Global mild solution for the Navier–Stokes–Nernst–Planck–Poisson system in Besov-weak-Herz spaces

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Abstract. We study a coupled Navier–Stokes–Nernst–Planck–Poisson system arising from electrohydrodynamics in critical Besov-weak-Herz spaces. When the initial value sufficiently small, we prove the existence and uniqueness of global mild solution to the cauchy problem in this spaces for $n \geq 3$. The spaces is larger than some other known critical spaces.

Keywords. Mild solution; Well-posedness; Besov-weak-Herz spaces; Critical.

1. introduction

In this paper, we study well-posedness of the following Navier–Stokes–Nernst–Planck–Poisson system modeling the motion of an isothermal, incompressible and viscous Newtonian fluid of charged particles in Besov-weak-Herz spaces

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = \Delta u + \Delta\phi\nabla\phi \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ v_t + (u \cdot \nabla)v = \nabla \cdot (\nabla v - v\nabla\phi) \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ w_t + (u \cdot \nabla)v = \nabla \cdot (\nabla w + w\nabla\phi) \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ \Delta\phi = v - w \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ (u, v, w)|_{t=0} = (u_0, v_0, w_0) \quad \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 3$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, $u(x,t) \in \mathbb{R}^n$ denotes the velocity field of the fluid, $p(x,t) \in \mathbb{R}$ denotes the pressure, $\phi(x,t) \in \mathbb{R}$ denotes the electrostatic potential, $\{v(x,t), w(x,t)\} \in \mathbb{R}$ denote the charge densities of the negatively and positively charged particles, respectively. $\Delta\phi\nabla\phi$ stand for Lorenz force exerted by charged particles.

If $v = w = 0$, system (1.1) reduces into the well-known nonhomogeneous incompressible Navier–Stokes equations. It has been paid great attentions for many years.

If $u = 0$, system (1.1) reduces into the Nernst–Planck–Poisson system, this system can be regard as degenerate parabolic equations coupled with

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elliptic equation, it has drawn much attention of analysts during the past years.

For system (1.1), the results are much less, the local smooth solution has be established in [3], weak solution and regularity in bounded domain be established in [4] and [5], the global well-posedness for small initial data in negative-order Besov spaces has be studied in [1], in the modulation and Lebesgue spaces can refer to [2] and [6], recently, some logarithmical regularity criteria results be obtained in [7].

Besov-weak-Herz spaces (BWH-spaces) was first introduced in [8](2018) for deal with the global well-posedness of incompressible Navier-Stokes equations. In fact, one has the chain of critical spaces given by the continuous embedding for appropriate index

$$\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n) \hookrightarrow L^n(\mathbb{R}^n) \hookrightarrow L^{n,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{\frac{n}{p}-1}_{p,\infty}(\mathbb{R}^n) \hookrightarrow BMO^{-1}(\mathbb{R}^n),$$

rough speaking, BMO^{-1} and Besov morrey space $\mathcal{N}^{\frac{n}{r}-1}_{r,q,\infty}$ are maximal critical spaces in the sense that it is not known a larger critical space small solutions are globally well-posed for Navier-Stokes equations. We also do not known whether there are inclusion relations between Besov-weak-Herz sapces and BMO^{-1} or between Besov-weak-Herz sapces and $\mathcal{N}^{\frac{n}{r}-1}_{r,q,\infty}$, for more detail see [8]. It clearly that

$$\dot{B}^{\frac{n}{p}-1}_{p,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}(\mathbb{R}^n),$$

for appropriate index, so Besov-weak-Herz sapces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ can be regard as expanse critical spaces.

From $(1.1)_5$, we have

$$\phi = (-\Delta)^{-1}(w-v) = C\int_{R^n} \frac{(w(y)-v(y))}{|x-y|^{n-2}}dy \quad \text{for} \quad n \geq 3, \tag{1.2}$$

$$\begin{aligned} \nabla\phi = \nabla(-\Delta)^{-1}(w-v) &= C_1\int_{R^n} \frac{(x-y)(w(y)-v(y))}{|x-y|^n}dy \\ &\leq C_1\int_{R^n} \frac{(w(y)-v(y))}{|x-y|^{n-1}}dy \quad \text{for} \quad n \geq 3, \end{aligned} \tag{1.3}$$

where $(-\Delta)^{-1}$ and $\nabla(-\Delta)^{-1}$ are fourier multiplication operators with symbol $|\xi|^{-2}$ and $\xi/|\xi|^{-2}$, they can be regard as Calderón–Zygmund singular integral operators. Consider a detail, where $C = 2^{-2}\pi^{-\frac{n}{2}}\Gamma(\frac{n-2}{2})/\Gamma(1)$ and $\Gamma(\cdot)$ is the Gamma functions, C_1 is a similar constant. Applying the Leray–Hopf projector $\mathbb{P}$, free divergence conditions and using Duhamel's principle, the Cauchy problem (1.1) can be reduced to the integral system:

$$\begin{cases} u = G(t)u_0 + B_1(u,u) + B_2(w-v, w-v), \\ v = G(t)v_0 + B_3(u,v) + B_4(v, w-v), \\ w = G(t)w_0 + B_3(u,w) - B_4(w, w-v), \end{cases} \tag{1.4}$$

where

$$B_1(u,u) = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau,$$
$$B_2(v,w) = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(v\nabla(-\Delta)^{-1}w)(\tau)d\tau,$$
$$B_3(u,v) = -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(uv)(\tau)d\tau,$$
$$B_4(v,w) = -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(v\nabla(-\Delta)^{-1}w)(\tau)d\tau,$$

we note $a\otimes b := (a_ib_j)_{1\le i,j\le n}$ is tensor product function. The operator $\mathbb{P}$ can be expressed as $\mathbb{P} = (\mathbb{P}_{i,j})_{n\times n}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + \mathcal{R}_i\mathcal{R}_j$, $\delta_{i,j}$ is the Kronecker delta and $\mathcal{R}_i = (-\Delta)^{-1/2}\partial_i$ is the i-th Riesz transform. $G(t) = e^{t\Delta}$ be the pseudo-differential operator with symbol $e^{-t|\xi|^2}$. By critical, which mean that we want to solve the system in functional spaces with norm independent of the changes of scales which leaves the system invariant. Note that if u, v, w are smooth solutions for (1.1)(or (1.4)), then

$$\begin{aligned} u_\lambda(x,t) &:= \lambda u(\lambda x, \lambda^2 t), \quad \phi_\lambda(x,t) := \phi(\lambda x, \lambda^2 t), \\ v_\lambda(x,t) &:= \lambda^2 v(\lambda x, \lambda^2 t), \quad w_\lambda(x,t) := \lambda^2 w(\lambda x, \lambda^2 t), \end{aligned} \tag{1.5}$$

are also solutions with initial data

$$\begin{aligned} (u_0)_\lambda(x) &= \lambda u(\lambda x), \quad (\phi_0)_\lambda(x) = \phi(\lambda x), \\ (v_0)_\lambda(x) &= \lambda^2 v(\lambda x), \quad (w_0)_\lambda(x) = \lambda^2 w(\lambda x). \end{aligned} \tag{1.6}$$

Compare to the Navier–Stokes equations, system (1.1) is very complicated to treat, by take advantage of the theorems of Besov-weak-Herz space and weak-Herz space, heat semigroup estimates, embedding theorems and interpolation properties, we overcome some difficulties of complex estimates, proved the existence and uniqueness of global mild solution of system (1.1) for small initial value.

We stat main results as follows.

Theorem 1.1. *Let $n \ge 3$, $1 \le q \le \infty$, $\frac{n}{2} < p < n$, $0 \le \alpha < 1 - \frac{n}{2p}$ and $\frac{1}{p} + \frac{1}{n} = \frac{1}{\tilde p}$. There exist $\epsilon > 0$ and $\delta > 0$ such that if $u_0 \in \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}$ with $\nabla\cdot u_0 = 0$, $v_0, w_0 \in \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}$ and $\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}} + \|v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}} + \|w_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}} \le \delta$, then there exists a unique mild solution*

$$u \in L^\infty((0,\infty); \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}), \quad \{v,w\} \in L^\infty((0,\infty); \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}),$$

for system (1.1) such that

$$\|u\|_X + \|v\|_Y + \|w\|_Y \le 2\epsilon,$$

where space X and Y are defined in section 3. Moreover $u(t) \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{-1}_{\infty,\infty}$, $v(t) \overset{*}{\rightharpoonup} v_0$, $w(t) \overset{*}{\rightharpoonup} w_0$ in $\dot{B}^{-2}_{\infty,\infty}$, as $t \to 0^+$.

The paper is written as follows. In section 2, we will establish some preliminary Lemmas and estimates. In section 3, we will prove the main results.

2. Preliminaries

In this section, we introduce homogeneous weak-Herz spaces, Sobolev-weak-Herz spaces and Besov-weak-Herz spaces.

For an integer $k \in \mathbb{Z}$, we define the set A_k as

$$A_k = \{x \in \mathbb{R}^n; 2^{k-1} \leq |x| < 2^k\},$$

observe that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k\in\mathbb{Z}} A_k$, weak-Herz space is defined as follows.

Definition 2.1. *Let $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$. The Homogeneous weak-Herz space $W\dot{K}^{\alpha}_{p,q} = W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined as the set of all measurable functions such that the following quantity is finite*

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}} \triangleq \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^{p,\infty}(A_k)}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha}\|f\|_{L^{p,\infty}(A_k)} & \text{if } q = \infty. \end{cases}$$

Here, $L^{p,\infty}$ noted the weak L^p spaces. If (p, q, α) satisfy the assumed conditions of above definition, space $W\dot{K}^{\alpha}_{p,q}$ with the norm $\|f\|_{W\dot{K}^{\alpha}_{p,q}}$ is a Banach space. Hölder inequality also holds for homogeneous Weak-Herz spaces, that is

$$\|fg\|_{W\dot{K}^{\alpha}_{p,q}} \leq C\|f\|_{W\dot{K}^{\alpha_1}_{p_1,q_1}}\|g\|_{W\dot{K}^{\alpha_2}_{p_2,q_2}}, \tag{2.1}$$

for any $1 < p, p_1, p_2 \leq \infty$, $1 \leq q, q_1, q_2 \leq \infty$, and $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$, such that $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, $\alpha = \alpha_1 + \alpha_2$.

$$W\dot{K}^{\alpha}_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty} \tag{2.2}$$

for $1 < p < \infty$ and $0 \leq \alpha \leq n(1 - 1/p)$. For more details see [9].

We define the homogeneous Sobolev-weak-Herz spaces.

Definition 2.2. *Let $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $\alpha, s \in \mathbb{R}$. Recall the Riesz operator $\widehat{I^s f} = |\xi|^s\hat{f}$. The homogeneous Sobolev-weak-Herz spaces $W\dot{K}^{\alpha,s}_{p,q} = W\dot{K}^{\alpha,s}_{p,q}(\mathbb{R}^n)$ are defined as*

$$W\dot{K}^{\alpha,s}_{p,q} = \Big\{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} < \infty\Big\}.$$

For the homogeneous Besov-weak-Herz spaces.

Definition 2.3. *Let $1 < p \leq \infty$, $1 \leq q, r \leq \infty$ and $\alpha, s \in \mathbb{R}$. The homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}(\mathbb{R}^n)$ are defined as*

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \Big\{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} < \infty\Big\},$$

where

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \triangleq \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\right)^{\frac{1}{r}} & \textit{if } 1 \le r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} & \textit{if } r = \infty. \end{cases} \tag{2.3}$$

(1). The spaces $W\dot{K}^{\alpha,s}_{p,q}$ and $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ are Banach spaces endowed with the norms $\|\cdot\|_{W\dot{K}^{\alpha,s}_{p,q}}$ and $\|\cdot\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}$, respectively.

(2). The continuous inclusion $\dot{B}^s_{p,r}(\mathbb{R}^n) \subset \dot{B}W\dot{K}^{0,s}_{p,\infty,r}$ holds for all $s \in \mathbb{R}$, $1 < p \le \infty$, and $1 \le r \le \infty$, where $\dot{B}^s_{p,r}$ stands for homogeneous Besov spaces. For that, it is sufficient to recall the definition of Besov spaces, (2.3) and the inclusion $L^p \subset W\dot{K}^0_{p,\infty}$ that is going to be showed in the lemma below:

The estimate for Pseudodifferential operator is important for us.

Lemma 2.1. [8] *Let $1 < p < \infty$, $1 \le q, r \le \infty$, $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$ and $m, s \in \mathbb{R}$. Let $P \in C^n(\mathbb{R}^n\backslash\{0\})$ be a function such that $\left|\partial^\beta_\xi P(\xi)\right| \le C|\xi|^{(m-|\beta|)}$ for all multi-index β satisfying $|\beta| \le n$. Then*

$$\|P(D)f\|_{\dot{B}W\dot{K}^{\alpha,s-m}_{p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}. \tag{2.4}$$

In what follows we present some inclusions involving Sobolev-weak-Herz and Besov-weak-Herz spaces.

Lemma 2.2. [8] *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$ and $m, s \in \mathbb{R}$. We have the following continuous inclusions*

$$\dot{B}W\dot{K}^{\alpha,0}_{p,q,1} \subset W\dot{K}^{\alpha}_{p,q} \subset \dot{B}W\dot{K}^{\alpha,0}_{p,q,\infty} \tag{2.5}$$

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,1} \subset W\dot{K}^{\alpha,s}_{p,q} \subset \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty} \tag{2.6}$$

Now we present an embedding theorem of Sobolev type, we will give the proof of this key Lemma.

Lemma 2.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q, r \le \infty$, $p \le p_1 < \infty$, $1 < p_2 \le p_1$ and $-\frac{n}{p} < \alpha < n(1+\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p})$. Then*

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha+n(\frac{1}{p}-\frac{1}{p_1}),s+n(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,q,r}}, \tag{2.7}$$

the proof can refer to [8].

In particular, for $n \ge 3$, $\frac{n}{2} < p < \infty$, $0 \le \alpha < \min\left\{1-\frac{n}{2p}, \frac{n}{2p}\right\}$ and $\frac{1}{\tilde{p}} = \frac{1}{p} + \frac{1}{n}$, we have follows importance conclusion

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{2p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+s+\frac{n}{2p}}_{p,q,r}}, \tag{2.8}$$

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{2\tilde{p},q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+s+\frac{n}{2\tilde{p}}}_{\tilde{p},q,r}}. \tag{2.9}$$

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{2\tilde{p},q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+s+\frac{n}{2\tilde{p}}-\frac{1}{2}}_{\frac{2n\tilde{p}}{2n-\tilde{p}},q,r}}. \tag{2.10}$$

Proof. Here, first take the transform $p \to 2p$ in (2.7), we choose $\alpha = n(\frac{1}{2p} - \frac{1}{p_1})$, $p_2 = p$, obviously, $\alpha < \frac{n}{2p}$, if we suppose $\alpha < \frac{n}{2} - \frac{n}{2p}$, then we have

$$\frac{n}{2} - \frac{n}{2p} - \alpha = \Big(\frac{n}{2} - \frac{n}{2p}\Big) - \Big(\frac{n}{2p} - \frac{n}{p_1}\Big) = \frac{n}{2} - \frac{n}{p} + \frac{n}{p_1} > 0,$$

which mean (above inequality guarantee the thirdly inequality hold)

$$\alpha \leq \frac{n}{2} - \frac{n}{2p} < n\Big(1 + \frac{1}{p_1} - \frac{1}{p} - \frac{1}{2p}\Big),$$

by the same way, we take the transform $p \to 2\tilde{p}$ in (2.7), we choose $\alpha = n(\frac{1}{2\tilde{p}} - \frac{1}{\tilde{p}_1})$, $p_2 = \tilde{p}$, where $\tilde{p_1}$ is another p_1 as a differentiate, then we have

$$\alpha \leq \frac{n}{2} - \frac{n}{2\tilde{p}} < n\Big(1 + \frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}} - \frac{1}{2\tilde{p}}\Big),$$

for the third one, by the same way we have $\alpha \leq \frac{n}{2} - \frac{n}{2\tilde{p}} + \frac{1}{4}$.

For α

$$\alpha < \min\Big\{(\frac{n}{2} - \frac{n}{2p}), (\frac{n}{2} - \frac{n}{2\tilde{p}})\Big\} = (\frac{n}{2} - \frac{n}{2\tilde{p}}) = (\frac{n}{2} - \frac{1}{2} - \frac{n}{2p}),$$

we have

$$\alpha < 1 - \frac{n}{2p} \leq (\frac{n}{2} - \frac{1}{2} - \frac{n}{2p}) \quad \text{for} \quad n \geq 3$$

and

$$\alpha = n(\frac{1}{2p} - \frac{1}{p_1}) \Rightarrow \alpha < \frac{n}{2p},$$

we also suppose $\alpha \geq 0$, then we obtain $0 \leq \alpha < \min\Big\{1 - \frac{n}{2p}, \frac{n}{2p}\Big\}$, and $0 < 1 - \frac{n}{2p} \Rightarrow \frac{n}{2} < p < \infty$, the proof of this particular case is thus complete. □

Lemma 2.4. [8] *Let $s_0, s_1, s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$. If $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$ with $\theta \in (0,1)$, then*

$$\Big(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}\Big)_{\theta,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}.$$

The Beta function is a basic tool, we will give the proof of this Lemma.

Lemma 2.5. *Beta function is defined as:*

$$\mathcal{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

for $a > 0, b > 0$, $\mathcal{B}(a,b)$ is finite positive constant. Then we have

$$\int_0^t \tau^{a-1}(t-\tau)^{b-1}d\tau = t^{a+b-1}\mathcal{B}(a,b) \quad \textit{for } a > 0, b > 0.$$

Proof. Let $x = \tau/t$, then $d\tau = tdx$, $0 < x < 1$, $\tau^{a-1} = t^{a-1}x^{a-1}$, $(t-\tau)^{b-1} = t^{b-1}(1-x)^{b-1}$, we have

$$\int_0^t \tau^{a-1}(t-\tau)^{b-1}d\tau = t^{(a-1)+(b-1)+1}\int_0^1 x^{a-1}(1-x)^{b-1}dx = t^{a+b-1}\mathcal{B}(a,b).$$

□

We now give the heat kernel estimates in Besov-weak-Herz spaces.

Lemma 2.6. [8] *Let $s,\sigma \in \mathbb{R}$, $s \le \sigma$, $1 < p < \infty$, $1 \le q,r \le \infty$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$. Then, there is $C>0$ (independent of f) such that*

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,r}} \le Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}, \tag{2.11}$$

for all $t>0$. Moreover, if $s<\sigma$, then we have the estimate

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1}} \le Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}, \tag{2.12}$$

for all $t>0$.

Lemma 2.7. [9] *Let $1 < p < \frac{n}{\gamma}$, $0 < q \le \infty$, $0 \le \gamma < n$ and $-\frac{n}{p}+\gamma < \alpha < n(1-\frac{1}{p})$. Let $\frac{1}{p}-\frac{1}{r}=\frac{\gamma}{n}$ and T_γ be a bounded operator from $L^{p,\infty}$ to $L^{r,\infty}$ $\|T_\gamma u\|_{L^{r,\infty}} \le C_1\|u\|_{L^{p,\infty}}$ satisfying*

$$|T_\gamma(f)(x)| \le C_2\int \frac{|f(y)|}{|x-y|^{n-\gamma}}dy \tag{2.13}$$

for $f \in L^1_{loc}$ with x not belong to supp f. Then T_γ is also a bounded operator from $W\dot{K}^{\alpha}_{p,q}$ to $W\dot{K}^{\alpha}_{r,q}$ with $\|T_\gamma\| \le c(C_1+C_2)$.

3. proof of main results

3.1. Function space

Let us define the space X and Y as

$$X := \Big\{u : (0,\infty) \to \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty} \cap W\dot{K}^{\alpha}_{2p,2q} \text{ with } \nabla\cdot u = 0 \text{ such that } \|u\|_X < \infty\Big\},$$

$$Y := \Big\{v : (0,\infty) \to \dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty} \cap W\dot{K}^{\alpha}_{2\tilde{p},2q} \text{ such that } \|v\|_Y < \infty\Big\},$$

where

$$\|u\|_X := \|u\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty})} + \sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}},$$

$$\|v\|_Y := \|v\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty})} + \sup_{t>0} t^{1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\|v\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}.$$

For readability, we now explain how to choose parameters of spaces X and Y, with the scale

$$u_\lambda(x,t) := \lambda u(\lambda x,\lambda^2 t), \quad v_\lambda(x,t) := \lambda^2 v(\lambda x,\lambda^2 t),$$

and analyze the structure of space $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$, it is easy obtain

$$\|u_\lambda(x,t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}} = \lambda^{-\alpha+s-\frac{n}{p}+1}\|u(x,t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}},$$

$$\|v_\lambda(x,t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{\tilde{p},q,\infty}} = \lambda^{-\alpha+s-\frac{n}{\tilde{p}}+2}\|v(x,t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{\tilde{p},q,\infty}},$$

by critical, we have $s=\alpha+\frac{n}{p}-1$ and $s=\alpha+\frac{n}{\tilde{p}}-2$, respectively.

For the second time-depandent part of spaces X and Y, notice the time scale $t \longrightarrow \lambda^2 t$, we have

$$(\lambda^{-2}t)^\beta\|u_\lambda(x,t)\|_{W\dot{K}^{\alpha}_{2p,2q}} = \lambda^{-2\beta-\alpha-\frac{n}{2p}+1}t^\beta\|u(x,t)\|_{W\dot{K}^{\alpha}_{2p,2q}},$$

$$(\lambda^{-2}t)^{\beta}\|v_\lambda(x,t)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}} = \lambda^{-2\beta-\alpha-\frac{n}{2\tilde{p}}+2}t^{\beta}\|v(x,t)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}},$$

by critical, we have $\beta = \frac{1}{2} - (\frac{\alpha}{2} + \frac{n}{4p})$ and $\beta = 1 - (\frac{\alpha}{2} + \frac{n}{4\tilde{p}})$, respectively.

Now, we can define a product space for solution (u, v, w)

$$X_u \times Y_v \times Y_w := \{(u, v, w) \in X \times Y \times Y \mid \|u\|_X + \|v\|_Y + \|w\|_Y < \infty\}. \tag{3.1}$$

And recalling the equations

$$\begin{cases} u = G(t)u_0 + B_1(u, u) + B_2(w - v, w - v), \\ v = G(t)v_0 + B_3(u, v) + B_4(v, w - v), \\ w = G(t)w_0 + B_3(u, w) - B_4(w, w - v), \end{cases}$$

where

$$B_1(u, u) = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(\tau)d\tau,$$
$$B_2(v, w) = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(v\nabla(-\Delta)^{-1}w)(\tau)d\tau,$$
$$B_3(u, v) = -\int_0^t e^{(t-\tau)\Delta}\nabla \cdot (uv)(\tau)d\tau,$$
$$B_4(v, w) = -\int_0^t e^{(t-\tau)\Delta}\nabla \cdot (v\nabla(-\Delta)^{-1}w)(\tau)d\tau.$$

3.2. Estimate for initial value

(1) Let $n \geq 3$. We now estimate the initial value u_0. Using (2.5), the heat kernel estimates (2.12) with the conditions $\boxed{\alpha + \frac{n}{2p} - 1 < 0}$ and the embedding theorem (2.7) with above condition, we have

$$\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{W\dot{K}^{\alpha}_{2p,2q}} \leq C\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}} \tag{3.2}$$
$$\leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{2p}-1}_{2p,2q,\infty}} \leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,2q,\infty}} \leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}.$$

Moreover, using heat kernel estimates (2.12), we have

$$\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}} \leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}},$$

from this estimate and (3.2), we get

$$\|G(t)u_0\|_X \leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}.$$

(2) We estimate the initial value v_0 and w_0. By a same way, using (2.5), the heat kernel estimates (2.12) with $\alpha + \frac{n}{2\tilde{p}} - 2 < 0$ (which can be obtained from $\alpha + \frac{n}{2p} - 1 < 0$, $\frac{1}{p} + \frac{1}{n} = \frac{1}{\tilde{p}}$) and the embedding theorem (2.7) with above condition, we have

$$\sup_{t>0} t^{1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\|G(t)v_0\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}} \leq C\sup_{t>0} t^{1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\|G(t)v_0\|_{BW\dot{K}^{\alpha,0}_{2\tilde{p},2q,1}} \tag{3.3}$$

$$\leq C\|v_0\|_{BW\dot{K}^{\alpha,\alpha+\frac{n}{2\tilde{p}}-2}_{2\tilde{p},2q,\infty}} \leq C\|v_0\|_{BW\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},2q,\infty}} \leq C\|v_0\|_{BW\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}}.$$

Moreover, using heat kernel estimates (2.12), we have

$$\|G(t)v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}} \leq C\|v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}},$$

from this estimate and (3.3), we get

$$\|G(t)v_0\|_Y \leq C\|v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}},$$

we ensure v and w are in the same space Y, by a same way, we have

$$\|G(t)w_0\|_Y \leq C\|w_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}},$$

Remark 3.1. *Notice the condition of kernel estimate* (2.12)*, it is easy obtain* $0 \leq \alpha < n(1-\frac{1}{p})$, $0 \leq \alpha < n(1-\frac{1}{\tilde{p}})$ *for* $n \geq 3$ *with conditions* $\boxed{\frac{n}{2} < p < \infty}$ *and* $0 \leq \alpha < 1 - \frac{n}{2p}$*, where* $(\frac{1}{p} + \frac{1}{n} = \frac{1}{\tilde{p}})$*, we can take advantage of the heat kernel estimate* (2.12)*, it is the reason why we chose* $\boxed{n \geq 3}$.

3.3. Bilinear estimate

(1) The second part of space X for $B_1(u,u)$, using (2.5), (2.8) and (2.12), we have

$$\begin{aligned}
&\|B_1(u,u)\|_{W\dot{K}^{\alpha}_{2p,2q}} \leq \|B_1(u,u)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}} \leq \|B_1(u,u)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{2p}}_{p,2q,1}} \\
&\leq C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{2p}}_{p,2q,1}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|\mathbb{P}\nabla\cdot(u\otimes u)\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,2q,\infty}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes u\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes u\|_{W\dot{K}^{2\alpha}_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\tau^{-2\left(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})\right)}\|u\|_X^2 d\tau \\
&\leq Ct^{-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})}\mathcal{B}\left(\alpha+\frac{n}{2p}, \frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})\right)\|u\|_X^2,
\end{aligned}$$

we note the conditions $\boxed{0 \leq \alpha < \min\left\{1-\frac{n}{2p}, \frac{n}{2p}\right\}}$ can guarantee $\alpha + \frac{n}{2p} > 0$ and $\frac{1}{2} - (\frac{\alpha}{2} + \frac{n}{4p}) > 0$, the previous estimate leads us to

$$\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|B_1(u,u)\|_{W\dot{K}^{\alpha}_{2p,2q}} \leq C\|u\|_X\|u\|_X. \tag{3.4}$$

The first part of space X for $B_1(u,u)$, we have

$$\begin{aligned}\|B_1(u,u)\|_{\dot{B}W\dot{K}_{p,q,\infty}^{\alpha,\alpha+\frac{n}{p}-1}} &\le C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\|_{\dot{B}W\dot{K}_{p,q,\infty}^{\alpha,\alpha+\frac{n}{p}-1}}d\tau\\ &\le C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)\|_{\dot{B}W\dot{K}_{p,q,\infty}^{2\alpha,2\alpha+\frac{n}{p}-1}}d\tau\\ &\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|\mathbb{P}\nabla\cdot(u\otimes u)\|_{\dot{B}W\dot{K}_{p,q,\infty}^{2\alpha,-1}}d\tau\\ &\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes u\|_{\dot{B}W\dot{K}_{p,q,\infty}^{2\alpha,0}}d\tau\\ &\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes u\|_{W\dot{K}_{p,q}^{2\alpha}}d\tau\\ &\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\|_{W\dot{K}_{2p,2q}^{\alpha}}\|u\|_{W\dot{K}_{2p,2q}^{\alpha}}d\tau\\ &\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\tau^{-2\left(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})\right)}\|u\|_X^2 d\tau\\ &\le C\mathcal{B}\Big(\alpha+\frac{n}{2p},1-(\alpha+\frac{n}{2p})\Big)\|u\|_X^2,\end{aligned}$$

$0\le\alpha<\min\left\{1-\frac{n}{2p},\frac{n}{2p}\right\}$ can guarantee $\alpha+\frac{n}{2p}>0$ and $1-(\alpha+\frac{n}{2p})>0$, therefore, we obtain the estimate

$$\|B_1(u,u)\|_{L^\infty((0,\infty);\dot{B}W\dot{K}_{p,q,\infty}^{\alpha,\alpha+\frac{n}{p}-1})}\le C\|u\|_X\|u\|_X, \tag{3.5}$$

the estimates (3.4) and (3.5) together give

$$\|B_1(u,u)\|_X\le K\|u\|_X\|u\|_X, \tag{3.6}$$

for some positive constant K.

(2) Now, we suppose $\boxed{\frac{1}{n}+\frac{1}{p}=\frac{1}{\tilde{p}}}$ and $\boxed{p<n}$, notice the expression (1.3)

$$\begin{aligned}\nabla(-\Delta)^{-1}(w-v) &= C_1\int_{R^n}\frac{(x-y)(w(y)-v(y))}{|x-y|^n}dy\\ &\le C_1\int_{R^n}\frac{(w(y)-v(y))}{|x-y|^{n-1}}dy \quad \text{for} \quad n\ge 3,\end{aligned}$$

with satisfy the condition (2.13) of Lemma 2.7 for $\gamma=1$, we have

$$\|\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{\alpha}_{\frac{2p\tilde{p}}{2\tilde{p}-p},2q}}\le C\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}, \tag{3.7}$$

in other words, $p<n$ is obtain from the constrained condition of Lemma 2.7.

Using the Hölder type inequality (2.1) for weak-Herz spaces, (2.4) and (3.7), we have

$$\begin{aligned}&\|B_2(w-v,w-v)\|_{W\dot{K}^{\alpha}_{2p,2q}}\le\|B_2(w-v,w-v)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}}\\ &\le\|B_2(w-v,w-v)\|_{\dot{B}W\dot{K}_{p,2q,1}^{2\alpha,\alpha+\frac{n}{2p}}}\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{2p}}_{p,2q,1}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4p})}\|(w-v)\nabla(-\Delta)^{-1}(w-v)\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4p})}\|(w-v)\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{2\alpha}_{p,q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4p})}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{\alpha}_{\frac{2p\tilde{p}}{2\tilde{p}-p},2q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4p})}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4p})}\tau^{-2\left(1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})\right)}\|(w-v)\|_Y^2 d\tau\\
&\le Ct^{-1-\frac{n}{4p}+\frac{\alpha}{2}+\frac{n}{2\tilde{p}}}\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde{p}}),1-(\frac{\alpha}{2}+\frac{n}{4p})\Big)\|(w-v)\|_Y^2\\
&\le Ct^{-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})}\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde{p}}),1-(\frac{\alpha}{2}+\frac{n}{4p})\Big)\|(w-v)\|_Y^2,
\end{aligned}
$$

for $-1+(\alpha+\frac{n}{2\tilde{p}})>0$, we only request $\boxed{\alpha>\frac{1}{2}-\frac{n}{2p}}$. The condition $0\le\alpha<\min\left\{1-\frac{n}{2p},\frac{n}{2p}\right\}$ can guarantee $1-(\frac{\alpha}{2}+\frac{n}{4p})>0$ hold. Above estimate leads to

$$
\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|B_2(w-v,w-v)\|_{W\dot{K}^{\alpha}_{2p,2q}}\le C\|(w-v)\|_Y^2. \tag{3.8}
$$

The first part of space X for $B_2(w-v,w-v)$, be the same way, we have

$$
\begin{aligned}
&\le \|B_2(w-v,w-v)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}\\
&\le C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}d\tau\\
&\le C\int_0^t \|e^{(t-\tau)\Delta}\mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{2\alpha,2\alpha+\frac{n}{p}-1}_{p,q,\infty}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p}-\frac{1}{2})}\|(w-v)\nabla(-\Delta)^{-1}(w-v)\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,q,\infty}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p}-\frac{1}{2})}\|(w-v)\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{2\alpha}_{p,q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p}-\frac{1}{2})}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|\nabla(-\Delta)^{-1}(w-v))\|_{W\dot{K}^{\alpha}_{\frac{2p\tilde{p}}{2\tilde{p}-p},2q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p}-\frac{1}{2})}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|(w-v))\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p}-\frac{1}{2})}\tau^{-2\left(1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})\right)}\|(w-v)\|_Y^2 d\tau
\end{aligned}
$$

$$\le Ct^{-\frac{1}{2}-\frac{n}{2p}+\frac{n}{2\tilde p}}\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde p}),\frac{3}{2}-(\alpha+\frac{n}{2p})\Big)\|(w-v)\|_Y^2$$
$$\le C\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde p}),\frac{3}{2}-(\alpha+\frac{n}{2p})\Big)\|(w-v)\|_Y^2,$$

the conditions $(\frac{1}{2}-\frac{n}{2p})<\alpha<\min\Big\{1-\frac{n}{2p},\frac{n}{2p}\Big\}$ can guarantee the variates of Beta function positive, therefore, we obtain the estimate

$$\|B_2(w-v,w-v)\|_{L^\infty((0,\infty);\dot{B}W\dot{K}_{p,q,\infty}^{\alpha,\alpha+\frac{n}{p}-1})}\le C\|w-v\|_Y\|w-v\|_Y, \tag{3.9}$$

the estimates (3.8) and (3.9) together give

$$\|B_2(w-v,w-v)\|_X\le K\|w-v\|_Y\|w-v\|_Y. \tag{3.10}$$

The second part of space Y for $B_3(u,v)$, using (2.5), (2.7) ($p_2=\frac{2n\tilde p}{2n-\tilde p}$), we have

$$\|B_3(u,v)\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}}\le\|B_3(u,v)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2\tilde p,2q,1}}\le\|B_3(u,v)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{2\tilde p}-\frac{1}{2}}_{\frac{2n\tilde p}{2n-\tilde p},2q,1}}$$
$$\le C\int_0^t\|e^{(t-\tau)\Delta}\nabla\cdot(uv)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{2\tilde p}-\frac{1}{2}}_{\frac{2n\tilde p}{2n-\tilde p},2q,1}}d\tau$$
$$\le C\int_0^t(t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4\tilde p}-\frac{1}{4})}\|\nabla\cdot(uv)\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{\frac{2n\tilde p}{2n-\tilde p},2q,\infty}}d\tau$$
$$\le C\int_0^t(t-\tau)^{-(\frac{1}{4}+\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|uv\|_{\dot{B}W\dot{K}^{2\alpha,0}_{\frac{2n\tilde p}{2n-\tilde p},2q,\infty}}d\tau$$
$$\le C\int_0^t(t-\tau)^{-(\frac{1}{4}+\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|uv\|_{W\dot{K}^{2\alpha}_{\frac{2n\tilde p}{2n-\tilde p},q}}d\tau$$
$$\le C\int_0^t(t-\tau)^{-(\frac{1}{4}+\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}\|v\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}}d\tau$$
$$\le C\int_0^t(t-\tau)^{-(\frac{1}{4}+\frac{\alpha}{2}+\frac{n}{4\tilde p})}\tau^{-\big(1-(\frac{\alpha}{2}+\frac{n}{4\tilde p})+\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})\big)}\|u\|_X\|v\|_Y d\tau$$
$$\le C\int_0^t(t-\tau)^{-(\frac{1}{4}+\frac{\alpha}{2}+\frac{n}{4\tilde p})}\tau^{-\big(\frac{3}{2}-(\alpha+\frac{n}{4\tilde p}+\frac{n}{4p})\big)}\|u\|_X\|v\|_Y d\tau$$
$$\le Ct^{-\frac{3}{4}+\frac{\alpha}{2}+\frac{n}{4p}}\mathcal{B}\Big(-\frac{1}{2}+(\alpha+\frac{n}{4p}+\frac{n}{4\tilde p}),\frac{3}{4}-(\frac{\alpha}{2}+\frac{n}{4\tilde p})\Big)\|u\|_X\|v\|_Y$$
$$\le Ct^{-1+\frac{\alpha}{2}+\frac{n}{4\tilde p}}\mathcal{B}\Big(-\frac{1}{4}+(\alpha+\frac{n}{2p}),\frac{3}{4}-(\frac{\alpha}{2}+\frac{n}{4\tilde p})\Big)\|u\|_X\|v\|_Y,$$

the conditions $(\frac{1}{2}-\frac{n}{2p})<\alpha<\min\Big\{1-\frac{n}{2p},\frac{n}{2p}\Big\}$ can guarantee $-\frac{1}{4}+(\alpha+\frac{n}{2p})>0$ and $\frac{3}{4}-(\frac{\alpha}{2}+\frac{n}{4\tilde p})>0$.

Above estimate leads to

$$\sup_{t>0}t^{1-\frac{\alpha}{2}+\frac{n}{4\tilde p}}\|B_3(u,v)\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}}\le C\|u\|_X\|v\|_Y,$$

for some positive constant K, we obtain

$$\|B_3(u,v)\|_Y \le K\|u\|_X\|v\|_Y. \tag{3.11}$$

Moreover, for the first part norm of Y, we have

$$\begin{aligned}
&\|B_3(u,v)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}} \le C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot(uv)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde p}-2}_{\tilde p,q,\infty}} d\tau\\
&\le C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot(uv)\|_{\dot{B}W\dot{K}^{2\alpha,2\alpha+\frac{n}{\tilde p}-\frac{5}{2}}_{\frac{2n\tilde p}{2n-\tilde p},q,\infty}} d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{3}{4}-(\alpha+\frac{n}{2\tilde p})}\|\nabla\cdot(uv)\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{\frac{2n\tilde p}{2n-\tilde p},q,\infty}} d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{3}{4}-(\alpha+\frac{n}{2\tilde p})}\|uv\|_{\dot{B}W\dot{K}^{2\alpha,0}_{\frac{2n\tilde p}{2n-\tilde p},q,\infty}} d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{3}{4}-(\alpha+\frac{n}{2\tilde p})}\|uv\|_{W\dot{K}^{2\alpha}_{\frac{2n\tilde p}{2n-\tilde p},q}} d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{3}{4}-(\alpha+\frac{n}{2\tilde p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}\|v\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}} d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{3}{4}-(\alpha+\frac{n}{2\tilde p})}\tau^{-\left(\frac{3}{2}-(\alpha+\frac{n}{4\tilde p}+\frac{n}{4p})\right)}\|u\|_X\|v\|_Y d\tau\\
&\le Ct^{\frac{1}{4}+\frac{n}{4p}-\frac{n}{4\tilde p}}\mathcal{B}\Big(-\frac{3}{4}+(\alpha+\frac{n}{2\tilde p}),\frac{7}{4}-(\alpha+\frac{n}{2\tilde p})\Big)\|u\|_X\|v\|_Y\\
&\le C\mathcal{B}\Big(-\frac{1}{4}+(\alpha+\frac{n}{2p}),\frac{5}{4}-(\alpha+\frac{n}{2p})\Big)\|u\|_X\|v\|_Y.
\end{aligned}$$

By the same way, we consider the estimate for $B_4(v,w-v)$, using (2.5) and (2.7)

$$\begin{aligned}
&\|B_4(v,w-v)\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}} \le \|B_4(v,w-v)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2\tilde p,2q,1}}\\
&\le \|B_4(v,w-v)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{p}-\frac{n}{2\tilde p}}_{p,2q,1}}\\
&\le C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot(v\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+\frac{n}{p}-\frac{n}{2\tilde p}}_{p,2q,1}} d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{2p}-\frac{n}{4\tilde p})}\|\nabla\cdot(v\nabla(-\Delta)^{-1}(w-v))\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,2q,\infty}} d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|v\nabla(-\Delta)^{-1}(w-v)\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}} d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|v\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{2\alpha}_{p,q}} d\tau\\
&\le C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4\tilde p})}\|v\|_{W\dot{K}^{\alpha}_{2\tilde p,2q}}\|\nabla(-\Delta)^{-1}(w-v)\|_{W\dot{K}^{\alpha}_{\frac{2p\tilde p}{2\tilde p-p},2q}} d\tau
\end{aligned}$$

$$\leq C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\|v\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|(w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\tau^{-2\left(1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})\right)}\|v\|_Y\|(w-v)\|_Y d\tau$$
$$\leq Ct^{-1+(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})}\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde{p}}),1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})\Big)\|v\|_Y\|(w-v)\|_Y,$$

the conditions $(\frac{1}{2}-\frac{n}{2p})<\alpha<\min\left\{1-\frac{n}{2p},\frac{n}{2p}\right\}$ can guarantee $-1+(\alpha+\frac{n}{2\tilde{p}})>0$ and $1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})>0$.

Above estimate leads to

$$\sup_{t>0} t^{1-\frac{\alpha}{2}+\frac{n}{4\tilde{p}}}\|B_4(v,w-v)\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\leq C\|v\|_Y\|(w-v)\|_Y,$$

for the first part, we have

$$\|B_4(v,w-v)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}}$$
$$\leq C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot(v\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}}d\tau$$
$$\leq C\int_0^t \|e^{(t-\tau)\Delta}\nabla\cdot(v\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot{B}W\dot{K}^{2\alpha,2\alpha+\frac{n}{\tilde{p}}-3}_{p,q,\infty}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\|\nabla\cdot(v\nabla(-\Delta)^{-1}(w-v))\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,2q,\infty}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\|(v\nabla(-\Delta)^{-1}(w-v))\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\|(v\nabla(-\Delta)^{-1}(w-v))\|_{W\dot{K}^{2\alpha}_{p,2q}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\|v\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|\nabla(-\Delta)^{-1}(w-v))\|_{W\dot{K}^{\alpha}_{\frac{2p\tilde{p}}{2\tilde{p}-p},2q}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\|v\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}\|w-v\|_{W\dot{K}^{\alpha}_{2\tilde{p},2q}}d\tau$$
$$\leq C\int_0^t (t-\tau)^{1-(\alpha+\frac{n}{2\tilde{p}})}\tau^{-2\left(1-(\frac{\alpha}{2}+\frac{n}{4\tilde{p}})\right)}\|v\|_Y\|w-v\|_Y d\tau$$
$$\leq C\mathcal{B}\Big(-1+(\alpha+\frac{n}{2\tilde{p}}),2-(\alpha+\frac{n}{2\tilde{p}})\Big)\|v\|_Y\|w-v\|_Y,$$

by the same way, for some positive constant K, we obtain

$$\|B_4(v,w-v)\|_Y\leq K\|v\|_Y\|w-v\|_Y. \tag{3.12}$$

Combining bilinear estimate (3.6), (3.10), (3.11), (3.12) and initial value estimate (3.2), (3.3), we obtain some necessary conditions: $\frac{n}{2}<p<n$, $(\frac{1}{2}-\frac{n}{2p})<\alpha<\min\left\{1-\frac{n}{2p},\frac{n}{2p}\right\}$ and $\frac{1}{p}+\frac{1}{n}=\frac{1}{\tilde{p}}$, For $\frac{n}{2} < p < n$, we have $(\frac{1}{2} - \frac{n}{2p}) < 0$ and $\min\left\{1 - \frac{n}{2p}, \frac{n}{2p}\right\} = 1 - \frac{n}{2p}$, so we obtain the last condition $\mathbb{P}$: $n \geq 3$, $1 \leq q \leq \infty$, $\frac{n}{2} < p < n$, $0 \leq \alpha \leq 1 - \frac{n}{2p}$ and $\frac{1}{p} + \frac{1}{n} = \frac{1}{\tilde{p}}$.

3.4. Existence and uniqueness

Note the expression of integral equations (1.4) and the definition of space (3.1):

$$\begin{cases} u = G(t)u_0 + B_1(u,u) + B_2(w-v, w-v), \\ v = G(t)v_0 + B_3(u,v) + B_4(v, w-v), \\ w = G(t)w_0 + B_3(u,w) - B_4(w, w-v), \end{cases}$$

$$X_u \times Y_v \times Y_w := \{(u,v,w) \in X \times Y \times Y \mid \|u\|_X + \|v\|_Y + \|w\|_Y < \infty\}.$$

We suppose the initial value satisfy

$$\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}} + \|v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}} + \|w_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}} \leq \delta.$$

Denote

$$\Phi(\vec{S}) : (u_0, v_0, w_0) \to (u,v,w), \quad \text{with} \quad \bar{B}(0, 2\epsilon) \to \bar{B}(0, 2\epsilon)$$

is the map from initial value to solutions, where $\bar{B}(0,\epsilon)$ denote the closed ball in $X_u \times Y_v \times Y_w$. Then, for any $\vec{S} := (u,v,w) \in X_u \times Y_v \times Y_w$, with sufficient small δ and ϵ satisfy $0 < \epsilon < \frac{1}{60K}$ and $0 < \delta < \frac{\epsilon}{3C}$, where $C = \max\{C_1, C_2, C_3\}$ as follows

$$\begin{aligned} \|u\|_X &\leq \|G(t)u_0\|_X + \|B_1(u,u)\|_X + \|B_2(w-v, w-v)\|_X \\ &\leq C_1\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}} + K\|u\|_X\|u\|_X + K\|w-v\|_Y\|w-v\|_Y \\ &\leq \frac{\varepsilon}{3} + 4K\epsilon^2 + 16K\epsilon^2 \leq \frac{2\varepsilon}{3}, \end{aligned}$$

$$\begin{aligned} \|v\|_Y &\leq \|G(t)v_0\|_Y + \|B_3(u,v)\|_Y + \|B_4(v, w-v)\|_Y \\ &\leq C_2\|v_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}} + K\|u\|_X\|v\|_Y + K\|v\|_Y\|w-v\|_Y \\ &\leq \frac{\varepsilon}{3} + 4K\epsilon^2 + 8K\epsilon^2 \leq \frac{2\varepsilon}{3}, \end{aligned}$$

$$\begin{aligned} \|w\|_Y &\leq \|G(t)w_0\|_Y + \|B_3(u,w)\|_Y + \|B_4(w, w-v)\|_Y \\ &\leq C_3\|w_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+\frac{n}{\tilde{p}}-2}_{\tilde{p},q,\infty}} + K\|u\|_X\|w\|_Y + K\|w\|_Y\|w-v\|_Y \\ &\leq \frac{\varepsilon}{3} + 4K\epsilon^2 + 8K\epsilon^2 \leq \frac{2\varepsilon}{3}, \end{aligned}$$

it follows that

$$\|u\|_X + \|v\|_Y + \|w\|_Y \leq 2\epsilon,$$

we obtain the solutions $(u,v,w) \in \bar{B}(0, 2\epsilon)$ in $X_u \times Y_v \times Y_w$, so Φ is well-defined.

On the other hand, for any $\vec{S} = (u, v, w)$ and $\vec{S_1} = (u_1, v_1, w_1)$ belong to $\bar{B}(0, 2\epsilon)$, for some bilinear B, it is easy obtain

$$\begin{aligned}&\|B(x,y) - B(\bar{x},\bar{y})\| = \|B(x,y) - B(\bar{x},y) + B(\bar{x},y) - B(\bar{x},\bar{y})\|\\ &= \|B(x-\bar{x},y) + B(\bar{x},y-\bar{y})\|,\end{aligned}$$

Then, we have

$$\begin{aligned}&\|u-u_1\|_X \leq \|B_1(u-u_1,u)\|_X + \|B_1(u_1,u-u_1)\|_X\\ &+ \|B_2((w-w_1)-(v-v_1),w-v)\|_X + \|B_2(w_1-v_1,(w-w_1)-(v-v_1))\|_X\\ &\leq K\|u-u_1\|_X\|u\|_X + K\|u_1\|_X\|u-u_1\|_X + K\|w-w_1\|_Y\|w-v\|_Y\\ &+ K\|v-v_1\|_Y\|w-v\|_Y + K\|w-w_1\|_Y\|w_1-v_1\|_Y + K\|v-v_1\|_Y\|w_1-v_1\|_Y\\ &\leq 20K\epsilon(\|u-u_1\|_X + \|v-v_1\|_Y + \|w-w_1\|_Y),\end{aligned}$$

by the same way, we obtain

$$\|v-v_1\|_Y \leq 20K\epsilon(\|u-u_1\|_X + \|v-v_1\|_Y + \|w-w_1\|_Y)$$

and

$$\|w-w_1\|_Y \leq 20K\epsilon(\|u-u_1\|_X + \|v-v_1\|_Y + \|w-w_1\|_Y),$$

this mean

$$\|\Phi(\vec{S}) - \Phi(\vec{S_1})\|_{X_u\times Y_v\times Y_w} \leq 60K\epsilon\|\vec{S} - \vec{S_1}\|_{X_u\times Y_v\times Y_w},$$

since $60K\epsilon < 1$, we get that $\Phi(\vec{S})$ is a contraction map, by the Banach fixed-point theorem, we can prove the system (1.1) existence a uniqueness global mild solution.

3.5. Time-weak continuity

Lemma 3.1. *For every real number s and $u_0 \in \dot{B}^s_{\infty,\infty}$, we have $G(t)u_0 \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^s_{\infty,\infty}$ as $t \to 0^+$.*

The proof can see [10].

Lemma 3.2. *Let $u \in X$, $v, w \in Y$. We have $B_1(u,u)$, $B_2(w-v.w-v)$ converges to 0 in the weak$-*$ topology of $\dot{B}^{-1}_{\infty,\infty}$, $B_3(u,v)$, $B_4(v.w-v)$ converges to 0 in the weak$-*$ topology of $\dot{B}^{-2}_{\infty,\infty}$ as $t \to 0^+$.*

Proof. Let $\phi \in \dot{B}^1_{1,1}$ and $\epsilon > 0$ an arbitrary number. We can choose $\tilde{\phi} \in \mathcal{S}$ such that $\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} < \epsilon$. Then, we have that

$$\begin{aligned}&\left|\left\langle B_1(u,u)(t), \phi - \tilde{\phi}\right\rangle\right| \leq \|B_1(u,u)(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\|\phi-\tilde{\phi}\|_{\dot{B}^1_{1,1}}\\ &\leq \|B_1(u,u)(t)\|_{\dot{BWK}^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}\|\phi-\tilde{\phi}\|_{\dot{B}^1_{1,1}} \leq K\|u\|^2_X\epsilon \leq C\epsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left|\left\langle B_1(u,u)(t), \tilde{\phi}\right\rangle\right|\\ &\leq \int_0^t \left|\left\langle G(t-\tau)\mathbb{P}\mathrm{div}(u\otimes u)(\tau), \tilde{\phi}\right\rangle\right| d\tau \leq \int_0^t \left|\left\langle \mathbb{P}\mathrm{div}(u\otimes u)(\tau), G(t-\tau)\tilde{\phi}\right\rangle\right| d\tau\end{aligned}$$

$$\leq \int_0^t \|\mathrm{div}(u\otimes u)(\tau)\|_{\dot B^{-1-2\alpha-n/p}_{\infty,\infty}} \|G(t-\tau)\tilde\phi\|_{\dot B^{1+2\alpha+n/p}_{1,1}} d\tau$$
$$\leq C_{\tilde\phi}\int_0^t \|(u\otimes u)(\tau)\|_{\dot B^{-2\alpha-n/p}_{\infty,\infty}} \leq C_{\tilde\phi}\int_0^t \|(u\otimes u)(\tau)\|_{W\dot K^{2\alpha}_{p,q}}$$
$$\leq C_{\tilde\phi}\int_0^t \tau^{2\cdot\left[-\frac12+(\frac{\alpha}{2}+\frac{n}{4p})\right]}\tau^{2\cdot\left[\frac12-(\frac{\alpha}{2}+\frac{n}{4p})\right]}\|u(\tau)\|^2_{W\dot K^{\alpha}_{2p,2q}}$$
$$\leq C_{\tilde\phi}\|u\|_X^2\int_0^t \tau^{-1+\alpha+\frac{n}{2p}}d\tau \leq C_{\tilde\phi}\|u\|_X^2 t^{\alpha+\frac{n}{2p}},$$

we obtain

$$0\leq \limsup_{t\to0^+}\left|\left\langle B_1(u,u)(t),\phi\right\rangle\right| \leq \limsup_{t\to0^+}\left|\left\langle B_1(u,u)(t),\phi-\tilde\phi\right\rangle\right|$$
$$+\limsup_{t\to0^+}\left|\left\langle B_1(u,u)(t),\tilde\phi\right\rangle\right| \leq C\epsilon+0.$$

Since $\epsilon>0$ is arbitrary, we conclude that $\limsup_{t\to0^+}\left|\left\langle B_1(u,u)(t),\phi\right\rangle\right|=0$. we used that $\phi\in\dot B^1_{1,1}$ is arbitrary.

$$\left|\left\langle B_2(w-v,w-v)(t),\phi-\tilde\phi\right\rangle\right| \leq \|B_2(w-v,w-v)(t)\|_{\dot B^{-1}_{\infty,\infty}}\|\phi-\tilde\phi\|_{\dot B^1_{1,1}}$$
$$\leq \|B_2(w-v,w-v)(t)\|_{\dot BW\dot K^{\alpha,\alpha+\frac{n}{p}-1}_{p,q,\infty}}\|\phi-\tilde\phi\|_{\dot B^1_{1,1}} \leq K\|w-v\|_Y^2\epsilon\leq C\epsilon.$$

On the other hand,

$$\left|\left\langle B_2(w-v,w-v)(t),\tilde\phi\right\rangle\right|$$
$$\leq \int_0^t \left|\left\langle G(t-\tau)\mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau),\tilde\phi\right\rangle\right|d\tau$$
$$\leq \int_0^t \left|\left\langle \mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau),G(t-\tau)\tilde\phi\right\rangle\right|d\tau$$
$$\leq \int_0^t \|\mathbb{P}((w-v)\nabla(-\Delta)^{-1}(w-v))(\tau)\|_{\dot B^{-2\alpha-n/p}_{\infty,\infty}}\|G(t-\tau)\tilde\phi\|_{\dot B^{2\alpha+n/p}_{1,1}}d\tau$$
$$\leq C_{\tilde\phi}\int_0^t \|(w-v)\nabla(-\Delta)^{-1}(w-v)(\tau)\|_{\dot B^{-2\alpha-n/p}_{\infty,\infty}}$$
$$\leq C_{\tilde\phi}\int_0^t \|(w-v)\nabla(-\Delta)^{-1}(w-v)(\tau)\|_{W\dot K^{2\alpha}_{p,q}}$$
$$\leq C_{\tilde\phi}\int_0^t \|(w-v)\|_{W\dot K^{\alpha}_{2\tilde p,2q}}\|\nabla(-\Delta)^{-1}(w-v))\|_{W\dot K^{\alpha}_{\frac{2p\tilde p}{2\tilde p-p},2q}}d\tau$$
$$\leq C_{\tilde\phi}\int_0^t \|(w-v)\|_{W\dot K^{\alpha}_{2\tilde p,2q}}\|(w-v))\|_{W\dot K^{\alpha}_{2\tilde p,2q}}d\tau$$
$$\leq C_{\tilde\phi}\|w-v\|_Y^2\int_0^t \tau^{-2+\alpha+\frac{n}{2\tilde p}}d\tau \leq C_{\tilde\phi}\|w-v\|_Y^2 t^{-1+\alpha+\frac{n}{2\tilde p}},$$

therein

$$(-1+\alpha+\frac{n}{2\tilde{p}}) = (\alpha+\frac{n}{2p}-\frac{1}{2}) > 0, \text{ for } p < n,$$

we have used (2.2): $W\dot{K}^{2\alpha}_{p,q} \hookrightarrow W\dot{K}^{2\alpha}_{p,\infty} \hookrightarrow \dot{B}^{-(2\alpha+n/p)}_{\infty,\infty}$ with the condition $\mathbb{P}$.

Then, we obtain

$$0 \leq \limsup_{t\to 0^+} \left|\left\langle B_2(w-v,w-v)(t),\phi\right\rangle\right| \leq \limsup_{t\to 0^+} \left|\left\langle B_2(w-v,w-v)(t),\phi-\tilde{\phi}\right\rangle\right|$$
$$+\limsup_{t\to 0^+} \left|\left\langle B_2(w-v,w-v)(t),\tilde{\phi}\right\rangle\right| \leq C\epsilon + 0,$$

by the same way, we conclude that $\limsup_{t\to 0^+} \left|\left\langle B_2(w-v,w-v)(t),\phi\right\rangle\right| = 0$.

For $B_3(u,v)$ and $B_4(v,w-v)$, we can only choose $\phi \in \dot{B}^2_{1,1}$, using the bilinear estimate for B_3 and B_4, we can obtain the same consequence. The proof is now complete. □

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