Green H-relation of the square matrices over a local ring whose maximal ideal is generated by a nilpotent element

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Abstract

Let R be a commutative local ring whose maximal ideal is generated by a nilpotent element, and $\mathrm{Mat}(n, R)$ be the multiplicative monoid of the square matrices of order n over R. In this article, we provide (1) the construction of the Green's H-equivalence classes in $\mathrm{Mat}(n, R)$, and (2) the enumeration of the Green's H-equivalence classes in $\mathrm{Mat}(n, \mathbb{Z}/p^d\mathbb{Z})$.

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1 Introduction

We firstly review the Green's relations defined by J. A. Green in [1].

Definition 1.1. Let M be a monoid, and $a, b \in M$. We write $a \leq_L b$ if there exists $m \in M$ such that $a = mb$, and write $a =_L b$ if both $a \leq_L b$ and $b \leq_L a$. Similarly, we write $a \leq_R b$ if there exists $m \in M$ such that $a = bm$, and write $a =_R b$ if both $a \leq_R b$ and $b \leq_R a$. And we write $a \leq_H b$ (resp., $a =_H b$) if both $a \leq_L b$ and $a \leq_R b$ (resp., both $a =_L b$ and $a =_R b$).

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We can check that $\leq_L$, $\leq_R$ and $\leq_H$ are partial orders on M, which are called the Green's L-order, R-order and H-order, respectively. We can also see that $=_L$, $=_R$ and $=_H$ are equivalence relations on M, which are called the Green's L-relation, R-relation and H-relation, respectively.

Remark 1.2. The notations aLb, aRb and aHb of the Green's relations are more standard than $a =_L b$, $a =_R b$ and $a =_H b$, respectively. But this article mainly discusses the Green's relations of matrices, and we denote matrices by capital letters according to tradition. Therefore the standard notations of the Green's relations would look quite confused.

Green's relations not only play a significant role for understanding the structure of a monoid or a semi-group [2, 3, 4], but also have important applications in many fields [5, 6, 7]. In particular, lots of such applications involve only the H-relation. The notion of the inverse along an element, introduced by X. Mary in [8], is a classical example.

Let R be a commutative local ring whose maximal ideal is generated by a nilpotent element. In this article, we mainly consider the H-equivalence classes (H-classes for short) in the monoid $\mathrm{Mat}(n, R)$ of square matrices of order n over R, Such kind of rings R include (1) $\mathbb{Z}/p^d\mathbb{Z}$ with $d \geq 2$ (the integers module by a prime power) which can be regarded as the most typical case, (2) the quotient $K[x]/\langle x^2 \rangle$ where K is a field, (3) the localization $(\mathbb{Z}/p^d q\mathbb{Z})_{\bar{q}}$ where p and q are distinct primes and $d \geq 2$, and so on.

After some preliminaries, we discuss the H-classes and its enumeration in $\mathrm{Mat}(n, K)$ for a field K in §2. This is a good comparison of our main theorem in §3 which describes the construction of the H-classes in $\mathrm{Mat}(n, R)$. In §4, we give a necessary and sufficient condition for the invertibility of the matrix over R, in term of the row or column vectors. As an application, we provide the size of each H-class in $\mathrm{Mat}(n, \mathbb{Z}/(p^d\mathbb{Z}))$ for prime p and positive integer d in §5.

2 Preliminaries and notations

Lemma 2.1. *Let M be a monoid, and $a, b, u, v \in M$. We assume that u, v are invertible elements. Then we have*

(1) $a \leq_L b$, $a \leq_R b$ and $a \leq_H b$ if and only if $uav \leq_L ubv$, $uav \leq_R ubv$ and $uav \leq_H ubv$, respectively;

(2) $a =_L b$, $a =_R b$ and $a =_H b$ if and only if $uav =_L ubv$, $uav =_R ubv$ and $uav =_H ubv$, respectively;

(3) the map $x \mapsto uxv$ is a bijection from the L-class (resp., R-class, H-class) of a to the L-class (resp., R-class, H-class) of uav.

Proof. If $a \leq_L b$, then by definition there exists $m \in M$ such that $a = mb$. Thus we have $uav = (umu^{-1})ubv$, so $uav \leq_L ubv$. The converse is obvious since $a = u^{-1}(uav)v^{-1}$ and $b = u^{-1}(ubv)v^{-1}$. The proofs for the R-order and H-order in (1) are similar. (2) and (3) simply follow from (1). □

Through the rest of this article, we use the following notations: Let R be a commutative local ring such that its maximal ideal is generated by a nilpotent element ξ. We denote by 0 and 1 the zero and identity of R, respectively. We write d for the nilpotent index of ξ, that is, d is the smallest positive integer such that $\xi^d = 0$. We also adopt the convention that $r^0 = 1$ for any $r \in R$. Furthermore, we denote by E_n and O_n the identity matrix and the zero matrix of order n over R, respectively.

Lemma 2.2. *For the ring R, we have*

(1) *every element r in R can be written in the form $u\xi^k$ for some unit $u \in R$ and some $k \in \{0, 1, \dots, d\}$, and the k is uniquely determined by r;*

(2) *writing $r = u\xi^k$ as in (1), we have $\xi^{d-1}r = 0$ implies $k \geq 1$, and $\xi^{d-1}r = \xi^{d-1}$ implies $k = 0$ (namely r is a unit);*

(3) *every ideal I of R can be generated by ξ^k for some $k \in \{0, 1, \dots, d\}$, so R is a principle ideal ring;*

(4) *for any matrix $A \in \mathrm{Mat}(n, R)$, there exist invertible matrices $P, Q \in \mathrm{Mat}(n, R)$ such that PAQ is a diagonal matrix whose diagonal entries are $1, \xi, \xi^2, \dots, \xi^{d-1}$ or 0, i.e., a matrix with the following form*

$$D = \begin{pmatrix} E_{r_0} & & & & & \\ & \xi E_{r_1} & & & & \\ & & \xi^2 E_{r_2} & & & \\ & & & \ddots & & \\ & & & & \xi^{d-1} E_{r_{d-1}} & \\ & & & & & O_s \end{pmatrix},$$

where $r_0, r_1, \dots, r_{d-1}, s \geq 0$, $r_0 + r_1 + \cdots + r_{d-1} + s = n$, and by $r_i = 0$ (resp., $s = 0$) we means that ξ^i (resp., 0) does not occur in the main diagonal;

(5) *using the notations of (4), the map $X \mapsto PXQ$ is a bijection from the H-class of A to the H-class of D.*

Proof. We have a chain $0 = \langle \xi^d \rangle \subsetneq \langle \xi^{d-1} \rangle \subsetneq \cdots \langle \xi \rangle \subsetneq \langle \xi^0 \rangle = R$ of ideals in R. Take a nonzero $r \in R$, we have $r \in \langle \xi^k \rangle$ but $r \notin \langle \xi^{k+1} \rangle$ for some $k \in \{0, \dots, d-1\}$, so $r = u\xi^k$ with $u \in R$. Since R is a local ring, if u was not a unit, then u would be contained in the maximal ideal $\langle \xi \rangle$, namely $u = v\xi$ and $r = v\xi^{k+1} \in \langle \xi^{k+1} \rangle$. This contradiction proves (1), and (2) follows from (1) immediately.

For a non-zero ideal I, we see from (1) that $\{i \in \{0, 1, \ldots, d-1\} : \xi^i \in I\}$ is not an empty set. By taking k to be the minimum number of this set, it is easy to check that $I = \langle \xi^k \rangle$, which proves (3). Since R is a principle ideal ring, (4) follows from [9, Theorem 4.1] and (1). Then (5) follows from Lemma 2.1 and (4). □

Definition 2.3. The matrix D in Lemma 2.2 is called the *Smith normal form* of A.

To close this section, we consider the H-classes in the monoid $\mathrm{Mat}(n, K)$ of square matrices of order n over an arbitrary field K. This discussion is a little aside from the subject of this article. But the reader may find that the result and proof in the case of $\mathrm{Mat}(n, K)$ provide good inspirations for those in the case of $\mathrm{Mat}(n, R)$.

For any matrix $A \in \mathrm{Mat}(n, K)$, there exist invertible matrices $P, Q \in \mathrm{Mat}(n, K)$ such that PAQ is a diagonal matrix whose diagonal entries are 1 or 0, i.e., a matrix with the form $\mathrm{diag}(E_r, O_s)$, where $r, s \geq 0$ and $r + s = n$. By Lemma 2.1, the map $X \mapsto PXQ$ is a bijection from the Green class of A to the corresponding Green class of $\mathrm{diag}(E_r, O_s)$. Hence, it suffices to consider the Green classes of the matrices with the form $\mathrm{diag}(E_r, O_s)$.

Theorem 2.4. *Let $A \in \mathrm{Mat}(n, K)$. The following conditions are equivalent:*

(1) $A =_H \mathrm{diag}(E_r, O_s)$;

(2) $A = \mathrm{diag}(A_{11}, O_s)$ for some invertible matrix A_{11} over K of order r.

When $r = 0$ and $s = n$, both $\mathrm{diag}(E_r, O_s)$ and $\mathrm{diag}(A_{11}, O_s)$ should be read as $A = O_n$.

Proof. Firstly, assume that the condition (2) holds. Then it is obvious that

$$\begin{aligned} \mathrm{diag}(A_{11}, O_s) &= \mathrm{diag}(A_{11}, O_s)\,\mathrm{diag}(E_r, O_s), \\ \mathrm{diag}(E_r, O_s) &= \mathrm{diag}(A_{11}^{-1}, O_s)\,\mathrm{diag}(A_{11}, O_s). \end{aligned}$$

Hence $A =_L \mathrm{diag}(E_r, O_s)$ by definition. And we can show $A =_R \mathrm{diag}(E_r, O_s)$ by a similar argument.

Conversely, assume that the condition (1) holds. Then by definition there are $W, Y \in \mathrm{Mat}(n, R)$ such that $A = \mathrm{diag}(E_r, O_s)W$ and that $\mathrm{diag}(E_r, O_s) = YA$. We partition the matrices A, W, Y into 2×2 blocks in the same way as $\mathrm{diag}(E_r, O_s)$. Then the equation $A = \mathrm{diag}(E_r, O_s)W$ reads

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathrm{diag}(E_r, O_s) \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ O & O_s \end{pmatrix}.$$

Thus A_{21} and A_{22} are both zero matrices. And the equation $\mathrm{diag}(E_r, O_s) = YA$ reads

$$\mathrm{diag}(E_r, O_s) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & O_s \end{pmatrix} = \begin{pmatrix} Y_{11}A_{11} & Y_{11}A_{12} \\ Y_{21}A_{11} & Y_{21}A_{12} \end{pmatrix}.$$

In particular, we have $Y_{11}A_{11} = E_r$, so both Y_{11} and A_{11} are invertible. Then we deduce from $Y_{11}A_{12} = O$ that $A_{12} = Y_{11}^{-1}O = O$. □

Corollary 2.5. *Let K be a finite field of order q, and r be the rank of the matrix $A \in \mathrm{Mat}(n, K)$. Then the H-class of A in $\mathrm{Mat}(n, K)$ consists of $\prod_{i=0}^{r-1}(q^r - q^i)$ matrices. When $r = 0$, we adopt the convention that $\prod_{i=0}^{0-1}(q^0 - q^i) = 1$.*

Proof. Combine Theorem 2.4 with the fact that there are exactly $\prod_{i=0}^{r-1}(q^r - q^i)$ inverse matrices of order r over a finite field of order q. □

3 The Green H-classes of matrices over R

By Lemma 2.2, it suffices to consider the H-classes of the matrices of the form $D = \mathrm{diag}(E_{r_0}, \xi E_{r_1}, \ldots, \xi^{d-1}E_{r_{d-1}}, O_s)$ with $r_0, \ldots, r_{d-1}, s \geq 0$ and $r_0 + \cdots + r_{d-1} + s = n$.

The following theorem is our main result.

Theorem 3.1. *Let $A \in \mathrm{Mat}(n, R)$, and $D = \mathrm{diag}(E_{r_0}, \xi E_{r_1}, \ldots, \xi^{d-1}E_{r_{d-1}}, O_s)$. Then $A =_H D$ if and only if A can be partitioned in the following form:*

$$\begin{pmatrix} B_{11} & \xi B_{12} & \xi^2 B_{13} & \cdots & \xi^{d-1}B_{1d} & O \\ \xi B_{21} & \xi B_{22} & \xi^2 B_{23} & \cdots & \xi^{d-1}B_{2d} & O \\ \xi^2 B_{31} & \xi^2 B_{32} & \xi^2 B_{33} & \cdots & \xi^{d-1}B_{3d} & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{d-1}B_{d1} & \xi^{d-1}B_{d2} & \xi^{d-1}B_{d3} & \cdots & \xi^{d-1}B_{dd} & O \\ O & O & O & \cdots & O & O_s \end{pmatrix}, \quad \text{i.e., the coefficient of } B_{ij} \text{ is } \xi^{\max\{i,j\}-1}, \qquad (*)$$

where $B_{11}, B_{22}, \ldots, B_{dd}$ are invertible matrices of rank $r_0, r_1, \ldots, r_{d-1}$, respectively. If some $r_{i-1} = 0$, then the ith row partition and the ith column partition vanish, so we don't consider whether such B_{ii} is invertible.

Lemma 3.2. *Let $A \in \mathrm{Mat}(n, R)$, and E_n be the identity matrix of order n. If $\xi^{d-1}A = \xi^{d-1}E_n$, then A is invertible in $\mathrm{Mat}(n, R)$.*

Proof. The proof is by induction on n. It holds obviously for $n = 1$, and assuming it to hold for $n - 1$, we will prove it for n. Since $\xi^{d-1}A = \xi^{d-1}E_n$, we deduce from Lemma 2.2 that

$$A = \begin{pmatrix} c_{11} & \xi c_{12} & \cdots & \xi c_{1n} \\ \xi c_{21} & c_{22} & \cdots & \xi c_{2n} \\ \vdots & \vdots & & \vdots \\ \xi c_{n1} & \xi c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

where $c_{11}, c_{22}, \ldots, c_{nn}$ are unit in R. Applying the Laplace expansion along the 1st row of A, we obtain

$$\det A = c_{11}\det(A_{11}) + \sum_{i=2}^{n}(-1)^{i+1}\xi c_{1i}\det(A_{1i}),$$

where A_{1i} is the submatrix formed by deleting the 1st row and ith column of A. By the inductive hypothesis A_{11} is invertible. Hence $\det A_{11}$ is a unit in R, and so is $c_{11}\det(A_{11})$. And the rest part of the Laplace expansion above is contained in the maximal ideal $\langle \xi \rangle$ of R, so $\det A$ is still a unit in R. Therefore A is invertible. $\square$

Now we can give the proof of Theorem 3.1 as follows.

Proof. For the necessity of Theorem 3.1, assume that $A =_H D$ for some $A \in \mathrm{Mat}(n, R)$. Then there exist $W, X, Y, Z \in \mathrm{Mat}(n, R)$ such that $A = DW = XD$ and $D = AY = ZA$. We partition these matrices into $(d+1) \times (d+1)$ blocks in the same way as D. For example,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1d} & A_{1,d+1} \\ A_{21} & A_{22} & \cdots & A_{2d} & A_{2,d+1} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{d1} & A_{d2} & \cdots & A_{dd} & A_{d,d+1} \\ A_{d+1,1} & A_{d+1,2} & \cdots & A_{d+1,d} & A_{d+1,d+1} \end{pmatrix}.$$

Calculating $A = DW = XD$ in the form of the block matrices, we conclude that $A_{ij} = \xi^{i-1}W_{ij} = \xi^{j-1}X_{ij}$ for $i, j \in \{1, \ldots, d\}$, and that $A_{ij} = O$ for $i = d+1$ or $j = d+1$. Thus there exists $r_{i-1} \times r_{j-1}$ matrix B_{ij} such that $A_{ij} = \xi^{\max\{i,j\}-1}B_{ij}$ for $i, j \in \{1, \ldots, d\}$, that is, A can be written in the form of $(*)$.

It remains to show that all of $B_{11}, B_{22}, \ldots, B_{dd}$ are invertible over R. For this, we set $\Delta = \mathrm{diag}(\xi^{d-1}E_{r_0}, \xi^{d-2}E_{r_1}, \ldots, E_{r_{d-1}}, O_s)$. Then $\Delta D = \mathrm{diag}(\xi^{d-1}E_{n-s}, O_s)$. Then we deduce from $D = AY$ that

$$\mathrm{diag}(\xi^{d-1}E_{n-s}, O_s) = \begin{pmatrix} \xi^{d-1}B_{11} & O & O & \cdots & O & O \\ \xi^{d-1}B_{21} & \xi^{d-1}B_{22} & O & \cdots & O & O \\ \xi^{d-1}B_{31} & \xi^{d-1}B_{32} & \xi^{d-1}B_{33} & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{d-1}B_{d1} & \xi^{d-1}B_{d2} & \xi^{d-1}B_{d3} & \cdots & \xi^{d-1}B_{dd} & O \\ O & O & O & \cdots & O & O_s \end{pmatrix} Y.$$

Denote by Y_0 the submatrix of Y formed by deleting the last s rows and the last

s columns. Then we deduce from the equation above that

$$\xi^{d-1}E_{n-s} = \xi^{d-1}\begin{pmatrix} B_{11} & O & O & \cdots & O \\ B_{21} & B_{22} & O & \cdots & O \\ B_{31} & B_{32} & B_{33} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{d1} & B_{d2} & B_{d3} & \cdots & B_{dd} \end{pmatrix} Y_0.$$

Write B for the lower triangular matrix in the middle term of the right side of the equality above. By Lemma 3.2, the matrix BY_0 is invertible, and so is B. Then all of the diagonal blocks $B_{11}, \ldots, B_{dd}$ must be invertible, which completes the proof of the necessity for Theorem 3.1.

For the sufficiency of Theorem 3.1, let A be the matrix given by $(*)$ in Theorem 3.1. Then the following equalities yield $A \leq_H D$.

$$\begin{aligned} A =& D \begin{pmatrix} A_{11} & \xi A_{12} & \xi^2 A_{13} & \cdots & \xi^{d-1}A_{1d} & O \\ A_{21} & A_{22} & \xi A_{23} & \cdots & \xi^{d-2}A_{2d} & O \\ A_{31} & A_{32} & A_{33} & \cdots & \xi^{d-3}A_{3d} & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{d1} & A_{d2} & A_{d3} & \cdots & A_{dd} & O \\ O & O & O & \cdots & O & O_s \end{pmatrix} \\ =& \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1d} & O \\ \xi A_{21} & A_{22} & A_{23} & \cdots & A_{2d} & O \\ \xi^2 A_{31} & \xi A_{32} & A_{33} & \cdots & A_{3d} & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{d-1}A_{d1} & \xi^{d-2}A_{d2} & \xi^{d-3}A_{d3} & \cdots & A_{dd} & O \\ O & O & O & \cdots & O & O_s \end{pmatrix} D. \end{aligned}$$

To show $D = \operatorname{diag}(E_{r_0}, \xi E_{r_1}, \ldots, \xi^{d-1}E_{r_{d-1}}, O_s) \leq_H A$, we only need to find matrices $Y, Z \in \operatorname{Mat}(n, R)$ such that $D = AY = ZA$. Review that the r_i's may be zero, and let k be the number of positive integers in $\{r_0, r_1, \ldots, r_{d-1}\}$. The proof is by induction on k. When $k = 0$, we have $D = O_n$, the conclusion holds obviously.

Assume that $k \geq 1$, and that the conclusion holds for $k - 1$, then we will prove it for k. Let $\delta = \max\{i : 1 \leq i \leq d, r_{i-1} \neq 0\}$, and A_0 be the submatrix of A formed by the first $\delta - 1$ row partitions and first $\delta - 1$ column partitions, which is a square matrix of order $t := r_1 + \cdots + r_{\delta-2} = n - s - r_{\delta-1}$. Then A can be written in the following form

$$A = \begin{pmatrix} A_0 & \xi^{\delta-1}U & O \\ \xi^{\delta-1}V & \xi^{\delta-1}B_{\delta\delta} & O \\ O & O & O_s \end{pmatrix},$$

where

$$U = (B_{1\delta}, B_{2\delta}, \ldots, B_{\delta-1,\delta})^T, \quad V = (B_{\delta 1}, B_{\delta 2}, \ldots, B_{\delta,\delta-1}).$$

By the inductive hypothesis, we have $A_0 =_H \operatorname{diag}(E_{r_0}, \xi E_{r_1}, \ldots, \xi^{\delta-2} E_{r_{\delta-2}})$. Let D_0 denote the latter diagonal matrix. Then by definition there exist square matrices Y_0 and Z_0 of order t such that $A_0 Y_0 = Z_0 A_0 = D_0$.

Now let Δ_0 be the diagonal matrix $\operatorname{diag}(\xi^{\delta-2} E_{r_0}, \xi^{\delta-3} E_{r_1}, \ldots, E_{r_{\delta-2}})$ of order t. Then we have $\Delta_0 D_0 = D_0 \Delta_0 = \xi^{\delta-2} E_t$. Combining the arguments above, one can check that

$$\begin{aligned} D &= \begin{pmatrix} A_0 & \xi^{\delta-1} U & O \\ \xi^{\delta-1} V & \xi^{\delta-1} B_{\delta\delta} & O \\ O & O & O_s \end{pmatrix} \begin{pmatrix} Y_0 + \xi Y_0 \Delta_0 U B_{\delta\delta}^{-1} V Y_0 & -\xi Y_0 \Delta_0 U B_{\delta\delta}^{-1} & O \\ -B_{\delta\delta}^{-1} V Y_0 & B_{\delta\delta}^{-1} & O \\ O & O & O_s \end{pmatrix} \\ &= \begin{pmatrix} Z_0 + \xi Z_0 U B_{\delta\delta}^{-1} V \Delta_0 Z_0 & -Z_0 U B_{\delta\delta}^{-1} & O \\ -\xi B_{\delta\delta}^{-1} V \Delta_0 Z_0 & B_{\delta\delta}^{-1} & O \\ O & O & O_s \end{pmatrix} \begin{pmatrix} A_0 & \xi^{\delta-1} U & O \\ \xi^{\delta-1} V & \xi^{\delta-1} B_{\delta\delta} & O \\ O & O & O_s \end{pmatrix}. \end{aligned}$$

This shows $D \leq_H A$ and completes the proof of Theorem 3.1. □

Example 3.3. Let $A \in \operatorname{Mat}(n, \mathbb{Z}/4\mathbb{Z})$. Then $A =_H \operatorname{diag}(E_{r_0}, 2E_{r_1}, O_s)$ if and only if A has the following partitioned form:

$$\begin{pmatrix} A_{11} & 2A_{12} & O \\ 2A_{21} & 2A_{22} & O \\ O & O & O_s \end{pmatrix},$$

where A_{11} and A_{22} are invertible matrices over $\mathbb{Z}/4\mathbb{Z}$ of order r_0 and r_1, respectively.

Remark 3.4. In general, the H-class in $\operatorname{Mat}(n, R)$ is a proper subset of the L or R-class. For instance, we see from Example 3.3 that the H-class of $\operatorname{diag}(2, 0)$ in $\operatorname{Mat}(2, \mathbb{Z}/4\mathbb{Z})$ only consists of itself. But we have

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the L-class of $\operatorname{diag}(2, 0)$ contains $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$, and the R-class of $\operatorname{diag}(2, 0)$ contains $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ by taking the transpose in the equations above.

4 The construction of invertible matrices over R

Let R° denote the subset of R consisting of 0 and all units, namely, $R^\circ = (R \backslash \langle \xi \rangle) \cup \{0\}$.

Theorem 4.1. *Let $A = (a_{ij})$ be a square matrix of order n over R. We denote by $r_1, \ldots, r_n$ and $c_1, \ldots, c_n$ the rows and columns of A, respectively. That is, $r_i = (a_{i1}, \ldots, a_{in})$ and $c_i = (a_{1i}, \ldots, a_{ni})^T$. Then the following conditions are equivalent:*

(1) A is invertible;

(2) if there exist $\lambda_1, \ldots, \lambda_n \in R^\circ$ and an n-dimensional row vector r over R such that $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi r$, then $\lambda_1 = \cdots = \lambda_n = 0$;

(3) if there exist $\lambda_1, \ldots, \lambda_n \in R^\circ$ and an n-dimensional column vector c over R such that $\lambda_1 c_1 + \cdots + \lambda_n c_n = \xi c$, then $\lambda_1 = \cdots = \lambda_n = 0$.

Proof. We give the proof for the equivalence of (1) and (2), the proof for that of (1) and (3) is similar.

Firstly, suppose that the condition (2) does not hold. Then there exist coefficients $\lambda_1, \ldots, \lambda_n \in R^\circ$, at least one of them is nonzero, and a row vector r such that $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi r$. We will show that $\det A$ is not a unit of R, and consequently A is not an invertible matrix. Since neither whether $\det A$ is unit nor whether the condition (2) holds is affected if interchanging the rows of A, we may assume that $\lambda_1 \neq 0$, i.e., λ_1 is a unit of R. Let A_1 be the matrix formed by replacing the 1st row r_1 of A by $r_1 + \lambda_1^{-1}\lambda_2 r_2 + \cdots + \lambda_1^{-1}\lambda_n r_n = \xi\lambda_1^{-1} r$. Then $\det A = \det A_1$, and every entry in the 1st row $\xi\lambda_1^{-1} r$ of A_1 has a factor ξ. Thus $\det A \in \langle \xi \rangle$ and hence not a unit of R.

Conversely, suppose that the condition (2) holds. We will prove $\det A$ is a unit in R by induction on n, and consequently A is an invertible matrix. For $n = 1$, it follows from Lemma 2.2 that $\det A$ is a unit. Then assume that $\det A$ is a unit for $n-1$, and we will prove it for n. We can see that at least one of $a_{n1}, \ldots, a_{nn}$ is not contained in $\langle \xi \rangle$, for otherwise it would lead a contradiction by taking $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n = 1$. Since neither whether $\det A$ is unit nor whether the condition (2) holds is affected if interchanging the columns of A, we may assume that a_{nn} is not contained in $\langle \xi \rangle$ and hence is a unit in R. Let A_2 be the matrix formed by replacing the row r_i of A by $r_i - a_{nn}^{-1} a_{in} r_n$ $(1 \leq i \leq n-1)$. Then the entries in the nth column of A_2 are all 0 except a_{nn}. By the Laplace expansion, we obtain $\det A = \det A_2 = a_{nn} \det A_3$, where A_3 is the submatrix of A_2 formed by deleting the last row and the last column.

Since a_{nn} is a unit of R, it suffices to show that so is $\det A_3$. Let r_i' be the ith row of A_3 $(1 \leq i \leq n-1)$. Then we have $(r_i', 0) = r_i - a_{nn}^{-1} a_{in} r_n$. Assume that there exist $\lambda_1, \ldots, \lambda_{n-1} \in R^\circ$ and an $(n-1)$-dimensional row vector r' such that $\lambda_1 r'_1 + \cdots + \lambda_{n-1} r'_{n-1} = \xi r'$. Then we deduce that

$$\begin{aligned}\xi\big(r', 0\big) =& \big(\sum_{i=1}^{n-1} \lambda_i r'_i, 0\big) = \sum_{i=1}^{n-1} \lambda_i \big(r'_i, 0\big) = \sum_{i=1}^{n-1} \lambda_i r_i - a_{nn}^{-1} \big(\sum_{i=1}^{n-1} \lambda_i a_{in}\big) r_n \\ =& \sum_{i=1}^{n} \lambda_i r_i, \qquad \text{by setting } \lambda_n = -a_{nn}^{-1}\big(\sum_{i=1}^{n-1} \lambda_i a_{in}\big).\end{aligned}$$

Review that $\lambda_1, \ldots, \lambda_{n-1} \in R^\circ$. In the case where $\lambda_n \in R^\circ$, we get $\lambda_1 = \cdots = \lambda_n = 0$ by the condition (2). And in the case where $\lambda_n \notin R^\circ$, say $\lambda_n = \xi t$, we have $\sum_{i=1}^{n-1} \lambda_i r_i + 0 \cdot r_n = \xi\big[(r', 0) - t r_n\big]$, so $\lambda_1 = \cdots = \lambda_{n-1} = 0$ by the condition (2) again. In either case, we can conclude from the inductive hypothesis that the matrix A_3 is invertible, so $\det A_3$ is a unit in R, which completes the proof. □

Corollary 4.2. *Use the notations of Theorem 4.1, and let $k \in \{0, 1, \ldots, d-1\}$, where d is the nilpotent index of ξ defined in Section 2. We assume that all of the a_{ij}'s are contained in the ideal $\langle \xi^k \rangle$. Then the following conditions are equivalent:*

(1) there is a invertible matrix B such that $A = \xi^k B$;

(2) if there exist $\lambda_1, \ldots, \lambda_n \in R^\circ$ and an n-dimensional row vector r over R such that $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi^{k+1} r$, then $\lambda_1 = \cdots = \lambda_n = 0$;

(3) if there exist $\lambda_1, \ldots, \lambda_n \in R^\circ$ and an n-dimensional column vector c over R such that $\lambda_1 c_1 + \cdots + \lambda_n c_n = \xi^{k+1} c$, then $\lambda_1 = \cdots = \lambda_n = 0$.

Proof. We need only to consider the case where $k \in \{1, 2, \ldots, d-1\}$, since when $k = 0$, this is just Theorem 4.1. We give the proof for the equivalence of (1) and (2), the proof for that of (1) and (3) is similar. Firstly suppose that the condition (1) holds. Let s_i be the ith row of B, then $r_i = \xi^k s_i$. Assume there are $\lambda_1, \ldots, \lambda_n \in R^\circ$ and a row vector r such that $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi^{k+1} r$. Then we have $\xi^k(\lambda_1 s_1 + \cdots + \lambda_n s_n - \xi r) = 0$. Since $0 < k < d$, there exists a row vector s such that $\lambda_1 s_1 + \cdots + \lambda_n s_n - \xi r = \xi s$. By Theorem 4.1, we have $\lambda_1 = \cdots = \lambda_n = 0$. Conversely, suppose that the condition (2) holds. Since $a_{ij} \in \langle \xi^k \rangle$ for any i and j, there is a matrix $B = (b_{ij})$ such that $A = \xi^k B$. We still denote by s_i the ith row of B, i.e., $r_i = \xi^k s_i$. Assume there are $\lambda_1, \ldots, \lambda_n \in R^\circ$ and a row vector s such that $\lambda_1 s_1 + \cdots + \lambda_n s_n = \xi s$. Then $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi^{k+1} r$. By the condition (2), we have $\lambda_1 = \cdots = \lambda_n = 0$. Hence B is invertible by Theorem 4.1. □

5 An application: enumeration over $\mathbb{Z}/p^d\mathbb{Z}$

In this section, we consider the typical case of $R = \mathbb{Z}/p^d\mathbb{Z}$, where p is prime and d is positive integer, and provide the size of the each H-class in $\mathrm{Mat}(n, \mathbb{Z}/p^d\mathbb{Z})$.

Note that the conditions in the notations in §2 are satisfied only if $d \geq 2$, while $\mathbb{Z}/p\mathbb{Z}$ is a finite field and hence the discussions in §2 are applicable.

Lemma 5.1. *Let $k \in \{0, 1, \ldots, d-1\}$, A be a square matrix of order n over $\mathbb{Z}/p^d\mathbb{Z}$ ($d \geq 1$), and $r_1, \ldots, r_n$ be the rows of A.*

(1) *There exists an invertible matrix B such that $A = p^k B$ if and only if all $a_{ij} \in \langle p^k \rangle$ and for any $\lambda_1, \ldots, \lambda_n \in \{0, 1, \ldots, p-1\}$ and any row vector r such that $\lambda_1 r_1 + \cdots + \lambda_n r_n = p^{k+1} r$, we must have $\lambda_1 = \cdots = \lambda_n = 0$.*

(2) *Let $m \in \{1, 2, \ldots, n\}$, $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m \in \{0, 1, \ldots, p-1\}$, and r and s be row vectors. In the case where the condition of (1) holds, we have $\lambda_1 r_1 + \cdots + \lambda_m r_m + p^{k+1} r = \mu_1 r_1 + \cdots + \mu r_m + p^{k+1} s$ if and only if $\lambda_1 = \mu_1$, $\cdots$, $\lambda_m = \mu_m$ and $p^{k+1} r = p^{k+1} s$.*

Proof. When $d = 1$, this is well known. When $d \geq 2$, the proof of (1) is based on Corollary 4.2 and the following observation. Any $\lambda_i \in (\mathbb{Z}/p^d\mathbb{Z})^\circ$ in the condition (2) of Corollary 4.2 can be written in the form $\lambda_i = \lambda_i' + p\mu_i$ with $\lambda_i' \in \{0, 1, \ldots, p-1\}$. And since every entry of A is contained in $\langle p^k \rangle$, every r_i can be written in the form $r_i = p^k s_i$. Hence, the formula $\lambda_1 r_1 + \cdots + \lambda_n r_n = \xi^{k+1} r$ in the condition (2) of Corollary 4.2 read as

$$\lambda_1' r_1 + \cdots + \lambda_n' r_n = p^{k+1}(r - \mu_1 s_1 - \cdots - \mu_n s_n), \quad \lambda_1', \ldots, \lambda_n' \in \{0, 1, \ldots, p-1\},$$

which is just what we need. And (2) is a simple consequence of (1) since $\lambda_1 r_1 + \cdots + \lambda_m r_m + p^{k+1} r = \mu_1 r_1 + \cdots + \mu r_m + p^{k+1} s$ can be reformulated as $(\lambda_1 - \mu_1) r_1 + \cdots + (\lambda_m - \mu_m) r_m = p^{k+1}(s - r)$. ☐

Lemma 5.2. *Let $k \in \{0, 1, \ldots, d-1\}$. The number of matrices over $\mathbb{Z}/p^d\mathbb{Z}$ ($d \geq 1$) with the form of $p^k B$ where B is an invertible matrix of order n is equal to*

$$\prod_{i=0}^{n-1} \left(p^{(d-k)n} - p^{(d-k-1)n+i}\right).$$

Proof. The proof consists in the following construction of such matrix $p^k B$ using Lemma 5.1. Let $r_i = (a_{i1}, \ldots, a_{in})$ be the ith row of $p^k B$ with all $a_{ij} \in \langle p^k \rangle$. And we know that the ideal $\langle p^k \rangle$ consists of p^{d-k} elements.

Firstly, we construct r_1. By (1) of Lemma 5.1, the only restriction for r_1 is that r_1 cannot be equal to $p^{k+1} r$ for some row vector r. Thus there are $p^{(d-k)n} - p^{(d-k-1)n}$ choices for r_1. Secondly, we construct r_2. By (1) of Lemma 5.1, the restriction for r_2 is that r_2 cannot be equal to $\lambda_1 r_1 + p^{k+1} r$ for some $\lambda \in \{0, 1, \ldots, p-1\}$ and some row vector r. And by (2) of Lemma 5.1, for a fixed r_1, there are $p^{(d-k)n} - p^{(d-k-1)n+1}$ choices for r_2. Continuing this process, by (1) of Lemma 5.1, the restriction for r_i is that r_i cannot be equal to $\lambda_1 r_1 + \cdots + \lambda_{i-1} r_{i-1} + p^{k+1} r$ for some $\lambda_1, \ldots, \lambda_{i-1} \in \{0, 1, \ldots, p-1\}$ and some row vector r; and by (2) of Lemma 5.1, for fixed $r_1, \ldots, r_{i-1}$, there are $p^{(d-k)n} - p^{(d-k-1)n+i-1}$ choices for r_i. Therefore, the assertion follows from the rule of product of combinatorics. $\square$

Corollary 5.3. *Let $A \in \mathrm{Mat}(n, \mathbb{Z}/p^d\mathbb{Z})$, and $\mathrm{diag}(E_{r_0}, pE_{r_1}, \ldots, p^{d-1}E_{r_{d-1}}, O_s)$ be its Smith normal form. The H-class of A consists of $\prod_{i=1}^{d-1} M(i) \prod_{i=0}^{d-1} N(i)$ elements, where*

$$M(i) = \begin{cases} p^{2r_i(r_0+\cdots+r_{i-1})(d-i)} & r_i \neq 0, \\ 1 & r_i = 0, \end{cases}$$
$$N(i) = \begin{cases} \prod_{j=0}^{r_i-1} \left(p^{(d-i)r_i} - p^{(d-i-1)r_i+j}\right) & r_i \neq 0, \\ 1 & r_i = 0. \end{cases}$$

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