New imperssive regulations for the non-fractional order and the time-fractional order of the biological population models

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Abstract

In this article, imperssive exact solutions and hence effective regulations to the non-fractional order and the time-fractional order of the biological population models are achevied for the first time in the framwork of the Paul-Painlevé approach. When the variables appearing in the exact solutions take specific values, the solaitry wave solutions will be easily satisfied.The realized results prove the efficiency of this technique.

Keywords : The (2+1)-dimentional non-fractional Zoomeron equation; the time-fractional biological population model; the Paul-Painlevé approach; Traveling wave solutions.

1. Introduction

The biological population process is one of the most important principal pillars on which the population regulation process depends. The principal axes that identfy this scheme and play a vital role to the population regulation process are position, time and density.Two important mathematical models are invented to study these population process, namely:
(i)-The (2+1)-dimentional non-fractional Zoomeron model (NFZM) mentioned at [1], [2] which is a forward stretch for the well knowen (1+1)-NFZM "extensively investigated in the literature " that describes special cases of solitons achieved in different branches of physics.
(ii)-The time-fractional biological population model (TFBPM) mentioned at [3] which is the other well known model that represents the population processes. Particullary, Gurney [4] gives a good model to the TFBPM for animals which is a special case of the (2+1)- NFZM.
Many scientists invented several methods to study the nonlinear partial differential equations. The majority of these methods are included in the refrences [5-21]. Recently in the literature, there are few works which study these suggested equations. Especially Abazari [22] obtained periodic and soliton solutions to Zoomeron equation by means of (G'/G)-expansion method, Kamruzzaman Khan, M. Ali Akbar, Md. Abdus Salam, Md. Hamidul Islam [23] using the (G'/G)-expansion method to find the traveling wave solutions to the (2+1)-dimensional Zoomeron equation.

The main target of this article is to realize new impressive regulations for the biological population models through obtaining the exact solution to these two models (which containe some variables) using the Paul-Painlevé approach [24]. When these variables take specific values, we will achieve the solaitry wave solutions.

2. Technique description of the Paul-Painlevé approach

To propose this approach , let us firstly propose the general forlasim of the nonlinear evolution equation, let us introduce R as a function of (x,t) and its partial derivatives as,

$$R(\varphi, \varphi_x, \varphi_t, \varphi_{xx}, \varphi_{tt}, ...) = 0, \tag{1}$$

that invoves the highest order derivatives and nonlinear terms. With the aid of the transformation $\varphi(x,t) = \varphi(\zeta), \zeta = x - C_0 t$ equation (2) can be reduced to the following ODE:

$$S(\varphi', \varphi'', \varphi''', ...) = 0, \tag{2}$$

Where, S is a function in $\varphi(\zeta)$ and its total derivatives, while $' = \frac{d}{d\xi}$.
According to Paul-Painlevé [24], the exact solution to the nonlinear ordinary differential equation can be written in the following form,

$$\varphi(\zeta) = A_0 + W(X)\, e^{-N\zeta}, X = R(\zeta), \tag{3}$$

Or

$$\varphi(\zeta) = A_0 + A_1 W(X) e^{-N\zeta} + A_2 W^2(X) e^{-2N\zeta}, X = R(\zeta), \tag{4}$$

Where $X = R(\zeta) = C_1 - \frac{e^{-N\zeta}}{N}$, and $W(X)\ in\ E$q. (4) satisfies the Riccati-equation in the form $W_X - AW^2 = 0,$ one can solve this equation to get,

$$W(X) = \frac{1}{AX + X_0} \tag{5}$$

Consequently,

$$\varphi_\zeta = -N\, e^{-N\zeta} W(X) + R_\zeta e^{-N\zeta} W_X, \tag{6}$$

$$\varphi_\zeta = -NA_1 e^{-N\zeta} W - (AA_1 + 2A_2) e^{-2N\zeta} W^2 - 2AA_2 e^{-3N\zeta} W^3, \tag{7}$$

$$\begin{aligned} \varphi_{\zeta\zeta} = N^2 A_1 e^{-N\zeta} W + (3AA_1 N + 4A_2 N) e^{-2N\zeta} W^2 + \\ (2A_1 A^2 + 8AA_2 + 2A_2 N) e^{-3N\zeta} W^3 + 4A_2 A^2\, e^{-4N\zeta} W^4, \end{aligned} \tag{8}$$

3. The exact solution to (2+1)-NFZM -equation

In this section, we will apply the Paul-Painlevé as a new technique to realize the exact solution to the (2+1)- NFZM [1], "in terms of some variables". Hence, we can easily obtain the traveling wave solutions when these variables take specific values. According to [1], [2] the (2+1)-NFZM -equation can be written as,

$$\left(\frac{\varphi_{xy}}{\varphi}\right)_{tt} - \left(\frac{\varphi_{xy}}{\varphi}\right)_{xx} + 2\left(\varphi^2\right)_{xt} = 0 \tag{9}$$

Where $\varphi(x, y, t) = \varphi(\zeta), \zeta = x + y - kt$ indicates the amplitude of the relative wave mode.Under this transformation equation (9) becomes,

$$k^2\left(\frac{\varphi''}{\varphi}\right)'' - \left(\frac{\varphi''}{\varphi}\right)'' - 2\left(\varphi^2\right)'' = 0 \tag{10}$$

Integrating twice with respect to ζ we get,

$$(k^2-1)\varphi'' - 2k\varphi^3 + k_1\varphi = 0 \tag{11}$$

Where k_1 indicates the constancy of integration

Substituting $\varphi, \varphi_\zeta, \varphi_{\zeta\zeta}$ at Eq. (10) and equating the coefficients of different powers of $W(\zeta)\, e^{-N\zeta}$ to zero, this system of equations implies,

$$\begin{aligned} &A^2(k^2-1) - k = 0, \\ &AN(k^2-1) - 2kA_0 = 0, \\ &N^2(k^2-1) - 6kA_0^2 + k_1 = 0, \\ &k_1 - 2kA_0^2 = 0, \end{aligned} \tag{12}$$

When one solves this system, the following results are achieved,

$$\begin{aligned} &(1)\, A = \frac{-\sqrt{k}}{\sqrt{-1+k^2}}, k_1 = \frac{1}{2}(-1+k^2)N^2, A_0 = \frac{\sqrt{k}\,N}{\sqrt{-1+k^2}} - \frac{k^{\frac{5}{2}}\,N}{\sqrt{-1+k^2}}, \\ &(2) A = \frac{\sqrt{k}}{\sqrt{-1+k^2}}, k_1 = \frac{1}{2}(-1+k^2)N^2, A_0 = -\frac{\sqrt{k}\,N}{\sqrt{-1+k^2}} + \frac{k^{\frac{5}{2}}\,N}{\sqrt{-1+k^2}}, \end{aligned} \tag{13}$$

Choose $k = 2, k_1 = 3$ these two results become,

$$\begin{aligned} &(1)\, A = -\sqrt{\frac{2}{3}}, N = \pm\sqrt{2}, A_0 = -\sqrt{6}N, \\ &(2) A = \sqrt{\frac{2}{3}}, N = \pm\sqrt{2}, A_0 = \sqrt{6}N, \end{aligned} \tag{14}$$

Now, the suggested solution according to the proposed method is;

$$\varphi(\zeta) = A_0 + e^{-N\zeta}W(X),\; X(\zeta) = R(\zeta) = C_1 - \frac{e^{-N\zeta}}{N}, \tag{15}$$

which can be written as,

$$\varphi(\zeta) = A_0 + \frac{e^{-N\zeta}}{AX + X_0}\;, \tag{16}$$

As for the obtained result, it becomes,

$$\varphi(\zeta) = \pm\sqrt{12} + \frac{e^{\pm\sqrt{2}\,\zeta}}{\pm\sqrt{\frac{2}{3}}\left(1 - \frac{e^{-N\zeta}}{\pm\sqrt{2}}\right) + X_0}\;, \tag{17}$$

From which we get these cases;

Case (1): $A_0 = \sqrt{12}\,, A = \sqrt{\frac{2}{3}}, N = \sqrt{2}$

$$\varphi(\zeta) = 3.5 + \frac{1.7\,e^{-(1.4)\,(x+y-2t)}}{3.1 - e^{-(1.4)\,(x+y-2t)}}\,, \tag{18}$$

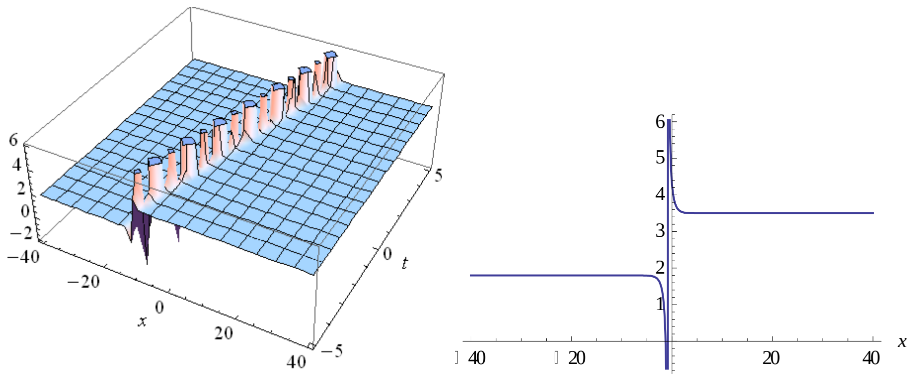

Figure 1. The plot of Eq.(18) in 2D and 3D with values: $A_0 = 3.5, A = \sqrt{\frac{2}{3}}, N = 1.4, X_0 = 1, k = 2, k_1 = 3$

Case (2): $A_0 = 3.5, A = \sqrt{\frac{2}{3}}, N = -\sqrt{2}$

$$\varphi(\zeta) = 3.5 + \frac{\sqrt{3}\,e^{(1.4)\,(x+y-2t)}}{\left(\sqrt{2} + e^{(1.4)\,(x+y-2t)}\right) + \sqrt{3}}\,, \tag{19}$$

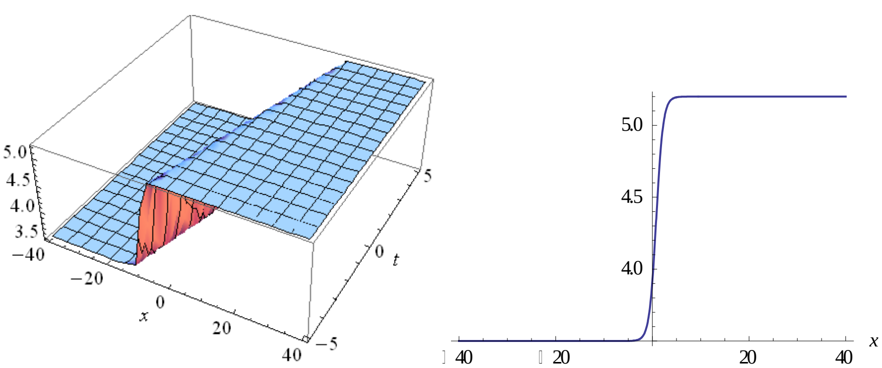

Figure 2. The plot of Eq.(19) in 2D and 3D with values: $A_0 = 3.5, A = \sqrt{\frac{2}{3}}, N = -1.4, X_0 = 1, k = 2, k_1 = 3$

Case (3): $A_0 = \sqrt{12}\,, A = -\sqrt{\frac{2}{3}}, N = -\sqrt{2}$

$$\varphi(\zeta) = \sqrt{12} - \frac{\sqrt{3}\, e^{(1.4)\,(x+y-2t)}}{\left(\sqrt{2} + e^{(1.4)\,(x+y-2t)}\right) + \sqrt{3}}\ , \tag{20}$$

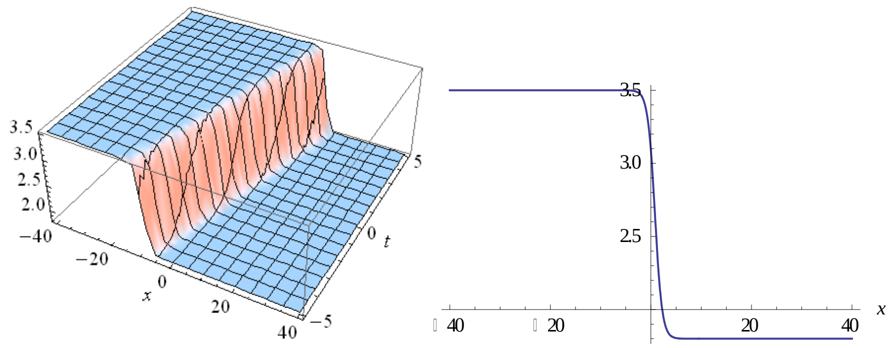

Figure 3. The plot of Eq.(20) in 2D and3D with values: $A_0 = 3.5,\ A = -\sqrt{\frac{2}{3}},\ N = -1.4,\ X_0 = 1,\ k = 2,\ k_1 = 3$

Case (4): $A_0 = -\sqrt{12}\,,\ A = \sqrt{\frac{2}{3}},\ N = \sqrt{2}$

$$\varphi(\zeta) = -3.5 + \frac{1.7\, e^{-(1.4)\,(x+y-2t)}}{3.1 - e^{-(1.4)\,(x+y-2t)}}\ , \tag{21}$$

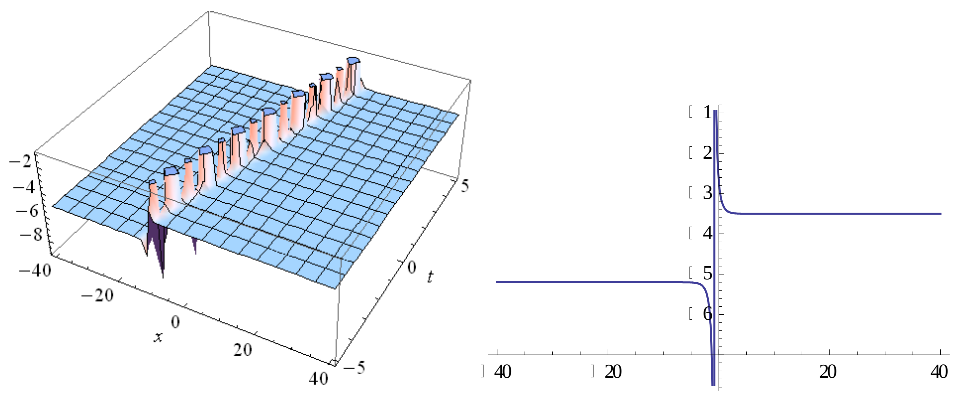

Figure 4. The plot of Eq.(21) in 2D and 3D with values: $A_0 = -3.5,\ A = \sqrt{\frac{2}{3}},\ N = 1.4,\ X_0 = 1,\ k = 2,\ k_1 = 3$

Case (5): $A_0 = -\sqrt{12}\,,\ A = \sqrt{\frac{2}{3}},\ N = -\sqrt{2}$

$$\varphi(\zeta) = -\sqrt{12} + \frac{\sqrt{3}\, e^{(1.4)\,(x+y-2t)}}{\left(\sqrt{2} + e^{(1.4)\,(x+y-2t)}\right) + \sqrt{3}}\ , \tag{22}$$

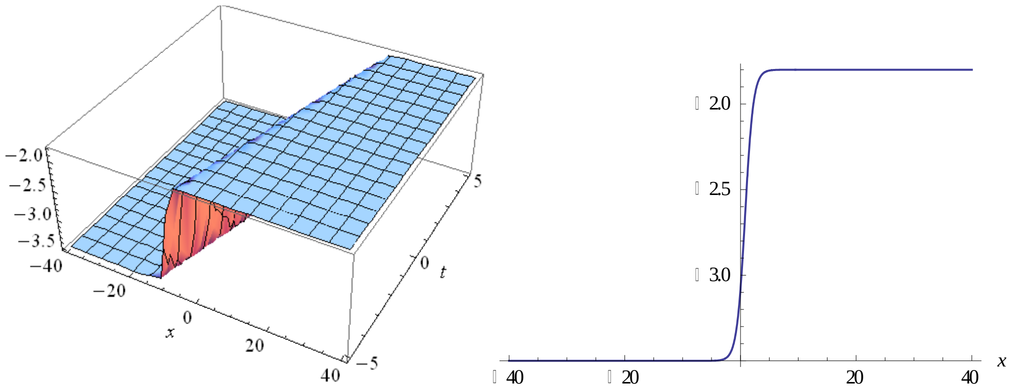

Figure 5. The plot of Eq.(22) in 2D and 3D with value $A_0 = -3.5, A = \sqrt{\dfrac{2}{3}}, N = -1.4, X_0 = 1, k = 2, k_1 = 3$

Case (6): $A_0 = -\sqrt{12}, A = -\sqrt{\dfrac{2}{3}}, N = \sqrt{2}$

$$\varphi(\zeta) = -3.5 - \frac{1.7\, e^{-(1.4)\,(x+y-2t)}}{3.1 - e^{-(1.4)\,(x+y-2t)}}\,, \tag{23}$$

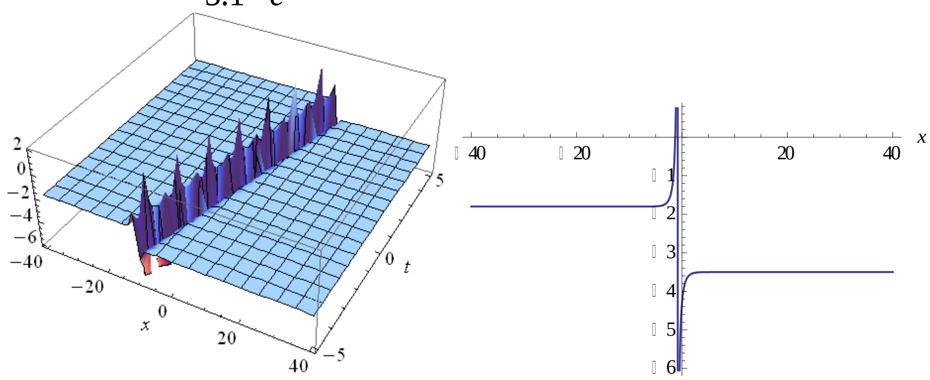

Figure 6. The plot of Eq.(23) in 2D and 3D with value $A_0 = -3.5, A = \sqrt{\dfrac{2}{3}}, N = -1.4, X_0 = 1, k = 2, k_1 = 3$

4. Some knots about fractional calculus

Before we apply the Paul-Painlevé approach mentioned above to the TFBPM [3], we firstly give some knots about the fractional calculus [25],

Caputo's fractional derivative [26-34]

Although the fractional calculus is a powerful tool to describe the physical phenomena systems which have long-term memory and long-range spatial interaction, there exists distortion between fractional mild and strong solutions. This confusion due to the fact that the fractional differential equation introduced a non-differentiable solution, this makes solution process difficult, but an approximate continuous solution can be realized if we observe the discontinuous solution. In this section we build Preliminary notes and remind some distinctive relations to Riemann–Liouville derivatives

DEFINITION 1:

A real function $g(x), x \succ 0$ is supposed to be in space $D_\alpha, \alpha \in \Re$ if there exists a real number $q \succ \alpha$, Such that $g(x) = x^q g_1(x)$, Where $g_1(x) \in D[0,\infty)$

DEFINITION 2:

A real function $g(x), x \succ 0$ is supposed to be in space $D_\alpha^m, m \in N \bigcup \{0\}$ if $g^m \in D_\alpha$. (24)

DEFINITION 3:

Let $g \in D_\alpha$ *and* $\alpha \geq -1$, then the left hand side of Riemann–Liouville integral of order $\mu, \mu \succ 0$ is given by

$$I_t^\mu g(x,t) = \frac{1}{\Gamma(\mu)}\int (t-\tau)^{\mu-1} g(x,\tau)\, d\tau, \quad \tau \succ 0, \tag{25}$$

DEFINITION 4:

The (left-sided) Caputo partial fractional derivative of g with respect to t, $g \in D_{-1}^m, m \in N \bigcup \{0\}$, is defined as

$$D_t^\mu g(x,t) = \frac{\partial^m}{\partial t^m} g(x,t), \mu = m \tag{26}$$

$$D_t^\mu g(x,t) = I_t^{\mu-m}\frac{\partial^m}{\partial t^m} g(x,t), \quad m-1 \prec \mu \prec m, \; m \in N \tag{27}$$

$$I_t^\mu D_t^\mu g(x,t) = g(x,t) - \sum_{k=0}^{m-1}\frac{\partial^k g}{\partial t^k}(x,0)\left(\frac{t^k}{k!}\right), \quad m-1 \prec \mu \prec m, \; m \in N \tag{28}$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu} \tag{29}$$

Furthermore, according to Grünwald–Letnikov,Caputo and Riemann–Liouville [26, 27] and the Modified Riemann- Liouville derivative by Jumarie [28, 29]. These definitions and some properties for the Jumarie's derivative of order α are listed as follows,

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-1} \tag{30}$$

$$D_t^\alpha [f(t)g(t)] = f(t) D_t^\alpha [g(t)] + g(t) D_t^\alpha [f(t)] \tag{31}$$

$$D_t^\alpha f[g(t)] = f'[g(t)] D_t^\alpha [g(t)] = D_t^\alpha [g(t)](g'(t))^\alpha \tag{32}$$

$$D_t^\alpha (c) = 0 \tag{33}$$

Moreover, the operator D_t^α satisfies the following basic properties:

(a) $D_t^\alpha D_t^\beta g(x) = D_t^{\alpha+\beta} g(x)$ (34)

(b) $D_t^\alpha x^\gamma = \left(\dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}\right) x^{\gamma-\alpha}$ (35)

(c) $D_t^\alpha I_t^\beta g(t) = g(t)$ (36)

(d) $I_t^\beta D_t^\alpha g(t) = g(t) - \sum_{k=0}^{m} g^k(0^+)\left(\dfrac{t^k}{k!}\right)$ (37)

Also, let us introduce the cconformable fractional derivative which is stated as follows,
If the function $\varphi:[0,\infty)\to\mathbb{R}$, thus the conformable fractional derivative of $\varphi(x), x\succ 0$ is defined as

$$D_x^{\gamma}(\varphi(x))=\lim_{H\to 0}\frac{\varphi(x+Hx^{1-\gamma})-\varphi(x)}{H}\quad \gamma,\in\quad (0 \tag{38}$$

For which all original differentiation rules that applied for the ordinary functions have been realized [40].
Now, Let us propose the general form of the NLFPDE in terms of the modified Riemann-Liouville derivatives $D_t^{\alpha}, D_x^{\alpha}$ as,

$$R(\varphi, D_t^{\alpha}\varphi, D_x^{\alpha}\varphi, D_{t^2}^{2\alpha}\varphi, D_{x^2}^{2\alpha}\varphi, ...)=0, 0\prec\alpha\leq 1 \tag{39}$$

According to [33, 34] the fractal derivative can be generally defined as,

$$\frac{\partial t}{\partial x^{\alpha}}=\Gamma(1+\alpha)\lim_{\substack{x\to x_0\\ \Delta x-x-x_0\to 0}}\frac{t-t_0}{(x-x_0)^{\alpha}}, \tag{40}$$

wher x_0 is the smallest scale beyond which there is no physical understanding. Now, using the fractional complex nonlinear transformation[30,34],

$$\varphi(x,t)=\varphi(\zeta), \zeta=\frac{k\,x^{\alpha}}{\Gamma(1+\alpha)}+\frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}+\zeta_0 \tag{41}$$

(where k, c and ζ_0 are constant with $k, c\neq 0$), we will transform equation (39) to this ordinary differential equation (ODE) with integer order, namely:

$$S(\varphi', \varphi'', \varphi''', ...)=0, \text{ where } (\ '=\frac{d}{d\xi}\ ) \tag{42}$$

According to [3], the TFBPM is given as:

$$\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}=\frac{\partial^2}{\partial x^2}(\varphi^2)+\frac{\partial^2}{\partial t^2}(\varphi^2)+\lambda\left(\varphi^2-r\right)=0 \tag{43}$$

$t>0,\ 0<\alpha\leq 1,\ x, y\in\mathbb{R}$, while$u$ and $\lambda(\varphi^2-r)$ represent the density and the princple support of population respictvely bulit on births and deaths while λ, and r are constants.

5. The exact solution to the time-fraction order of the biological population model

To apply the cosructed approach to the time-fraction order of the biological population model (TFBPM),
let us consider this complex transformation,

$$\varphi(x,y,t)=\varphi(\zeta), \zeta=x+iy-\frac{c\,t^{\alpha}}{\Gamma(1+\alpha)}, \text{ c is costant and } i^2=-1, \tag{44}$$

Thus, according to this transformation Eq. (33) becomes,

$$c\varphi'+\lambda\varphi^2-\lambda r=0,\quad \varphi'=\frac{d\varphi}{d\zeta} \tag{45}$$

Hence, according to the proposed method the suggested solution is;

$$\varphi(\zeta) = A_0 + e^{-N\zeta} W(X),\ X(\zeta) = R(\zeta) = C_1 - \frac{e^{-N\zeta}}{N}, \tag{46}$$

Substituting φ^2, φ_ζ, at Eq. (45) and equating the coefficients of different powers of $W(\zeta)\, e^{-N\zeta}$ to zero, this system of equations implies,

$$\begin{aligned} &-cA + \lambda = 0, \\ &cN + 2\lambda A_0 = 0, \\ &\lambda A_0^2 - \lambda r = 0, \end{aligned} \tag{47}$$

When one solves this system, the following results are achieved,

$$A = \frac{\lambda}{c}, N = \pm \frac{2\lambda\sqrt{r}}{c}, A_0 = \pm\sqrt{r} \tag{48}$$

According to the proposed method the solution is,

$$\varphi(\zeta) = A_0 + \frac{e^{-N\zeta}}{AX + X_0}, \tag{49}$$

As for the obtained result, it becomes,

$$\varphi(\zeta) = \pm\sqrt{r} + \frac{e^{\pm\frac{2\lambda\sqrt{r}}{c}\zeta}}{\frac{\lambda}{c}\left(1 - \frac{e^{\pm\frac{2\lambda\sqrt{r}}{c}\zeta}}{\pm\frac{2\lambda\sqrt{r}}{c}}\right) + X_0}, \tag{50}$$

From which we get these cases;

Case (1): $A_0 = \sqrt{r}, A = \sqrt{\frac{2}{3}}, N = \frac{2\lambda\sqrt{r}}{c}$

$$\varphi(\zeta) = \sqrt{r} + \frac{e^{\frac{-2\lambda\sqrt{r}}{c}\zeta}}{\frac{\lambda}{c}\left(1 - \frac{e^{\frac{-2\lambda\sqrt{r}}{c}\zeta}}{\frac{2\lambda\sqrt{r}}{c}}\right) + X_0}, \tag{51}$$

Choose $r = 4, c = 1, \lambda = \frac{1}{2}, A = \sqrt{\frac{2}{3}}, N = 2, X_0 = 1, k = 2$

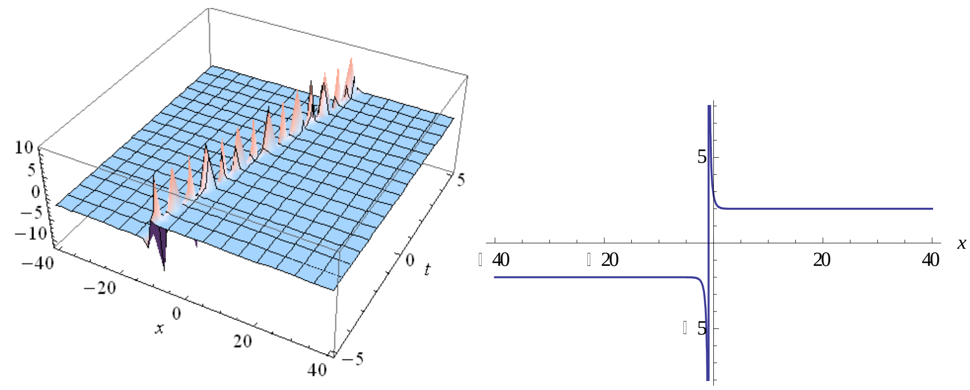

Figure 7. The plot of Eq.(51) in 2D and 3D with value $A_0 = 2$, $A = 0.5, N = 2, X_0 = 1, k = 2, r = 4, \lambda = 2,$

Case (2): $A_0 = \sqrt{r}\,, A = \sqrt{\frac{2}{3}}, N = \frac{-2\lambda\sqrt{r}}{c}$

$$\varphi(\zeta) = \sqrt{r} + \frac{e^{\frac{2\lambda\sqrt{r}}{c}\zeta}}{\frac{\lambda}{c}\left(1 + \frac{e^{\frac{2\lambda\sqrt{r}}{c}\zeta}}{\frac{2\lambda\sqrt{r}}{c}}\right) + X_0}\,, \tag{52}$$

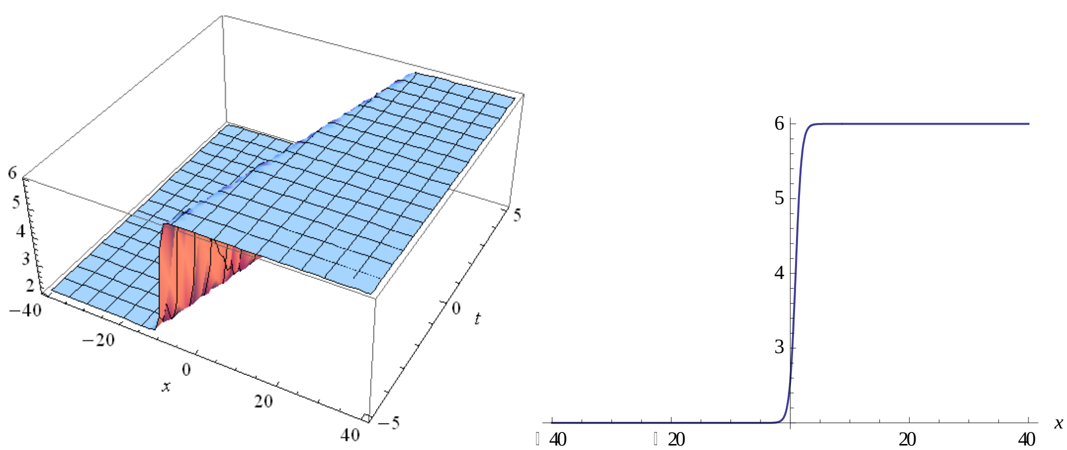

Figure 8. The plot of Eq.(52) in 2D and 3D with value $A_0 = 2$, $A = 0.5, N = -2, X_0 = 1, k = 2, r = 4, \lambda = 2,$

Case (3): $A_0 = -\sqrt{r}\,, A = \sqrt{\frac{2}{3}}, N = \frac{2\lambda\sqrt{r}}{c}$

$$\varphi(\zeta) = -\sqrt{r} + \frac{e^{\frac{-2\lambda\sqrt{r}}{c}\zeta}}{\frac{\lambda}{c}\left(1 - \frac{e^{\frac{-2\lambda\sqrt{r}}{c}\zeta}}{\frac{2\lambda\sqrt{r}}{c}}\right) + X_0}\,, \tag{53}$$

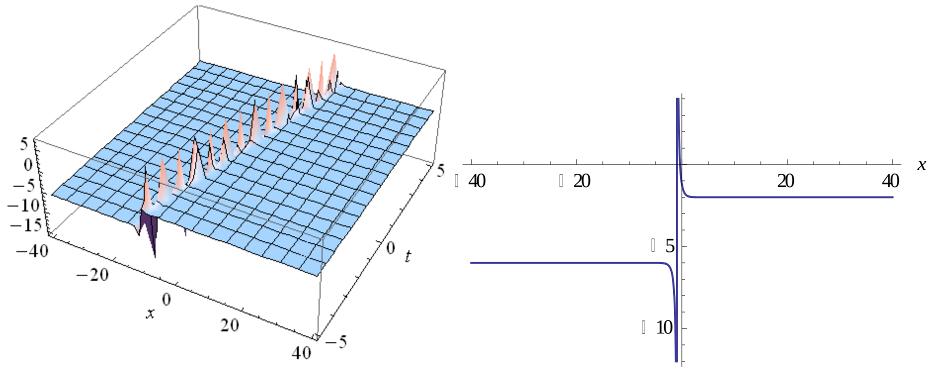

Figure 9. The plot of Eq.(53) in 2D and 3D with value $A_0 = -2, A = 0.5, N = 2, X_0 = 1, k = 2, r = 4, \lambda = 2,$

Case (4): $A_0 = -\sqrt{r}\,, A = \sqrt{\dfrac{2}{3}}, N = \dfrac{-2\lambda\sqrt{r}}{c}$

$$\varphi(\zeta) = -\sqrt{r} + \frac{e^{\frac{2\lambda\sqrt{r}}{c}\zeta}}{\dfrac{\lambda}{c}\left(1 + \dfrac{e^{\frac{2\lambda\sqrt{r}}{c}\zeta}}{\frac{2\lambda\sqrt{r}}{c}}\right) + X_0}\,, \tag{54}$$

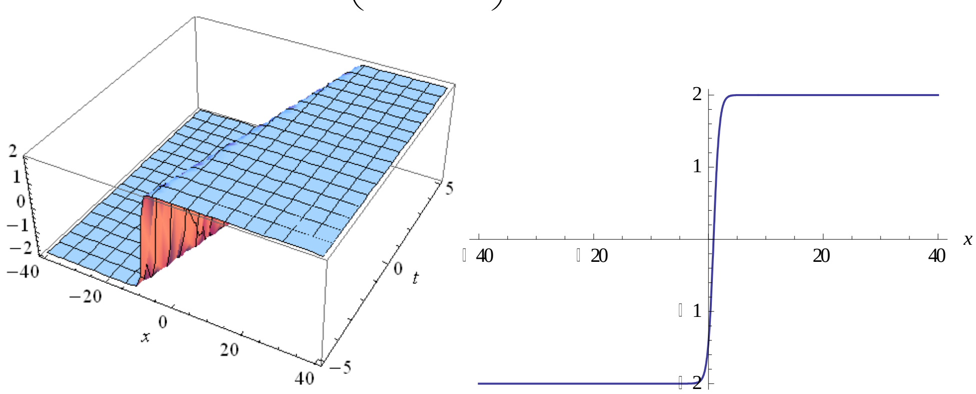

Figure 10. The plot of Eq.(54) in 2D and 3D with values; $A_0 = -2, A = 0.5, N = -2, X_0 = 1, k = 2, r = 4, \lambda = 2,$

6.Results and discussion

(a) Firstly, for the (2+1)-dimensional Zoomeron model according to the obtained results and the corresponding figures (1-6) we notice that the proposed technique achieved solutions which weren't satisfied by [22], [23] whose using the (G'/G)-expansion method. In adtion to, there are many solutions realized using the proposed method which weren't obtained using [41] which use the exp-function method. Furthermore, the simulation of the solitary solutions is achieved as [42] for the bounded traveling wave solution of this model using the bifurcation method of dynamical system and numerical simulation method. Also some of the obtained solutions are agreements with the solution achieved by [43] in some cases and increase in other cases.

(b) For the time fraction population equation, some of the achieved solutions using the constructed technique are more accurate than obtained by [36] who use the Homotopy analysis sumudu transformation method and [39],[40] whose use Homotopy perturbation method and Homotopy analysis respectivlyand the other are new.

7. Conclusion

The Paul-Painlevé approach has been effectively used for the first time to find new imperssive regulations for the biological population models with its different forms which are the non-fractional order biological population (2+1)-Zoomeron model (NFBPZM) Figures (1-6) and the time-fractional order biological population model (TFBPM) Figures (7-10). Some of the realized results agree with the results previously obtained by other authors [22, 23] for the NFBPZM and [36], [38-40] for TFBPM in some cases and increase in other cases. The satisfying results of this new accurate regulations prove the efficiency of this tichnique.

References

[1] Calogero F, Degasperis A., (1976) Nonlinear evolution equations solvable by the inverse spectral transform I, Nuovo Cimento B,.32; 201–42.

[2] Shakeri E., Dehghan M., (2007) Numerical solution of a biological population model using Hes variatiional iteration method, Computer And mathematics with Applications, 54, 1197-1207

[3] Bekir, A.,Guner, O., (2013) Exact solutions of nonlinear fractional differential equations by (G'/G) −expansion method, Chin. Phys. B 22, 11,

[4] Jabbari, A., Kheiri, H., Bekir, A., (2011) Exact solutions of the coupled Higgs equation and the Maccari system using He's semi-inverse method and (G'/G)-expansion method, Computers & Mathematics with Applications, 62, (5) 2177-2186

[5] Younis, M.A., (2014) New approach for the exact solutions of nonlinear equations of fractional order via modified simple equation method. Applied Mathematics, 5, 13, 1927

[6] Wang, M.L, Xiangzheng L., Jinliang Z., (2008) The (G'/G) −expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Physics Letters A 372,4, 417-423.

[7] Zahran, E H. M, Khater, M.M.A., (2014) Exact Travelling wave solutions for the system of shallow water wave Equation Modified Liouville Equation using Extended Jacobi Elliptic function Expansion Method. American Journal of computational Mathematics, 4, 455-463.

[8] Zhang, S., Jing-Lin, T., Wei, W., (2008) A generalized expansion method for the mKdV equation with variable coefficients. Physics Letters A 372, 13, 2254-2257.

[9] Zahran, E H. M, Khater, M.M.A., (2015) The Two Variable $(G'/G, 1/G)$ −Expansion Method for Solving Nonlinear Dynamics of Microtubules-A New Model. Global Journal of Science Frontier Research Vol. 15, Issue 2

[10] Zahran, E H. M., (2015) Travelling wave solutions of Non Linear Evolution Equation Via Modified $exp\left(-\phi(\xi)\right)-$Expansion method, Journal of Computational and Theoretical Nano Science, 12, 5716-5724.

[11] Shehata, M S. M., (2015) Extended Jacobi Elliptic function Expansion Method and its Applications for Solving some Nonlinear Evolution Equations in Mathematical Physics, International Journal of Computer Applications, 109-No.12.

[12] Zahran, E H. M., (2015) Exact traveling wave solutions for nonlinear fractional partial differential equations arising in soliton using the $exp(-\phi(\xi))-$expansion method. International Journal of Computer Applications. 109, 13.

[13] Aslan, I., (2011). Application of the exp-function equations method to (2+1) dimension Boiti-Lean Pempinelli equation using symbolic computation. International Journal of Computer Mathematics, 88, 4 747-761.

[14] Shehata, M S. M., Rezazadeh, H., Zahran , E H. M., Tala-Tebue, E., and Bekir, A., (2019) New Optical Soliton Solutions of the Perturbed Fokas-Lenells Equation, Commun. Theor. Phys.71, 1275–1280

[15] Bekir, A., Boz, A., (2007) Exact solutions for a class of nonlinear partial differential equations using exp-function method. International Journal of Nonlinear Sciences and Numerical Simulation 8,4, 505-512.

[16] Shehata, M S. M., (2016) Exact Traveling Wave Solutions for Nonlinear Evolutions Equation, Journal of Computational and Theoretical Nano science, 13, 1 534-538.

[17] Wazwaz, A.M., (2004) The tanh method for traveling wave solutions of nonlinear equations. Appl. Math.Comput., 154(3) 713-723.

[18] Fan, E., (2000) Extended tanh-function method and its applications to nonlinear equations.Physics Letters.A. 277, 4, 212-218.

[19] Elwakil, S. A., et al., (2002) Modified extended tanh-function method for solving nonlinear partial differential equations. Physics Letters A, 299, 2, 179-188.

[20] Zahran, E H. M, Khater, M.M.A., (2016) Modified extended tanh-function method and its applications to the Bogoyavlenskii equation. Applied Mathematical Modeling 40, 3, 1769-1775.

[21] Lu, D., Khater, Khater, M.M.A., Zahran, E H. M, (2017) Solitary wave solutions of the Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law nonlinearity, Appl. Math &Inf.Sci.11(5):1-5

[22] Abazari, R., (2011) The solitary wave solutions of Zoomeron equation, Appl Math Sci. 5(59), 2943–9.

[23] Khan, K., Akbar, M.A., Salam, A., Islam, H., (2014) A note on enhanced (G'/G)-expansion method in

nonlinear Physics; Ain Shams Engineering Journal

[24] Kudryashov N.A., (2019) The Painlevé approach for finding solitary wave solutions of nonlinear non-integrable differential equations, Optik 183, 642–649

[25] Caputo, M., Mainardi, F., (1971). Linear models of dissipation in anelastic solids.Rivista del Nuovo Cimento 161-198.

[26] Caputo, M. (1967) Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. Royal Astronom. Soc, 13529-539.

[27] Samko, S. G. Kilbas, A. AMarichev, . O. I. (1993) Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Switzerland

[28] Jumarie, G. (2006). Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput. Math. Appl., 51, 1367-1376.

[29] Jumarie, G. (2009) Table of some basic fractional calculus formulae derived from a modified Riemann-Liouvillie derivative for nondifferentiable functions, Appl. Math.. Lett., 22, 378-385.

[30] Millar, KS, Ross, B., (1993) An introduction to the fractional calculus and fractional differential equations; New York: Wiley

[31] Kilbas, A.A., Srivastava H.M., Trujillo, J.J. (2006) ”Theory and Applications of Fractional Differential Equations”, North-Holland Mathematical Studies, Vol. 204, Elsevier, Amsterdam,.

[32] Bekir, A., Guner, O., (2014). The (G′/G)-expansion method using modified Riemann-Liouville derivative form space-time fractional differential equations, Ain Shams Eng. Journal, 5, 959-965.

[33] He, J-H., Elagan, S.K. and Li, Z.B.. (2012). Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus; Phys. Lett. A,376;257-259

[35] Ain, Q.T., He, J-H, (2019). Two scale mathematics and fractional calculus for thermo dynamics, Thermal Science, 23(3B) 1707-1712

[35] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., (2014). A new definition of fractional derivative. J Comput. Appl. Math. 265, 65–701

[36] Gupta, V.G, Kumar, P., (2013) Approximate solutions of fractional Biological population model by Homotopy analysis sumudu transformation method; International Journal of scince and research Vol.6,14.

[37] Gurtin, M.E., Maccamy, R.C., (1977). On the diffusion of biological population; Math Biol Sci 54 (33), 35-49.

[38] Lu, YG., (2000) HöIder estimate of solutions of biological population equations; Appl Math Lett , 13, 123-126.

[39] Liu, Y., Li, Z., Zhang, Y., (2011). Homotopy perturbation method to fractional biological population equation. Fract Diff Calc, 1, 117-124.

[40] Arafa, A.A., Rida, S.Z., Mohamed, H., (2011). Homotopy analysis method for solving biological population model. Commun Theor Phys., 56, 7972800.

[41] Khan, K., Akbar, M.A., (2014) Traveling wave solutions of the (2 + 1)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the Exp-function method Ain Shams Engineering Journal, 5, 247-256

[42] Zhou, Y., Cai, S. and Liu, Q., (2015) Bounded Traveling Waves of the (2+1)-Dimensional Zoomeron Equation; Mathematical Problem In Engineering, 2015, 163597

[43] Manafian,J., Lakestani,M., Bekir,A., (2016) Study of the Analytical Treatment of the (2+1)-Dimensional Zoomeron, the Duffing and the SRLW Equations via a New Analytical Approach, International Journal of Applied and Computational Mathematics; 2, 243–268