SUMUDU TRANSFORM FOR SOME APPLICATIONS WITH CONSTANT PROPORTIONAL CAPUTO DERIVATIVE

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Abstract. In this study, we obtain the solutions of some interesting problems by Sumudu transform. We investigate the economic models based on market equilibrium with constant proportional Caputo derivative. We prove the efficiency of the Sumudu transform by some applications.

1. Introduction

Many studies have been conducted on the Bivariate Mittag-Leffler function and fractional derivatives. Some of them are as follows: Garg et al. [1] have investigated the Mittag-Leffler-type function of two variables. Özarslan and Kürt [2] have studied Bivariate Mittag-Leffler functions arising in the solutions of convolution integral equation with 2D-Laguerre-Konhauser polynomials in the kernel. Özarslan, [3] has worked on a singular integral equation including a set of multivariate polynomials suggested by Laguerre polynomials. The unified rheological model for cells and cellularised materials have been researched by Bonfanti et al. [4]. Kürt et al. [5] have studied on a certain bivariate Mittag-Leffler function analysed from a fractional-calculus point of view. Saxena et al. [6] have searched the multivariate analogue of generalized Mittag-Leffler function. Jayakumar et al. [7] has worked the generalization to bivariate Mittag–Leffler and bivariate discrete Mittag–Leffler autoregressive processes. Mainardi [8] has investigated some properties of the Mittag-Leffler function. Abdeljawad et al. [9–13] have solved many fractional differential equations with different technicques.

Sumudu transformation is a promising transformation due to its basic formula and features resulting from this formula. It can contribute to the solution of many complex problems. Weerakoon [19] has studied the partial difference equations for Sumudu transformations. Then Watinal [16] has constructed some works related to this transform. The study of Watugala's user [17] showed that the Sumudu transformation is effective for solving differential equations. Watugala [18] extended this transformation to bivariate partial differential equations. In this study, firstly we used the Sumudu transform with constant propotional Caputo derivative for some examples. Secondly we deal with economic model which is solved by Sumudu transform with new derivative.

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2. Main theorems and applications

Definition 2.1. We define [20]:

$$A = \{h(x) \mid \exists N, \tau_1, \tau_2 > 0, \mid h(x) \mid < N \exp(\mid x \mid / \tau_k, \text{ if } x \in (-1)^k \times [0, \infty)\}, \tag{2.1}$$

and

$$H(s) = S[h(x)] = \int_0^\infty h(sx) \exp(-x) dx, \quad s \in (-\tau_1, \tau_2). \tag{2.2}$$

Authors showed the Sumudu transform which has units preserving properties, and moreover they can utilize to investigate the problems without resorting to the frequency domain. There are many strength points of this new transform, particularly with respect to implementations in problems with physical dimensions. Actually, the Sumudu transform is linear, protects linear functions, and also especially does not alter units [17].

Lemma 2.2. *Let h(x) be in A, and let $H_k(s)$ indicates the Sumudu transform of the kth derivative, $h^{(k)}(x)$ of f(x), then for $j \geq 1$, [20]*

$$H_k(s) = \frac{H(s)}{s^k} - \sum_{l=0}^{k-1} \frac{h^{(l)}(0)}{s^{k-l}} \tag{2.3}$$

Lemma 2.3. *We obtain the Sumudu transform of the power series function as [20],*

$$h(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{2.4}$$

its power series function is as:

$$H(s) = \sum_{k=0}^{\infty} k! a_k s^k. \tag{2.5}$$

Definition 2.4. The Caputo fractional derivative is given as [21]:

$${}_a^C D^\sigma h(x) = \frac{1}{\Gamma(1-\sigma)} \int_a^x (x-\tau)^{-\sigma} h'(\tau) d\tau \tag{2.6}$$

Definition 2.5. The Rieamann-Liouville integral is defined as [21];

$${}_a^{RL} I^\sigma h(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-\tau)^{\sigma-1} h(\tau) d\tau \tag{2.7}$$

in here $h(x)$ is an integrable function and $\sigma > 0$.

From above definitons,

$${}_a^C D^\sigma h(x) = {}_a^{RL} I^{1-\sigma} h'(x) \tag{2.8}$$

is written.

Definition 2.6. The constant proportional Caputo (CPC) derivative is defined by [21]

$${}_0^{CPC} D_x^\alpha h(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \left(k_1(\alpha) h(\tau) + k_0(\alpha) h'(\tau)\right) (x-\tau)^{-\alpha} d\tau \tag{2.9}$$

Definition 2.7. Let $\xi, \vartheta : [0,\infty) \to \mathfrak{R}$, then the convolution of ξ, ϑ is

$$(\xi * \eta) = \int_0^x \xi(x-u)\eta(u)du \tag{2.10}$$

and assume that $\xi, \eta : [0,\infty) \to \mathfrak{R}$, then we have:

$$S\{(\xi\eta)(x)\} = sS\{\xi(x)\}S\{\eta(x)\}. \tag{2.11}$$

Definition 2.8. The classical Mittag-Leffler function which has one parameter $E_\alpha(\nu)$ is given as:

$$E_\alpha(\nu) = \sum_{l=0}^{\infty} \frac{\nu^l}{\Gamma(\alpha l + 1)} \quad (\nu \in \mathbb{C}, Re(\alpha) > 0), \tag{2.12}$$

Also, the Mittag-Leffler function which has two parameters is presented as:

$$E_{\alpha,\beta}(\nu) = \sum_{l=0}^{\infty} \frac{\nu^l}{\Gamma(\alpha l + \beta)} \quad (\nu, \beta \in \mathbb{C}, Re(\alpha) > 0), \tag{2.13}$$

it is worth, $E_{\alpha,\beta}(\nu)$ corresponds to the Mittag-Leffler function Eq.(2.12) when $\beta = 1$.

Theorem 2.9. *The Sumudu transform of constant proportional Caputo (CPC) derivative is found as:*

$$S\{{}_0^{CPC}D_x^\alpha h(x)\} = k_1(\alpha)S\{h(x)\}s^{1-\alpha} + k_0(\alpha)[S\{h(x)\} - h(0)]s^{-\alpha} \tag{2.14}$$

Proof. We have

$$\begin{aligned} {}_0^{CPC}D_x^\alpha h(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x \left(k_1(\alpha)h(\tau) + k_0(\alpha)h'(\tau)\right)(h-\tau)^{-\alpha} d\tau \\ &= k_1(\alpha)\left\{h(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\} + k_0(\alpha)\left\{h'(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\} \end{aligned}$$

Then we apply the Sumudu transform to the above equation:

$$\begin{aligned} S\left[{}_0^{CPC}D_x^\alpha h(x)\right] &= S\left[k_1(\alpha)\left\{h(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\} + k_0(\alpha)\left\{h'(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}\right] \\ &= s\left[k_1(\alpha)S\{h(x)\}S\left\{\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\} + k_0(\alpha)S\{h'(x)\}S\left\{\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}\right] \\ &= s\left[k_1(\alpha)S\{h(x)\}s^{-\alpha} + k_0(\alpha)\left(\frac{S\{h(x)\} - h(0)}{s}\right)s^{-\alpha}\right] \\ &= k_1(\alpha)S\{h(x)\}s^{1-\alpha} + k_0(\alpha)[S\{h(x)\} - h(0)]s^{-\alpha}. \end{aligned}$$

This completes the proof. □

Now, we take into consideration the following example;

$${}_0^{CPC}D_x^\alpha h(x) = 0, \quad h(0) = A, \tag{2.15}$$

Using the Sumudu transform to this equation, we can write

$$S\{{}_0^{CPC}D_x^\alpha h(x)\} = 0 \tag{2.16}$$

If the expression Theorem 2.9 is utilized, then we obtain as:

$$k_1(\alpha)S\{h(x)\}s^{1-\alpha} + k_0(\alpha)[S\{h(x)\} - h(0)]s^{-\alpha} = 0, \tag{2.17}$$

and

$$S\{h(x)\} = \frac{Ak_0(\alpha)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha} + k_0(\alpha)s^{-\alpha}} = \frac{A}{1 + \frac{k_1(\alpha)}{k_0(\alpha)}}, \tag{2.18}$$

by taking the inverse Sumudu transform of the last equation gives:

$$h(x) = AS^{-1}\left\{\frac{1}{1 + \frac{k_1(\alpha)}{k_0(\alpha)}}\right\}, \tag{2.19}$$

then the solution is acquired

$$h(x) = A\exp\left(\frac{-k_1(\alpha)}{k_0(\alpha)}x\right). \tag{2.20}$$

Let consider another example:

$${}_0^{CPC}D_x^{\alpha}h(x) = \lambda h(x), \quad h(0) = 1, \tag{2.21}$$

When the Sumudu transform is applied to the above equation, we will get

$$S\{{}_0^{CPC}D_x^{\alpha}h(x)\} = \lambda S\{h(x)\}, \tag{2.22}$$

If the expression Theorem 2.9 is utilized then we find:

$$k_1(\alpha)S\{h(x)\}s^{1-\alpha} + k_0(\alpha)[S\{h(x)\} - h(0)]s^{-\alpha} = \lambda S\{h(x)\}, \tag{2.23}$$

and

$$\begin{aligned}
S\{h(x)\} &= \frac{k_0(\alpha)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha} + k_0(\alpha)s^{-\alpha} - \lambda} \\
&= \frac{1}{1 - \frac{\lambda}{k_0(\alpha)}s^{\alpha} + \frac{k_1(\alpha)}{k_0(\alpha)}s} \\
&= \left[1 - \frac{\lambda s^{\alpha} - k_1(\alpha)s}{k_0(\alpha)}\right]^{-1} \\
&= \sum_{j=0}^{\infty}\left[\frac{\lambda s^{\alpha} - k_1(\alpha)s}{k_0(\alpha)}\right]^{j} \\
&= \sum_{j=0}^{\infty}\frac{1}{k_0(\alpha)^j}\sum_{r=0}^{j}\binom{j}{r}[\lambda s^{\alpha}]^{j-r}[-k_1(\alpha)s]^{r} \\
&= \sum_{j=0}^{\infty}\sum_{r=0}^{j}\frac{-k_1(\alpha)^r\lambda^{j-r}}{k_0(\alpha)^j}\binom{j}{r}s^{\alpha j-\alpha r+r}.
\end{aligned}$$

Then, we apply the inverse Sumudu transform and obtain:

$$h(x) = \sum_{j=0}^{\infty}\sum_{r=0}^{j}\frac{-k_1(\alpha)^r\lambda^{j-r}}{k_0(\alpha)^j}\binom{j}{r}\frac{x^{\alpha j-\alpha r+r}}{\Gamma(\alpha j-\alpha r+r+1)}. \tag{2.24}$$

When we take $p = j - r$, we will get

$$\begin{aligned}
h(x) &= \sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{-k_1(\alpha)^r\lambda^p}{k_0(\alpha)^{r+p}}\frac{(r+p)!}{r!p!}\frac{x^{\alpha p+r}}{\Gamma(\alpha p+r+1)} \\
&= \sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(r+p)!}{r!p!}\left[\frac{-k_1(\alpha)}{k_0(\alpha)}x\right]^{r}\left[\frac{\lambda}{k_0(\alpha)}x^{\alpha}\right]^{r}\frac{x^{\alpha p+r}}{\Gamma(\alpha p+r+1)}.
\end{aligned}$$

We can write this series as [22]:

$$h(x) = E^{1}_{\alpha,1,1}\left(\frac{\lambda}{k_0(\alpha)}x^{\alpha}, \frac{-k_1(\alpha)}{k_0(\alpha)}x\right). \tag{2.25}$$

3. Applications of the economic model

A wide variety of applications such as optimization problems, balance or static analysis, comparative static have been encountered recently. The authors have begun using a number of new tools in differential analysis for all these fields to develop economic models. It will be understood that economists use mathematical economic models to get strong predictions about maximum profit. Each economic model has maximum benefit for buyers, maximum profit for sellers, and limited pricing in a balance model. How to reach balance point in economy, price, supply and demand, price supply and demand and supply and demand curves. A competitive market is based on the competitive balance. We have

$$q_d(x) = d_0 - d_1 h(x), \quad q_s(x) = -s_0 + s_1 h(x) \tag{3.1}$$

where h the price of goods, d_0, s_0, d_1, s_1 are positive constants. For $q_d(x) = q_s(x)$, when the demanded quantity equals the supplied quantity, the equilibrium price is found as; $h^* = \frac{d_0+s_0}{d_1+s_1}$. Now, considering [24]:

$$h'(x) = k(q_d - q_s) \tag{3.2}$$

where $k > 0$. Then, we acquire

$$h'(x) + k(d_1 + s_1)h(x) = k(d_0 + s_0) \tag{3.3}$$

When we solved the first order ordinary differential equation, the solution can be obtained:

$$h(x) = \frac{d_0 + s_0}{d_1 + s_1} - \left[h(0) + \frac{d_0 + s_0}{d_1 + s_1}\right]\exp(-k(d_1 + s_1)x, \tag{3.4}$$

where $h(0)$ is the price at the time $x = 0$.

Let us take into consideration the following example:

$${}^{CPC}_{0}D^{\alpha}_{x}h(x) + k(d_1 + s_1)h(x) = k(d_0 + s_0), \tag{3.5}$$

If we apply the Sumudu transform to the equation, we obtain

$$S\{{}^{CPC}_{0}D^{\alpha}_{x}h(x)\} + k(d_1 + s_1)S\{h(x)\} = k(d_0 + s_0), \tag{3.6}$$

If we use the expression Theorem 2.9, then we find:

$$k_1(\alpha)S\{h(x)\}s^{1-\alpha} + k_0(\alpha)[S\{h(x)\} - h(0)]s^{-\alpha} + k(d_1 + s_1)S\{h(x)\} = k(d_0 + s_0), \tag{3.7}$$

and

$$\begin{aligned}
S\{h(x)\} &= \frac{k(d_0+s_0)}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}+k(d_1+s_1)} \\
&+\frac{k_0(\alpha)h(0)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}+k(d_1+s_1)} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\left[1-\frac{-k_1(\alpha)s^{1-\alpha}-k_0\alpha s^{-\alpha}}{k(d_0+s_0)}\right]^{-1} \\
&+h(0)\left[1-\frac{-k_1(\alpha)s-k(d_1+s_1)s^{\alpha}}{k_0(\alpha)}\right]^{-1} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{j=0}^{\infty}\left[\frac{-k_1(\alpha)s^{1-\alpha}-k_0(\alpha)s^{-\alpha}}{k(d_0+s_0)}\right]^{j} \\
&+h(0)\sum_{j=0}^{\infty}\left[\frac{-k_1(\alpha)s-k(d_1+s_1)s^{\alpha}}{k_0(\alpha)}\right]^{j} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{j=0}^{\infty}\frac{1}{k^j(d_0+s_0)^j}\sum_{r=0}^{j}\binom{j}{r}[-k_1(\alpha)s^{1-\alpha}]^{j-r}[-k_0(\alpha)s^{-\alpha}]^{r} \\
&+h(0)\sum_{j=0}^{\infty}\frac{1}{k_0(\alpha)^j}\sum_{r=0}^{j}\binom{j}{r}[-k_1(\alpha)s]^{j-r}[-k(d_1+s_1)s^{\alpha}]^{r} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^j\frac{k_1(\alpha)^{j-r}k_0(\alpha)^r}{k^j(d_0+s_0)^j}\binom{j}{r}s^{(1-\alpha)(j-r)-\alpha r} \\
&+h(0)\sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^j\frac{k_1(\alpha)^{j-r}k^r(d_1+s_1)^r}{k_0(\alpha)^j}\binom{j}{r}s^{j-r+\alpha r},
\end{aligned}$$

Then, we apply the inverse Sumudu transform and get:

$$\begin{aligned}
h(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^j\frac{k_1(\alpha)^{j-r}k_0(\alpha)^r}{k^j(d_0+s_0)^j}\binom{j}{r}\frac{x^{(1-\alpha)(j-r)-\alpha r}}{\Gamma((1-\alpha)(j-r)-\alpha r+1)} \\
&+h(0)\sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^j\frac{k_1(\alpha)^{j-r}k^r(d_1+s_1)^r}{k_0(\alpha)^j}\binom{j}{r}\frac{x^{(j-r)+\alpha r}}{\Gamma((j-r)+\alpha r+1)},
\end{aligned}$$

When we take $p=j-r$ we will get:

$$\begin{aligned}
h(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(r+p)!}{r!p!}\frac{(-k_1(\alpha))^p(-k_0(\alpha))^r}{k^{p+r}(d_0+s_0)^{p+r}}\frac{x^{(1-\alpha)p-\alpha r}}{\Gamma((1-\alpha)p-\alpha r+1)} \\
&+h(0)\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(r+p)!}{r!p!}\frac{(-k_1(\alpha))^p(-k(d_1+s_1))^r}{k_0(\alpha)^{p+r}}\frac{x^{p+\alpha r}}{\Gamma(p+\alpha r+1)},
\end{aligned}$$

$$\begin{aligned}
h(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(r+p)!}{r!p!}\left[\frac{-k_0(\alpha)}{k(d_0+s_0)}x^{-\alpha}\right]^r\left[\frac{-k_1(\alpha)}{k(d_0+s_0)}x^{1-\alpha}\right]^p\frac{1}{\Gamma((1-\alpha)p-\alpha r+1)} \\
&+h(0)\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{(r+p)!}{r!p!}\left[\frac{-k(d_1+s_1)}{k_0(\alpha)}x^{\alpha}\right]^r\left[\frac{-k_1(\alpha)}{k_0(\alpha)}x\right]^p\frac{1}{\Gamma(p+\alpha r+1)},
\end{aligned}$$

We can write this series as [22]:

$$\begin{aligned} h(x) &= \frac{(d_0+s_0)}{(d_1+s_1)} E^1_{1-\alpha,-\alpha,1}\left(\frac{-k_1(\alpha)}{k(d_0+s_0)}x^{1-\alpha}, \frac{-k_0(\alpha)}{k(d_0+s_0)}x^{-\alpha}\right) \\ &\quad + h(0) E^1_{1,\alpha,1}\left(\frac{-k_1(\alpha)}{k_0(\alpha)}x, \frac{-k(d_1+s_1)}{k_0(\alpha)}x^{\alpha}\right). \end{aligned}$$

4. Conclusions

In this paper, we presented some details of Sumudu transform. Then, we applied it to the general problems. We constructed the economic model and we solved this model by the Sumudu transform. We concluded that the Sumudu transform is very effective for solving such problems.

Conflict of interest

The authors declare that they do not have any conflict of interest.

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