GLOBAL WELL-POSEDNESS FOR THE GENERALIZED NAVIER-STOKES-CORIOLIS EQUATIONS WITH HIGHLY OSCILLATING INITIAL DATA

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ABSTRACT. We study the small initial date Cauchy problem for the generalized incompressible Navier-Stokes-Coriolis equations in critical hybrid-Besov space $\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}(\mathbb{R}^3)$ with $1/2 < \alpha < 2$ and $2 \leq p \leq 4$. We prove that hybrid-Besov spaces norm of a class of highly osillating initial velocity can be arbitrarily small. and we prove the estimation of highly frequency L^p smoothing effect for generalized Stokes-Coriolis semigroup with $1 \leq p \leq \infty$, At the same time, we prove space-time norm estimation of hybrid-Besov spaces for Stokes-Coriolis semigroup. From this result we deduce bilinear estimation in our work space. Our method relies upon Bony's high and low frequency decomposition technology and Banach fixed point theorem.

1. INTRODUCTION

We consider the initial value problem for the generalized incompressible Navier-Stokes-Coriolis equations(GNSC):

$$
\begin{cases}
\partial_t u + \nu(-\Delta)^\alpha u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0, & x \in \mathbb{R}^3,\ t \in (0,\infty),\\
\operatorname{div} u = 0, & x \in \mathbb{R}^3,\ t \in (0,\infty),\\
u(0,x) = u_0(x), & x \in \mathbb{R}^3,
\end{cases}
\tag{1.1}
$$

where number $\alpha > 0$ represents the 'strength of dissipation'. $u = u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ denotes the unknown velocity vector field of the fluid and $p = p(t,x)$ denotes the unknown scalar pressure at the point $(t,\,x) \in (0,\infty) \times \mathbb{R}^3$, while $u_0(x)$ denotes the given initial velocity vector field with $\operatorname{div} u_0 = 0$. The constant $\Omega \in \mathbb{R}$ is the Coriolis parameter, which represents the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$ and ν denotes the kinematic viscosity coefficient of the fluid. "$\times$" denotes the exterior product. Moreover, ∂_t and $\Delta = \sum_{j=1}^3 \partial_{x_j}^2$ are the partial derivative with respect to t and the Laplacian with respect to $x = (x_1, x_2, x_3)$, respectively. $(-\Delta)^\alpha (\alpha > 0)$ denotes the fractional Laplacian which is defined

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as

$$\mathcal{F}((-\Delta)^{\alpha}u)(t,\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(t,\xi),$$

where $\mathcal{F}u$ is the Fourier transform of u with respect to space variable x. Sun and Ding [24] proved the time-local existence and uniqueness of mild solution for every $\Omega \in \mathbb{R} \setminus 0$ and $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $1/4 < \alpha \leq 3/2$, $3/2 - \alpha < s \leq 5/4$.

If $\Omega = 0$, the equations (1.1) become the generalized incompressible Navier-Stokes equations(GNS):

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha}u + (u\cdot\nabla)u + \nabla p = 0, & x \in \mathbb{R}^3,\, t \in (0,\infty), \\ \mathrm{div} u = 0, & x \in \mathbb{R}^3,\, t \in (0,\infty), \\ u(0,x) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

When $\alpha = 1$, the equation (1.2) become the classical Navier-Stokes equations. For $\alpha \geqslant 5/4$, Lions [21] proved the global existence of the classical solutions of (1.2).

For $x \in \mathbb{R}^n$, $n \geqslant 2$, there are many results on (1.2) in recent years. Wu [28] proved (1.2) has a unique global solution with small data in $\dot{B}^{n/p-2\alpha+1}_{p,q}(1 \leqslant q \leqslant \infty)$ for $\alpha > 1/2$, $p = 2$ or $1/2 < \alpha \leqslant 1$, $2 < p < \infty$. Li, Zhai, Xiao and Yang [18–20] obtained the well-posedness of (1.2) in the case $1/2 < \alpha < 1$ in some Q-spaces and diagonal Besov-Q spaces. Yu and Zhai [29] showed the global existence and uniqueness with small initial data in $\dot{B}^{1-2\alpha}_{\infty,\infty}$ for (1.2) in the case $1/2 < \alpha < 1$. Recently, Huang and Wang [14] proved the global well-posedness for (1.2) ($\alpha = 1/2$) with small data in critical Besov spaces $\dot{B}^{n/p}_{p,1}(1 \leqslant p \leqslant \infty)$ and proved the similar results for all $\alpha \in (1/2, 1)$. Sun and Liu [25] proved uniqueness of the weak solution to the fractional anisotropic Navier-Stokes system with only horizontal dissipation.

If $\alpha = 1$, (1.1) is the well-known Navier-Stokes-Coriolis equation (NSC) which have been extensively studied in recent years due to its importance in applications to geophysical flows, cf. [1, 2, 4–7, 10–13, 15, 17, 22, 26]. More specifically, (1.1) allows a global mild solution for arbitrary large data in the L^2-setting provided the speed Ω of rotation is fast enough, see [1], [2] and [4]. Hieber and Shibata further proved that (1.1) possess a unique global mild solution for arbitrary speed of rotation provided the initial data u_0 is small enough in the $H^{\frac{1}{2}}_{\sigma}$-norms, see [13]. Iwabuchi and Takada proved the existence of global unique solutions in the homogeneous Sobolev spaces $\dot{H}^s$ for $1/2 < s < 3/4$ if the speed of rotation is sufficiently large, see [15]. They also proved the local in time existence and uniqueness of the mild solution for every $\Omega \in \mathbb{R}\backslash\{0\}$ and $u_0 \in \dot{H}^s$ with $1/2 < s < 5/4$, see [16]. Koniecany and Yoneda proved that (1.1) exists a unique global solution in the Fourier Besov framework when initial data be small enough, see [17]. Ohyama showed the unique existence of global in time mild solutions for small initial data belonging to function spaces of the Besov type characterized by the time evolution semigroup associated with the linear Stokes-Coriolis operator, see [22]. Sun, Yang and Cui [9] proved uniqueness existence of global mild solutions to (1.1) in $\dot{B}^s_{p,r}(\mathbb{R}^3)$ with $p \in (\frac{3}{2}, 2)$ in the case $|\Omega| > \Omega_0 > 0$, see [26]. Chen, Miao and Zhang proved the global well-posedness in the hybrid-Besov spaces $\dot{\mathscr{B}}^{\frac{1}{2}, \frac{3}{p}-1}_{2,p}$ for $p \in [2, 4]$, see [7]. Let us emphasize that this result allows us to construct global solutions for highly oscillating initial data which may have a large norm in $\dot{H}^{1/2}$. A typical example is

$$u_0(x) = \sin \frac{x_3}{\varepsilon} \left(-\partial_{x_2}\phi(x), \partial_{x_1}\phi(x), 0\right),$$

where $\phi \in C_0^\infty$ and $\varepsilon > 0$. At the same time, there is a large literature on existence of solutions of (1.1) in the setting of nondecaying initial data by Giga et al., see [10–12]. In fact, there are many exact solutions which is unbounded at space infinity in some practical flow problem. In this paper, we mainly consider the case $\alpha > 0$. i.e. the generalized incompressible Navier-Stokes-Coriolis equations is the main object of study.

The goal of this paper is to prove the global existence of a solution of (1.1) for a class of highly oscillating initial velocity. Motivated by the concept of hybrid-Besov spaces in [7], we consider the global well-posedness of (1.1) in the appropriate hybrid-Besov spaces.

Throughout this paper, $C \geqslant 1$; $0 < c \leqslant 1$ will denote constants which can be different at different places, we will use $A \lesssim B$ to denote $A \leqslant CB$. We denote $L^p(\mathbb{R}^3)$ the Lebesgue function space with the norm $\|\cdot\|_p$. For $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$, we denote $\|u(t,x)\|_p = (\|u^1(t,x)\|_p^2 + \|u^2(t,x)\|_p^2 + \|u^3(t,x)\|_p^2)^{1/2}$ provided $1 \leqslant p < \infty$, and we do usual modifications for $p = \infty$. Let X be a quasi-Banach space. For any $I \subset [0, \infty)$, we denote

$$\|u\|_{L^r(I;X)} = \left(\int_I \|u(t,\cdot)\|_X^r \,\mathrm{d}t\right)^{1/r}.$$

2. Function spaces and main results

Now let us recall the definition of dyadic decomposition in Littlewood-Paley theory. We denote $\varphi \in \mathcal{S}(\mathbb{R}^3)$ a smooth cut-off function supported in $\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \ \xi \neq 0.$$

We also introduce the following functions:

$$\varphi_j(\xi) = \varphi(2^{-j}\xi) \ \text{ and } \ \psi_j(\xi) = \sum_{k\leqslant j-1} \varphi_k(\xi).$$

For $f \in \mathcal{S}'$, we define the standard localization operators:

$$\Delta_j f = \varphi_j(D) f, \quad S_j f = \sum_{k \leqslant j-1} \Delta_k f = \psi_j(D) f, \quad j \in \mathbb{Z}, \quad D = \frac{\nabla_x}{i}. \tag{2.1}$$

It is then easy to verify the following identities:

$$\Delta_j \Delta_k f = 0, \quad \text{if } \ |j-k| \geqslant 2. \tag{2.2}$$

$$\Delta_j(S_{k-1}f\Delta_k f)=0,\quad \text{if}\ \ |j-k|\geqslant 5. \tag{2.3}$$

Moreover, we have the following Bony decomposition:

$$fg=T_f g+T_g f+R(f,g), \tag{2.4}$$

where

$$T_f g=\sum_{j\in\mathbb{Z}}S_{j-1}f\Delta_j g,\quad R(f,g)=\sum_{j\in\mathbb{Z}}\Delta_j f\tilde{\Delta}_j g,\quad \tilde{\Delta}_j g=\sum_{|j'-j|\leqslant 1}\Delta_{j'}g. \tag{2.5}$$

Definition 2.1. Let $s\in\mathbb{R}$, $1\leqslant p\leqslant\infty, 1\leqslant q<\infty$. the homogeneous Besov spaces $\dot{B}^s_{p,q}$ is defined

$$\dot{B}^s_{p,\,q}(\mathbb{R}^n)=\left\{f\in\mathcal{S}'(\mathbb{R}^3)/\mathscr{P}:\|f\|_{\dot{B}^s_{p,\,q}}=\left(\sum_{k\in\mathbb{Z}}2^{ksq}\|\Delta_k f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}<\infty\right\}.$$

$$\dot{B}^s_{p,\infty}(\mathbb{R}^n)=\left\{f\in\mathcal{S}'(\mathbb{R}^3)/\mathscr{P}:\|f\|_{\dot{B}^s_{p,\infty}}=\sup_{k\in\mathbb{Z}}2^{ks}\|\Delta_k f\|_{L^p(\mathbb{R}^n)}<\infty\right\},$$

where $\mathcal{S}'$ be a tempered distribution space, $\mathscr{P}$ be a space of polynomial functions.

Definition 2.2. (hybrid-Besov space) Let $s,\sigma\in\mathbb{R}, 1\leqslant p\leqslant\infty$. The hybrid-Besov space $\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p}$ is defined by

$$\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p}:=\{f\in\mathcal{S}'(\mathbb{R}^3)/\mathscr{P}:\|f\|_{\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p}}<+\infty\},$$

where

$$\|f\|_{\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p}}:=\sup_{2^k\leqslant\Omega}2^{ks}\|\Delta_k f\|_{L^2}+\sup_{2^k>\Omega}2^{k\sigma}\|\Delta_k f\|_{L^p}.$$

The norm of the space $\tilde{L}^r_T(\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p})$ is defined by

$$\|f\|_{\tilde{L}^r_T(\dot{\mathscr{B}}^{s,\,\sigma}_{2,\,p})}:=\sup_{2^k\leqslant\Omega}2^{ks}\|\Delta_k f\|_{L^r_T L^2}+\sup_{2^k>\Omega}2^{k\sigma}\|\Delta_k f\|_{L^r_T L^p}.$$

Lemma 2.1. ([5]) *Let $1\leqslant p\leqslant q\leqslant+\infty$. Then for any $\beta,\gamma\in(\mathbb{N}\cup\{0\})^3$, there exist a constant C independent of f, j such that, for any $f\in L^p$,*

$$\text{supp }\hat{f}\subset\{\xi:|\xi|\leqslant A_0 2^j\}\Rightarrow\|\partial^\gamma f\|_{L^q}\leqslant C2^{j|\gamma|+3j(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$

$$\text{supp }\hat{f}\subset\{\xi:A_1 2^j\leqslant|\xi|\leqslant A_2 2^j\}\Rightarrow\|f\|_{L^p}\leqslant C2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta f\|_{L^p}.$$

Lemma 2.2. *Let $\phi\in C_0^\infty(\mathbb{R}^3)$, $\alpha>1/2$, and $p\geqslant\max\{\frac{3}{2\alpha-1},1\}$. If $\phi_\varepsilon(x):=e^{i\frac{x_1}{\varepsilon}}\phi(x)$, then for any $0<\varepsilon\leqslant\Omega^{-2}(\Omega\gg 1)$,*

$$\|\phi_\varepsilon\|_{\dot{\mathscr{B}}^{\max\{\frac{5}{2}-2\alpha,\alpha-\frac{1}{2}\},\frac{3}{p}-2\alpha+1}_{2,\,p}}\leqslant C\varepsilon^{\alpha-\frac{3}{2p}-\frac{1}{2}},$$

where C is a constant independent of ε.

Proof. First of all, we consider the estimate of $\|\Delta_j\,\phi_\varepsilon\|_{L^q}$ for $q \geqslant 1$:

Noting that $e^{i\frac{x_1}{\varepsilon}} = (-i\varepsilon\partial_1)^N e^{i\frac{x_1}{\varepsilon}}$ for any $N \in \mathbb{N}$. By integration by parts we have

$$\Delta_j\,\phi_\varepsilon(x) = (i\varepsilon)^N 2^{3j} \int_{\mathbb{R}^3} e^{i\frac{y_1}{\varepsilon}} \partial_{y_1}^N \left(h(2^j(x-y))\phi(y)\right)\,\mathrm{d}y, \quad h(x) := (\mathcal{F}^{-1}\varphi)(x).$$

Using the Leibnitz formula, we can now derive the following estimates:

$$|\Delta_j\,\phi_\varepsilon(x)| \leqslant C\varepsilon^N 2^{3j} \sum_{k=0}^{N} 2^{kj} \int_{\mathbb{R}^3} |(\partial_{y_1}^k h)(2^j(x-y))||\partial_{y_1}^{N-k}\phi(y)|\,\mathrm{d}y.$$

For $j \geqslant 0$, we utilize Young's inequality to show that

$$\|\Delta_j\,\phi_\varepsilon\|_{L^q} \leqslant C\varepsilon^N \sum_{k=0}^{N} 2^{kj} 2^{3j} \|(\partial_{y_1}^k h)(2^j y)\|_{L^1} \|\partial_{y_1}^{N-k}\phi(y)\|_{L^q} \leqslant C\varepsilon^N 2^{jN},$$

and for $j \leqslant 0$,

$$\|\Delta_j\,\phi_\varepsilon\|_{L^q} \leqslant C\varepsilon^N \sum_{k=0}^{N} 2^{kj} 2^{3j} \|(\partial_{y_1}^k h)(2^j y)\|_{L^q} \|\partial_{y_1}^{N-k}\phi(y)\|_{L^1} \leqslant C\varepsilon^N 2^{(1-\frac{1}{q})3j}.$$

Let $j_0 \in \mathbb{N}$ with $\Omega \leqslant 2^{j_0} \sim \varepsilon^{-1/2}$. According to Lemma 2.1, it follows that

$$\sup_{j \geqslant j_0} 2^{j(\frac{3}{p}-2\alpha+1)} \|\Delta_j \phi_\varepsilon\|_{L^p} \leqslant C 2^{j_0(\frac{3}{p}-2\alpha+1)} \leqslant C\varepsilon^{\alpha-\frac{3}{2p}-\frac{1}{2}}.$$

$$\sup_{\Omega < 2^j < 2^{j_0}} 2^{j(\frac{3}{p}-2\alpha+1)} \|\Delta_j\,\phi_\varepsilon\|_{L^p} \leqslant C\varepsilon^N 2^{(\frac{3}{p}-2\alpha+1+N)j_0} \leqslant C\varepsilon^{\frac{N}{2}+(\alpha-\frac{3}{2p}-\frac{1}{2})},$$

For any $\alpha > 1/2$,

$$\sup_{2^j < \Omega, j \geqslant 0} 2^{j(\frac{5}{2}-2\alpha)} \|\Delta_j\,\phi_\varepsilon\|_{L^2} \leqslant C\Omega^{\frac{5}{2}-2\alpha+N}\varepsilon^N \leqslant C\varepsilon^{\frac{N}{2}-(\frac{5}{4}-\alpha)}.$$

$$\sup_{2^j < \Omega, j \geqslant 0} 2^{j(\alpha-\frac{1}{2})} \|\Delta_j\,\phi_\varepsilon\|_{L^2} \leqslant C\Omega^{\alpha-\frac{1}{2}+N}\varepsilon^N \leqslant C\varepsilon^{\frac{N}{2}-(\frac{\alpha}{2}-\frac{1}{4})}.$$

$$\sup_{2^j < \Omega, j \leqslant 0} 2^{j(\alpha-\frac{1}{2})} \|\Delta_j\,\phi_\varepsilon\|_{L^2} \leqslant C\Omega^{\alpha+1}\varepsilon^N \leqslant C\varepsilon^{N-\frac{\alpha+1}{2}}.$$

For $\frac{1}{2} < \alpha \leqslant 1$,

$$\sup_{2^j < \Omega, j \leqslant 0} 2^{j(\frac{5}{2}-2\alpha)} \|\Delta_j\,\phi_\varepsilon\|_{L^2} \leqslant C\Omega^{4-2\alpha}\varepsilon^N \leqslant C\varepsilon^{N-(2-\alpha)}.$$

We choose N large enough, then the above estimates yields that

$$\|\phi_\varepsilon\|_{\dot{\mathscr{B}}_{2,\,p}^{\max\{\frac{5}{2}-2\alpha, \alpha-\frac{1}{2}\}, \frac{3}{p}-2\alpha+1}} \leqslant C\varepsilon^{\alpha-\frac{3}{2p}-\frac{1}{2}}.$$

We have thus proved the Lemma 2.2. $\square$

Definition 2.3. Let $1 \leqslant p \leqslant \infty$, we denote by $E_{p,\,\alpha}$ the space of functions such that

$$E_{p,\,\alpha} = \{u : \operatorname{div} u = 0,\ \|u\|_{E_{p,\alpha}<+\infty}\},$$

where

$$\|u\|_{E_{p,\,\alpha}} := \|u\|_{\tilde{L}^\infty\left(\mathbb{R}^+;\, \dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}\right)} + \|u\|_{\tilde{L}^1\left(\mathbb{R}^+;\, \dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2},\, \frac{3}{p}+1}\right)}.$$

Definition 2.4. We denote by $C_*\big([0,\,\infty); \dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}\big)$ the set of functions u such that u is continuous from $(0,\,\infty)$ to $\dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}$, but weakly continuous at $t = 0$, i.e.,

$$\lim_{t\to 0^+} \sup_{0<s<t} \langle u(s,\cdot), g(\cdot)\rangle = 0 \ \text{ for all } g \in \mathcal{S} \text{ with } \|g\|_{\dot{\mathscr{B}}_{2,\,p'}^{-(\frac{5}{2}-2\alpha),\, -(\frac{3}{p}-2\alpha+1)}} \leqslant 1$$

Here $\mathcal{S}$ is the set of Schwartz functions.

Our main results are the following theorems.:

Theorem 2.3. *Let $1/2 < \alpha < 2$, $2 \leqslant p \leqslant 4$ and $\frac{1}{p} > \max\big\{\frac{\alpha-1}{3}, \frac{5-4\alpha}{6}\big\}$. If there exists a positive constant c independent of Ω such that $\|u_0\|_{\dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha, \frac{3}{p}-2\alpha+1}} \leqslant c$, then there exists a unique solution $u \in E_{p,\,\alpha}$ of (1.1) such that*

$$u \in C^*([0,\,\infty); \dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}).$$

Remark 2.5. Due to the inclusion relation

$$\dot{H}^{\frac{5}{2}-2\alpha} \subset \dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha, \frac{3}{p}-2\alpha+1} \quad \text{for} \quad p \geqslant 2,$$

Theorem 2.3 is an improvement on Theorem A. In particular, Theorem 2.3 allows us to construct global solutions of (1.1) for a class of highly oscillating initial velocity u_0, for example,

$$u_0(x) = \sin \frac{x_3}{\varepsilon}\,(-\partial_2\phi(x),\, \partial_1\phi(x), 0)$$

where $\phi \in C_0^\infty(\mathbb{R}^3)$ and $\varepsilon > 0$ is small enough. This type of data is large in Sobolev spaces; however, it is small in hybrid-Besov spaces.

If $u_0 \in \dot{H}^{\frac{5}{2}-2\alpha}$, we can obtain the following global well-posedness result.

Theorem 2.4. *Let $1/2 < \alpha < 5/4$, $2 \leqslant p \leqslant 4$ and $\frac{1}{p} > \max\big\{\frac{\alpha-1}{3}, \frac{5-4\alpha}{6}\big\}$. For $u_0 \in \dot{H}^{\frac{5}{2}-2\alpha}$, if there exists a positive constant c independent of Ω such that $\|u_0\|_{\dot{\mathscr{B}}_{2,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}} \leqslant c$, then there exists a unique global solution of (1.1) in $C(\mathbb{R}^+,\, \dot{H}^{\frac{5}{2}-2\alpha})$.*

Remark 2.6. Since we only impose the smallness condition of the initial data in the norm of $\dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}$, this allows us to obtain the global well-posedness of (1.1) for a class of highly oscillating initial velocity u_0. Moreover, the uniqueness holds in the class $C(\mathbb{R}^+, \dot{H}^{\frac{5}{2}-2\alpha})$ with $1/2 < \alpha < 5/4$, i.e., it is unconditional.

In order to solve (1.1), we consider the following integral equations:

$$u(t) = S_\Omega(t)u_0 - \int_0^t S_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u](\tau)\,\mathrm{d}\tau, \tag{IGNSC}$$

where $\mathbb{P}$ denotes the Helmholtz projection onto the divergence-free vector fields and $S_\Omega(\cdot)$ denotes the semigroup corresponding to the linear problem of (1.1), which is given explicitly by

$$S_\Omega(t)u_0 = \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^{2\alpha}}I\widehat{u_0} + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^{2\alpha}}R(\xi)\widehat{u_0}\right],$$

for $t \geqslant 0$ and divergence-free vector fields u_0. Here I is the identity matrix on $\mathbb{R}^3$ and $R(\xi)$ is the skew-symmetric matrix symbol on $\mathbb{R}^3$, which is defined by

$$R(\xi) = \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^3\setminus\{0\}.$$

For the derivation of the explicit form of $S_\Omega(\cdot)$, we refer to Babin, Mahalov and Nicolaenko [1] [2], Giga, Inui, Mahalov and Matsui [10], Hieber and Shibata [13] and Iwabuchi and Takada [15]. We call that u is a mild solution to (1.1) if u satisfies (IGNSC) in some appropriate function space.

This paper is organized as follows. In Section 3, we recall some results concerning the generalized Stokes-Coriolis semigroup's regularizing effect. Section 4 is devoted to the important bilinear estimates. In Section 5, we prove Theorem 2.3 and Theorem 2.4.

3. Regularizing effect of the generalized Stokes-Coriolis semigroup

We consider the linear system

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + \Omega e_3\times u + \nabla p = 0, & x\in\mathbb{R}^3,\ t>0, \\ \operatorname{div} u = 0, & x\in\mathbb{R}^3,\ t>0, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^3. \end{cases} \tag{3.1}$$

From the Proposition 2.1 of [13], we know that

$$\hat{u}(t,\xi) = \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^{2\alpha}}I\widehat{u_0}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^{2\alpha}}R(\xi)\widehat{u_0}(\xi), \tag{3.2}$$

for $t\geqslant 0$ and $\xi\in\mathbb{R}^3$. The generalized Stokes-Coriolis semigroup is explicitly represented by

$$S_\Omega(t)u = [\cos(\Omega R_3 t)I + \sin(\Omega R_3 t)R]e^{-\nu t(-\Delta)^\alpha}u, \text{ for } t\geqslant 0, u\in L^p, \text{and } \operatorname{div} u = 0, \tag{3.3}$$

where $\widehat{R_3 u}(\xi) = (\xi_3/|\xi|)\hat{u}(\xi)$ for $\xi\neq 0$.

Theorem 3.1. (smoothing effect of the generalized Stokes-Coriolis semigroup) *Suppose that* $\alpha > 1/2$, $\mathscr{C} := \{\xi \in \mathbb{R}^3 : 1/2 \leqslant |\xi| < 2\}$ *be a ring centered at* 0 *in* $\mathbb{R}^3$, $u = (u^1, u^2, u^3)$. *If* $\operatorname{supp} \hat{u^i} \subset \lambda\mathscr{C}, i = 1, 2, 3$, *then there exist positive constants* c *and* C *depending only on* ν *such that:*

(i) *for any* $\lambda > 0$,

$$\|S_\Omega(t)u\|_{L^2} \leqslant Ce^{-c\lambda^{2\alpha}t}\|u\|_{L^2}; \tag{3.4}$$

(ii) *if* $\lambda \gtrsim \Omega$, *then, for any* $1 \leqslant p \leqslant \infty$,

$$\|S_\Omega(t)u\|_{L^p} \leqslant Ce^{-c\lambda^{2\alpha}t}\|u\|_{L^p}. \tag{3.5}$$

Proof. (i) Since $\|u\|_p := (\|u^1\|_p^2 + \|u^2\|_p^2 + \|u^3\|_p^2)^{1/2}$, it suffices to prove that the conclusion holds for scalar function f. Thanks to (3.2) and the Plancherel theorem, we get

$$\|S_\Omega(t)f\|_{L^2} = \|\widehat{S_\Omega}(t,\xi)\hat{f}(\xi)\|_{L^2} \leqslant C\|e^{-\nu|\xi|^{2\alpha}t}\hat{f}(\xi)\|_2 \leqslant Ce^{-c\lambda^{2\alpha}t}\|f\|_{L^2}.$$

(ii) Let $\phi \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$ be a nonnegative radial function, $\operatorname{supp}\phi \subset \{\xi \in \mathbb{R}^3 : 1/4 \leqslant |\xi| < 4\}$, $\phi(\xi) = 1$ for all $1/2 \leqslant |\xi| < 2$. Set

$$g(t,x) := (2\pi)^{-3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\phi(\lambda^{-1}\xi)\widehat{S_\Omega}(t,\xi)\,\mathrm{d}\xi.$$

To prove (3.5), it suffices to show

$$\|g(t,x)\|_{L^1} \leqslant Ce^{-c\lambda^{2\alpha}t}. \tag{3.6}$$

Thanks to (3.3), we infer that

$$\begin{aligned} &\int_{|x|\leqslant\lambda^{-1}} |g(x,t)|\,\mathrm{d}x \\ \leqslant\ & C\int_{|x|\leqslant\lambda^{-1}}\int_{\mathbb{R}^3} |\phi(\lambda^{-1}\xi)||\widehat{S_\Omega}(t,\xi)|\,\mathrm{d}\xi\,\mathrm{d}x \leqslant Ce^{-c\lambda^{2\alpha}t}\int_{|x|\leqslant\lambda^{-1}}\lambda^3\,\mathrm{d}\lambda \leqslant Ce^{-c\lambda^{2\alpha}t}. \end{aligned} \tag{3.7}$$

Set $L := x\cdot\nabla_\xi/(i|x|^2)$. Noting that $L(e^{ix\cdot\xi}) = e^{ix\cdot\xi}$. Using integration by parts, we have

$$g(x,t) = C\int_{\mathbb{R}^3} L^N(e^{ix\cdot\xi})\phi(\lambda^{-1}\xi)\widehat{S_\Omega}(t,\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^3} e^{ix\cdot\xi}(L^*)^N(\phi(\lambda^{-1}\xi)\widehat{S_\Omega}(t,\xi))\,\mathrm{d}\xi,$$

where $N \in \mathbb{N}$ is chosen later. Using the Leibnitz formula, it is easy to verify that

$$|\partial^\gamma(e^{\pm i\Omega\frac{\xi_3}{|\xi|}t})| \leqslant C|\xi|^{-|\gamma|}(1+\Omega t)^{|\gamma|}, \quad |\partial^\gamma(e^{-\nu|\xi|^{2\alpha}t})| \leqslant C|\xi|^{-|\gamma|}e^{-\frac{|\gamma|}{2}|\xi|^{2\alpha}t},$$

$$\left|\partial^\gamma\left(\frac{\xi_i}{|\xi|}\right)\right| \leqslant C|\xi|^{-\gamma}. \quad i = 1,2,3.$$

Thus we obtain

$$
\begin{aligned}
&|(L^*)^N(\phi(\lambda^{-1}\xi)\widehat{S_\Omega}(t,\xi))|\\
\leqslant\ & C|x|^{-N}\sum_{|k_1|+|k_2|+|k_3|=|k|\leqslant N}\lambda^{-N+|k|}\left|(\nabla^{N-|k|}\phi)(\frac{\xi}{\lambda})\partial^{k_1}(e^{\pm i\Omega\frac{\xi_3}{|\xi|}t})\partial^{k_2}(e^{-\nu|\xi|^{2\alpha}t})\partial^{k_3}(I+R(\xi))\right|\\
\leqslant\ & C|\lambda x|^{-N}\sum_{|k_1|+|k_2|+|k_3|=|k|\leqslant N}\lambda^{|k|}|(\nabla^{N-|k|}\phi)(\lambda^{-1}\xi)||\xi|^{-|k_1|-|k_2|-|k_3|}e^{-\frac{\nu}{2}|\xi|^{2\alpha}t}(1+\Omega t)^{|k_1|}.
\end{aligned}
$$

Taking $N=4$. For any $\xi\in\{\xi:2^{-1}\lambda\leqslant|\xi|\leqslant2\lambda\}$ and $\lambda\gtrsim\Omega$, we obtain

$$|(L^*)^4(\phi(\lambda^{-1}\xi)\widehat{S_\Omega}(t,\xi))|\leqslant C|\lambda x|^{-4}e^{-\frac{\nu}{4}|\xi|^{2\alpha}t},$$

which implies that

$$\int_{|x|\geqslant\frac{1}{\lambda}}|g(x,t)|\,\mathrm{d}x\leqslant Ce^{-c\lambda^{2\alpha}t}\lambda^3\int_{|x|\geqslant\frac{1}{\lambda}}|\lambda x|^{-4}\,\mathrm{d}x\leqslant Ce^{-c\lambda^{2\alpha}t},$$

which, together with (3.7), we finally have (3.6). Then the inequality (3.5) is proved. □

We now turn to time-space norm estimation of hybrid-Besov spaces to the generalized Stokes-Coriolis semigroup

Theorem 3.2. *Let $\alpha>0,s,\sigma\in\mathbb{R}$, and $(p,q)\in[1,\infty]$. Then, for any $u(x)\in\dot{\mathscr{B}}_{2,p}^{s-\frac{2\alpha}{q},\sigma-\frac{2\alpha}{q}}$, we have*

$$\|S_\Omega(t)u\|_{\tilde{L}^q_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})}\leqslant C\|u\|_{\dot{\mathscr{B}}_{2,p}^{s-\frac{2\alpha}{q},\sigma-\frac{2\alpha}{q}}},\tag{3.8}$$

and for any $u(t,x)\in\tilde{L}^1_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})$, we have

$$\left\|\int_0^t S_\Omega(t-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{\tilde{L}^q_T\left(\dot{\mathscr{B}}_{2,p}^{s+\frac{2\alpha}{q},\sigma+\frac{2\alpha}{q}}\right)}\leqslant C\|u(t)\|_{\tilde{L}^1_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})}.\tag{3.9}$$

Proof. Here we only prove (3.9). (3.8) can be proved by the method analogous to (3.9). For any $2^j\geqslant\Omega$, we get by Theorem 3.1 that

$$\left\|\Delta_j\int_0^t S_\Omega(t-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{L^p}\leqslant C\int_0^t e^{-c(t-\tau)2^{2\alpha j}}\|\Delta_ju(\tau)\|_{L^p}\,\mathrm{d}\tau.$$

Using Young's inequality, we have

$$\left\|\Delta_j\int_0^t S_\Omega(t-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{L^q_TL^p}\leqslant C\|e^{-ct2^{2\alpha j}}\|_{L^q_T}\|\Delta_ju(\tau)\|_{L^1_TL^p}\leqslant C2^{-\frac{2\alpha}{q}j}\|\Delta_ju(\tau)\|_{L^1_TL^p}.\tag{3.10}$$

Similarly, for $2^j<\Omega$, we also have

$$\left\|\Delta_j\int_0^t S_\Omega(t-\tau)u(\tau)\,\mathrm{d}\tau\right\|_{L^q_TL^2}\leqslant C\|e^{-ct2^{2\alpha j}}\|_{L^q_T}\|\Delta_ju(\tau)\|_{L^1_TL^2}\leqslant C2^{-\frac{2\alpha}{q}j}\|\Delta_ju(\tau)\|_{L^1_TL^2}.\tag{3.11}$$

Then the inequality (3.9) follows from (3.10) and (3.11). □

4. Bilinear estimates

Let us first define the space $E_{p,\alpha,T}$ whose norm is defined by

$$\|u\|_{E_{p,\alpha,T}} := \|u\|_{\tilde{L}^\infty\left(0,T;\dot{\mathscr{B}}_{2,p}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}\right)} + \|u\|_{\tilde{L}^1\left(0,T;\dot{\mathscr{B}}_{2,p}^{\frac{5}{2},\frac{3}{p}+1}\right)}.$$

Set

$$B(u,v) := \int_0^t S_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes v)\,\mathrm{d}\tau,$$

where $\mathbb{P}$ denotes the Helmholtz projection.

Theorem 4.1. *Let $1/2<\alpha<5/2$, $2\leqslant p\leqslant 4$ and $\frac{1}{p}>\max\left\{\frac{\alpha-1}{3},\frac{5-4\alpha}{6}\right\}$. Assume that $u,v\in E_{p,\alpha,T}$. There exists a constant C independent of Ω, such that for any $T>0$,*

$$\|B(u,v)\|_{E_{p,\alpha,T}} \leqslant C\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}}. \tag{4.1}$$

Proof. Due to Theorem 3.2, we need only prove

$$\|uv\|_{\tilde{L}^1_T\dot{\mathscr{B}}_{2,p}^{\frac{7}{2}-2\alpha,\,\frac{3}{p}-2\alpha+2}} \leqslant C\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}}. \tag{4.2}$$

Using Bony's decomposition (2.4) and (2.5), we get

$$\begin{aligned}\Delta_j(uv) &= \sum_{|k-j|\leqslant 4}\Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\leqslant 4}\Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\geqslant j-2}\Delta_j(\Delta_k u\tilde{\Delta}_k v)\\ &=: I_j + II_j + III_j.\end{aligned}$$

Put $J_j := \{(k',k) : |k-j|\leqslant 4, k'\leqslant k-2\}$. Then for $2^j>\Omega$,

$$\begin{aligned}\|I_j\|_{L^1_TL^p} &\leqslant \sum_{J_j}\|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_TL^p}\\ &\leqslant \left(\sum_{J_{j,ll}}+\sum_{J_{j,lh}}+\sum_{J_{j,hh}}\right)\|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_TL^p} := I_{j,1}+I_{j,2}+I_{j,3},\end{aligned}$$

where

$$J_{j,ll} = \{(k',k)\in J_j : 2^{k'}\leqslant\Omega, 2^k\leqslant\Omega\},$$

$$J_{j,lh} = \{(k',k)\in J_j : 2^{k'}\leqslant\Omega, 2^k>\Omega\},$$

$$J_{j,hh} = \{(k',k)\in J_j : 2^{k'}>\Omega, 2^k>\Omega\}.$$

Using Lemma 2.1, we obtain

$$
\begin{aligned}
I_{j,1} &\leqslant C\sum_{(k',\,k)\in J_{j,ll}}\|\Delta_{k'}u\|_{L^\infty_T L^\infty}2^{k(\frac{3}{2}-\frac{3}{p})}\|\Delta_k v\|_{L^1_T L^2}\\
&\leqslant C\sum_{(k',\,k)\in J_{j,ll}}2^{k'\left(\frac{5}{2}-2\alpha\right)}\|\Delta_{k'}u\|_{L^\infty_T L^2}2^{k'(2\alpha-1)}2^{\frac{5}{2}k}\|\Delta_k v\|_{L^1_T L^2}2^{-k\left(1+\frac{3}{p}\right)}\\
&\leqslant C\|u\|_{\tilde L^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde L^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\sum_{(k',\,k)\in J_{j,ll}}2^{(k'-k)(2\alpha-1)}2^{-k(\frac{3}{p}-2\alpha+2)}\\
&\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{\tilde L^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde L^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}},
\end{aligned}
$$

where we used in the last inequality the fact that

$$
\sum_{(k',\,k)\in J_{j,ll}}2^{(k'-k)(2\alpha-1)}2^{-k(\frac{3}{p}-2\alpha+2)}\leqslant\sum_{|k-j|\leqslant 4}2^{-k(\frac{3}{p}-2\alpha+2)}\sum_{k'\leqslant k-2}2^{(k'-k)(2\alpha-1)}\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)},
$$

with C independent of j. A similar argument, we get

$$
\begin{aligned}
I_{j,2} &\leqslant C\sum_{(k',\,k)\in J_{j,lh}}\|\Delta_{k'}u\|_{L^\infty_T L^\infty}\|\Delta_k v\|_{L^1_T L^p}\\
&\leqslant C\sum_{(k',\,k)\in J_{j,lh}}2^{k'\left(\frac{5}{2}-2\alpha\right)}\|\Delta_{k'}u\|_{L^\infty_T L^2}2^{k'(2\alpha-1)}\|\Delta_k v\|_{L^1_T L^p}\\
&\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{\tilde L^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde L^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{j,\,3} &\leqslant C\sum_{(k',\,k)\in J_{j,\,hh}}\|\Delta_{k'}u\|_{L^\infty_T L^\infty}\|\Delta_k v\|_{L^1_T L^p}\\
&\leqslant C\sum_{(k',\,k)\in J_{j,\,hh}}2^{k'\left(\frac{3}{p}-2\alpha+1\right)}\|\Delta_{k'}u\|_{L^\infty_T L^p}2^{k'(2\alpha-1)}\|\Delta_k v\|_{L^1_T L^p}\\
&\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{\tilde L^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde L^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}.
\end{aligned}
$$

For $2^j\leqslant\Omega$, we have

$$
\begin{aligned}
\|I_j\|_{L^1_T L^2} &\leqslant \sum_{J_j}\|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_T L^2}\\
&\leqslant \left(\sum_{J_{j,ll}}+\sum_{J_{j,lh}}+\sum_{J_{j,hh}}\right)\|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_T L^2}:=I_{j,4}+I_{j,5}+I_{j,6}.
\end{aligned}
$$

Using the same argument as above, we obtain

$$\begin{aligned} I_{j,4} &\leqslant C \sum_{(k',k)\in J_{j,ll}} 2^{k'(\frac{5}{2}-2\alpha)} \|\Delta_{k'}u\|_{L_T^\infty L^2} 2^{k'(2\alpha-1)} 2^{\frac{5k}{2}} \|\Delta_k v\|_{L_T^1 L^2} 2^{-\frac{5k}{2}} \\ &\leqslant C 2^{-j(\frac{7}{2}-2\alpha)} \|u\|_{\tilde{L}_T^\infty \dot{\mathscr{B}}_{2,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}} \|v\|_{\tilde{L}_T^1 \dot{\mathscr{B}}_{2,p}^{\frac{5}{2},\frac{3}{p}+1}}. \end{aligned}$$

Noting that $2 \leqslant p \leqslant 4$ and $\frac{1}{p} > \frac{5-4\alpha}{6}$, we have

$$\begin{aligned} I_{j,5} &\leqslant C \sum_{(k',k)\in J_{j,lh}} \|\Delta_{k'}u\|_{L_T^\infty L^{\frac{2p}{p-2}}} \|\Delta_k v\|_{L_T^1 L^p} \\ &\leqslant C \sum_{(k',k)\in J_{j,lh}} 2^{k'(\frac{5}{2}-2\alpha)} \|\Delta_{k'}u\|_{L_T^\infty L^2} 2^{k'(\frac{3}{p}-\frac{5}{2}+2\alpha)} \|\Delta_k v\|_{L_T^1 L^p} \\ &\leqslant C 2^{-j(\frac{7}{2}-2\alpha)} \|u\|_{\tilde{L}_T^\infty \dot{\mathscr{B}}_{2,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}} \|v\|_{\tilde{L}_T^1 \dot{\mathscr{B}}_{2,p}^{\frac{5}{2},\frac{3}{p}+1}}, \end{aligned}$$

and

$$\begin{aligned} I_{j,6} &\leqslant C \sum_{(k',k)\in J_{j,hh}} \|\Delta_{k'}u\|_{L_T^\infty L^{\frac{2p}{p-2}}} \|\Delta_k v\|_{L_T^1 L^p} \\ &\leqslant C \sum_{(k',k)\in J_{j,hh}} 2^{k'(\frac{3}{p}-2\alpha+1)} \|\Delta_{k'}u\|_{L_T^\infty L^p} 2^{k'(\frac{3}{p}+2\alpha-\frac{5}{2})} \|\Delta_k v\|_{L_T^1 L^p} \\ &\leqslant C 2^{-j(\frac{7}{2}-2\alpha)} \|u\|_{\tilde{L}_T^\infty \dot{\mathscr{B}}_{2,p}^{\frac{5}{2}-2\alpha,\, \frac{3}{p}-2\alpha+1}} \|v\|_{\tilde{L}_T^1 \dot{\mathscr{B}}_{2,p}^{\frac{5}{2},\frac{3}{p}+1}}. \end{aligned}$$

Summing up the estimates for $I_{j,1}$ through $I_{j,6}$ yields that

$$\sup_{2^j>\Omega} 2^{j(\frac{3}{p}-2\alpha+2)} \|I_j\|_{L_T^1 L^p} + \sup_{2^j<\Omega} 2^{j(\frac{7}{2}-2\alpha)} \|I_j\|_{L_T^1 L^2} \leqslant C\|u\|_{E_{p,\alpha,T}} \|v\|_{E_{p,\alpha,T}}. \tag{4.3}$$

Using the same argument as in the proof of (4.3), we can easily obtain

$$\sup_{2^j>\Omega} 2^{j(\frac{3}{p}-2\alpha+2)} \|II_j\|_{L_T^1 L^p} + \sup_{2^j<\Omega} 2^{j(\frac{7}{2}-2\alpha)} \|II_j\|_{L_T^1 L^2} \leqslant C\|u\|_{E_{p,\alpha,T}} \|v\|_{E_{p,\alpha,T}}. \tag{4.4}$$

Put $K_j := \{(k,\, k') : k \geqslant j-3, |k-k'| \leqslant 1\}$. We have

$$III_j = \left(\sum_{K_{j,ll}} + \sum_{K_{j,lh}} + \sum_{K_{j,hl}} + \sum_{K_{j,hh}} \right) \Delta_j(\Delta_k u \Delta_{k'} v) := III_{j,1} + III_{j,2} + III_{j,3} + III_{j,4},$$

where

$$\begin{aligned} K_{j,ll} &= \{(k,\, k') \in K_j : 2^k \leqslant \Omega, 2^{k'} \leqslant \Omega\}, \\ K_{j,lh} &= \{(k,\, k') \in K_j : 2^k \leqslant \Omega, 2^{k'} > \Omega\}, \\ K_{j,hl} &= \{(k,\, k') \in K_j : 2^k > \Omega, 2^{k'} \leqslant \Omega\}, \\ K_{j,hh} &= \{(k,\, k') \in K_j : 2^k > \Omega, 2^{k'} > \Omega\}. \end{aligned}$$

Noting that $1/2<\alpha<5/2$, we get by Lemma 2.1 that

$$\begin{aligned}
\|III_{j,1}\|_{L^1_TL^p} &\leqslant C2^{3j(1-\frac{1}{p})}\sum_{(k,k')\in K_{j,ll}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^1}\\
&\leqslant C2^{3j(1-\frac{1}{p})}\sum_{(k,k')\in K_{j,ll}}2^{k(\frac{5}{2}-2\alpha)}\|\Delta_k u\|_{L^\infty_TL^2}2^{-k(\frac{5}{2}-2\alpha)}2^{k'\frac{5}{2}}\|\Delta_{k'}v\|_{L^1_TL^2}2^{-k'\frac{5}{2}}\\
&\leqslant C2^{3j(1-\frac{1}{p})}\|u\|_{\tilde{L}^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde{L}^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\sum_{(k,k')\in K_{j,ll}}2^{-k(\frac{5}{2}-2\alpha)-\frac{5k'}{2}}\\
&\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{\tilde{L}^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde{L}^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\sum_{k\geqslant j-3}2^{(2\alpha-5)(k-j)}\\
&\leqslant C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{E_{p,\,\alpha,T}}\|v\|_{E_{p,\,\alpha,T}},
\end{aligned}$$

and

$$\begin{aligned}
\|III_{j,1}\|_{L^1_TL^2} &\leqslant C2^{\frac{3j}{2}}\sum_{(k,k')\in K_{j,ll}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^1}\\
&\leqslant C2^{\frac{3j}{2}}\sum_{(k,k')\in K_{j,ll}}2^{k(\frac{5}{2}-2\alpha)}\|\Delta_k u\|_{L^\infty_TL^2}2^{-k(\frac{5}{2}-2\alpha)}2^{k'\frac{5}{2}}\|\Delta_{k'}v\|_{L^1_TL^2}2^{-k'\frac{5}{2}}\\
&\leqslant C2^{\frac{3j}{2}}\|u\|_{\tilde{L}^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde{L}^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\sum_{(k,k')\in K_{j,ll}}2^{-k(\frac{5}{2}-2\alpha)-\frac{5k'}{2}}\\
&\leqslant C2^{-j(\frac{7}{2}-2\alpha)}\|u\|_{\tilde{L}^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde{L}^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\sum_{k\geqslant j-3}2^{(2\alpha-5)(k-j)}\\
&\leqslant C2^{-j(\frac{7}{2}-2\alpha)}\|u\|_{E_{p,\,\alpha,T}}\|v\|_{E_{p,\,\alpha,T}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|III_{j,2}+III_{j,3}\|_{L^1_TL^p}\\
\leqslant\ & C2^{\frac{3j}{2}}\sum_{(k,k')\in K_{j,lh}\cup K_{j,hl}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{2p}{2+p}}}\\
\leqslant\ & C2^{\frac{3j}{2}}\left(\sum_{(k,k')\in K_{j,lh}}\|\Delta_k u\|_{L^\infty_TL^2}\|\Delta_{k'}v\|_{L^1_TL^p}+\sum_{(k,k')\in K_{j,hl}}\|\Delta_k u\|_{L^1_TL^p}\|\Delta_{k'}v\|_{L^\infty_TL^2}\right)\\
\leqslant\ & C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{\tilde{L}^\infty_T\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\|v\|_{\tilde{L}^1_T\dot{\mathscr{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,\,p}}\left(2\times\sum_{k\geqslant j-3}2^{(j-k)(\frac{7}{2}-2\alpha+\frac{3}{p})}\right)\\
\leqslant\ & C2^{-j(\frac{3}{p}-2\alpha+2)}\|u\|_{E_{p,\,\alpha,T}}\|v\|_{E_{p,\,\alpha,T}},
\end{aligned}$$

and

$$\begin{aligned}
&\|III_{j,2}+III_{j,3}\|_{L^1_TL^2}\\
\leqslant\ & C2^{\frac{3j}{p}}\sum_{(k,k')\in K_{j,lh}\cup K_{j,hl}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{2p}{2+p}}}\\
\leqslant\ & C2^{\frac{3j}{p}}\left(\sum_{(k,k')\in K_{j,lh}}\|\Delta_k u\|_{L^\infty_TL^2}\|\Delta_{k'}v\|_{L^1_TL^p}+\sum_{(k,k')\in K_{j,hl}}\|\Delta_k u\|_{L^1_TL^p}\|\Delta_{k'}v\|_{L^\infty_TL^2}\right)\\
\leqslant\ & C2^{-j(\frac72-2\alpha)}\|u\|_{\tilde L^\infty_T\dot{\mathscr B}^{\frac52-2\alpha,\,\frac3p-2\alpha+1}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathscr B}^{\frac52,\frac3p+1}_{2,p}}\left(2\times\sum_{k\geqslant j-3}2^{(j-k)(\frac72-2\alpha+\frac3p)}\right)\\
\leqslant\ & C2^{-j(\frac72-2\alpha)}\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}}.
\end{aligned}$$

Finally, thanks to $2\leqslant p\leqslant 4$, it follows that

$$\begin{aligned}
&\|III_{j,4}\|_{L^1_TL^p}\\
\leqslant\ & C2^{\frac{3j}{p}}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac p2}}\\
\leqslant\ & C2^{\frac{3j}{p}}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\|_{L^\infty_TL^p}\|\Delta_{k'}v\|_{L^1_TL^p}\\
\leqslant\ & C2^{-j(\frac3p-2\alpha+2)}\|u\|_{\tilde L^\infty_T\dot{\mathscr B}^{\frac52-2\alpha,\,\frac3p-2\alpha+1}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathscr B}^{\frac52,\frac3p+1}_{2,p}}\sum_{k\geqslant j-3}2^{(j-k)(2-2\alpha+\frac6p)}\\
\leqslant\ & C2^{-j(\frac3p-2\alpha+2)}\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}},
\end{aligned}$$

and

$$\begin{aligned}
&\|III_{j,4}\|_{L^1_TL^2}\\
\leqslant\ & C2^{3j(\frac2p-\frac12)}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac p2}}\\
\leqslant\ & C2^{3j(\frac2p-\frac12)}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\|_{L^\infty_TL^p}\|\Delta_{k'}v\|_{L^1_TL^p}\\
\leqslant\ & C2^{-j(\frac72-2\alpha)}\|u\|_{\tilde L^\infty_T\dot{\mathscr B}^{\frac52-2\alpha,\,\frac3p-2\alpha+1}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathscr B}^{\frac52,\frac3p+1}_{2,p}}\sum_{k\geqslant j-3}2^{(j-k)(2-2\alpha+\frac6p)}\\
\leqslant\ & C2^{-j(\frac72-2\alpha)}\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}}.
\end{aligned}$$

Taking the estimates of $III_{j,1}-III_{j,4}$ into cosideration, we obtain

$$\sup_{2^j>\Omega}2^{j(\frac3p-2\alpha+2)}\|III_j\|_{L^1_TL^p}+\sup_{2^j<\Omega}2^{j(\frac72-2\alpha)}\|III_j\|_{L^1_TL^2}\leqslant C\|u\|_{E_{p,\alpha,T}}\|v\|_{E_{p,\alpha,T}}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we are confident that the inequality (4.2) holds. The proof is completed. ☐

In order to obtain the uniqueness of the solution in $C(\mathbb{R}^+;\dot{H}^{\frac{5}{2}-2\alpha})$, we need consider the following new bilinear estimate in the weighted time-space Besov spaces.

Theorem 4.2. *Assume that* $\frac{1}{2}<\alpha<\frac{5}{4}$ *and* $u,v\in\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}$. *Then, for any* $T>0$, *we have*

$$\|B(u,v)\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\leqslant C\|u\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\big\|\omega_{j,T}2^{j(\frac{5}{2}-2\alpha)}\|\Delta_j v\|_{L^\infty_T L^2}\big\|_{l^\infty},$$

where

$$\omega_{j,T}:=\sup_{k\geqslant j}e_{k,T}2^{(\frac{5}{2}-2\alpha)(j-k)},\quad e_{j,T}:=1-e^{-c2^{2j\alpha}T}.$$

Proof. Since $e_{j,T}\leqslant\omega_{j,T}$ for any $j\in\mathbb{Z}$ and that

$$\omega_{j,T}\leqslant 2^{(\frac{5}{2}-2\alpha)(j-j')}\omega_{j',T}\ \text{ if } j'\leqslant j,\quad \omega_{j,T}\leqslant 2\omega_{j',T}\ \text{ if } j\leqslant j'. \tag{4.6}$$

From Theorem 3.1, thus we conclude that

$$\begin{aligned}
\|B(u,\,v)\|_{\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}} &\leqslant \sup_{j\in\mathbb{Z}}2^{j(\frac{5}{2}-2\alpha)}\int_0^t\|S_\Omega(t-\tau)\Delta_j\mathbb{P}\nabla\cdot(u\otimes v)(\tau)\|_{L^2}\,\mathrm{d}\tau\\
&\leqslant \sup_{j\in\mathbb{Z}}2^{j(\frac{7}{2}-2\alpha)}\big\|e^{-c2^{2j\alpha}t}\big\|_{L^1_T}\|\Delta_j(u\otimes v)\|_{L^\infty_T L^2}\\
&\leqslant C\sup_{j\in\mathbb{Z}}2^{j(\frac{7}{2}-4\alpha)}e_{j,T}\|\Delta_j(uv)\|_{L^\infty_T L^2}.
\end{aligned}\tag{4.7}$$

Thanks to Bony's decomposition to estimate $\|\Delta_j(uv)\|_{L^\infty_T L^2}$ and (4.6), we infer that

$$\begin{aligned}
&\sum_{|k-j|\leqslant 4}\|\Delta_j(S_{k-1}u\Delta_k v)\|_{L^\infty_T L^2}\\
\leqslant\ & C\|u\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\sum_{|k-j|\leqslant 4}2^{k(2\alpha-1)}\|\Delta_k v\|_{L^\infty_T L^2}\\
\leqslant\ & C\omega^{-1}_{j,T}2^{j(4\alpha-\frac{7}{2})}\|u\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\big\|\omega_{k,T}2^{k(\frac{5}{2}-2\alpha)}\|\Delta_k v\|_{L^\infty_T L^2}\big\|_{l^\infty}
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
\|S_{k-1}v\|_{L^\infty}&\leqslant\sum_{k'\leqslant k-2}\|\Delta_{k'}v\|_{L^2}2^{\frac{3}{2}k'}\leqslant\|\omega_{k',T}2^{k'(\frac{5}{2}-2\alpha)}\|\Delta_{k'}v\|_{L^\infty_T L^2}\|_{l^\infty}\sum_{k'\leqslant k-2}2^{k'(2\alpha-1)}\omega^{-1}_{k'T}\\
&\leqslant 2^{k(2\alpha-1)}\omega^{-1}_{k,T}\|\omega_{k',T}2^{k'(\frac{5}{2}-2\alpha)}\|\Delta_{k'}v\|_{L^\infty_T L^2}\|_{l^\infty},
\end{aligned}$$

it follows that

$$
\begin{aligned}
&\sum_{|k-j|\leqslant 4}\|\Delta_j(S_{k-1}v\Delta_k u)\|_{L^\infty_T L^2}\\
\leqslant\;& C\omega_{j,T}^{-1}2^{j(4\alpha-\frac{7}{2})}\|u\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\left\|\omega_{k',T}2^{k'(\frac{5}{2}-2\alpha)}\|\Delta_{k'}v\|_{L^\infty_T L^2}\right\|_{l^\infty}.
\end{aligned}\tag{4.9}
$$

As the remainder term,

$$
\begin{aligned}
&\sum_{k\geqslant j-2}\|\Delta_j(\Delta_k u\tilde{\Delta}_k v)\|_{L^\infty_T L^2}\\
\leqslant\;& \sum_{k\geqslant j-2}2^{\frac{3}{2}j}\left\|\Delta_k u\tilde{\Delta}_k v\right\|_{L^\infty_T L^1}\\
\leqslant\;& C\sum_{k\geqslant j-2}2^{\frac{3}{2}j}\|\Delta_k u\|_{L^\infty_T L^2}\left\|\tilde{\Delta}_k v\right\|_{L^\infty_T L^2}\\
\leqslant\;& C\omega_{j,T}^{-1}2^{j(4\alpha-\frac{7}{2})}\|u\|_{L^\infty_T\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}}\left\|\omega_{k,T}2^{k(\frac{5}{2}-2\alpha)}\|\Delta_k v\|_{L^\infty_T L^2}\right\|_{l^\infty}.
\end{aligned}\tag{4.10}
$$

Combining (4.8), (4.9), (4.10) and (4.7), which completes the proof of Theorem 4.2. □

5. Proofs of Theorem 2.3 and Theorem 2.4

The trick of the proof to Theorem 2.3 is the following classical lemma.

Lemma 5.1. ([3]) *Let $(X,\|\cdot\|_X)$ be a Banach space and $B: X\times X\to X$ a bounded bilinear form satisfying $\|B(x_1,x_2)\|_X\leqslant\eta\|x_1\|_X\|x_2\|_X$ for all $x_1,x_2\in X$ and a constant $\eta>0$. Then if $0<\epsilon<1/(4\eta)$ and if $y\in X$ such that $\|y\|_X<\epsilon$, the equation $x=y+B(x,x)$ has a solution in X such that $\|x\|_X\leqslant 2\epsilon$. This solution is the only one in the ball $\bar{B}(0,2\epsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_X\leqslant\epsilon, \tilde{x}=\tilde{y}+B(\tilde{x},\tilde{x})$, and $\|\tilde{x}\|_X\leqslant 2\epsilon$ then*

$$\|x-\tilde{x}\|_X\leqslant\frac{1}{1-4\eta\epsilon}\|y-\tilde{y}\|_X.$$

Proof of Theorem 2.3. Noting that the integral form to (1.1)

$$u(x,t)=S_\Omega(t)u_0-\int_0^t S_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)\,\mathrm{d}\tau:=S_\Omega(t)u_0+B(u,u).\tag{5.1}$$

Thanks to Theorem 3.2, we infer that

$$\|S_\Omega u_0\|_{E_{p,\alpha}}\leqslant C\|u_0\|_{\dot{\mathscr{B}}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}_{2,\,p}}\leqslant Cc.$$

Clearly, $B(u,v)$ is bilinear, and we obtain by Theorem 4.1 that

$$\|B(u,v)\|_{E_{p,\alpha}}\leqslant C\|u\|_{E_{p,\alpha}}\|v\|_{E_{p,\alpha}}.$$

Set c such that $4C^2c < \frac{3}{4}$, Lemma 5.1 ensures that the equation

$$u = S_\Omega(t)u_0 + B(u,u)$$

has a unique solution in the ball $\{u \in E_{p,\alpha} : \|u\|_{E_{p,\alpha}} \leqslant \frac{1}{4C}\}$.

Proof of Theorem 2.4. We define a Banach space $F_{p,\alpha}$ whose norm is defined by

$$\|u\|_{F_{p,\alpha}} := \|u\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{5}{2}-2\alpha})} + \|u\|_{E_{p,\alpha}}.$$

Step 1: Existence in $F_{p,\alpha}$.

Let $\mathscr{T}u := S_\Omega(t)u_0 + B(u,u)$. Our purpose now is to prove that the map $\mathscr{T}$ has a unique fixed point in the ball

$$B_A := \{u \in F_{p,\alpha} : \|u\|_{E_{p,\alpha}} \leqslant Ac, \|u\|_{F_{p,\alpha}} \leqslant A\|u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}}\}$$

if c is small enough for some $A > 0$ to be determined later. Using Theorem 3.2 and Theorem 4.1, we get

$$\|\mathscr{T}u\|_{E_{p,\alpha}} \leqslant C\|u_0\|_{\dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}} + C\|u\|^2_{E_{p,\alpha}}. \tag{5.2}$$

On the other hand, we get by Theorem 3.1 that

$$\begin{aligned}
&\|B(u,u)\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{5}{2}-2\alpha})}\\
\leqslant\ & \left\| \int_0^t S_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\,\mathrm{d}\tau \right\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{5}{2}-2\alpha})}\\
\leqslant\ & C\left(\sum_{j\in\mathbb{Z}} 2^{j(5-4\alpha)}\left(\sup_{t\in\mathbb{R}^+}\int_0^t \|S_\Omega(t-\tau)\Delta_j\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^2}\,\mathrm{d}\tau\right)^2\right)^{\frac{1}{2}}\\
\leqslant\ & C\left\| 2^{j(\frac{7}{2}-2\alpha)} \sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j\alpha}(t-\tau)}\|\Delta_j(u\otimes u)\|_{L^2}\,\mathrm{d}\tau\right\|_{l^2}.
\end{aligned}\tag{5.3}$$

We get by Lemma 2.1 that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j\alpha}(t-\tau)}\|\Delta_j(T_u u)\|_{L^2}\,\mathrm{d}\tau\\
\leqslant\ & \|e^{-\tilde{c}2^{2j\alpha}t}\|_{L^1(\mathbb{R}^+)} \sum_{|k-j|\leqslant 4} \|\Delta_j(S_{k-1}u\Delta_k u)\|_{L^\infty(\mathbb{R}^+;L^2)}\\
\leqslant\ & C2^{-2j\alpha}\sum_{|k-j|\leqslant 4}\|S_{k-1}u\|_{L^\infty(\mathbb{R}^+;L^\infty)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\\
\leqslant\ & C\|u\|_{\tilde{L}^\infty\left(\mathbb{R}^+;\dot{\mathscr{B}}_{2,\,p}^{\frac{5}{2}-2\alpha,\,\frac{3}{p}-2\alpha+1}\right)} 2^{-2j\alpha}\sum_{|k-j|\leqslant 4} 2^{k(2\alpha-1)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\\
\leqslant\ & C2^{-j(\frac{7}{2}-2\alpha)}\|u\|_{E_{p,\alpha}}\sum_{|k-j|\leqslant 4} 2^{(k-j)(4\alpha-\frac{7}{2})}2^{k(\frac{5}{2}-2\alpha)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}.
\end{aligned}\tag{5.4}$$

The remainder term of uv is estimated by

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j\alpha}(t-\tau)}\|\Delta_j R(u,u)\|_{L^2}\,\mathrm{d}\tau\\
\leqslant\ & \|e^{-\tilde{c}2^{2j\alpha}t}\|_{L^\infty(\mathbb{R}^+)}\sum_{k\geqslant j-2}\|\Delta_j(\Delta_k u\tilde{\Delta}_k u)\|_{L^1(\mathbb{R}^+;L^2)}\\
\leqslant\ & C\sum_{k\geqslant j-2}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\|\tilde{\Delta}_k u\|_{L^1(\mathbb{R}^+;L^\infty)}\\
\leqslant\ & C\|u\|_{\tilde{L}^1\left(\mathbb{R}^+;\dot{\mathscr{B}}^{\frac52,\,\frac3p+1}_{2,\,p}\right)}\sum_{k\geqslant j-2}2^{-k}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}.
\end{aligned}\tag{5.5}$$

Combining (5.4) and (5.5) with (5.3) yields that

$$\|B(u,u)\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac52-2\alpha})}\leqslant C\|u\|_{E_{p,\alpha}}\|u\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac52-2\alpha})},$$

where we used the fact that

$$\begin{aligned}
&\Big\|2^{j(\frac72-2\alpha)}\sum_{k\geqslant j-2}2^{-k}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\Big\|_{l^2}\\
=\ & \Big\|2^{j(\frac72-2\alpha)}\sum_{k\geqslant j-2}2^{-k(\frac72-2\alpha)}2^{k(\frac52-2\alpha)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\Big\|_{l^2}\\
=\ & \Big[\sum_j\big(\sum_{k\geqslant j-2}2^{-(k-j)(\frac72-2\alpha)}2^{k(\frac52-2\alpha)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\big)^2\Big]^{1/2}\\
=\ & \Big[\sum_j\big(\sum_{k'\geqslant -2}2^{-k'(\frac72-2\alpha)}2^{(j+k')(\frac52-2\alpha)}\|\Delta_{j+k'} u\|_{L^\infty(\mathbb{R}^+;L^2)}\big)^2\Big]^{1/2}\\
\leqslant\ & \sum_{k'\geqslant -2}\big(\sum_j 2^{2(j+k')(\frac52-2\alpha)}\|\Delta_{j+k'}u\|^2_{L^\infty(\mathbb{R}^+;L^2)}\big)^{1/2}2^{-k'(\frac72-2\alpha)}\\
\leqslant\ & \sum_{k'\geqslant -2}2^{-k'(\frac72-2\alpha)}\big(\sum_{j'}2^{2j'(\frac52-2\alpha)}\|\Delta_{j'}u\|^2_{L^\infty(\mathbb{R}^+;L^2)}\big)^{1/2}\\
\leqslant\ & C\|u\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac52-2\alpha})}.
\end{aligned}$$

It is straightforward to verify that

$$\|S_\Omega(t)u_0\|_{\tilde{L}^\infty_T\dot{H}^{\frac52-2\alpha}}\leqslant C\|u_0\|_{\dot{H}^{\frac52-2\alpha}}.$$

Hence by (5.2) and the estimate

$$\|u_0\|_{\dot{\mathscr{B}}^{\frac52-2\alpha,\,\frac3p-2\alpha+1}_{2,\,p}}\leqslant C\|u_0\|_{\dot{H}^{\frac52-2\alpha}},$$

we obtain

$$\|\mathscr{T}u\|_{F_{p,\alpha}}\leqslant C\|u_0\|_{\dot{H}^{\frac52-2\alpha}}+C\|u\|_{E_{p,\alpha}}\|u\|_{F_{p,\alpha}}.\tag{5.6}$$

Taking $A = 2C$ and $c > 0$ such that $2C^2c \leqslant \frac{1}{2}$, it follows from (5.2) and (5.6) that the map $\mathscr{T}$ is a map from B_A to B_A. Similarly, it can be proved that $\mathscr{T}$ is also a contraction in B_A. Thus, the Banach fixed point theorem ensures that the map $\mathscr{T}$ has a unique fixed point in B_A.

Step 2: Uniqueness in $C(\mathbb{R}^+; \dot{H}^{\frac{5}{2}-2\alpha})$.

Let u_1 and u_2 be two solutions of (1.1) in $C(\mathbb{R}^+; \dot{H}^{\frac{5}{2}-2\alpha})$ with the same initial value u_0. A routine computation

$$\begin{aligned} & u_1 - u_2 \\ = \ & B\left(u_1 - S_\Omega(t)u_0, u_1 - u_2\right) + B\left(S_\Omega(t)u_0, u_1 - u_2\right) \\ + \ & B\left(u_1 - u_2, u_2 - S_\Omega(t)u_0\right) + B\left(u_1 - u_2, S_\Omega(t)u_0\right). \end{aligned}$$

It follows by Theorem 4.2 that

$$\begin{aligned} & \sup_{t\in[0,T]} \|(u_1 - u_2)(t)\|_{\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}} \\ \leqslant \ & C \sup_{t\in[0,T]} \|(u_1 - u_2)(t)\|_{\dot{B}^{\frac{5}{2}-2\alpha}_{2,\infty}} \Big(\big\|\omega_{j,T} 2^{j(\frac{5}{2}-2\alpha)} \|\Delta_j u_0\|_2 \big\|_{l^\infty} \\ + \ & \sup_{t\in[0,T]} \|u_1(t) - S_\Omega(t)u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + \sup_{t\in[0,T]} \|u_2(t) - S_\Omega(t)u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \Big), \end{aligned} \tag{5.7}$$

where we used the fact $\omega_{j,T} \leqslant 1$ so that

$$\big\|\omega_{j,T} 2^{j(\frac{5}{2}-2\alpha)} \|\Delta_j u\|_{L^\infty_T L^2} \big\|_{l^\infty} \leqslant \sup_{t\in[0,T]} \|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}}.$$

Because of the fact that $\omega_{j,0} = 0$ and $u_0 \in \dot{H}^{\frac{5}{2}-2\alpha}$, we have

$$\big\|\omega_{j,T} 2^{j(\frac{5}{2}-2\alpha)} \|\Delta_j u_0\|_2 \big\|_{l^\infty} \leqslant \frac{1}{3C}$$

for T small enough. On the other hand, since $u_1, u_2 \in C(\mathbb{R}^+; \dot{H}^{\frac{5}{2}-2\alpha})$, we also get

$$\sup_{t\in[0,T]} \|u_1(t) - S_\Omega(t)u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} + \sup_{t\in[0,T]} \|u_2(t) - S_\Omega(t)u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \leqslant \frac{1}{3C}$$

for T small enough. Then (5.7) ensures that $u_1(t) = u_2(t)$ for T small enough. Using a standard continuity argument, we can derive that $u_1 = u_2$ on $[0, \infty)$. The proof is completed. □

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