Fractional SZIR Model of Zombies Infection

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Abstract

The paper is concerned with the SIZR mathematical model for an outbreak of zombie infection with time-dependent infection rate. This class of the SIZR model involves equations that relate the susceptible S(t), the infected I(t), the zombie Z(t), and removed population R(t). The well poseness of the model is presented. The proposed model is then outstretched to the fractional order mathematical model with three different derivative operators i.e., Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivative operator. The conditions under which the model has a unique solution are established for different derivative operators. Using the numerical scheme which was proposed by Atangana and Toufik the numerical solutions are presented for the different fractional derivative operators.

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1 Introduction

Fictional characters have been something that intrigues humans the most. From movies to TV shows to video games, these fictional characters somehow become a part of us. Creators manifest these characters showing a possibility that someday they might turn into reality. One such fictional character that is quite popular and most sought after among writers and creators is 'zombie'. The word 'zombie' originated from the West African word 'nZambi' or 'Zumbi'. It was taken into the English dictionary as Zombie in the year 1819. A zombie is a walking cadaver that feeds on human life. Zombies can be perceived as the victim of some serious infectious disease outbreak. They have their brains almost dead and their sole purpose is to run after other people's life, bite them and infect them as well.

Zombies gained popularity from movies like 'Resident Evil', 'Zombieland' etc, while TV shows like "The Walking Dead' increased their popularity. In movies, TV shows, and video games, zombies are depicted as dead corpses grunting and limping around the city in search of human bodies to relish their flesh and converting them to one of their kind. They are deemed walking dead. Although they possess the body of a dead person, they are still alive with their only objective being to infect other humans. Their treatment is not possible by any means. The only way to quell their proliferation is to kill them. Zombies are an exaggerated description of an infectious disease. This disease is so severe that once a person is infected there is no turning back [1].

In the movie 'World War Z', it is shown that how a zombie outbreak takes place and spreads around the whole globe. The protagonist embarks on an adventure to discover a cure of the outbreak while tackling boisterous zombies and eventually succeeds in his mission.

Zombies apart from a fictional story can be seen as a viral disease that might become a challenge for the human race. Its spread can be considered as that of any other viral disease. Mathematical models are capable of decision making, saving lives, assisting in policy, and many more. These are helpful in understanding the conditions needed to sustain lives and provide us ways to study and predict the behavior of the spread. The concept of derivatives and integrals pays a lot in the formulation of these mathematical models. In this work, we will study the SIZR mathematical model for the outbreak of zombies attack. A SIZR model determines the number of people infected with a transmissible infection in a closed population over a while. These models are acquired in such a way that they involve equations that correlate number of susceptible people S(t), number of people infected I(t), number of people who have transformed to zombie, and who have removed R(t).

2 SIZR Model For Outbreak of zombie With Time Dependent Infection Rate.

In the **SIZR** model for zombie infection, we have considered the four different classes of individuals as **S:** Susceptible group of individuals **I:** Infected symptomatically group of individuals **Z:** Individuals who have become infected, **R:** Removed group of individuals. We assume that when zombies attack the susceptible individuals, they leave wound containing zombie's slobber in it, this slobbering fluid get mix with the blood of susceptible individuals and therefore infecting them. The susceptible class of individuals after the zombie's attack first move to the class of infected individuals. The infected individuals can either die natural death or else become zombie. Zombies move to the removed group by demolishing their brains or by removing their heads. This removed group is comprised of the humans who have died either through the natural death or by zombie attack. Individuals in the removed class can also restore to life and again become a zombie. The following assumptions are also made

- Zombies have thirst for human beef only and can transmit infection only to the human beings.
- The new zombies can only arise through (a) the susceptible individuals who came in contact with zombie, (b) those who have restored to life from the removed class of individuals.

Further, a time dependence on the parameter β the transmission rate [2], is introduced in this model, so the effect of transmission will be given as $\beta_0 \, e^{-\varsigma t}$. This choice of decreasing exponential function is justified by step by step implementation of rules and by rise in consciousness and awareness in individuals. Here, β_0 is the infection rate at the beginning of the zombie infection and ς determines the change in the time of infection [2]. Keeping in mind the aforementioned assumptions the SIZR model is formulated as [3]

$$
\begin{aligned}
\frac{dS}{dt} &= N - \beta_0 \, e^{-\varsigma \, t} \, S(t) \, Z(t) - \omega \, S(t), && S(0) = S_0, \\
\frac{dI}{dt} &= \beta_0 \, e^{-\varsigma \, t} \, S(t) \, Z(t) - \chi \, I(t) - \omega \, I(t), && I(0) = I_0, \\
\frac{dZ}{dt} &= \chi \, I(t) + \delta \, R(t) - \rho \, S(t) \, Z(t), && Z(0) = Z_0, \\
\frac{dR}{dt} &= \omega \, S(t) + \omega \, I(t) + \rho \, S(t) \, Z(t) - \delta \, R(t), && R(0) = R_0. \quad (2.1)
\end{aligned}
$$

Where, $N, \omega, \delta, \rho, \chi$ represents the birth rate, death rate, rate by which removed individuals resurrect and become a zombie, the rate by which zombie move to removed class by destroying their brains and demolishing their heads, the rate by which infected individuals become zombie respectively. The SIZR model is demonstrated in fig 1.

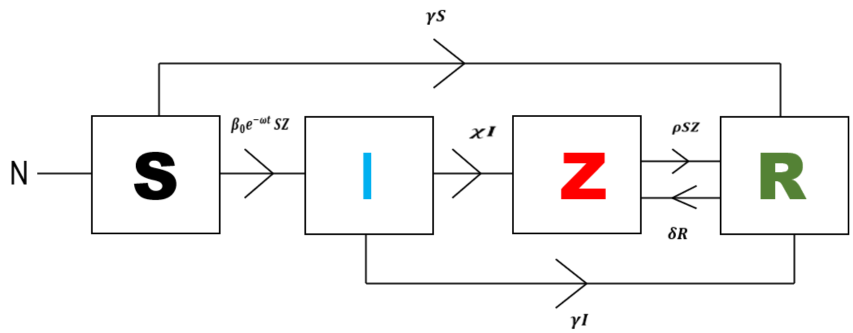

Figure 1: The SIZR model for zombie attack

Theorem 1. *[24] Assuming that all the parameters which are defined in the above model are positive, the solution of the model $S(t), I(t), Z(t), R(t)$ with non-negative initial conditions are positive $\forall t > 0$.*

Proof. Define Δ=$Sup\{t > 0 : S_0, I_0, Z_0, R_0 \geq 0\}$. Consider the following equation

$$\frac{dS}{dt} = N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t),\ \geq -\omega S(t)$$

$\Longrightarrow$

$$\frac{dS}{dt} = \geq -\omega S(t). \tag{2.2}$$

On solving, we get $S(t) = A_1\, e^{-\omega t}$, where A_1 is constant of integration.
Using the fact that $S(0) = S_0 > 0$, we get

$$S(t) = S_0\, e^{-\omega\, t} \geq 0. \tag{2.3}$$

Next Consider,

$$\frac{dI}{dt} \quad = \quad \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), \quad \geq -(\chi + \omega) I(t)$$

$\Longrightarrow$

$$\frac{dI}{dt} \geq -(\chi + \omega) I(t). \tag{2.4}$$

On solving, we get $I(t) = A_2\, e^{(\chi+\omega)t}$, where A_2 is constant of integration.
Using the fact that $I(0) = I_0 > 0$, we get

$$I(t) = I_0\, e^{-(\chi+\omega)t} \geq 0. \tag{2.5}$$

In a similar way, we get

$$Z(t) = Z_0\, e^{-\rho t S(t)} \geq 0. \tag{2.6}$$

$$R(t) = R_0\, e^{-(\delta)t} \geq 0. \tag{2.7}$$

$\square$

3 Fractional SIZR model of Zombie Infection

We will now extends the SIZR model of zombie infection to the fractional mathematical model of zombie infection using different fractional derivative operators. The concept of fractional derivatives ([5]-[8]) are very useful in making better understanding of the real world problems which exhibit non-local behaviours. Different fractional derivative operators have their own significance. The definition with singular kernel was proposed by Riemann-Liouville [5] and this was the first most accepted definition in the area of fractional calculus, after that new developments were made in the area of fractional calculus which involves definition of fractional derivative with non-singular kernel, one of which was presented by Caputo [5]. This definition is based on the conception of power law, so is does not work for the problems that exhibit the fading memory process. Next improvement in this area was made by Caputo and Fabrizio [9] by suggesting the definition that can deal with the process that exhibits fading memory process due to the exponential decay accompanied by Delta-Dirac characteristic. Lastly Atangana and Baleanu [10] introduced the definition of fractional derivatives and integrals that can easily handle the process by exhibiting a passage from fading memory to power law. For the detail applications of fractional calculus we can be see [11]-[30]. We will now discuss the detail analysis of the proposed fractional mathematical SIZR model in the sense of Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivative operators. The existence and uniqueness of the solution of the SIZR model and their numerical schemes in the sense of different fractional derivative operators are briefed in the next sections. In section 4, 5, 6 we have analysed the SIZR model in sense of Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivative respectively.

4 SIZR model of zombie infection with Caputo fractional derivative

$$\begin{aligned}
{}^{C}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), && S(0) = S_0,\\
{}^{C}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), && I(0) = I_0,\\
{}^{C}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), && Z(0) = Z_0,\\
{}^{C}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), && R(0) = R_0. \quad (4.1)
\end{aligned}$$

here, ${}^{C}\zeta_t^{\eta}$ denotes the fractional Caputo derivative of order η. which is defined as

Definition 1. *[5] Let f on $\mathbb{R}$ be an integrable function, $0 < \eta < 1$ the fractional Caputo derivative of order η is given as*

$$ {}^{C}\zeta_t^{\eta} f(t) = \frac{1}{\omega(1-\eta)} \int_0^t \frac{1}{(t-\tau)^{\eta}} \frac{d}{d\tau} f(\tau) d\tau. \quad (4.2)$$

Where, ${}^{C}\zeta_t^{\eta}$ denotes the fractional Caputo derivative of order η.

Theorem 2. *[31] Assuming that there exists positive constants L and $\bar{L}$ such that the following holds*

- *Lipschitz Condition:* $\forall v_1, v_2 \in \mathbb{R}$ *and* $\forall t \in [t_0, T]$
$$|g(t, v_1(t)) - g(t, v_2(t))| \leq L|v_1 - v_2|, \tag{4.3}$$
- *Linear growth condition:* $\forall (v, t) \in \mathbb{R} \times [[t_0, T]]$
$$|g(t, v)|^2 \leq \bar{L}(1 + |v|^2). \tag{4.4}$$

Then the Cauchy problem with Caputo derivative admits a unique solution.

4.1 Existence and Uniqueness of the fractional SIZR Model.

We will now prove the existence and the uniqness for SIZR mathematical model and for convenience, we write the SIZR model as

$$\begin{aligned} {}^{\mathcal{C}}\zeta_t^{\eta} S(t) &= f_1(t, S) \\ {}^{\mathcal{C}}\zeta_t^{\eta} I(t) &= f_2(t, I) \\ {}^{\mathcal{C}}\zeta_t^{\eta} Z(t) &= f_3(t, Z) \\ {}^{\mathcal{C}}\zeta_t^{\eta} R(t) &= f_4(t, R) \end{aligned} \tag{4.5}$$

where, D denotes the Caputo fractional derivative and

$$\begin{aligned} f_1(t, S) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), \\ f_2(t, I) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), \\ f_3(t, Z) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), \\ f_4(t, R) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t). \end{aligned} \tag{4.6}$$

To prove the existence and the uniqueness of the solution of the SIZR model, we use the concept which was recently proposed by Atangana [31] and hence prove the following theorem.

Theorem 3. *[25] Assuming that there exists positive constants C_1, C_2, C_3, C_4 and $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4$, such that the following holds*
(i)

$$\begin{aligned} |f_1(t, S(t)) - f_1(t, S_1(t))| &\leq C_1|S - S_1| \\ |f_2(t, I(t)) - f_2(t, I_1(t))| &\leq C_2|I - I_1| \\ |f_3(t, Z(t)) - f_3(t, Z_1(t))| &\leq C_3|Z - Z_1| \\ |f_4(t, R(t)) - f_4(t, R_1(t))| &\leq C_4|R - R_1| \end{aligned}$$

(ii)

$$\begin{aligned} |f_1(t, S(t))|^2 &\leq \bar{C}_1(1 + |S|^2) \\ |f_2(t, I(t))|^2 &\leq \bar{C}_2(1 + |I|^2) \\ |f_3(t, Z(t))|^2 &\leq \bar{C}_3(1 + |Z|^2) \\ |f_4(t, R(t))|^2 &\leq \bar{C}_4(1 + |R|^2) \end{aligned}$$

Proof. Consider

$$|f_1(t,S)-f_1(t,S_1)|=\left|\left(\beta_0\, e^{-\varsigma t}Z(t)-\omega\right)(S(t)-S_1(t))\right|. \tag{4.7}$$

Define the norm as $\|q\|_\infty = Sup_{t\in[0,T]}|q|$, we get

$$|f_1(t,S)-f_1(t,S_1)| \;\le\; (\beta_0\|Z\|_\infty+\omega)|S-S_1|. \tag{4.8}$$

Taking $(\beta_0\|Z\|_\infty+\omega)=C_1$, we have

$$|f_1(t,S)-f_1(t,S_1)\le C_1|S-S_1|. \tag{4.9}$$

Next consider

$$|f_2(t,I)-f_2(t,I_1)|=|(-(\chi+\omega))\,(I(t)-I_1(t))|. \tag{4.10}$$

Again defining the norm as $\|q\|_\infty = Sup_{t\in[0,T]}|q|$,

$$|f_2(t,I)-f_2(t,I_1)|\le(\chi+\omega)\,|I-I_1|. \tag{4.11}$$

Take $(\chi+\omega)=C_2$,

$$|f_2(t,I)-f_2(t,I_1)|=C_2|I-I_1|. \tag{4.12}$$

Next consider,

$$|f_3(t,Z)-f_3(t,Z_1)|=|(-\rho S(t))\,(Z(t)-Z_1(t))|. \tag{4.13}$$

$$|f_3(t,Z)-f_3(t,Z_1)|\le\rho\|S\|_\infty|Z-Z_1| \tag{4.14}$$

On taking $\rho\|S\|_\infty = C_3$

$$|f_3(t,Z)-f_3(t,Z_1)|=C_3|Z-Z_1|, \tag{4.15}$$

Similarly,

$$|f_4(t,R)-f_4(t,R_1)|=\delta|R-R_1|, \tag{4.16}$$

Taking $\delta=C_4$ We get

$$|f_4(t,R)-f_4(t,R_1)|=C_4|R-R_1|, \tag{4.17}$$

We now prove the second part of the above stated theorem. We first show $|f_1(t,S(t))|\le\bar{L_1}(1+|S|^2)$. Consider

$$\begin{aligned}
|f_1(t,S)|^2 &= |N-\beta_0 e^{-\varsigma t}\,S(t)\,Z(t)-\omega S(t)|^2\\
&= |N+(-\beta_0 e^{-\varsigma\, t}Z(t)-\omega)S(t)|^2\\
&\le 2|N|^2+(2\beta_0^2 e^{-2\varsigma\, t}|Z|^2+2\omega^2)|S|^2\\
&\le \|N\|_\infty^2+(2\beta_0^2\|Z\|_\infty^2+\omega^2)|S|^2\\
&= \|N\|_\infty^2\left[1+\left(\frac{2\beta_0^2\|Z\|_\infty^2+\omega^2}{\|N\|_\infty^2}\right)|S|^2\right]
\end{aligned} \tag{4.18}$$

On taking $\bar{C}_1 = \|N\|_\infty^2$ and under the condition

$$\left(\frac{2\beta_0^2\|Z\|_\infty^2 + \omega^2}{\|N\|_\infty^2}\right) \leq 1$$

we get

$$|f_1(t,S)|^2 \leq \bar{C}_1(1+|S|^2). \tag{4.19}$$

We now show

$$|f_2(t,I(t))|^2 \leq \bar{C}_2(1+|I(t)|^2). \tag{4.20}$$

Consider

$$\begin{aligned}
|f_2(t,I)|^2 &= |\beta_0 e^{-\varsigma t} S(t)Z(t) - (\chi+\omega)I(t)|^2 \\
&\leq 2\beta_0^2 e^{-2\varsigma t}|S|^2|Z|^2 + 2(\chi+\omega)^2|I(t)|^2 \\
&\leq 2\beta_0^2\|S\|_\infty^2\|Z\|_\infty^2 + 2(\chi+\omega)^2|I(t)|^2 \\
&= 2\beta_0^2\|S\|_\infty^2\|Z\|_\infty^2\left[1 + \left(\frac{2(\chi+\omega)^2}{2\beta_0^2\|S\|_\infty^2\|Z\|_\infty^2}\right)|I(t)|^2\right] \qquad (4.21)
\end{aligned}$$

On taking $\bar{C}_2 = 2\beta_0^2\|S\|_\infty^2\|Z\|_\infty^2$, and under the condition

$$\left(\frac{2(\chi+\omega)^2}{2\beta_0^2\|S\|_\infty^2\|Z\|_\infty^2}\right) \leq 1.$$

We get $|f_2(t,I(t))|^2 \leq \bar{C}_2(1+|I(t)|^2)$.
Next to show $|f_3(t,Z(t))|^2 \leq \bar{C}_3(1+|Z(t)|^2)$.
Consider

$$\begin{aligned}
|f_3(t,Z)|^2 &= |\chi I(t) + \delta R(t) - \rho S(t)Z(t)|^2 \\
&\leq 3\chi^2|I|^2 + 3\delta^2|R|^2 + 3\rho^2|S|^2|Z|^2 \\
&\leq 3\chi^2\|I\|_\infty^2 + 3\delta^2\|R\|_\infty^2 + 3\rho^2\|S\|_\infty^2\|Z\|_\infty^2 \\
&= \leq 3\chi^2\|I\|_\infty^2 + 3\delta^2\|R\|_\infty^2\left[1 + \left(\frac{3\rho^2\|S\|_\infty^2}{\leq 3\chi^2\|I\|_\infty^2 + 3\delta^2\|R\|_\infty^2}\right)|Z|^2\right] \qquad (4.22)
\end{aligned}$$

On taking $\bar{C}_3 = \leq 3\chi^2\|I\|_\infty^2 + 3\delta^2\|R\|_\infty^2$, and under the condition

$$\left(\frac{3\rho^2\|S\|_\infty^2}{\leq 3\chi^2\|I\|_\infty^2 + 3\delta^2\|R\|_\infty^2}\right) \leq 1.$$

We get $|f_3(t,Z(t))|^2 \leq \bar{C}_3(1+|Z(t)|^2)$.
Next to show $|f_4(t,R(t))|^2 \leq \bar{C}_4(1+|R(t)|^2)$.
Consider

$$\begin{aligned}
|f_4(t,R)|^2 &= |\omega S(t) + \omega I(t) + \rho S(t)Z(t) - \delta R(t)|^2 \\
&\leq 4\omega^2|S|^2 + 4\omega^2|I|^2 + 4\rho^2|S|^2|Z|^2 + 4\delta^2|R|^2 \\
&\leq 4\omega^2\|S\|_\infty^2 + 4\omega^2\|I\|_\infty^2 + 4\rho^2\|S\|_\infty^2\|Z\|_\infty^2 + 4\delta^2\|R\|_\infty^2 \\
&= (4\omega^2\|S\|_\infty^2 + 4\omega^2\|I\|_\infty^2 + 4\rho^2\|S\|_\infty^2\|Z\|_\infty^2) \times \\
&\quad \left[1 + \left(\frac{4\delta^2}{4\omega^2\|S\|_\infty^2 + 4\omega^2\|I\|_\infty^2 + 4\rho^2\|S\|_\infty^2\|Z\|_\infty^2}\right)|R|^2\right] \qquad (4.23)
\end{aligned}$$

On taking $\bar{C_4} = 4\omega^2\|S\|^2_\infty + 4\omega^2\|I\|^2_\infty + 4\rho^2\|S\|^2_\infty\|Z\|^2_\infty$ and under the condition

$$\left(\frac{4\delta^2}{4\omega^2\|S\|^2_\infty + 4\omega^2\|I\|^2_\infty + 4\rho^2\|S\|^2_\infty\|Z\|^2_\infty}\right) \leq 1.$$

We get $|f_4(t, R(t))|^2 \leq \bar{C_4}(1 + |R(t)|^2)$.

Hence, by using Theorem 2, proof for the existence and uniqueness of the SIZR model is completed. □

4.2 Numerical scheme for the SIZR model in frame of Caputo Derivative

Consider the SIZR model for Zombie infection in frame of Caputo fractional derivative operator

$$\begin{aligned}
{}^{C}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), & S(0) &= S_0,\\
{}^{C}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), & I(0) &= I_0,\\
{}^{C}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), & Z(0) &= Z_0,\\
{}^{C}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), & R(0) &= R_0. \qquad (4.24)
\end{aligned}$$

here, ${}^{C}\zeta_t^{\eta}$ denotes the fractional Caputo derivative of order η.
To find the numerical solution ([32]-[33]), consider

$${}_0^C\zeta_t^{\eta}\psi(t) = k(t, \psi(t)), \quad t \geq 0, \quad \psi(0) = \psi_0. \qquad (4.25)$$

Using the fundamental theorem, we rewrite the above equation as

$$\psi(t) = \psi(0) + \frac{1}{\omega(\eta)}\int_0^t (t-\tau)^{\eta-1} k(\psi, \tau)d\tau, \qquad (4.26)$$

at $t = t_{p+1}$, we have

$$\psi_{p+1} = \psi(t_{p+1}) = \psi(0) + \frac{1}{\omega(\eta)}\int_0^{t_{p+1}} (t_{p+1}-\tau)^{\eta-1} k(\psi, \tau)d\tau, \qquad (4.27)$$

at $t = t_p$, we have

$$\psi_p = \psi(t_p) = \psi(0) + \frac{1}{\omega(\eta)}\int_0^{t_p} (t_p-\tau)^{\eta-1} k(\psi, \tau)d\tau, \qquad (4.28)$$

From the above two equations, we get

$$\begin{aligned}
\psi(t_{p+1}) - \psi(t_p) = \frac{1}{\omega(\eta)}\Bigg[&\int_0^{t_{p+1}} (t_{p+1}-\tau)^{\eta-1} k(\psi, \tau)\, d\tau \\
&- \int_0^{t_p} (t_p-\tau)^{\eta-1} k(\psi, \tau)d\tau\Bigg] \qquad (4.29)
\end{aligned}$$

Now, applying Atangana and Toufik numerical scheme with Lagrange polynomial interpolation, we have the following numerical scheme

$$\begin{aligned} S_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_1(t_m, S_m) \, [(p-m+1)^\eta (p-m+2+2\eta) \\ - (p-m)^\eta (p-m+2+2\eta)] \\ -\frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_1(t_{m-1}, S_{m-1}) \left[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)\right], \end{aligned} \tag{4.30}$$

where,

$$f_1(t,S) = N - \beta_0 \, e^{-\varsigma\, t} \, S(t) \, Z(t) - \omega \, S(t). \tag{4.31}$$

$$\begin{aligned} I_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_2(t_m, I_m) \, [(p-m+1)^\eta (p-m+2+2\eta) \\ - (p-m)^\eta (p-m+2+2\eta)] \\ -\frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_2(t_{m-1}, I_{m-1}) \left[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)\right], \end{aligned} \tag{4.32}$$

where,

$$f_2(t,I) = \beta_0 \, e^{-\varsigma\, t} \, S(t) \, Z(t) - \chi \, I(t) - \omega \, I(t) \tag{4.33}$$

$$\begin{aligned} Z_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_3(t_m, Z_m) \, [(p-m+1)^\eta (p-m+2+2\eta) \\ - (p-m)^\eta (p-m+2+2\eta)] \\ -\frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_3(t_{m-1}, Z_{m-1}) \left[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)\right] \end{aligned} \tag{4.34}$$

,

$$f_3(t,Z) = \chi \, I(t) + \delta \, R(t) - \rho \, S(t) \, Z(t) \tag{4.35}$$

$$\begin{aligned} R_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_4(t_m, R_m) \, [(p-m+1)^\eta (p-m+2+2\eta) \\ - (p-m)^\eta (p-m+2+2\eta)] \\ -\frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_4(t_{m-1}, R_{m-1}) \left[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)\right], \end{aligned} \tag{4.36}$$

$$f_4(t, R) = \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t). \tag{4.37}$$

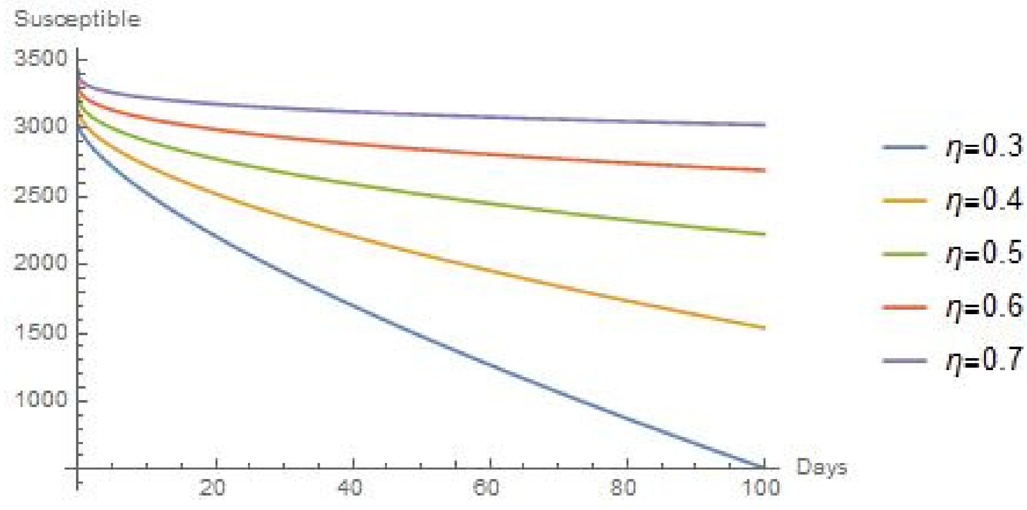

Figure 2: Simulation of S(t) for different ordered fractional derivatives using Caputo fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on decreasing the order of fractional derivative the susceptible individuals decreases well.

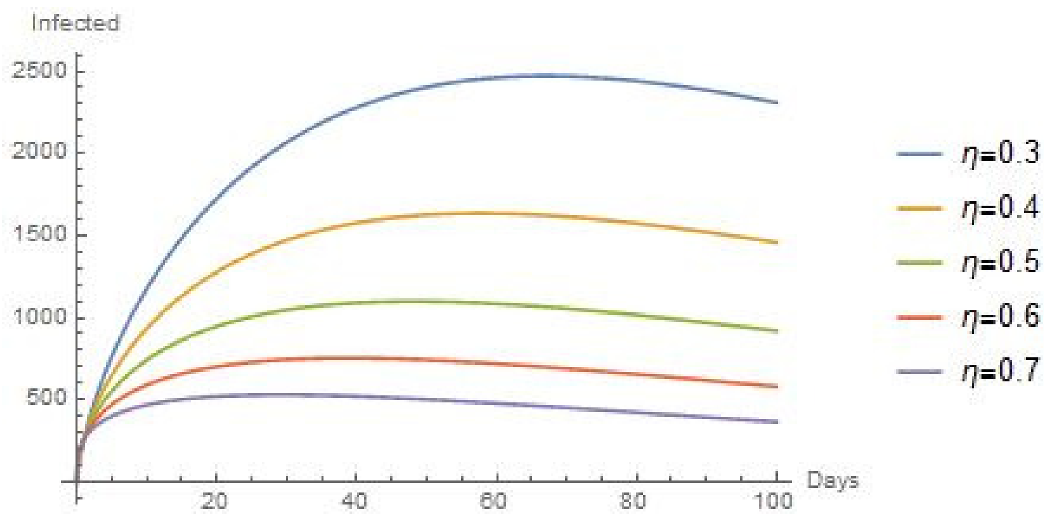

Figure 3: Simulation of I(t) for different ordered fractional derivatives using Caputo fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the infected individuals decreases well but the infection can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

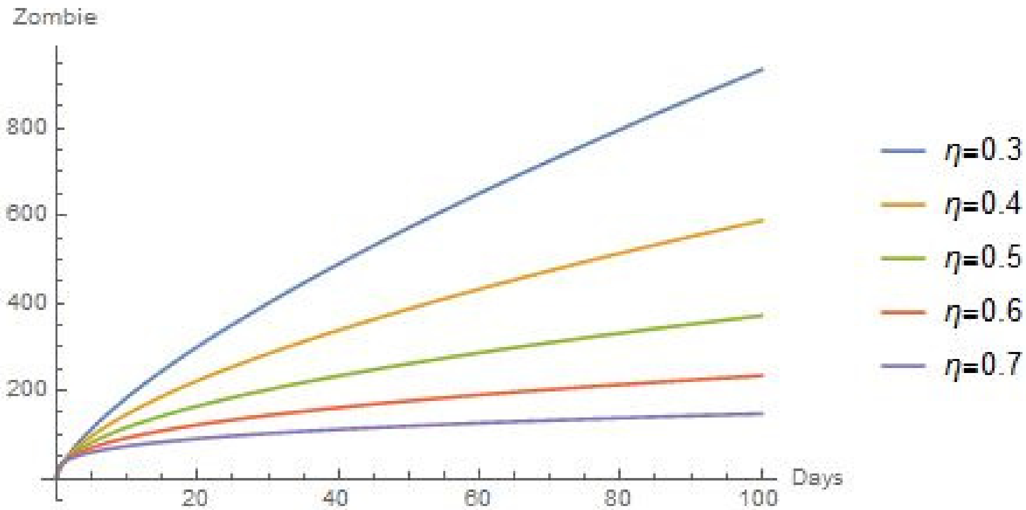

Figure 4: Numerical simulation of Z(t) for different ordered fractional derivatives using Caputo fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the zombies decrease but the zombie's group can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

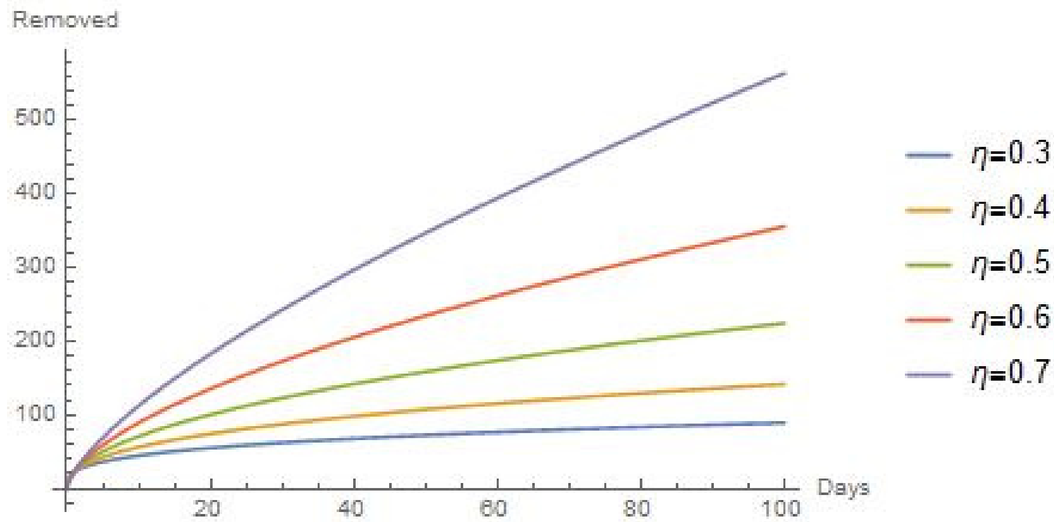

Figure 5: Numerical simulation of R(t) for different ordered fractional derivatives using Caputo fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the removed individuals keep on increasing, since the only way to get out of the infection is to destroy the zombies.

5 SIZR Model of Zombie Infection With Caputo Fabrizio Fractional Derivative

$$
\begin{aligned}
{}^{\mathcal{CF}}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), && S(0) = S_0,\\
{}^{\mathcal{CF}}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), && I(0) = I_0,\\
{}^{\mathcal{CF}}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), && Z(0) = Z_0,\\
{}^{\mathcal{CF}}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), && R(0) = R_0. \quad (5.1)
\end{aligned}
$$

here, ${}^{\mathcal{CF}}\zeta_t^{\eta}$ denotes the Caputo Fabrizio fractional derivative of order η. defined as

Definition 2. *[9] Let f on $\mathbb{R}$ be an integrable function, $t > 0, \quad 0 < \eta < 1$, the Caputo-Fabrizio fractional derivative of order η is defined as*

$$
{}_{0}^{CF}\zeta_t^{\eta}(f(t)) = \frac{N(\eta)}{1-\eta} \int_0^t exp\left(\frac{-\eta(t-\tau)}{1-\eta}\right) f'(\tau) d\tau. \tag{5.2}
$$

Where ${}_{0}^{CF}\zeta_t^{\eta}$ represents the fractional Caputo-Fabrizio derivative of order η, $N(\eta)$ is a normalization function such that $N(0) = N(1) = 1$.

Definition 3. *[9] Let f on $\mathbb{R}$ be an integrable function, $t > 0$, $0 < \eta < 1$, the Caputo-Fabrizio time fractional integral of order η is given as*

$$\mathcal{I}^{\mathcal{CF}}{}_t^{\beta_0}(f(t)) = \frac{2\,(1-\eta)}{(2-\eta)\,N(\eta)}\,f(t) + \frac{2\,\eta}{(2-\eta)\,N(\eta)}\int_0^t f(\tau)\,d\tau. \tag{5.3}$$

Where, $N(\eta)$ is the normalization function such that $N(0) = N(1) = 1$.

5.1 Numerical scheme for the SIZR model in frame of Caputo-Fabrizio Derivative

Consider the SIZR model for Zombie infection in frame of fractional Caputo-Fabrizio derivative

$$\begin{array}{rcll} {}^{\mathcal{CF}}\zeta_t^{\eta} S(t) &=& N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), & S(0) = S_0, \\ {}^{\mathcal{CF}}\zeta_t^{\eta} I(t) &=& \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), & I(0) = I_0, \\ {}^{\mathcal{CF}}\zeta_t^{\eta} Z(t) &=& \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), & Z(0) = Z_0, \\ {}^{\mathcal{CF}}\zeta_t^{\eta} R(t) &=& \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), & R(0) = R_0. \end{array} \tag{5.4}$$

here, ${}^{\mathcal{CF}}\zeta_t^{\eta}$ denotes the fractional Caputo Fabrizio derivative of order η.

To illustrate the method ([32]-[33]), consider

$${}_0^{CF}\zeta_t^{\eta}\phi(t) = k(t, \phi(t)), \;\; t \geq 0, \quad \phi(0) = \phi_0. \tag{5.5}$$

Using the fundamental theorem, we rewrite the above equation as

$$\psi(t) - \psi(0) = \frac{1-\eta}{G(\eta)}k(t, \psi(t)) + \frac{\eta}{G(\eta)}\int_0^t k(\tau, \psi(\tau))d\tau. \tag{5.6}$$

At $t = t_{p+1}$, for p= 0,1,2 $\cdots$, equation (5.6) is given as

$$\psi(t_{p+1}) - \psi(0) = \frac{1-\eta}{G(\eta)}k(t_p, \psi(t_p)) + \frac{\eta}{G(\eta)}\int_0^{t_{p+1}} k(\tau, \psi(\tau))d\tau. \tag{5.7}$$

At $t = t_p$, for p= 0,1,2 $\cdots$, equation (5.6) is given as

$$\psi(t_p) - \psi(0) = \frac{1-\eta}{G(\eta)}g(t_{p-1}, \psi(t_{p-1})) + \frac{\eta}{G(\eta)}\int_0^{t_p} k(\tau, \psi(\tau))d\tau. \tag{5.8}$$

From the above two equations, we get

$$\psi(t_{p+1}) - \psi(t_p) = \frac{1-\eta}{G(\eta)}[k(t_p, \psi(t_p)) - k(t_{p-1}, \psi(t_{p-1}))] + \frac{\eta}{G(\eta)}\int_{t_p}^{t_{p+1}} k(\tau, \psi(\tau))d\tau. \tag{5.9}$$

Considering k(τ, $\psi(\tau)$) through Lagrange polynomial interpolation,

$$k(\tau, \psi(\tau)) = \frac{\tau - t_{m-1}}{t_m - t_{m-1}} g(t_m, \psi_{t_m}) + \frac{\tau - t_m}{t_{m-1} - t_m} k(t_{m-1}, \psi_{t_{m-1}}). \quad (5.10)$$

where $\psi(t_m)$ is a function at time t_m and $\psi(t_{m-1})$ is a function at time t_{m-1}. Substituting the value of $k(\tau, \psi(\tau))$ in equation (5.9), we get

$$\psi_{p+1} - \psi_p = \frac{1-\eta}{G(\eta)}[k(t_p, \psi(t_p)) - k(t_{p-1}, \psi(t_{p-1}))] + \frac{\eta}{G(\eta)} \int_{t_p}^{t_{p+1}} \left(\frac{k(t_p, \psi_p)}{h}(\tau - t_{p-1}) - \frac{k(t_{p-1}, \psi_{p-1})}{h}(\tau - t_p) \right) d\tau. \quad (5.11)$$

Substituting $h = t_m - t_{m-1}$ and after solving, we have

$$\psi_{p+1} = \psi_0 + \left(\frac{1-\eta}{G(\eta)} + \frac{3h}{2\ G(\eta)} \right) k(t_p, \psi(t_p)) - \left(\frac{1-\eta}{G(\eta)} + \frac{\eta\ h}{2\ G(\eta)} \right) k(t_{p-1}, \psi(t_{p-1})). \quad (5.12)$$

Using this concept ([32]-[33]), the numerical scheme for the fractional model of zombie attack in the sense of fractional Caputo-Fabrizio derivative is given as

$$S_{p+1} = S_0 + \left(\frac{1-\eta}{G(\eta)} + \frac{3h}{2\ G(\eta)} \right) f_1(t_p, \psi(t_p)) - \left(\frac{1-\eta}{G(\eta)} + \frac{\eta\ h}{2\ G(\eta)} \right) f_1(t_{p-1}, \psi(t_{p-1})). \quad (5.13)$$

where,

$$f_1(t, S) = N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t).$$

$$I_{p+1} = I_0 + \left(\frac{1-\eta}{G(\eta)} + \frac{3h}{2\ G(\eta)} \right) f_2(t_p, \psi(t_p)) - \left(\frac{1-\eta}{G(\eta)} + \frac{\eta\ h}{2\ G(\eta)} \right) f_2(t_{p-1}, \psi(t_{p-1})). \quad (5.14)$$

where,

$$f_2(t, I) = \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t)$$

$$Z_{m+1} = Z_0 + \left(\frac{1-\eta}{G(\eta)} + \frac{3h}{2\ G(\eta)} \right) f_3(t_p, \psi(t_p)) - \left(\frac{1-\eta}{G(\eta)} + \frac{\eta\ h}{2\ G(\eta)} \right) f_3(t_{p-1}, \psi(t_{p-1})). \quad (5.15)$$

where,

$$f_3(t, Z) = \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t)$$

$$R_{p+1} = R_0 + \left(\frac{1-\eta}{G(\eta)} + \frac{3h}{2\,G(\eta)}\right) f_3(t_p, \psi(t_p)) - \left(\frac{1-\eta}{G(\eta)} + \frac{\eta\, h}{2\,G(\eta)}\right) f_3(t_{p-1}, \psi(t_{p-1})). \quad (5.16)$$

where,

$$f_4(t, R) = \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t).$$

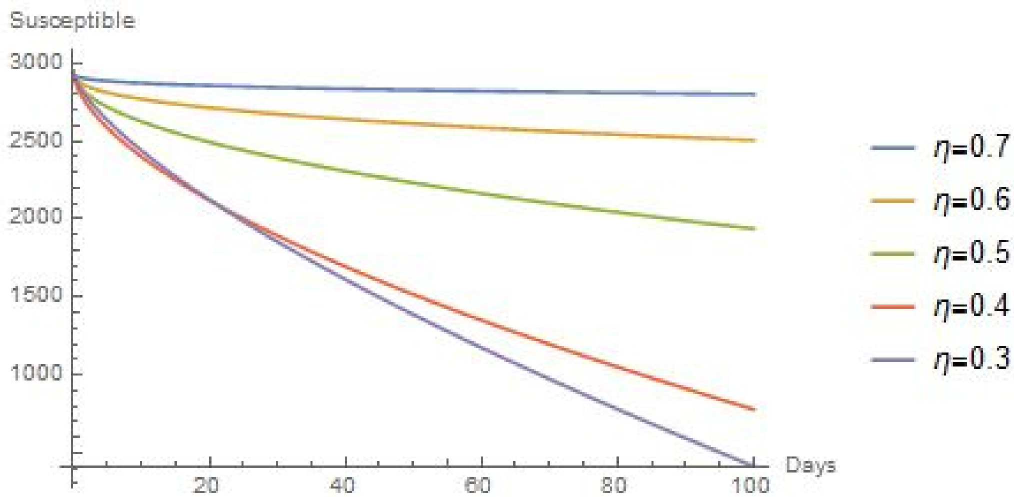

Figure 6: Simulation of S(t) for different ordered fractional derivatives using Caputo-Fabrizio fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on decreasing the order of fractional derivative the susceptible individuals decreases well.

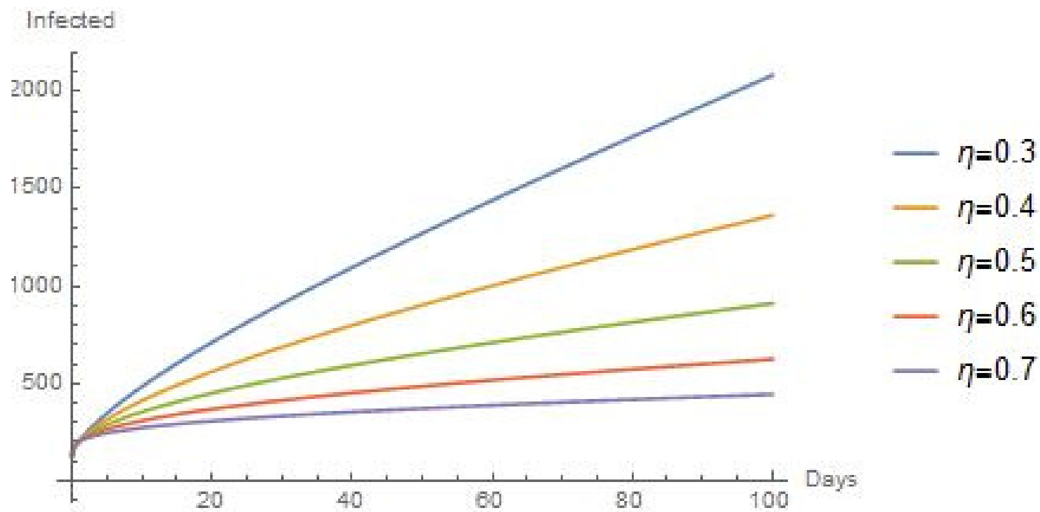

Figure 7: Simulation of I(t) for different ordered fractional derivatives using Caputo-Fabrizio fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the infected individuals decreases well but the infection can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

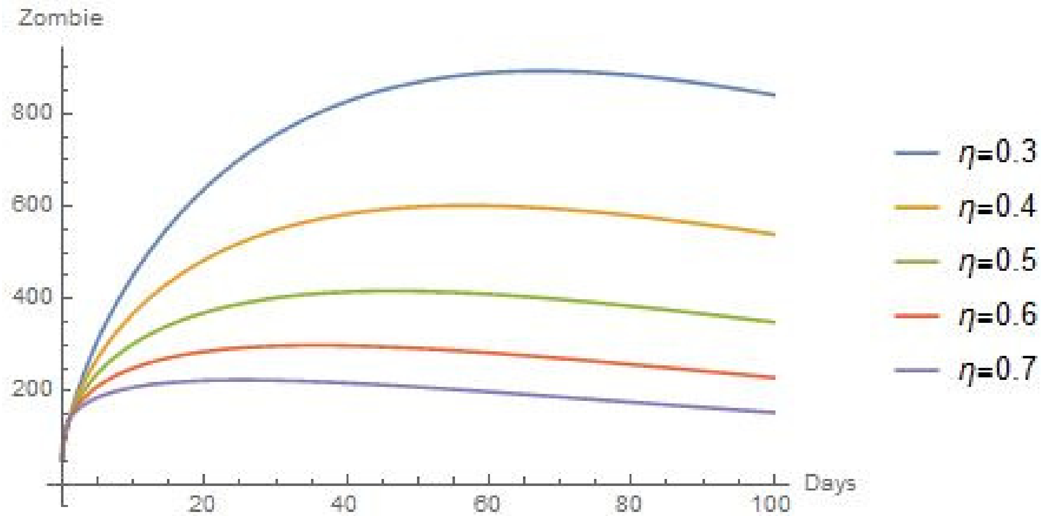

Figure 8: Numerical simulation of Z(t) for different ordered fractional derivatives using Caputo-Fabrizio fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the zombies decrease but the zombie's group can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

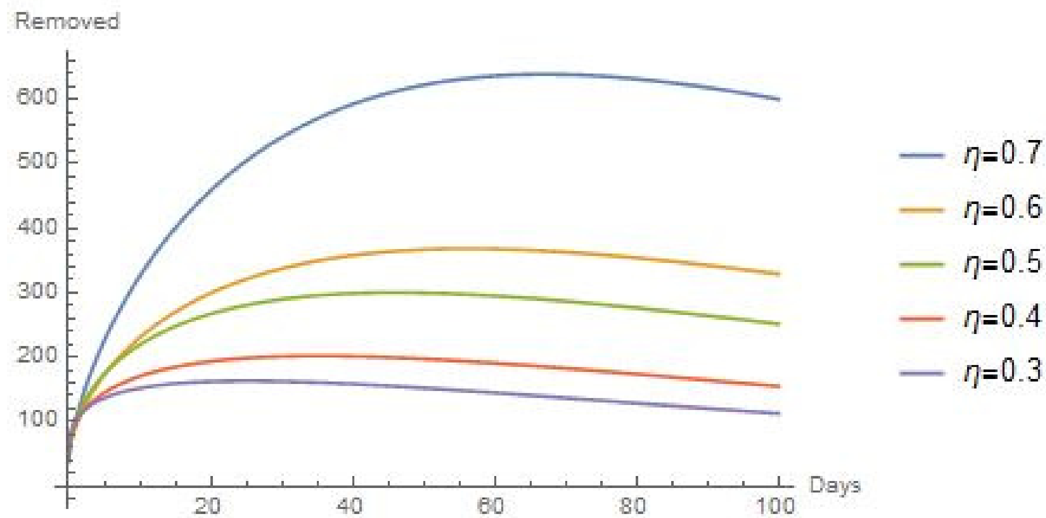

Figure 9: Numerical simulation of R(t) for different ordered fractional derivatives using Caputo-Fabrizio fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the removed individuals keep on increasing, since the only way to get out of the infection is to destroy the zombies.

6 SIZR Model of Zombie Infection With Atangana-Baleanu Fractional Derivative

$$\begin{aligned}
{}^{\mathcal{ABC}}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), & S(0) = S_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), & I(0) = I_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), & Z(0) = Z_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), & R(0) = R_0. \quad (6.1)
\end{aligned}$$

here, ${}^{\mathcal{ABC}}\zeta_t^{\eta}$ denotes the Atangana-Baleanu fractional derivative of order η. defined as

Definition 4. *[10] Let f on $\mathbb{R}$ be an integrable function, let $0 < \eta < 1$, then the Atangana-Baleanu fractional derivative is given as*

$$ {}_0^{ABC}\zeta_t^{\eta}(g(t)) = \frac{N(\eta)}{1-\eta} \int_0^t g'(\tau) E_\eta \left[-\eta \frac{(t-\tau)^\eta}{1-\eta} \right] d\tau. \quad (6.2)$$

Where, ${}_0^{ABC}\zeta_t^{\eta}$ is the Atangana-Baleanu fractional derivative of order η in Caputo sense, E_η is the Mittag-Leffler function and $N(\eta)$ is the normalization function such that $N(0) = N(1) = 1$.

Definition 5. *[10] Let f on $\mathbb{R}$ be an integrable function, the fractional integral of Atangana-Baleanu fractional derivative of order η is given as*

$$\mathcal{I}^{\mathcal{AB}}{}^{\eta}_{t}(g(t)) = \frac{1-\eta}{N(\eta)} g(t) + \frac{\eta}{N(\eta)\omega(\eta)} \int_0^t g(\tau)(t-\tau)^{\eta-1} d\tau. \tag{6.3}$$

Theorem 4. *[10] The fractional differential equation*

$$\mathcal{I}^{\mathcal{AB}}{}^{\eta}_{t}(g(t)) = w(t),$$

possesses a solution which is unique given as

$$g(t) = \frac{1-\eta}{M(\eta)} w(t) + \frac{\eta}{M(\eta)\omega(\eta)} \int_0^t w(\tau)(t-\tau)^{\eta-1} d\tau.$$

6.1 Existence and Uniqueness of the Solution

Theorem 5. *[25] The kernels*

$$\begin{aligned}
f_1(t,S) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t) \\
f_2(t,I) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t) \\
f_3(t,Z) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t) \\
f_4(t,R) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t)
\end{aligned}$$

satisfy the Lipschitz condition and contractions if following hold:

(i) $0 < C_1 < 1$

(ii) $0 < C_2 < 1$

(iii) $0 < C_3 < 1$

(iv) $0 < C_4 < 1$

Proof. The Lipschitz's condition is proved in theorem and if $0 < C_1 < 1$, $0 < C_2 < 1$, $0 < C_3 < 1$, $0 < C_4 < 1$ then this proves contraction for $f_1(t,S)$, $f_2(t,I)$, $f_3(t,Z)$, $f_4(t,R)$. □

Theorem 6. *[25] The following is the time fractional SIZR model of zombie infection*

$$\begin{aligned}
{}^{\mathcal{ABC}}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), & S(0) &= S_0, \\
{}^{\mathcal{ABC}}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), & I(0) &= I_0, \\
{}^{\mathcal{ABC}}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), & Z(0) &= Z_0, \\
{}^{\mathcal{ABC}}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), & R(0) &= R_0.
\end{aligned} \tag{6.4}$$

possesses a unique solution under the conditions that we are able to search t_{max} which satisfies

$$\frac{1-\eta}{N(\eta)} C_i + \frac{t_{max}^{\eta}}{N(\eta)\omega(\eta)} C_i < 1, \quad for\ i = 1,2,3. \tag{6.5}$$

Proof Consider the following equation

$$ {}_0^{ABC}\zeta_t^{\eta} S(t) = N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), \qquad S(0) = S_0 \tag{6.6} $$

Taking $f_1(t, S) = N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t)$
Then equation (6.6) can be written as

$$ {}_0^{ABC}\zeta_t^{\eta} S(t) = f_1(t, S). \tag{6.7} $$

Using theorem 4, we get

$$ S(t) = S_0 + \frac{1-\eta}{N(\eta)} f_1(t, S(t)) + \frac{\eta}{N(\eta)\omega(\eta)} \int_0^t (t-\tau)^{\eta-1} f_1(\tau, S(\tau)) d\tau. \tag{6.8} $$

Let $L = (0, T)$ and define an operator $X: \ C(L, \mathbb{R}^4) \longrightarrow C(L, \mathbb{R}^4)$ such that

$$ X[S(t)] = S_0 + \frac{1-\eta}{N(\eta)} f_1(t, S(t)) + \frac{\eta}{N(\eta)\omega(\eta)} \int_0^t (t-\tau)^{\eta-1} f_1(\tau, S(\tau)) d\tau. \tag{6.9} $$

So equation (6.8) can be seen as $X[S(t)] = S(t)$. Define the supremum norm on L as $\|S\| = Sup_{t\in J}|S(t)|$. Then $C(L, \mathbb{R}^4)$ and $\|.\|$ defines a Banach Space. Finally consider

$$ \begin{aligned} X[S_1(t)] - X[S_2(t)] = \frac{1-\eta}{N(\eta)} \left(f_1(t, S_1(t)) - f_2(t, S_2(t))\right) + \\ \frac{\eta}{N(\eta)\omega(\eta)} \int_0^t (t-\tau)^{\eta-1} \left(f_1(\tau, S_1(\tau)) - f_1(\tau, S_2(\tau))\right) d\tau. \end{aligned} \tag{6.10} $$

Now take the modulus on both sides of equation (6.10) and using triangle inequality we have

$$ \begin{aligned} |X[S_1(t)] - X[S_2(t)]| \le \frac{1-\eta}{N(\eta)} \left|\left(f_1(t, S_1(t)) - f_2(t, S_2(t))\right)\right| + \\ \frac{\eta}{N(\eta)\omega(\eta)} \int_0^t \left|(t-\tau)^{\eta-1} \left(f_1(\tau, S_1(\tau)) - f_1(\tau, S_2(\tau))\right) d\tau\right|. \end{aligned} \tag{6.11} $$

Lastly, knowing that the kernel $f_1(t, S(t))$ satisfies Lipschitz condition, we get

$$ |X(S_1) \quad - \quad X(S_2)| \le \qquad \left(\frac{1-\eta}{N(\eta)} p_1 + \frac{t_{max}^{\eta}}{N(\eta)\omega(\eta)} p_1\right) |S_1 - S_2|. \tag{6.12} $$

Equation (6.12) is a contraction if

$$ \frac{1-\eta}{N(\eta)} C_1 + \frac{t_{max}^{\eta}}{N(\eta)\omega(\eta)} C_1 < 1. \tag{6.13} $$

Hence, using the Banach Fixed Point theorem, we can show the existence of a unique solution for the fractional model of zombie attack in sense of Atangana-Baleanu derivative operator.

6.2 Numerical Scheme In Frame of ABC Derivative

Consider the SIZR model for Zombie infection in sense of Atangana-Baleanu fractional derivative operator

$$\begin{aligned}
{}^{\mathcal{ABC}}\zeta_t^{\eta} S(t) &= N - \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \omega\, S(t), & S(0) &= S_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} I(t) &= \beta_0\, e^{-\varsigma\, t}\, S(t)\, Z(t) - \chi\, I(t) - \omega\, I(t), & I(0) &= I_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} Z(t) &= \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t), & Z(0) &= Z_0,\\
{}^{\mathcal{ABC}}\zeta_t^{\eta} R(t) &= \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t), & R(0) &= R_0
\end{aligned} \tag{6.14}$$

here, ${}^{\mathcal{ABC}}\zeta_t^{\eta}$ denotes the Atangana-Baleanu fractional derivative of order η.

Using the Collocatation method which was proposed by Toufik and Atangana ([32]- [33]), for solving fractional derivatives that have non-singular and non-local kernel. To illustrate the method

$${}_{0}^{ABC}\zeta_t^{\eta}\chi(t) = k(t,\chi(t)), \quad t \geq 0, \quad \chi(0) = \chi_0. \tag{6.15}$$

Using Theorem 4, we rewrite the the above equation as

$$\psi(t) - \psi(0) = \frac{1-\eta}{G(\eta)} k(t,\psi(t)) + \frac{\eta}{G(\eta)\omega(\eta)} \int_0^t (t-\tau)^{\eta-1} k(\tau,\psi(\tau)) d\tau. \tag{6.16}$$

At $t = t_{p+1}$, for p= 0,1,2 $\cdots$, equation (6.16) becomes

$$\psi(t_{p+1}) - \psi(0) = \frac{1-\eta}{G(\eta)} k(t_p,\psi(t_p)) + \frac{\eta}{G(\eta)\omega(\eta)} \int_0^{t_{p+1}} (t_{p+1}-\tau)^{\eta-1} k(\tau,\psi(\tau)) d\tau. \tag{6.17}$$

$$\begin{aligned}
\psi_{p+1} = \psi(t_{p+1}) = \psi(0) + \frac{1-\eta}{G(\eta)} k(t_p,\psi(t_p))& \\
+ \frac{\eta}{G(\eta)\omega(\eta)} \sum_{m=0}^{r} \int_{t_m}^{t_{m+1}} (t_{p+1}-\tau)^{\eta-1} g(\tau,\psi(\tau)) d\tau.&
\end{aligned} \tag{6.18}$$

Considering k(τ, $\psi(\tau)$) through Lagrange polynomial interpolation,

$$\begin{aligned}
u_n &= k(\tau,\psi(\tau))\\
&= \frac{\tau - t_{m-1}}{t_m - t_{m-1}} k(t_m,\psi_{t_m}) + \frac{\tau - t_m}{t_{m-1} - t_m} k(t_{m-1},\psi_{t_{m-1}}).
\end{aligned} \tag{6.19}$$

Substituting the value of $k(\tau,\chi(\tau))$ in equation (6.18), we get

$$\begin{aligned}
\psi_{p+1} = \psi(0) + \frac{1-\eta}{G(\eta)} k(t_p,\psi(t_p)) + \frac{\eta}{G(\eta)\omega(\eta)}&\\
\sum_{m=0}^{r}\Bigg(\frac{k(t_m,\psi(t_m))}{h} \int_{t_m}^{t_{m+1}} (t - t_{m-1})(t_{p+1}-t)^{\eta-1} dt&\\
- \frac{g(t_{m-1},\psi(t_{m-1}))}{h} \int_{t_m}^{t_{m+1}} (t - t_{p-1})(t_{p+1}-t)^{\eta-1} dt \Bigg).&
\end{aligned} \tag{6.20}$$

Substituting $h = t_m - t_{m-1}$ and on solving, we get

$$\psi_{p+1} = \psi_0 + \frac{1-\eta}{G(\eta)} k(t_p, \psi(t_p)) + \frac{\eta}{G(\eta)}$$
$$\sum_{m=0}^{r} \left[\frac{h^\eta \ k(t_m, \psi_{t_m})}{\omega(\eta+2)} \left((p+1-m)^\eta \ (p-m+2+\eta) - (p-m)^\eta (p-m+2+2\eta)\right) \right.$$
$$\left. - \frac{h^\eta \ k(t_{m-1}, \eta(t_{m-1}))}{\omega(\eta+2)} \left((p+1-m)^{\eta+1} \ - (p-m)^\eta \ (p-m+1+\eta)\right) \right]. \tag{6.21}$$

Using the above illustrated concept, the numerical scheme for the fractional model of zombie infection in framework of Atangana-Baleanu derivative operator is given as

$$S_{p+1} = S_0 + \frac{1-\eta}{G(\eta)} f_1(t_p, S(t_p)) + \frac{\eta}{G(\eta)}$$
$$\sum_{m=0}^{r} \left[\frac{h^\eta \ f_1(t_m, S_{t_m})}{\omega(\eta+2)} \left((p+1-m)^\eta \ (p-m+2+\eta) - (p-m)^\eta (p-m+2+2\eta)\right) \right.$$
$$\left. - \frac{h^\eta \ f_1(t_{m-1}, S(t_{m-1}))}{\omega(\eta+2)} \left((p+1-m)^{\eta+1} \ - (p-m)^\eta \ (p-m+1+\eta)\right) \right] \tag{6.22}$$

where,

$$f_1(t, S) = N - \beta_0 \, e^{-\varsigma \, t} \, S(t) \, Z(t) - \omega \, S(t).$$

$$I_{p+1} = I_0 + \frac{1-\eta}{G(\eta)} f_2(t_p, I(t_p)) + \frac{\eta}{G(\eta)}$$
$$\sum_{m=0}^{r} \left[\frac{h^\eta \ f_2(t_m, I_{t_m})}{\omega(\eta+2)} \left((p+1-m)^\eta \ (p-m+2+\eta) - (p-m)^\eta (p-m+2+2\eta)\right) \right.$$
$$\left. - \frac{h^\eta \ f_2(t_{m-1}, I(t_{m-1}))}{\omega(\eta+2)} \left((p+1-m)^{\eta+1} \ - (p-m)^\eta \ (p-m+1+\eta)\right) \right]. \tag{6.23}$$

where,

$$f_2(t, I) = \beta_0 \, e^{-\varsigma \, t} \, S(t) \, Z(t) - \chi \, I(t) - \omega \, I(t).$$

$$Z_{p+1} = Z_0 + \frac{1-\eta}{G(\eta)} f_3(t_p, Z(t_p)) + \frac{\eta}{G(\eta)}$$
$$\sum_{m=0}^{r} \left[\frac{h^\eta \ f_3(t_m, Z_{t_m})}{\omega(\eta+2)} \left((p+1-m)^\eta \ (p-m+2+\eta) - (p-m)^\eta (p-m+2+2\eta)\right) \right.$$
$$\left. - \frac{h^\eta \ f_3(t_{m-1}, Z(t_{m-1}))}{\omega(\eta+2)} \left((p+1-m)^{\eta+1} \ - (p-m)^\eta \ (p-m+1+\eta)\right) \right]. \tag{6.24}$$

where,

$$f_3(t, Z) = \chi\, I(t) + \delta\, R(t) - \rho\, S(t)\, Z(t)$$

$$R_{p+1} = R_0 + \frac{1-\eta}{G(\eta)} f_4(t_p, R(t_p)) + \frac{\eta}{G(\eta)}$$
$$\sum_{m=0}^{r} \left[\frac{h^\eta \ f_4(t_m, R_{t_m})}{\omega(\eta+2)} \left((p+1-m)^\eta \ (p-m+2+\eta) - (p-m)^\eta (p-m+2+2\eta)\right) - \frac{h^\eta \ f_4(t_{m-1}, R(t_{m-1}))}{\omega(\eta+2)} \left((p+1-m)^{\eta+1} \ - (p-m)^\eta \ (p-m+1+\eta)\right) \right]. \tag{6.25}$$

where,

$$f_4(t, R) = \omega\, S(t) + \omega\, I(t) + \rho\, S(t)\, Z(t) - \delta\, R(t).$$

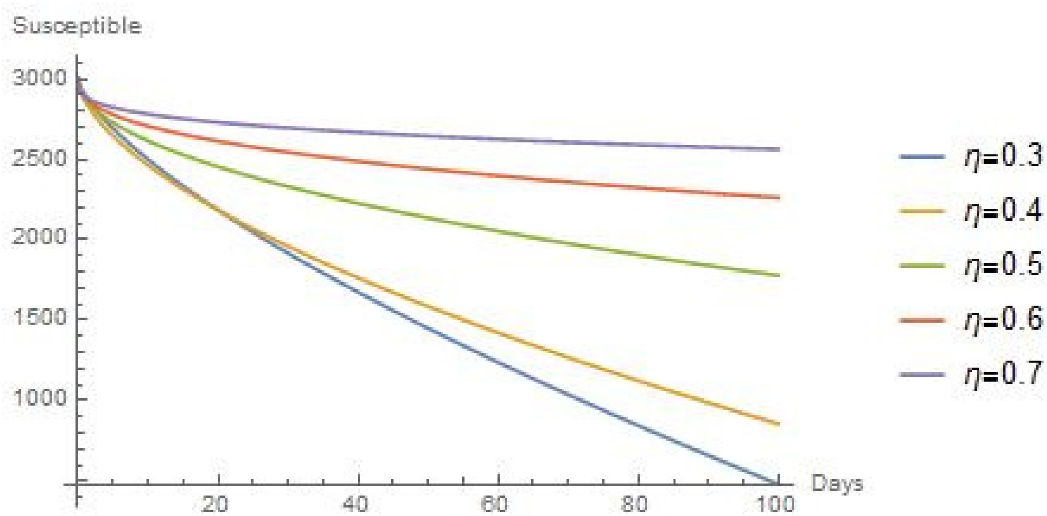

Figure 10: Simulation of S(t) for different ordered fractional derivatives using Atangana-Baleanu fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on decreasing the order of fractional derivative the susceptible individuals decreases well.

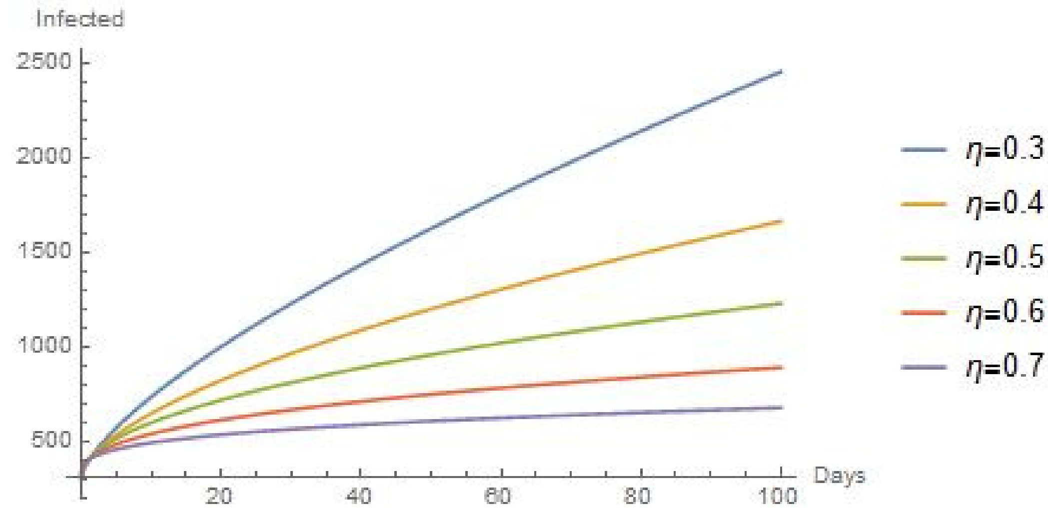

Figure 11: Simulation of I(t) for different ordered fractional derivatives using Atangana-Baleanu fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the infected individuals decreases well but the infection can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

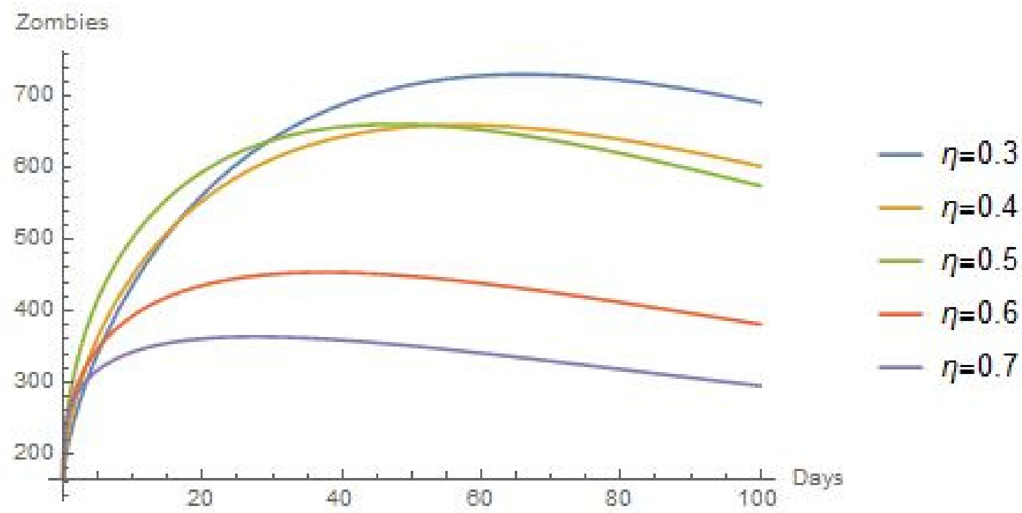

Figure 12: Numerical simulation of Z(t) for different ordered fractional derivatives using Atangana-Baleanu fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the zombies decrease but the zombie's group can not be completely eradicated since there is no availability of vaccine and this is also because the proposed model shows that the removed group of zombies can also restore to life.

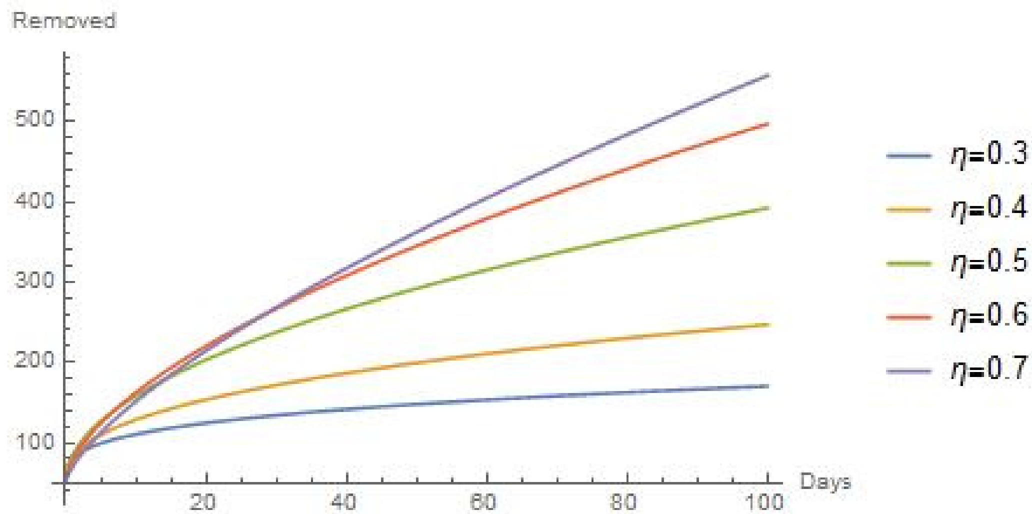

Figure 13: Numerical simulation of R(t) for different ordered fractional derivatives using Atangana-Baleanu fractional derivative for the parameter value $\varsigma = 0.1, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$. This shows that on increasing the order of fractional derivative (close to 1) the removed individuals keep on increasing, since the only way to get out of the infection is to destroy the zombies.

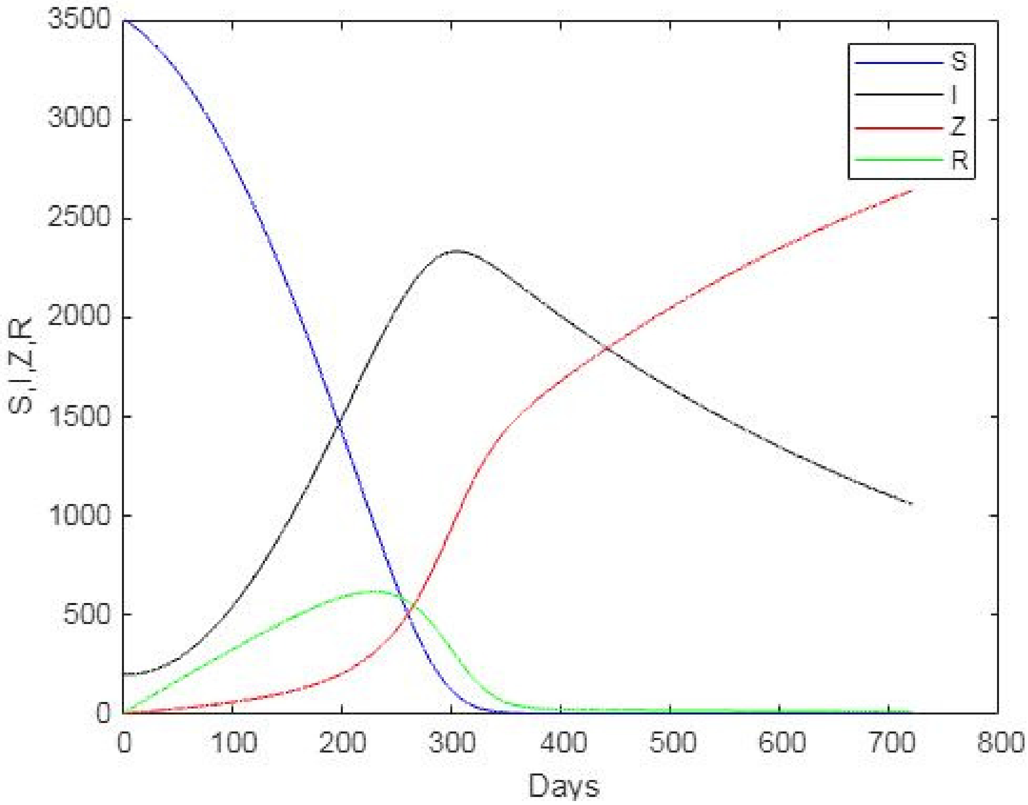

Figure 14: Numerical simulation of SIZR for $\varsigma = 0.01$ and $\eta = 0.5, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$ using Atangana-Baleanu fractional derivative. This shows that due to the unavailability of vaccine or any other source to eradicate the infection, the zombies keep on increasing and there is no solution except demolishing their head and moving them to the removed group. The graph also justifies the proposed model that the removed group can also restore to life and again become a zombie and hence the population of zombies keep on increasing.

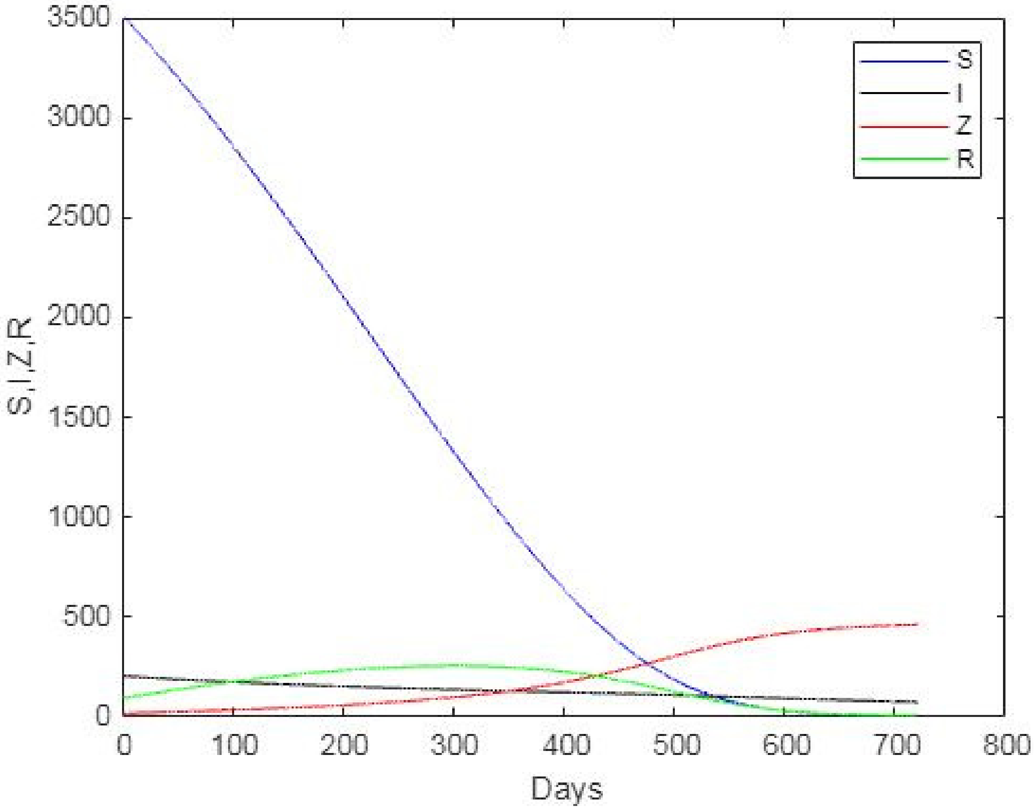

Figure 15: Numerical simulation of SIZR for $\varsigma = 0.1$ and $\eta = 0.7, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$ using Atangana-Baleanu fractional derivative. This shows that the the infection can not be eradicated but with the effect of time dependency on the infection rate and the continuous increase in the number of removed group, the infection can be minimized to great extent and hence the zombie's population can be controlled.

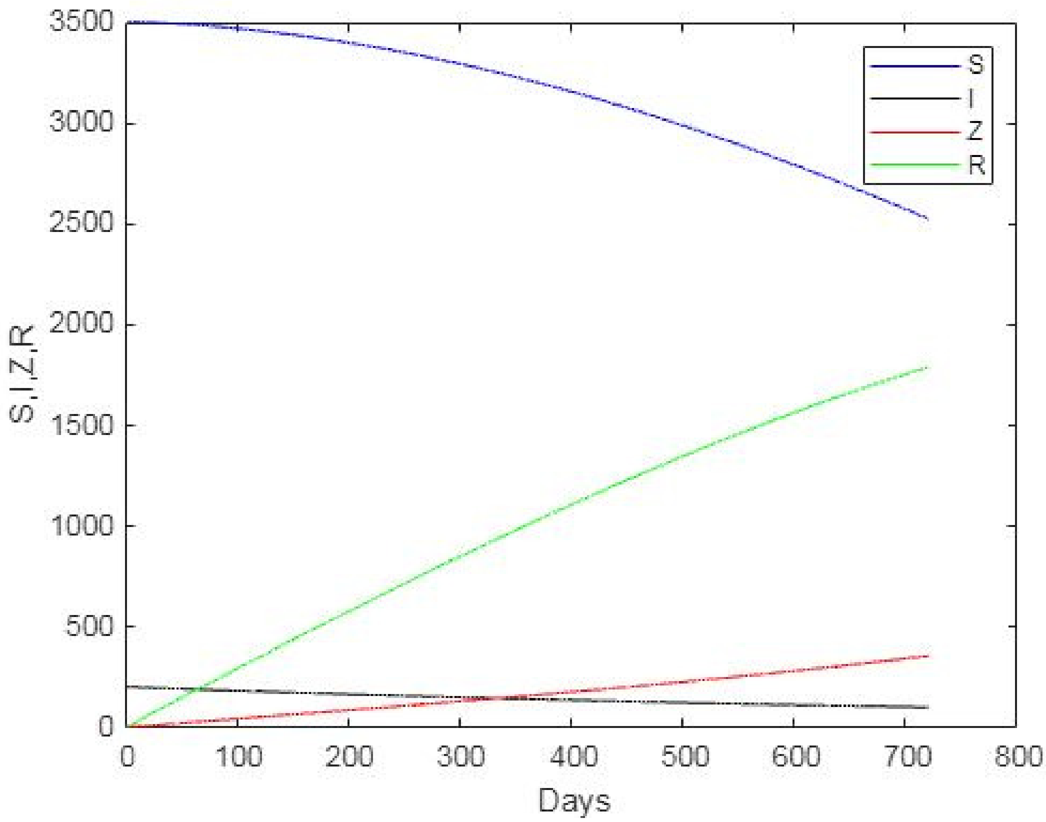

Figure 16: Numerical simulation of SIZR for $\varsigma = 0.1$ and $\eta = 0.8, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$ using Atangana-Baleanu fractional derivative. This shows that the the infection can not be eradicated but with the effect of time dependency on the infection rate and due to increasing fractional order there is continuous increase in the number of removed group, the infection can be minimized to great extent and hence the zombie's population can be controlled.

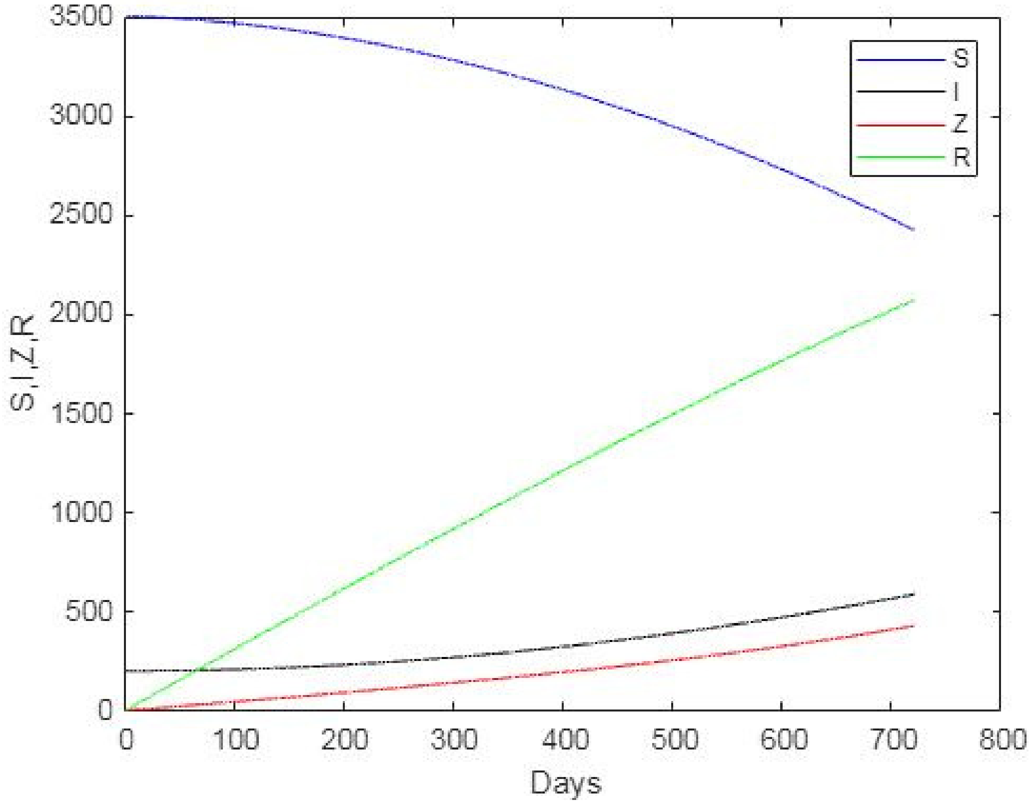

Figure 17: Numerical simulation of SIZR for $\varsigma = 0.1$ and $\eta = 0.9, \omega = 0.0001, \chi = 0.005, \delta = 0.0001, \rho = 0.005$ using Atangana-Baleanu fractional derivative. This shows that the the infection can not be eradicated but with the effect of time dependency on the infection rate and due to increasing fractional order there is continuous increase in the number of removed group, the infection can be minimized to great extent and hence the zombie's population can be controlled.

7 Conclusion and Future Directions

In this work, we have considered the SIZR model for the zombie infection with time dependent infection rate. The SIZR model is then extended to the fractional order using Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivative operators. The existence and uniqueness of the solution of the fractional SIZR model in the sense of each fractional derivative operator along with their numerical solutions are briefed. Graphical representations provide us with a better understanding of the fractional SIZR model. This SIZR model is very different from the other infectious disease SIR and SEIR models as in this SIZR model the removed zombies can again retort to life. This is an unexpected plot which are obviously not practical in real life but with this SIZR model we have tried to illustrate the significance of mathematical modeling in different circumstances and challenges in the field of medical science. Figure 2,3,4,5 represents the behaviour of the susceptible, infected, zombie and removed individuals for different fractional ordered derivative η in sense of Caputo fractional derivative operator. Figure 6,7,8,9 represents the behaviour of the susceptible, infected, zombie and removed individuals for different fractional ordered derivative η in sense of Caputo-Fabrizio fractional derivative operator. Figure 10,11,12,13 represents the behaviour of the susceptible, infected, zombie and removed individuals for different fractional ordered derivative η in sense of Atangana-Baleanu fractional derivative operator. In figure 14, 15, 16, 17 the impact of parameter ς along with different fractional order is represented. We have seen that for $\varsigma = 0.01$ the infection is very high and for $\varsigma = 0.1$, the infection rate has a sharp decline. This shows the effect of time dependency on the infection rate β_0. Furthermore, we concluded that the zombie's infection on humans is very disastrous. Although strong and rigid quarantine can be helpful in extirpating the infection. A cure with proper vaccination can also be a way in eliminating the infection. In the future, we can also consider SIZR model for zombie infection with parameters involving quarantine and vaccination. Lastly, as observed in movies, we conclude that zombie infection can open on to the destruction and collapse of human development and it is imperious to deal with zombies as early as possible, otherwise this will put the civilization in unresting and destructive circumstances.

8 Conflict of Interest:

The authors declare no conflict of interest.

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