad (\Re(\omega) > 0; x \geq 0), \tag{2}$$

respectively and also satisfy the following decomposition relation:

$$\gamma(\omega, x) + \Gamma(\omega, x) = \Gamma(\omega), \quad (\Re(\omega) > 0). \tag{3}$$

The incomplete Pochhammer symbols $(\omega; x)_n$ and $[\omega; x]_n$ can also be represented in terms of incomplete gamma functions $\gamma(\omega, x)$ and $\Gamma(\omega, x)$[20] as,

$$(\omega; x)_n := \frac{\gamma(\omega + n, x)}{\Gamma(\omega)}, \quad (\omega, n \in \mathbf{C}; x \geq 0), \tag{4}$$

†This is an example for title footnote.

and

$$[\omega; x]_n := \frac{\Gamma(\omega + n, x)}{\Gamma(\omega)}, \qquad (\omega, n \in \mathbf{C}; x \geq 0). \tag{5}$$

In the view of (3), (4) and (5) give the following decomposition relation:

$$(\omega; x)_n + [\omega; x]_n = (\omega)_n := \frac{\Gamma(\omega + n)}{\Gamma(\omega)} = \begin{cases} 1 & (n = 0; \omega \in \mathbf{C} \setminus \{0\}) \\ \omega(\omega + 1) \dots (\omega + n - 1) & (n \in \mathbf{N}; \omega \in \mathbf{C}). \end{cases} \tag{6}$$

where $(\omega)_n$ is the Pochhammer symbol[18]. In particular, the generalized pochhammer symbol $(\omega)_{kn}$ can be represented in the following form[18]:

$$(\omega)_{kn} = k^{kn} \left(\frac{\omega}{k}\right)_n \left(\frac{\omega + 1}{k}\right)_n \cdots \cdots \left(\frac{\omega + k - 1}{k}\right)_n. \tag{7}$$

The generalized hypergeometric function[7] is defined by

$${}_pF_q \left[\begin{matrix} u_1, u_2, \dots, u_p \\ v_1, v_2, \dots, v_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(u_1)_n (u_2)_n \dots (u_p)_n}{(v_1)_n (v_2)_n \dots (v_q)_n} \frac{z^n}{n!}, \quad |z| < 1, \tag{8}$$

where $p, q \in \mathbf{Z}^+ \cup \{0\}$ and $u_1, u_2, \dots, u_p \in \mathbf{C}$ and $v_1, v_2, \dots, v_q \in \mathbf{C} \setminus \mathbf{Z}_0^-$. Here, $(u)_n$ is a Pochhammer symbol defined in (6).

Recently, Desai and Shukla[5,6] introduced the ${}_pR_q(\alpha, \beta; z)$ function, this defined as,

$$\begin{aligned} {}_pR_q(\alpha, \beta; z) &= {}_pR_q \left[\begin{matrix} u_1, u_2, \dots, u_p \\ v_1, v_2, \dots, v_q \end{matrix} \middle| \alpha, \beta; z \right], \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{(u_1)_n \dots (u_p)_n}{(v_1)_n \dots (v_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{9}$$

where $\alpha, \beta \in \mathbf{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0$ and $\Re(u_i) > 0, \Re(v_j) > 0$; for all $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$. Here, $(u)_n$ is a Pochhammer symbol defined by (6).

The series (9) is defined when $v_j \in \mathbf{C} \setminus \mathbf{Z}_0^-$. If any numerator parameter u_i $(i = 1, 2, \dots, p)$ is zero or negative integer, then the series ceases to polynomial in z.

The ${}_pR_q(\alpha, \beta; z)$ function satisfies the following modified convergence criteria:

i. If $\Re(\alpha) \geq p - q$, the series converges for all finite values of z.

ii. If $\Re(\alpha) = p - q - 1$, the series converges for all $|z| < 1$ and diverges for $|z| > 1$.

iii. When $\Re(\alpha) = p - q - 1$ and $|z| = 1$, the series can converges on condition depending on the parameters. If $\alpha = p - q - 1$, the series in (9) is absolutely convergent on the circle $|z| = 1$ if, $\Re\left(\alpha + \sum_{j=1}^{q} v_j - \sum_{i=1}^{p} u_i\right) > 0$.

2 | THE INCOMPLETE EXPONENTIAL ${}_PR_Q(\alpha, \beta; Z)$ FUNCTION

In 2005, Chaudhry and Qadir[2] motivated to study the incomplete gamma function (Erdélyi et al.[8]). Further, they gave the incomplete exponential functions as mentioned below,

$$e\left((x, t); \mu\right) := \sum_{n=0}^{\infty} \frac{\gamma\left(\mu + n, x\right)}{\Gamma\left(\mu + n\right)} \frac{t^n}{n!}, \tag{10}$$

$$E\left((x, t); \mu\right) := \sum_{n=0}^{\infty} \frac{\Gamma\left(\mu + n, x\right)}{\Gamma\left(\mu + n\right)} \frac{t^n}{n!}. \tag{11}$$

From (10) and (11), one can obtain the following result,

$$e\left((x,t);\mu\right) + E\left((x,t);\mu\right) = e^{t}. \tag{12}$$

The following integral representations[2] of the incomplete exponential functions (10) and (11) are important for our further study.

Lemma 1. The following integral representations holds true for $e\left((x,t);\mu\right)$ and $E\left((x,t);\mu\right)$:

$$\begin{aligned} e\left((x,t);\mu\right) &= \frac{1}{\Gamma(\mu)}\int_0^x u^{\mu-1}\, e^{-u}\left(\sum_{n=0}^{\infty}\frac{1}{(\mu)_n}\frac{(ut)^n}{n!}\right)du \\ &= \frac{1}{\Gamma(\mu)}\int_0^x u^{\mu-1}e^{-u}{}_0F_1(-;\mu;tu)du, \end{aligned} \tag{13}$$

and

$$\begin{aligned} E\left((x,t);\mu\right) &= \frac{1}{\Gamma(\mu)}\int_x^\infty u^{\mu-1}\, e^{-u}\left(\sum_{n=0}^{\infty}\frac{1}{(\mu)_n}\frac{(ut)^n}{n!}\right)du \\ &= \frac{1}{\Gamma(\mu)}\int_x^\infty u^{\mu-1}e^{-u}{}_0F_1(-;\mu;tu)du, \end{aligned} \tag{14}$$

In sequel to the study of the generalized incomplete exponential functions (10) and (11), we define the following generalized incomplete exponential ${}_pR_q(\alpha,\beta;z)$ functions as,

$$\begin{aligned} {}_pe_q(x,\alpha,\beta;\upsilon) &= {}_pe_q\left[(x,\alpha,\beta;\upsilon)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right.\right], \\ &= \sum_{n=0}^{\infty}\frac{\gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\dots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\dots(\upsilon_q)_n}\frac{\upsilon^n}{n!}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} {}_pE_q(x,\alpha,\beta;\upsilon) &= {}_pE_q\left[(x,\alpha,\beta;\upsilon)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right.\right], \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\dots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\dots(\upsilon_q)_n}\frac{\upsilon^n}{n!}, \end{aligned} \tag{16}$$

where $\alpha,\beta \in \mathbf{C}$, $u_1,u_2,\dots,u_p \in \mathbf{C}$ and $\upsilon_1,\upsilon_2,\dots,\upsilon_q \in \mathbf{C}\setminus\mathbf{Z}_0^-$ provided that the series defined by (15) and (16) on r.h.s. converges.

From (15) and (16), we can obtain the following decomposition formula:

$${}_pe_q\left[(x,\alpha,\beta;\upsilon)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right.\right] + {}_pE_q\left[(x,\alpha,\beta;\upsilon)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right.\right] = {}_pF_q\left[\left.\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right|\upsilon\right], \tag{17}$$

where ${}_pF_q(\cdot)$ recognize as the generalized hypergeometric function defined by (8).

Remark 1. For $p=0, q=0$ and $\alpha=1$, (15) and (16) reduces to incomplete exponential functions (10) and (11):

$$\begin{aligned} {}_0e_0(x,1,\beta;\upsilon) &= {}_0e_0\left[(x,1,\beta;\upsilon)\left|\begin{matrix}-\\ -\end{matrix}\right.\right], \\ &= \sum_{n=0}^{\infty}\frac{\gamma(n+\beta,x)}{\Gamma(n+\beta)}\frac{\upsilon^n}{n!} \\ &= e\left((x,\upsilon);\beta\right), \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
{}_0E_0(x,1,\beta;\upsilon) &= {}_0E_0\left[(x,1,\beta;\upsilon)\middle|\begin{matrix}-\\-\end{matrix}\right],\\
&= \sum_{n=0}^{\infty}\frac{\Gamma(n+\beta,x)}{\Gamma(n+\beta)}\frac{\upsilon^n}{n!}\\
&= E\left((x,\upsilon);\beta\right). \qquad (19)
\end{aligned}
$$

Theorem 1 (First integral representation). The generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function satisfies the following integral representation:

$$
{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right] = \int_x^{\infty} u^{\beta-1}\,e^{-u}\;{}_pR_q\left[\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\middle|\alpha,\beta;\upsilon u^{\alpha}\right]du, \qquad (20)
$$

where $\alpha,\beta,u_i,\upsilon_j\in\mathbf{C}$ and $\Re(\alpha)>0,\Re(\beta)>0,\Re(u_i)>0,\Re(\upsilon_j)>0,\ \forall i=1,2,\dots,p,\ \forall j=1,2,\dots,q.$

Proof. On using the integral representation of incomplete gamma function defined by (2), we arrive at

$$
{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,\dots,u_p\\ \upsilon_1,\dots,\upsilon_q\end{matrix}\right] = \int_x^{\infty} u^{\alpha n+\beta-1}\,e^{-u}\left(\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n\dots(u_p)_n}{(\upsilon_1)_n\dots(\upsilon_q)_n}\frac{\upsilon^n}{n!}\right)du.
$$

Further simplification by reversing the order of summation and integration yields the r.h.s. of assertion (20). □

Corollary 1 ([2]). By setting $\alpha=1,\beta=c$, and $p=1,q=0$ i.e. $u_1=a$, (20) reduces to

$$
{}_1E_0\left[(x,1,c;\upsilon)\middle|\begin{matrix}a\\-\end{matrix}\right] = \frac{1}{\Gamma(c)}\int_x^{\infty} u^{c-1}\,e^{-u}\;{}_1F_1\left[\begin{matrix}a\\c\end{matrix}\middle|\upsilon u\right]du.
$$

Corollary 2. For ${}_pR_q(\alpha,\beta;z)$ function, the following integral representation holds true:

$$
{}_pR_q\left[\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\middle|\alpha,\beta;z\right] = \frac{1}{\Gamma(u_1)}\int_0^{\infty} u^{u_1-1}e^{-u}\;{}_{p-1}R_q\left[\begin{matrix}u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\middle|\alpha,\beta;uz\right]du.
$$

where $p\le q+1,\ \Re(u_1)>0$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(\upsilon_j)>0,\ \forall i=2,\dots,p,\ \forall j=2,\dots,q.$

Theorem 2 (Second integral representation). The generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function have the following integral representation:

$$
\begin{aligned}
{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\dots,u_p\\ \upsilon_1,\upsilon_2,\dots,\upsilon_q\end{matrix}\right] &= \frac{\Gamma(\upsilon_1)}{\Gamma(u_1)\Gamma(\upsilon_1-u_1)}\int_0^1 u^{u_1-1}(1-u)^{\upsilon_1-u_1-1}\\
&\quad\times{}_{p-1}E_{q-1}\left[(x,\alpha,\beta;u\upsilon)\middle|\begin{matrix}u_2,\dots,u_p\\ \upsilon_2,\dots,\upsilon_q\end{matrix}\right]du, \qquad (21)
\end{aligned}
$$

where $\Re(\upsilon_1)>\Re(u_1)>0$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(\upsilon_j)>0,\ \forall i=2,\dots,p,\ \forall j=2,\dots,q.$

Proof. By adopting the following elementary integral definition of Beta function $\mathbf{B}(\gamma,\delta)$:

$$
\frac{(l_1)_n}{(l_2)_n} = \frac{\mathbf{B}(l_1+n,l_2-l_1)}{\mathbf{B}(l_1,l_2-l_1)} = \frac{1}{\mathbf{B}(l_1,l_2-l_1)}\int_0^1 u^{l_1+n-1}\,(1-u)^{l_2-l_1-1}\,du,
$$

in the l.h.s. of (21), this yields

$$
{}_pE_q\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ v_1,v_2,\dots,v_q\end{matrix}\right.\right] = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n\cdots(u_p)_n}{(v_1)_n\dots(v_q)_n}\frac{v^n}{n!}
$$
$$
= \frac{1}{\mathbf{B}(u_1,v_1-u_1)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_2)_n\cdots(u_p)_n}{(v_2)_n\dots(v_q)_n}\frac{v^n}{n!}
$$
$$
\times\int_0^1 u^{u_1+n-1}\,(1-u)^{v_1-u_1-1}\,du.
$$

Further simplification by reversing the order of summation and integration yields the r.h.s. of assertion (21). □

Corollary 3 ([5]). For the ${}_pR_q(\alpha,\beta;z)$ function, we have the following integral representation:

$$
{}_pR_q\left[\begin{matrix}u_1,u_2,\dots,u_p\\ v_1,v_2,\dots,v_q\end{matrix}\left|\alpha,\beta;z\right.\right] = \frac{\Gamma(v_1)}{\Gamma(u_1)\Gamma(v_1-u_1)}\int_0^1 u^{u_1-1}(1-u)^{v_1-u_1-1}
$$
$$
\times\,{}_{p-1}R_{q-1}\left[\begin{matrix}u_2,\dots,u_p\\ v_2,\dots,v_q\end{matrix}\left|\alpha,\beta;uz\right.\right]du,
$$

where $p\le q+1$, $\Re(v_1)>\Re(u_1)>0$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(v_j)>0$, $\forall i=2,\dots,p$, $\forall j=2,\dots,q$.

Theorem 3 (Derivative formula). The generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;v)$ function have the following derivative formula:

$$
\frac{d^n}{dv^n}\left\{{}_pE_q\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ v_1,v_2,\dots,v_q\end{matrix}\right.\right]\right\} = \frac{(u_1)_n\dots(u_p)_n}{(v_1)_n\dots(v_q)_n}
$$
$$
\times\,{}_pE_q\left[(x,\alpha,\alpha+\beta;v)\left|\begin{matrix}u_1+n,\dots,u_p+n\\ v_1+n,\dots,v_q+n\end{matrix}\right.\right], \tag{22}
$$

where $\alpha,\beta,u_i,v_j\in\mathbf{C}$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(v_j)>0$, $\forall i=1,2,\dots,p$, $\forall j=1,2,\dots,q$.

Proof. Differentiating (16) with respect to v and replacing n by $n+1$, we arrive at

$$
\frac{d}{dv}\left\{{}_pE_q\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2,\dots,u_p\\ v_1,v_2,\dots,v_q\end{matrix}\right.\right]\right\} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\alpha+\beta,x)}{\Gamma(\alpha n+\alpha+\beta)}\frac{(u_1)_{n+1}\cdots(u_p)_{n+1}}{(v_1)_{n+1}\dots(v_q)_{n+1}}\frac{v^n}{n!}.
$$

Using the relation $(u)_{n+1}=u(u+1)_n$, we get

$$
\frac{d}{dv}\left\{{}_pE_q\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2,\dots,a_p\\ v_1,v_2,\dots,v_q\end{matrix}\right.\right]\right\} = \frac{u_1\dots u_p}{v_1\dots v_q}\times{}_pE_q\left[(x,\alpha,\alpha+\beta;v)\left|\begin{matrix}u_1+1,\dots,u_p+1\\ v_1+1,\dots,v_q+1\end{matrix}\right.\right].
$$

By repeating above procedure n-times yields the r.h.s. of assertion (22). □

Corollary 4. For the ${}_pR_q(\alpha,\beta;z)$ function, we have the following derivative formula:

$$
\frac{d^n}{dz^n}\left\{{}_pR_q\left[\begin{matrix}u_1,u_2,\dots,a_p\\ v_1,v_2,\dots,v_q\end{matrix}\left|\alpha,\beta;z\right.\right]\right\} = \frac{(u_1)_n\dots(u_p)_n}{(v_1)_n\dots(v_q)_n}
$$
$$
\times\,{}_pR_q\left[\begin{matrix}u_1+n,u_2+n,\dots,u_p+n\\ v_1+n,v_2+n,\dots,v_q+n\end{matrix}\left|\alpha,\alpha+\beta;z\right.\right],
$$

where $\alpha,\beta,u_i,v_j\in\mathbf{C}$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(v_j)>0$, $\forall i=1,2,\dots,p$, $\forall j=1,2,\dots,q$.

Theorem 4 (Partial Derivatives). For the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function, the following partial derivatives holds true:

$$\frac{\partial}{\partial \upsilon}\left\{{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right]\right\}=\frac{u_1\ldots u_p}{\upsilon_1\ldots\upsilon_q}\times{}_pE_q\left[(x,\alpha,\alpha+\beta;\upsilon)\middle|\begin{matrix}u_1+1,\ldots,u_p+1\\ \upsilon_1+1,\ldots,\upsilon_q+1\end{matrix}\right],\tag{23}$$

$$\frac{\partial}{\partial x}\left\{{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right]\right\}=-e^{-x}x^{\beta-1}{}_pR_q\left[\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\middle|\alpha,\beta;\upsilon x^{\alpha}\right],\tag{24}$$

where $\alpha,\beta,u_i,\upsilon_j\in\mathbf{C}$ and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(\upsilon_j)>0,\ \forall i=1,2,\ldots,p,\ \forall j=1,2,\ldots,q.$

Proof. Differentiating partially (16) with respect to υ and treating x as a constant, we have

$$\frac{\partial}{\partial \upsilon}\left\{{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right]\right\}=\frac{\partial}{\partial \upsilon}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\ldots(\upsilon_q)_n}\frac{\upsilon^n}{n!}\right\},$$
$$=\sum_{n=1}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\ldots(\upsilon_q)_n}\frac{\upsilon^{n-1}}{(n-1)!}.$$

By replacing n by $n+1$, this leads to proof of (23).

For the proof of (24), we differentiate partially first integral representation (20) with respect to x and treating υ as a constant. □

Next we evaluate the ${}_2E_1(x,\alpha,\beta;\upsilon)$ function which is a special case of generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function.

Theorem 5 (Estimation of ${}_2E_1(x,\alpha,\beta;\upsilon)$ function). Evaluation of the incomplete exponential ${}_2E_1(x,\alpha,\beta;\upsilon)$ function is given by

$${}_2E_1\left[(x,\alpha,\beta;1)\middle|\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\right]=\frac{\Gamma(\upsilon_1)\Gamma(\upsilon_1-u_1-u_2)}{\Gamma(\upsilon_1-u_1)\Gamma(\upsilon_1-u_2)}-\gamma(\alpha n+\beta,x)\times{}_2R_1(\alpha,\beta;\upsilon),\tag{25}$$

where $\alpha,\beta,u_1,u_2,\upsilon_1\in\mathbf{C}$ and $\Re(u_1),\Re(u_2),\Re(\upsilon_1)>0$ and $\Re(\alpha),\Re(\beta)>0$.

Proof. Setting $\upsilon=1,p=2,q=1$ in the decomposition formula (17), we get

$${}_2E_1\left[(x,\alpha,\beta;1)\middle|\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\right]={}_2F_1\left[\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\middle|1\right]-{}_2e_1\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\right],$$
$$={}_2F_1\left[\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\middle|1\right]-\int_0^x u^{\beta-1}e^{-u}\,{}_2R_1\left[\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\middle|\alpha,\beta;\upsilon u^{\alpha}\right]du,$$
$$={}_2F_1\left[\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\middle|1\right]-\int_0^x u^{\beta-1}e^{-u}\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n}{(\upsilon_1)_n}\frac{(\upsilon u^{\alpha})^n}{(n)!}du.$$

By using Gauss summation formula (see [18]) for $\upsilon=1$ and reversing the order of summation and integration, we arrive at

$${}_2E_1\left[(x,\alpha,\beta;1)\middle|\begin{matrix}u_1,u_2\\ \upsilon_1\end{matrix}\right]=\frac{\Gamma(\upsilon_1)\Gamma(\upsilon_1-u_1-u_2)}{\Gamma(\upsilon_1-u_1)\Gamma(\upsilon_1-u_2)}-\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n}{(\upsilon_1)_n}\frac{(\upsilon)^n}{(n)!}\times\int_0^x u^{\alpha n+\beta-1}e^{-u}du.$$

Further simplification by using (1) and (9), this leads to the right hand side of (25). □

Theorem 6 (**Addition formula).** For the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function, the following addition formula for addition of two argument is valid:

$$ {}_pE_q\left[(x,\alpha,\beta;y+\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right] = \frac{\Gamma(\upsilon_1)\ldots\Gamma(\upsilon_q)}{\Gamma(u_1)\ldots\Gamma(u_p)}\sum_{n=0}^{\infty}\frac{\Gamma(u_1+n)\ldots\Gamma(u_p+n)}{\Gamma(\upsilon_1+n)\ldots\Gamma(\upsilon_q+n)}\frac{\upsilon^n}{n!} \times {}_pE_q\left[(x,\alpha,\alpha n+\beta;y)\middle|\begin{matrix}u_1+n,\ldots,u_p+n\\ \upsilon_1+n,\ldots,\upsilon_q+n\end{matrix}\right]. \tag{26} $$

Proof. Proof of above theorem pursue from the derivative formula of ${}_pE_q(x,\alpha,\beta;\upsilon)$ function:

$$ \frac{d^n}{d\upsilon^n}\left\{{}_pE_q\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right]\right\} = \frac{(u_1)_n\cdots(u_p)_n}{(\upsilon_1)_n\cdots(\upsilon_q)_n} \times {}_pE_q\left[(x,\alpha,\alpha+\beta;\upsilon)\middle|\begin{matrix}u_1+n,\ldots,u_p+n\\ \upsilon_1+n,\ldots,\upsilon_q+n\end{matrix}\right]. $$

□

Theorem 7 (**Multiplication formula).** For the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function, the following multiplication formula for multiplication of two argument is valid:

$$ {}_pE_q\left[(x,\alpha,\beta;y\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right] = \frac{\Gamma(\upsilon_1)\ldots\Gamma(\upsilon_q)}{\Gamma(u_1)\ldots\Gamma(u_p)}\sum_{n=0}^{\infty}\frac{\Gamma(u_1+n)\ldots\Gamma(u_p+n)}{\Gamma(\upsilon_1+n)\ldots\Gamma(\upsilon_q+n)}\frac{y^n(\upsilon-1)^n}{n!} \times {}_pE_q\left[(x,\alpha,\alpha n+\beta;y)\middle|\begin{matrix}u_1+n,\ldots,u_p+n\\ \upsilon_1+n,\ldots,\upsilon_q+n\end{matrix}\right]. \tag{27} $$

Proof. Proof of above theorem is similar to Theorem 6. □

Theorem 8. For the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function, we have the following integral representation:

$$ \int_0^t u^{\alpha-1}(t-u)^{\beta-1}{}_pE_q\left[(x,\alpha,\beta;\lambda u^k)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right]du = \mathbf{B}(\alpha,\beta)\,t^{\alpha+\beta-1} \times {}_{p+k}E_{q+k}\left[(x,\alpha,\beta;\lambda t^k)\middle|\begin{matrix}u_1,u_2,\ldots,u_p,\Delta(k,\alpha)\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q,\Delta(k,\alpha+\beta)\end{matrix}\right], \tag{28} $$

where $\Delta(k,\alpha)$ represents the sequence of k parameters i.e.

$$ \frac{\alpha}{k},\frac{\alpha+1}{k},\frac{\alpha+2}{\alpha},\ldots,\frac{\alpha+k-1}{k}, $$

and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(\upsilon_j)>0,\ \forall i=1,2,\ldots,p,\ \forall j=1,2,\ldots,q.$

Proof. Let $\mathcal{A}_1$ be the l.h.s. of (28). Then, using (16), this gives

$$ \mathcal{A}_1 = \int_0^t u^{\alpha-1}(t-u)^{\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\cdots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\cdots(\upsilon_q)_n}\frac{(\lambda u^k)^n}{n!}du. $$

Substituting $u=tx$, we arrive at

$$ \begin{aligned} \mathcal{A}_1 &= t^{\alpha+\beta-1}\int_0^1 x^{\alpha+kn-1}(1-x)^{\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\cdots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\cdots(\upsilon_q)_n}\frac{(\lambda t^k)^n}{n!}dx,\\ &= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\cdots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\cdots(\upsilon_q)_n}\frac{(\lambda t^k)^n}{n!}\,\mathbf{B}(\alpha+kn,\beta),\\ &= t^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\cdots(u_p)_n}{(\upsilon_1)_n(\upsilon_2)_n\cdots(\upsilon_q)_n}\frac{(\lambda t^k)^n}{n!}\,\frac{\Gamma(\alpha+kn)\Gamma(\beta)}{\Gamma(\alpha+\beta+kn)}. \end{aligned} $$

Now using the property of Pochhammer symbol defined in (7), this leads to the right hand side of (28). □

Theorem 9. For the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;v)$ function, the following integral representation holds true:

$$\frac{1}{\mathbf{B}(\beta,\delta)}\int_t^x (x-u)^{\delta-1}(u-t)^{\beta-1}{}_pE_q\left[(x,\alpha,\beta;\lambda(u-t)^k)\left|\begin{matrix}u_1,u_2,\ldots,u_p\\ v_1,v_2,\ldots,v_q\end{matrix}\right.\right]du =$$

$$\times\,(x-t)^{\delta+\beta-1}{}_{p+k}E_{q+k}\left[(x,\alpha,\beta;\lambda(x-t)^k)\left|\begin{matrix}u_1,u_2,\ldots,u_p,\Delta(k,\beta)\\ v_1,v_2,\ldots,v_q,\Delta(k,\beta+\delta)\end{matrix}\right.\right], \tag{29}$$

where $\Delta(k,\beta)$ represents the sequence of k parameters i.e.

$$\frac{\beta}{k},\frac{\beta+1}{k},\frac{\beta+2}{\alpha},\ldots,\frac{\beta+k-1}{k},$$

and $\Re(\alpha)>0,\Re(\beta)>0$ and $\Re(u_i)>0,\Re(v_j)>0,\ \forall i=1,2,\ldots,p,\ \forall j=1,2,\ldots,q.$

Proof. Let $\mathcal{A}_2$ be the l.h.s. of (29), and using (16), we get

$$\mathcal{A}_2=\frac{1}{\mathbf{B}(\beta,\delta)}\int_t^x (x-u)^{\delta-1}(u-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(v_1)_n(v_2)_n\ldots(v_q)_n}\frac{(\lambda(u-t)^k)^n}{n!}du.$$

Substituting $m=\frac{u-t}{x-t}$, we arrive at

$$\begin{aligned}\mathcal{A}_2&=\frac{(x-t)^{\delta+\beta-1}}{\mathbf{B}(\beta,\delta)}\int_0^1 m^{kn+\beta-1}(1-m)^{\delta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(v_1)_n(v_2)_n\ldots(v_q)_n}\frac{(\lambda(x-t)^k)^n}{n!}dm,\\ &=\frac{(x-t)^{\delta+\beta-1}}{\mathbf{B}(\beta,\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(v_1)_n(v_2)_n\ldots(v_q)_n}\frac{(\lambda(x-t)^k)^n}{n!}\mathbf{B}(kn+\beta,\delta),\\ &=\frac{(x-t)^{\delta+\beta-1}}{\mathbf{B}(\beta,\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n\ldots(u_p)_n}{(v_1)_n(v_2)_n\ldots(v_q)_n}\frac{(\lambda(x-t)^k)^n}{n!}\frac{\Gamma(kn+\beta)\Gamma(\delta)}{\Gamma(kn+\beta+\delta)}.\end{aligned}$$

Now using the property of Pochhammer symbol defined in (7), this leads to the right hand side of (29). □

Theorem 10 (Recurrence relation). We have the following recurrence relation for the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;v)$ function:

$$(u_1-v_1+1)\,{}_2E_1\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2\\ v_1\end{matrix}\right.\right]=a\,{}_2E_1\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1+1,u_2\\ v_1\end{matrix}\right.\right]$$

$$-(v_1-1)\,{}_2E_1\left[(x,\alpha,\beta;v)\left|\begin{matrix}u_1,u_2\\ v_1-1\end{matrix}\right.\right]. \tag{30}$$

Proof. Let $\mathcal{R}_1$ be the l.h.s. of (30). Then, using (16), we get

$$\mathcal{R}_1=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{u_1(u_1+1)_n(u_2)_n}{(v_1)_n}\frac{v^n}{n!}-\sum_{n=0}^{\infty}\frac{\Gamma(\alpha n+\beta,x)}{\Gamma(\alpha n+\beta)}\frac{(u_1)_n(u_2)_n(v_1-1)}{(v_1-1)_n}\frac{v^n}{n!}. \tag{31}$$

Using the relations (see [18])

$$u_1(u_1+1)_n=(u_1+n)(u_1)_n,$$
$$(v_1-1)(v_1)_n=(v_1-1)_n(v_1+n-1),$$

this yields the left hand side of (30). □

3 | SOME GENERATING RELATIONS

In this section, we establish some linear generating relations for the generalized incomplete exponential ${}_pR_q(\alpha, \beta; z)$ functions i.e. (15) and (16) as asserted by Theorem 11.

We need the following result[4] for proving Theorem 11.

$$(\delta - k + 1)_n = (\delta + 1)_n \binom{k - \delta - 1}{k} \cdot \binom{k - \delta - n - 1}{k}^{-1}, \quad (k, n \in \mathbf{N}_0), \tag{32}$$

where

$$\binom{\delta}{k} = \frac{\Gamma(\delta + 1)}{\Gamma(k + 1)\Gamma(\delta - k + 1)}, \quad (k \in \mathbf{N_0}; \delta \in \mathbf{C}). \tag{33}$$

Theorem 11. For the generalized incomplete exponential ${}_pe_q(x, \alpha, \beta; v)$ and ${}_pE_q(x, \alpha, \beta; v)$ functions, the following generating function holds true:

$$\sum_{k=0}^{\infty} \binom{k - \delta - 1}{k} {}_pe_{q+1}\left[(x, \alpha, \beta; v)\middle| \begin{matrix} u_1, u_2, \dots, u_p \\ \delta - k + 1, v_1, v_2, \dots, b_q \end{matrix}\right] u^k = (1 - u)^{\delta} \times {}_pe_{q+1}\left[(x, \alpha, \beta; v(1 - u))\middle| \begin{matrix} u_1, u_2, \dots, u_p \\ 1 - \delta, v_1, v_2, \dots, v_q \end{matrix}\right], \tag{34}$$

$$\sum_{k=0}^{\infty} \binom{k - \delta - 1}{k} {}_pE_{q+1}\left[(x, \alpha, \beta; v)\middle| \begin{matrix} u_1, u_2, \dots, u_p \\ \delta - k + 1, v_1, v_2, \dots, v_q \end{matrix}\right] u^k = (1 - u)^{\delta} \times {}_pE_{q+1}\left[(x, \alpha, \beta; v(1 - u))\middle| \begin{matrix} u_1, u_2, \dots, u_p \\ 1 - \delta, v_1, v_2, \dots, v_q \end{matrix}\right], \tag{35}$$

where $x > 0; \delta \in \mathbf{C}$ and $|t| < 1$.

Proof. Let C_1 be the left hand side of (34) and applying (15), we arrive at

$$C_1 = \sum_{k=0}^{\infty} \binom{k - \delta - 1}{k} \left(\sum_{n=0}^{\infty} \frac{\gamma(\alpha n + \beta, x)}{\Gamma(\alpha n + \beta)} \frac{(u_1)_n (u_2)_n \dots (u_p)_n}{(\delta - k + 1)_n (v_1)_n (v_2)_n \dots (v_q)_n} \frac{v^n}{n!} \right) u^k.$$

By reversing the order of summation and using the result defined in (32), we obtain

$$C_1 = \sum_{n=0}^{\infty} \frac{\gamma(\alpha n + \beta, x)}{\Gamma(\alpha n + \beta)} \frac{(u_1)_n (u_2)_n \dots (u_p)_n}{(\delta + 1)_n (v_1)_n (v_2)_n \dots (v_q)_n} \frac{v^n}{n!} \cdot \sum_{k=0}^{\infty} \binom{k - \delta - n - 1}{k} u^k. \tag{36}$$

Now we find the inner sum in (36) by using binomial expansion

$$\sum_{k=0}^{\infty} \binom{k - \delta - 1}{k} u^k = (1 - u)^{\delta}, \quad |u| < 1. \tag{37}$$

Replacing the inner sum of (36) and using (37), this yields the r.h.s. of (34). Similarly, one can prove (35) as asserted by Theorem 11. □

3.1 | Generalization of the generating functions for the generalived incomplete exponential ${}_pR_q(\alpha, \beta; z)$ functions

In this subsection, we derive the generalization of the generating functions for the generalized incomplete exponential ${}_pR_q(\alpha, \beta; z)$ functions (15) and (16) asserted by Theorem 12.

For further extension of the generalized incomplete exponential ${}_pR_q(\alpha,\beta;z)$ functions (15) and (16), we consider two sequences $\{\Psi_r^{(\gamma,\eta)}(\upsilon)\}_{r=0}^{\infty}$ and $\{\Phi_r^{(\gamma,\eta)}(\upsilon)\}_{r=0}^{\infty}$ as,

$$\begin{aligned}\Psi_r^{(\gamma,\eta)}(\upsilon) &= \Psi_r^{(\gamma,\eta)}\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right],\\ &= {}_pe_{q+\eta}\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \Delta(\eta,1-\delta-r),\upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right],\end{aligned} \tag{38}$$

$$\begin{aligned}\Phi_r^{(\gamma,\eta)}(\upsilon) &= \Phi_r^{(\gamma,\eta)}\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}u_1,u_2,\ldots,u_p\\ \upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right],\\ &= {}_pE_{q+\eta}\left[(x,\alpha,\beta;\upsilon)\middle|\begin{matrix}\upsilon_1,\upsilon_2,\ldots,u_p\\ \Delta(\eta,1-\delta-r),\upsilon_1,\upsilon_2,\ldots,\upsilon_q\end{matrix}\right],\end{aligned} \tag{39}$$

where $\Delta(\eta,\delta)$ abbreviates the array of η parameters as follows:

$$\frac{\delta}{\eta},\frac{\delta+1}{\eta},\frac{\delta+2}{\eta},\ldots,\frac{\delta+\eta-1}{\eta},\quad (\delta\in\mathbf{C},\ \eta\in\mathbf{N}). \tag{40}$$

Theorem 12. The following generating functions holds true for the following two sequences $\{\Psi_r^{(\gamma,\eta)}(\upsilon)\}_{r=0}^{\infty}$ and $\{\Phi_r^{(\gamma,\eta)}(\upsilon)\}_{r=0}^{\infty}$:

$$\sum_{r=0}^{\infty}\binom{\delta+m+r-1}{r}\Psi_{m+r}^{(\gamma,\eta)}(\upsilon)\,u^n = (1-u)^{-\delta-m}\,\Psi_m^{(\gamma,\eta)}(\upsilon(1-u)^{\eta}), \tag{41}$$

$$\sum_{r=0}^{\infty}\binom{\delta+m+r-1}{r}\Phi_{m+r}^{(\gamma,\eta)}(\upsilon)\,u^n = (1-u)^{-\delta-m}\,\Phi_m^{(\gamma,\eta)}(\upsilon(1-u)^{\eta}), \tag{42}$$

where $x\geq 0; m\in\mathbf{N}_0; \delta\in\mathbf{C},\ \eta\in\mathbf{N};\ |u|<1$.

Proof. Proof of Theorem 12 is similar to Theorem 11. In aforesaid theorem, one can use the following identity,

$$(1-\delta-m-r)_{\eta n} = (1-\delta-m)_{\eta n}\binom{\delta+m+r-1}{r}\binom{\delta+m-\eta n+r-1}{r}^{-1}, \tag{43}$$

where $r,n\in\mathbf{N}_0;\eta\in\mathbf{N}$. □

On setting $\eta=1$ and replacing δ by $\delta-m$ in (41), this can easily reduce to the result (34).

4 | SOME APPLICATIONS

4.1 | Applications to ground water pumping modeling

Ground water pumping modeling is the technique and science of exploring, developing, and controlling ground water. Generally this modeling relates with specialized fields of oil science, geophysics, geology, mathematics, hydraulics, hydrology, mechanical and chemical engineering. It also concerns with ground-water behavior for the solution of engineering problems

Hantush[10,11] determined the connection between two integral functions M and M* and dynamics of groundwater mounding beneath recharge zones. Accurate algebraic expressions have been established in terms of a formal power series expansion for these two integral functions. During 1950s and 1960s, Hantush developed a mathematical structure for determining the features of draw-down and mounding processes in aquifers. These papers have very much importance due to integral quantities which have been pertained successfully to numerous hydrological research.

In 1964, Hantush[11] gave two functions $\mathrm{M}(\delta,\eta)$ and $\mathrm{M}^*(\delta,\eta)$ that arises with respect to ground water pumping modeling. The following integral are defined as,

$$\mathrm{M}(\delta,\eta)=\int_{\delta}^{\infty}\frac{e^{-u}\ \operatorname{erf}(\eta\sqrt{u})}{u}du, \tag{44}$$

and

$$\mathrm{M}^*(\delta,\eta)=\frac{\delta e^{-\eta}}{\pi}\int_{0}^{1}\frac{e^{-\delta^2\eta u}}{\sqrt{u}(1+\delta^2 u)}du, \tag{45}$$

respectively, for some constants δ and η, where error function $\operatorname{erf}(.)$[1] is given by

$$\operatorname{erf}(u)=\frac{2}{\sqrt{\pi}}\int_{0}^{u}e^{-t^2}dt.$$

This integral defined by (44) and (45) are often evaluated via numerical quadrature and useful in the studies of unsteady flow near partially penetrating wells.

$$\mathrm{M}(\delta,\eta)=\frac{2\eta}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(2\eta)^{2n}n!\ \Gamma(n+\frac{1}{2},\delta(1+\eta^2))}{(2n+1)!(1+\eta^2)^{n+\frac{1}{2}}}, \tag{46}$$

where $\Gamma(\cdot,\cdot)$ is the complementary incomplete gamma function defined in (3).

The expression on the r.h.s. of (46) can be written in terms of the incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function as,

$$\mathrm{M}(\delta,\eta)=\frac{2\eta}{\sqrt{1+\eta^2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2},\delta(1+\eta^2))}{\Gamma(n+\frac{1}{2})}\frac{\left(\frac{1}{2}\right)_n(1)n}{\left(\frac{3}{2}\right)_n}\frac{1}{n!}\left(\frac{\eta^2}{1+\eta^2}\right)^n, \tag{47}$$

$$=\frac{2\eta}{\sqrt{1+\eta^2}}\ {}_2E_1\left[\delta(1+\eta^2),1,1;\left(\frac{\eta^2}{1+\eta^2}\right)\middle|\begin{matrix}\left(\frac{1}{2},1\right)\\ \frac{3}{2}\end{matrix}\right],\ \forall\ \delta,\eta>0, \tag{48}$$

where ${}_2E_1(\cdot)$ is the generalized incomplete exponential ${}_pE_q(x,\alpha,\beta;\upsilon)$ function defined by (16).

Now, we can also represent the expression (45) as,

$$\begin{aligned}\mathrm{M}^*(\delta,\eta)&=\frac{\delta e^{-\eta}}{\pi}\int_{0}^{1}\frac{e^{-\delta^2\eta u}}{\sqrt{u}}\left(\sum_{n=0}^{\infty}(-\delta^2)^n u^n\right)du\\&=\frac{\delta e^{-\eta}}{\pi}\sum_{n=0}^{\infty}(-\delta^2)^n\int_{0}^{1}u^{n-\frac{1}{2}}\ e^{-\delta^2\eta u}\ du\\&=\frac{\delta e^{-\eta}}{\pi\sqrt{\eta}}\sum_{n=0}^{\infty}\left(-\frac{1}{\eta}\right)^n\gamma\left(n+\frac{1}{2},\delta^2\eta\right)\\&=\frac{\delta e^{-\eta}}{\sqrt{\pi\eta}}\sum_{n=0}^{\infty}\frac{\gamma\left(n+\frac{1}{2},\delta^2\eta\right)}{\Gamma(n+\frac{1}{2})}\left(\frac{1}{2}\right)_n\left(-\frac{1}{\eta}\right)^n\frac{(1)_n}{n!}\\&=\frac{\delta e^{-\eta}}{\sqrt{\pi\eta}}\ {}_2e_0\left[\delta\eta^2,1,\frac{1}{2};-\frac{1}{\eta}\middle|\begin{matrix}\left(\frac{1}{2},1\right)\\ -\end{matrix}\right],\ \forall\delta>0,\eta>1,\end{aligned}$$

where ${}_2e_0(\cdot)$ is the generalized incomplete exponential ${}_pe_q(x,\alpha,\beta;\upsilon)$ function defined by (15) .

4.2 | Application to probability theory

The Poisson process are appropiate models which are used in counting process. The generalized Poisson process and their distribution are generally useful in ecological modelling and are applied to biostatistical data. Ecological and genetical problems are commonly used in counting processes. Under some basic assumptions, Poisson process can represent a perfect theory. For further details, see[3,16]. By modification of Poisson process, Janardan[13] and Janardan et al.[14] established a continuous time Markov chain $\{X(t) : t \geq 0\}$ with $X(0) = 0$ and derived a integral representation which is very useful to derive the moments and other properties of probability mass function.

Janardan et al.[14] studied a stochastic process $\{X(t) : t \geq 0\}$ through the probability mass function given by

$$P_0(t) = e^{-\rho t}, \tag{49}$$

$$P_n(t) = \frac{\rho\mu^{n-1}}{(\mu-\rho)^n}\left\{e^{-\rho t} - e^{-\mu t}\sum_{k=0}^{n-1}\frac{(\mu-\rho)^k t^k}{k!}\right\}, \tag{50}$$

where $n \geq 1, \lambda > 0, \mu > 0$.

Consider the transition probabilities in the following form:

$$\lim_{h\to 0}\frac{P\{X(t+h) = j|X(t) = i\}}{h} = \begin{cases}\rho_j & \text{if } j = i+1\\ 0 & \text{if } j \neq i+1.\end{cases} \tag{51}$$

Using these basic assumption as defined by (51), an integral representation corresponding to (50) can be defined as

$$P_n(t) = \frac{\rho\mu^{n-1}e^{-\rho t}}{(n-1)!}\int_0^t u^{n-1}e^{-(\mu-\rho)u}\,du, \tag{52}$$

by assuming $\rho_j = \rho$ for $j = 0$ and $\rho_j = \mu, \forall j = 1, 2, \dots$. For details see[13].

During last few decades, many researchers connected these models to some special functions which help us to establish the properties in stochastic processes. Prabhakar[17], Saxena et al.[19] and Haubold et al.[12] established various generalization with applications. Here, we establish a connection of (52) with generalized incomplete exponential ${}_pR_q(\alpha, \beta; z)$ function in the form of Theorem 13.

Theorem 13. Let $\{X(t) : t \geq 0\}$ be a continuous time Markov chain with $X(0) = 0$ and the transition probabilities as in (51). Then, (52) can be derived in terms of generalized incomplete exponential function as

$$P_n(t) = \frac{\rho}{\mu}\,{}_0e_0\left[(\mu-\rho)t, 1, 0; \frac{\rho\mu t}{\rho-\mu}\middle| \begin{matrix} - \\ - \end{matrix}\right]. \tag{53}$$

Proof. From the integral representation (52), we have

$$\begin{aligned}
P_n(t) &= \frac{\rho\mu^{n-1}e^{-\rho t}}{(n-1)!}\int_0^t u^{n-1}e^{-(\mu-\rho)u}\,du\\
&= \frac{\rho\mu^{n-1}e^{-\rho t}}{(n-1)!}\cdot\frac{1}{(\mu-\rho)^n}\int_0^{(\mu-\rho)t} x^{n-1}e^{-x}\,dx\\
&= \frac{\rho\mu^{n-1}e^{-\rho t}}{(n-1)!}\cdot\frac{\gamma(n,(\mu-\rho)t)}{(\mu-\rho)^n}\\
&= \frac{\rho}{\mu}\sum_{n=0}^{\infty}\frac{\gamma(n,(\mu-\rho)t)}{\Gamma(n)}\left(\frac{\mu}{\mu-\rho}\right)^n\frac{(-\rho t)^n}{n!}\\
&= \frac{\rho}{\mu}\,{}_0e_0\left[(\mu-\rho)t, 1, 0; \frac{\rho\mu t}{\rho-\mu}\middle| \begin{matrix} - \\ - \end{matrix}\right],
\end{aligned}$$

where ${}_0e_0(\cdot)$ is the generalized incomplete exponential ${}_pe_q(x,\alpha,\beta;\upsilon)$ function defined by (15). This completes the proof. □

4.3 | Applications to non central chi-square distribution

The non-central chi-square distribution plays an important role in the area of statistics and applied mathematics. It have significant used in structural equation modeling (SEM). Many researchers used the noncentral chi-square distribution in the construction of fit indices, such as Steiger and LindâĂŹs[21] Root Mean Square Error of Approximation (RMSEA) and for the computation of statistical power for model hypothesis testing.

Consider the independent random variables $\left(Y_1, Y_2, \ldots, Y_\nu\right)$ that are normally distributed with means λ_i and unit variances. Then, the random variables

$$\sum_{i=1}^{\nu}\left(Y_i+\lambda_i\right)^2$$

distributed according to the non central chi-square distribution which depends on $\lambda_1, \lambda_2, \ldots, \lambda_\upsilon$ only through the sum of their squares. It is known as the non central chi-square distribution with ν degree of freedom and μ which defines the mean of random variable by $\mu=\sum_{i=1}^{\nu}\lambda_i$. Sometimes μ called the non centrality parameter. The probability density function of the chi-square distribution[15] is given by

$$f_Y(t;\nu,\mu)=\frac{e^{-\frac{1}{2}(\mu+t)}}{2^{\nu/2}}\sum_{i=0}^{\infty}\left(\frac{\mu}{4}\right)^i\frac{t^{\frac{\nu}{2}+i-1}}{i!\,\Gamma\left(\frac{\nu}{2}+1\right)}, \tag{54}$$

$$f_Y(t;\nu,\mu)=\frac{1}{2}\,e^{-\frac{\mu+t}{2}}\left(\frac{t}{\mu}\right)^{\frac{\nu-2}{4}}J_{\frac{\nu}{2}-1}(\sqrt{\mu t}), \tag{55}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν[9].

Venables[24] gave the representation of above probability density function,

$$f_Y(t;\nu,\mu)=e^{-\frac{\mu}{2}}\,{}_0F_1\left(-;\frac{\nu}{2};\frac{\mu t}{4}\right)\frac{e^{-\frac{t}{2}}\,t^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}, \tag{56}$$

where ${}_0F_1(\cdot)$ is the hypergeometric function.

On utilizing the Venables representation for chi-square distribution, Chaudhry and Qadir[2] gave the cumulative density function as,

$$\begin{aligned} F(t;\nu,\mu) &= \frac{e^{-\frac{\mu}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\int_0^t u^{\frac{\nu}{2}-1}e^{-\frac{u}{2}}\,{}_0F_1\left(-;\frac{\nu}{2};\frac{\mu u}{4}\right)du,\\ &= \frac{e^{-\frac{\mu}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\int_0^t u^{\frac{\nu}{2}-1}e^{-\frac{u}{2}}\sum_{n=0}^{\infty}\frac{1}{\left(\frac{\nu}{2}\right)_n}\left(\frac{\mu u}{4}\right)^n\frac{1}{n!}du,\\ &= \frac{e^{-\frac{\mu}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\sum_{n=0}^{\infty}\frac{1}{\left(\frac{\nu}{2}\right)_n}\left(\frac{\mu}{4}\right)^n\frac{1}{n!}\int_0^t u^{n+\frac{\nu}{2}-1}e^{-\frac{u}{2}}du. \end{aligned} \tag{57}$$

Substituting $\frac{u}{2}=\tau$ in (57) transforms the representation

$$F(t;\nu,\mu)=\frac{e^{-\frac{\mu}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\sum_{n=0}^{\infty}\frac{1}{\left(\frac{\nu}{2}\right)_n}\left(\frac{\mu}{4}\right)^n\frac{1}{n!}\,2^{n+\frac{\nu}{2}}\int_0^{t/2}\tau^{n+\frac{\nu}{2}-1}e^{-\tau}d\tau,$$
$$=e^{-\frac{\mu}{2}}\sum_{n=0}^{\infty}\frac{\gamma\left(n+\frac{\nu}{2},\frac{t}{2}\right)}{\Gamma\left(n+\frac{\nu}{2}\right)}\left(\frac{\mu}{2}\right)^n\frac{1}{n!},$$
$$=e^{-\frac{\mu}{2}}\,{}_0e_0\left[\frac{t}{2},1,\frac{\nu}{2};\frac{\mu}{2}\,\middle|\,\begin{matrix}-\\-\end{matrix}\right],$$

where ${}_0e_0(\cdot)$ is the generalized incomplete exponential ${}_pe_q(x,\alpha,\beta;\upsilon)$ function defined by (15) .

5 | CONCLUSION

In this article, we have established the generalization of ${}_pR_q(\alpha,\beta;z)$ function in terms of incomplete exponential functions. We also found several interesting properties and generating relations of ${}_pR_q(\alpha,\beta;z)$ incomplete exponential functions. Some applications related to area of ground water pumping modelling and probability theory have also been established.

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