A Computational Approach for the Classifications of 8-dimensional Nilsolitons

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Abstract

In this study, we present an algorithm in MATLAB to classify 8 dimensional non-abelian nilsoliton metric Lie algebras with singular Gram matrices. With this algorithm, we can compute Lie brackets, structure constants, index, rank and eigenvalues of the nilsoliton derivation.

This paper can be considered as a follow up paper to our previous study [12]. In our previous paper, we defined an algorithm that helps to classify algebras that admit simple derivations and singular Gram matrices U. Since the Gram matrices are singular, there exists infinitely many solutions to $Uv = [1]_m$, where the solutions are exactly the structure constants' squares. In order the algebra to be a Lie algebra, the structure constants has to satisfy the Jacobi identity. In our previous work, we did not present a methodology to classify algebras that satisfy Jacobi identity. But in this paper, we extend the capability in such a way that we are able to create and solve the Jacobi identity/identities with the help of computer algorithms for each index set. Therefore we completely classify all indecomposible nilsolitons in dimension 8. Several examples are provided for the illustration of the methodology.

1 Introduction

Symbolic computation is an area of mathematics which deals with developing, executing and applying the algorithms to manipulate and analyze the mathematical expressions or other mathematical objects. It concerns with the formulation of algorithms to find mostly exact solutions of symbolic mathematical problems and also concerns with the implementation of these algorithms in terms of the operations and control structures available in computer algebra programming languages such as MATLAB, GAP, Fortran etc. It is a useful tool since it does the computations more productively and accurately than doing by hand, or does the computations that are almost impossible to carry out by hand.

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Recently, symbolic computation methods have been used for Lie algebra theory which is mostly result highly complex symbolic expressions that are very difficult to carryout without the aid of computer algorithms. But introduction and implementation of new algorithms make it possible to work with Lie algebras that are too big to deal with by hand. These days, most of the fundamental objects occurring structure theory of Lie algebras can be constructed by computer algorithms. For example, for a given algebra one can develop algorithms for constructing quotient algebras, its centralizer, normalizer or minimal matrix representations etc.[4, 3, 7, 9, 21].

There are three methods to represent a Lie algebra and its related structures: Representing a Lie algebra as a linear Lie algebra, i.e. subalgebra of $gl(n)$, using table of its structure constants or using generators and relations [7]. In this paper, we use the table of structure constants related to a Lie algebra. We vary the Lie algebra structure by finding structure constants. Namely we determine a Lie algebra η with a fixed basis $\{X_i : 1 \leq i \leq n\}$ explicitly by given multiplication table, consisting of structure constants α_{ij}^k which are defined by the relations

$$[X_i, X_j] = \sum \alpha_{ij}^k X_k. \tag{1}$$

In order to use computer algebra systems, we encode nonzero structure constants by using index set $\Lambda = \{(i,j,k) | \alpha_{ij}^k \neq 0, i < j < k\}$ ignoring repetitions due to skew-symmetry. While indexing the structure constants, we use triples $(i,j,k) \in \Lambda$ such that $i < j < k$, and if $(i,j,k),(i,j,m) \in \Lambda$ then $k = m$ and $(i,j_1,k),(i,j_2,k) \in \Lambda$ then $j_1 = j_2$. Therefore we fix a basis $\{X_1, ..., X_n\}$ for a nilpotent Lie algebra η with $[Xi, Xj] = \sum \alpha_{ij}^k X_k \neq 0$ such that for every $i, j, \#\{k : \alpha_{ij}^k \neq 0\} \leq 1$, and for every $i, k, \#\{j : \alpha_{ij}^k \neq 0\} \leq 1$. Such basis $\{X_j\}$ is called nice and defined by Nikaloyevski in [16].

It is possible to define several different Riemannian metrics on Lie groups. Considering any Riemannian metrics, Einstein metrics are the most preferable metric, as the Ricci tensor complies the Einstein metric: $Ric = cg$ for some constant $c \in \mathbb{R}$. But, it is not possible to define Einstein metrics on non-abelian nilpotent Lie algebras, therefore we consider the following weaker condition on a left invariant metric g on a nilpotent Lie group G:

$$Ric_g = \beta I + D \tag{2}$$

for some $\beta \in \mathbb{R}$ and $D \in Der(\eta)$, where Ric_g denotes the Ricci operator of (η, g), η is the Lie algebra of G and $Der(\eta)$ denotes the Lie algebra of derivations of η. Equation 2 is called nilsoliton condition, D is called nilsoliton derivation, and β is called nilsoliton constant.

Nilsolitons are an important topic in mathematics for several reasons. First, nilsoliton metric Lie algebras are unique up to isometry and scaling. In [14], Theorem 2.11 states that a nilpotent Lie algebra η is an Einstein nilradical if and only if η admits a nilsoliton metric. Therefore it indicates that classification of nilsoliton metrics on a nilpotent Lie algebra is equivalent to the same of Einstein nilradicals. On the other hand, an Einstein solvmanifold δ can completely be determined by the Lie algebra $\eta = [\delta, \delta]$. Therefore the study of solvmanifolds are actually the study of nilsolitons. See [14] and [15] for a survey on nilsoliton metric Lie algebras.

Nilsoliton metrics are classified in different dimensions by several approaches [2, 5, 8, 13, 14, 19, 22] and [16]. In dimension 7 and 8 with non-singular Gram matrix in [10] and also dimension 7 in [5] and [6]. Also in [11] the ordered type of nilsoliton metrics in higher dimensions were classified. In this paper, we classify 8-dimensinal nilsoliton metric Lie algebras with simple nilsoliton derivation D corresponding to a singular Gram matrix. By using that any nilpotent Lie algebra of dimension less than or equal to 6 is an Einstein nil-radical, and that all nilsoliton metric Lie algebras are classified in dimension 7, by Theorem 2.2, one obtains that any decomposable 8-dimensional nilpotent Lie algebra is an Einstein nilradical and it is easy to give a nilsoliton metric in each case; therefore we focus on studying indecomposable algebras.

This paper can be considered as a continuation paper to our last paper. In our last paper, we defined an algorithm which prunes algebras with non-simple derivation and with non singular Gram matrix [12]. But we have not considered the Jacobi identity. This paper takes care of this issue. By this study, we complete the classification of nilsolitons in dimension 8, because we have already classified 8 dimensional nilsoliton metric Lie algebras with simple derivation, where the corresponding Gram matrix is nonsingular [10]. In dimension 8, the classifications of such nilsoliton metric Lie algebras corresponding to a singular Gram matrix can be found in the following theorem:

Theorem 1.1. *Let (η, Q) be a 8-dimensional nilsoliton metric Lie algebra with nilsoliton derivation D having distinct positive eigenvalues. Suppose that the canonical Gram matrix U is singular. Then (η, Q) is homothetic to precisely one of the 8-dimensional nilsoliton Lie algebras listed in Table I, Table II, Table III and Table IV.*

Proof. The proof of Theorem 1.1 rely on a computational procedure implemented in the computer algebra system Matlab. □

This paper consists of six sections. In the second section, we present preliminary background to construct the symbolic computation algorithm. In the third section, we prove necessary theorems that helps us to calculate structural elements of a nilsolitons with the help of the algorithm. In fourth section, we present the algorithm, and give some examples to illustrate the steps of it. In the fifth section, we present some remarks regarding to the computation results. And in sixth section, we present the classification tables, and the notations with the several examples.

The computational procedure was implemented using Matlab R2018a on Intel(R)Core(TM) i3-5015U CPU at 2.10 GHz processor and 4 GB of RAM.

2 Preliminaries

Let (η_μ, Q) be a metric algebra , where $\mu \in \Lambda^2\eta \otimes \eta^*$. Let $B = \{X_i\}_{i=1}^n$ be a Q-orthonormal basis of η_μ (We always assume that basis are ordered). The nil-Ricci endomorphism Ric_μ is defined as $< Ric_\mu X, Y >= ric_\mu(X, Y)$, where

$$ric_\mu(X,Y) = -\frac{1}{2}\sum_{i=1}^{n} < [X,X_i],[Y,X_i] > + \frac{1}{4}\sum_{i=1}^{n} < [X_i,X_j],X >< [X_i,X_j],Y > \qquad (3)$$

for $X, Y \in \eta$ (We often write an inner product $Q(.,.)$ as $< .,. >$.). When η is a nilpotent Lie algebra, the nil-Ricci endomorphism is the Ricci endomorphism. If all elements of the basis are eigenvectors for the nil-Ricci endomorphism Ric_μ, we call the orthonormal basis a Ricci eigenvector basis.

Let $Der(\eta)$ denote the derivation algebra of η . A maximal abelian subalgebra of $Der(\eta)$ comprised of semisimple elements is called a maximal torus. The dimension of a maximal torus is called the rank of η.

Let η be a nilpotent Lie algebra. The lower central series for the Lie algebra η is defined by $\eta^0 = \eta$, and $\eta^k = [\eta, \eta^{k-1}]$ for $k \geq 1$. If $\eta^r = 0$ and $\eta^{r-1} \neq 0$, then η is called an $r-$ step nilpotent Lie algebra, and r is called nilpotency index of the Lie algebra η.

Suppose that $\Lambda = \{(i,j,k)|\alpha_{ij}^k \neq 0, i < j < k\}$ a finite set which indexes the set of nonzero structure constants corresponding to a Lie algebra η, ignoring repetitions due to skew-symmetry. For $1 \leq i,j,k \leq n$, we define $1 \times n$ row vector y_{ij}^k to be $\epsilon_i^T + \epsilon_j^T - \epsilon_k^t$, where $\{\epsilon\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$. We call the vectors in $\{y_{ij}^k|(i,j,k) \in \Lambda\}$ *root vectors* for Λ. Let $y_1, y_2, ..., y_m$ (where $m = |\Lambda|$) be an enumeration of the root vectors in dictionary order. We define *root matrix* Y_Λ for Λ to be the $m \times n$ matrix whose rows are the root vectors $y_1, y_2, ..., y_m$. The *Gram matrix* U_Λ for Λ is the $m \times m$ matrix defined by $U_\Lambda = Y_\Lambda Y_\Lambda^T$; the (i,j) entry of U_Λ is the inner product of the i th and j th root vectors. From Theorem 5 in [18] we know that U is a symmetric matrix where its all diagonal entries are 3 and its off-diagonal entries are in the set $\{-2,-1,0,1,2\}$. Nikolayevski showed that every Lie algebra admitting a derivation with all the eigenvalues of multiplicity one has a nice basis [16], we use this type of basis in our classifications. This way our Gram matrices corresponding to metric nilpotent Lie algebras does not have a 2 as an entree (Lemma 2 in [10]).

Now, suppose that $|\Lambda| = m$ and $[1]_m$ represents a column vector $[1\,1\,1\,..\,1]^T$ in $\mathbb{R}^m$.

Theorem 2.1. *(Theorem 1 in[18]) Let η be a nonabelian metric algebra with Ricci eigenvector basis B . Let U and $[\alpha^2]$ be the Gram matrix and the structure vector for η with respect to B . Then η satisfies the nilsoliton condition with nilsoliton constant β if and only if $U[\alpha^2] = 2\beta[1]_m$.*

Above theorem indicates a Lie algebra η admits a nilsoliton metric iff there exists a solution $v \in \mathbb{R}^m$ of the Linear system $U_\Lambda v = [1]_m$ where all entries are positive real numbers.

Theorem 2.2. *(See Theorem 7 in [17]) Let a nilpotent Lie algebra η be the direct sum of η_1 and η_2. Then:*

(a) If D_i are pre-Einstein derivations for η_i, then $D = D_1 \oplus D_2$ derivation for η.

(b) The algebra η is an Einstein nilradical if and only if η_i are Einstein nilradicals.

Now we present the theorems which helps to create possible Jacobi identity equations:

Theorem 2.3. *([18]) Let η be an n-dimensional vector space, $B = \{X_i\}_{i=1}^n$ be a basis for η. Suppose that a set of nonzero structure constants $\alpha_{i,j}^k$ relative to B, indexed by Λ, defines a skew symmetric product on η. Assume that if $(i,j,k) \in \Lambda$, then $i < j < k$. Then the algebra is a Lie algebra if and only if whenever there exists m so that the inner product of root vectors $< y_{ij}^l, y_{lk}^m >= -1$ for triples (i,j,l) and (l,k,m) or (k,l,m) in Λ, the equation*

$$\sum_{s<m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{4}$$

holds. Furthermore, a term of form $\alpha_{i,j}^l \alpha_{l,k}^m$ is nonzero if and only if $< y_{i,j}^l, y_{l,k}^m >= -1$.

Lemma 2.4. *(Lemma 2.8 in [10]) Let η be an n-dimensional nonabelian nilpotent Lie algebra. Suppose that η admits a derivation D having distinct real positive eigenvalues. Let B be a basis consisting of eigenvectors for the derivation D, and let Λ index the nonzero structure constants with respect to B. Let Y be the $m \times n$ root matrix for Λ. If $rank(Y) = m$, then the following hold.*

1. $|\Lambda| \leq n - 1$.
2. *If $(i_1, j_1, k_1) \in \Lambda$ and $(i_2, j_2, k_2) \in \Lambda$, then $< y_{i_1 j_1}^{k_1}, y_{i_2 j_2}^{k_2} >\neq -1$*

Theorem 2.5. *(Theorem 3.2 in [12]) Let η be an n dimensional nonabelian nilpotent Lie algebra which admits a soliton derivation D having distinct real positive eigenvalues, where $n > 5$. Suppose that B is a basis consisting of eigenvectors for the derivation D, and Λ indexes the nonzero structure constants with respect to B. If the corresponding Gram-matrix U is singular, then $|\Lambda| \geq 5$.*

Considering Theorem 2.3 together with Lemma 2.4, one can see that the Jacobi identity is always satisfied when rank of the root matrix is 8 and therefore the Gram matrix is non-singular.

3 Computing the Structural Elements of the Nilsolitons

In this section, we introduce theorem-like environments for providing algorithmic approach to compute structural elements of the nilsoliton metric Lie algebras.

In order an algebra to be a Lie algebra, we need to satisfy the Jacobi identity. Using our index set Λ, the corresponding Jacobi identity turns into the Equation 4. Also, in the Jacobi identity, there has to be at least two product couples. Otherwise if there is one product couple in Jacobi identity, it leads to $\alpha_{i,j}^s \alpha_{s,k}^m = 0$, therefore it contradicts the fact that $(i,j,s), (s,k,m) \in \Lambda$.

In some cases, there can be more than two square root product couples in the Jacobi equation 4. In that case, we need to consider all the cases of the signs between the product couples $\alpha_{i,j}^s \alpha_{s,k}^m$, $\alpha_{j,k}^s \alpha_{s,i}^m$, and $\alpha_{k,i}^s \alpha_{s,j}^m$. Following proposition deals with this matter.

Lemma 3.1. *If there are k product couples in the Jacobi identity, then there are $2^{k-1}-1$ possible sign choices.*

Proof. Suppose that there are t product couples in the Jacobi identity for the same number m in Equation 4 , and

$$P_m = \{p_1, p_2, ..., p_t : \ \ p_s = \alpha_{ij}^{r_s}\alpha_{r_s,k}^{m}, \ \ 1 \leq s \leq t\} \tag{5}$$

is the set of all product couples related to the Jacobi identity for m. Therefore the Jacobi identity turns into $p_1 \mp p_2 \mp ... \mp p_t = 0$. Without loss of generality, we assume that $p_1 > 0$. For each $p_s \in P_m$, where $2 \leq s \leq t$ there are two possible sign choices $\{+, -\}$. Therefore we have 2^{t-1} possible sign choices. Since all the product couples are nonzero, then they can not all be $+$. Therefore we drop the case $(+, +, ..., +)$. Thus there are $2^{t-1} - 1$ possible sign choices for the set P_m. □

Sign Choices: By above lemma, if there are t product couples in the Jacobi identity for same m, to encode those $2^{t-1} - 1$ number of sign choices we define a $(2^{t-1} - 1) \times (t - 1)$ matrix $SC = [s_{js}]$ such that for each $j \in \{1, 2, ..., 2^{t-1} - 1\}$

$$s_{js} = \begin{cases} 1 & p_s > 0, \\ 0 & p_s < 0 \end{cases}$$

The jth Jacobi identity is computed by $p_1 + \sum_{i=1}^{t-1}(-1)^{s_{ji}}p_i = 0$. Therefore each Jacobi identity is of form $p_1 \mp p_2 \mp ... \mp p_t = 0$. For example, if there are 3 product couples, i.e.,if $p_1 = \alpha_{i_1,j_1}^{s_1}\alpha_{s_1,k_1}^{m}, p_2 = \alpha_{i_2,j_2}^{s_2}\alpha_{s_2,k_2}^{m}, p_3 = \alpha_{i_3,j_3}^{s_3}\alpha_{s_3,k_3}^{m}$ corresponds to an algebra η, then the matrix SC is as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Possible Jacobi identities are

$$\begin{aligned} p_1 + (-1)^{s_{11}}p_2 + (-1)^{s_{12}}p_3 &= 0 \\ p_1 + (-1)^{s_{21}}p_2 + (-1)^{s_{22}}p_3 &= 0 \\ p_1 + (-1)^{s_{31}}p_2 + (-1)^{s_{32}}p_3 &= 0. \end{aligned}$$

In order η to be a Lie algebra, at least one of the above equations has to be satisfied.

Additionally, if there is more than one array of number m's in equation 4 in the Theorem 2.3, then one needs to find common solutions of at least one Jacobi identity for each m. The following example is.....

The following proposition and its following corollary helps us to compute rank of a nilsoliton metric Lie algebra.

Proposition 3.2. *(See Proposition 4.7 in [20].) Let η be a nonabelian Lie algebra that admits a simple derivation D. Let $B = \{Xi\}_{i=1}^{n}$ be an eigenvector basis with index set , and let Y be*

the root matrix associated to . Then the rank of the nilsoliton metric Lie algebra η equals to the nullity of the root matrix Y.

Corollary 3.3. *Let η be a nonabelian $n-$ dimensional Lie algebra that admits a simple derivation. Let B be an eigenvector basis with index set , and let Y be the root matrix associated to . Then*

$$rank(\eta) = n + nullity(YY^t) - |\Lambda|. \tag{6}$$

Proof. Suppose that η is an $n-$ dimensional non-abelian Lie algebra admitting a simple derivation. Let B be an eigenvector basis with index set where $|\Lambda| = m$, and let Y be the root matrix associated to . By Proposition 3.2, the root matrix Y is an $m \times n$ matrix, whose nullity is $rank(\eta)$. Therefore, from rank-nullity theorem, $rank(\eta) = n - Rank(Y)$. We also know that $rank(Y) = rank(YY^T)$, therefore we have

$$rank(\eta) = n - Rank(YY^T).$$

On the other hand, YY^T is an $m \times m$ matrix. Therefore $Rank(YY^T) = m - Nullity(YY^T)$. Then we have

$$\begin{aligned} rank(\eta) &= n - (m - Nullity(YY^T)) \\ &= n + Nullity(YY^T) - |\Lambda|. \end{aligned}$$

□

As a result of above corollary, and $YY^T = U$, we compute the rank of nilsoliton metric Lie algebra by $rank(n) = 8 + nullity(U) - |\Lambda|$.

Definition 3.4. *Let $X \in \eta$, ad_X denotes the adjoint representation and η^* denotes the dual of the Lie algebra η. Then the skew symmetric bilinear form Ψ_f where $f \in \eta^*$ is defined by*

$$\begin{aligned} \Psi_f : \eta \times \eta \to & \quad \mathbb{R} \\ (X, Y) \to & \quad \Psi_f(X, Y) \quad = f([X, Y]). \end{aligned}$$

The index of a Lie algebra η is the integer $inf\{dim\eta_f : f \in \eta^*\}$ where $\eta_f = ker(\Psi_f)$ defined by $\eta_f = \{X \in \eta : f([X, Y]) = 0, \forall y \in \eta\}$.

Proposition 3.5. *(see Proposition 4 in [1]) The index of a n-dimensional Lie algebra η is the integer*

$$index_\eta = n - Rank_{R(\eta)}([X_i, X_j])_{1 \le i,j \le n} \tag{7}$$

where $R(\eta)$ is the quotient field of symmetric algebra $S(\eta)$.

Remark 3.6. *Above proposition tells us that the index of a Lie algebra is the nullity of the matrix $E_\eta = ([X_i, X_j])$.*

Example 3.7. *Suppose that* $\Lambda = \{(1,2,3),(1,3,4),(2,3,5)\}$*is the index set of an algebra* η *of dimension 6. Its corresponding Gram matrix is*

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3. \end{pmatrix}$$

Since there is no -1 *entry,* η *is a Lie algebra. The matrix* E_η *is as follows*

$$\begin{pmatrix} 0 & X_3 & X_4 & 0 & 0 & 0 \\ -X_3 & 0 & X_5 & 0 & 0 & 0 \\ -X4 & -X_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its rank is 2, therefore $index_\eta = 4$.

4 Algorithm

In our previous paper [12], we have eliminated the rows of W that correspond to a unique -1 inner product of root vectors, nonsingular Gram matrices, abelian algebras, and non-simple derivations. We also have pruned some of the rows of W that correspond to algebras that does not satisfy Jacobi identity in the 14th step of the algorithm. For each algebra in the row of W matrix, we first obtain the array of number m's in equation 4 in the Theorem 2.3. We call the array as Arrayofms. Different number m's correspond to different equations. Thus if there is a unique m for Λ, there exists a unique couple of indexes $\{(i,j,l),(l,k,m)\}$ or $\{(i,j,l),(k,l,m)\}$ in Λ. For this m, the Jacobi identity is $\alpha^l_{ij}\alpha^m_{lk} = 0$ which implies that at least one of α^l_{ij} or α^m_{lk} is zero. This means the algebra does not satisfy Jacobi identity for positive α^k_{ij} structure constant. We eliminate these cases in our algorithm. The following is an example of such case:

Example 4.1. *Let* $\Lambda = \{(1,2,3),(1,6,7),(3,4,7),(3,5,6),(3,6,8)\}$*, then* -1 *entries are coming from* $\{(1,2,3),(3,4,7)\}$*,* $\{(1,2,3),(3,5,6)\}$*,* $\{(1,2,3),(3,6,8)\}$ *and* $\{(1,6,7),(3,5,6)\}$*. Then,* $Arrayofms = \{7,6,8,6\}$ *respectively. As can be seen,* $m = 7$ *is coming only from indexes* $\{(1,2,3),(3,4,7)\}$*. So, corresponding Jacobi identity equation is* $\alpha^3_{12}.\alpha^7_{34} = 0$*. We eliminate this case.*

So far, we have eliminated the rows that correspond to zero structure constants. But this does not eliminate the rows that does not correspond to a nilsoliton metric Lie algebra. What we need to do is to compute Jacobi identity equation/equations-which are non-linear equations-for each algebra, and find out if there exists a common solution to the system of non-linear equations.

For this, we first start with dividing the W matrix in terms of its nullity type. The maximum nullity is 5, therefore we need to create five sub matrices of W, $WNullity1$ to $WNullity5$ which consist of the rows of W.

Now we present the algorithm that we have used for our classifications.

Input: The input is the dimension.

Output: The outputs of this algorithm are: index set, eigenvalue of the nilsoliton derivation, rank, step and the index of the Lie algebra, and structure constants of the nilsoliton metric Lie algebra.

The main algorithm of this study is the following one:

1. Enter the dimension η.

2. Compute Z matrix: The matrix whose rows are elements of the index set Λ.

3. Compute the root matrix Y and the Gram matrix U.

4. Use the algorithm in [12] to compute prunned W matrix.

5. Use Theorem 2.5 to eliminate the rows of W that corresponds to an index set Λ where $|\Lambda| < 5$

6. Divide W into 5 submatrices in terms of nullity of Gram matrices.

7. For each submatrix, compute Arrayofms, subindex set Λ_s

8. Compute Sign table of each structure constant in Jacobi identity.

9. Compute Jacobi identity for each different entries in Arrayofms.

10. Find common solution of Jacobi identities for each entree in Arrayofms.

11. Compute the rank, step, index and eigenvalues of the derivation of the Nilsoliton metric Lie algebra

If there is more than one entry in Arrayofms matrix, then we need to find a common solution of Jacobi identities for each m.

Example 4.2. *Let* $\Lambda = \{(1,2,4),(1,4,5),(1,5,7),(2,3,6),(2,7,8),(3,6,8),(4,5,8)\}$*, then* $m = 8$*, and the Jacobi identity is:* $\sqrt{4/9-x}\sqrt{1/3-x} - \sqrt{x}\sqrt{x} = 0$ *whose solution is* $x = 4/21$*. Therefore* Λ *corresponds to a nilsoliton metric Lie algebra with corresponding structure constant squares are as the following:*

$$((\alpha^4_{1,2})^2,(\alpha^5_{1,4})^2,(\alpha^7_{1,5})^2,(\alpha^6_{2,3})^2,(\alpha^8_{2,7})^2,(\alpha^8_{3,6})^2,(\alpha^8_{4,5})^2) = \frac{1}{63}(12,21,16,14,9,14,12),$$

and the eigenvalues of nilsoliton derivation are $D = \frac{1}{18}(2,4,5,6,8,9,10,14)$*. This nilsoliton metric Lie algebra appears in row 75 of Table 2.*

Now we give an example of an algebra with more than one entry in Arrayofms that does not satisfy Jacobi Identity:

Example 4.3. *Suppose that* $\Lambda = \{(1,4,5),(1,5,6),(1,6,7),(2,7,8),(3,4,6),(3,5,7)\}$*. Then* $Arrayofms = [7\ \ 8]$*, and the Jacobi identities are:*

- *for* $m = 7$*,* $\sqrt{25/33-x}\sqrt{58/99} = \sqrt{58/99}\sqrt{x}$*,*

- *for* $m = 8$, $\sqrt{25/33 - x}\sqrt{19/33 - x} = \sqrt{x - 1/19}\sqrt{x}$

The solution to the first equation is $x = 25/66$, *and the solution to the second equation is* $x = 475/1331$. *Clearly no common solution to the Jacobi identities. Therefore the index set does not correspond to a Lie algebra.*

5 Remarks

Remark 5.1. *After applying above algorithm, we find a counterexample of the reverse of Lemma 2.4. Therefore if non of the inner product of root vectors is* -1 *does not imply that the Gram matrix of the nilsoliton is nonsingular. Additionally, if the kardinality of the index set* $|\Lambda| \leq n - 1$ *does not imply that the Gram matrix is nonsingular. The following example is one of the examples of this case.*

Example 5.2. *Suppose that* η *be an algebra which is indexed by the following index set:*

$$\Lambda = \{(1,2,5),(1,3,6),(1,5,7),(1,6,8),(2,4,7),(2,5,8),(3,4,8)\}. \tag{8}$$

Its corresponding Gram matrix is as follows.

$$\begin{pmatrix} 3 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}. \tag{9}$$

It is a singular matrix with nullity $= 1$. *Also, it does not have* -1 *entry. From Theorem 2.3, it is a Lie algebra. Also, since the solution space of* $U.v = [1]$ *is* $\{v = (t + 1/19, 6/19 - t, t, 5/19 - t, 5/19 - t, 3/19, t) | 0 < t < 5/19\}$ *from theorem ... in [18] that it is a nilsoliton metric Lie algebra with the magnitudes of the structure constants* $|\alpha^5_{1,2}| = t + 1/19$, $|\alpha^6_{1,3}| = 6/19 - t$, $|\alpha^7_{1,5}| = t$, $|\alpha^8_{1,6}| = 5/19 - t$, $|\alpha^7_{2,4}| = 5/19 - t$, $|\alpha^8_{2,5}| = 3/19$ *and* $|\alpha^8_{3,4}| = t$ *with simple derivation of type* $7 < 10 < 13 < 14 < 17 < 20 < 24 < 27$ *with singular Gram matrix* U *and* $|\Lambda| = n - 1$. *We have two examples of this type which appear* #63 *in Table 2, and* #112 *in Table 3.*

Remark 5.3. *After running the algorithm, we could not find any 2-step nilpotent Lie algebra with soliton metric in dimension* 8 *where the corresponding Gram matrices are singular. As we combine this result with the results of our previous paper [10], we conclude that there is no 2-step nilpotent Lie algebra endowed with a soliton metric in dimension* 8.

6 Classifications

Classification results for dimensions 8 appear in Table 1 , Table 2 and in Table 3. By this classification, we complete all nilsoliton metric Lie algebras with simple nilsoliton derivation in

dimension 8. Please note that the Lie algebras appear in Table 1 and Table 2 are the ones whose nullity of the Gram matrix is 1. We use vector notation to represent Lie algebra structures, as we did in [10] and [11].

Notation:

1. Lie Bracket column illustrates both structure constants and how Lie bracket functions on a given ordered eigen-vector bases $\{X_i\}_{i=1}^{8}$. In the following, we give an example to illustrate how to read the tables in this section:

   Example 6.1. *In table 2, line 118, Lie bracket type:*

   $$(0, 0, 2\sqrt{3}.12, 0, 2\sqrt{5}.13, \sqrt{15}.15, 2\sqrt{3}.24, 2\sqrt{3}.26 + \sqrt{15}.35) \tag{10}$$

   shows structure constants together with Lie bracket as follows:

   $$\begin{aligned} &[X_1, X_2] = \sqrt{3}X_3, \qquad [X_1, X_3] = \sqrt{5}X_5, \qquad [X_1, X_5] = \sqrt{15}X_6 \\ &[X_2, X_4] = \sqrt{3}X_7, \qquad [X_2, X_6] = \sqrt{3}X_8, \qquad [X_3, X_5] = \sqrt{15}X_8. \end{aligned} \tag{11}$$

   The eigenvalues of the nilsoliton derivation are $\frac{1}{120}(13, 24, 37, 48, 50, 63, 72, 87)$. *Therefore the nilsoliton is of type* $13 < 24 < 37 < 48 < 50 < 63 < 72 < 87$.

2. AoM shows how many Jacobi identities that the Lie algebra satisfies. This is encoded as $arrayofms$ matrix which consists of the distinct entries from $Arrayofms$ as we define in Section 2. For example in Table 2, #77, AoM is 2, because there are two entries in the $arrayofms$ matrix [7, 8], which means that the number m's in the Equation 2.3 are 7 and 8. Therefore there are two Jacobi identities as the following: For $m = 7$, the Jacobi identity is

   $$\sqrt{\frac{13}{20} - x}\sqrt{\frac{4}{5} - x} = \sqrt{x - \frac{3}{20}}\sqrt{x}, \tag{12}$$

   and for m=8, the Jacobi identity is

   $$\sqrt{\frac{4}{5} - x}\sqrt{\frac{3}{5}} = \sqrt{\frac{3}{5}}\sqrt{x} \tag{13}$$

   in which both is satisfied for $x = 2/5$. Therefore since Jacobi identity is satisfied, it is a Lie algebra. On the other hand, this solution leads to the squares of the structure constants vector $v = (\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5})$. As can be seen all of the elements of the vector are positives. Thus the Lie algebra is nilsoliton. With the rescaling we have $(0, 0, 0, \sqrt{15}.12, 2\sqrt{5}.14, \sqrt{15}.15, 2\sqrt{6}.26 + 2\sqrt{6}.45, 6.37)$.

3. The E/VPA column, the letter E shows that the solution of the Jacobi identity is exact. VPA shows that the solution is approximated using VPA (Variable-precision arithmetic).

Please note that VPA evaluates each element of the symbolic input x to at least 32 significant digits. There is also $-$ symbol appears in #63 in Table 2, and #112 in Table 3, meaning that there is no -1 entry in their Gram matrix, therefore there is the Jacobi identity is always satisfied for any number such that all entries of the solution are positive numbers.

4. The Rank column illustrates the dimension of maximal torus of derivations.

5. The Index column illustrates the index of the Lie algebra.

6. The NI column illustrates the nilpotency index of the Lie algebra, i.e. the length of the lower central series for the Lie algebra.

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	Lie Bracket	Derivation	AoM	E/VPA	Rank	Index	NI
1	$(0,0,0,0,\sqrt{414}.13,\sqrt{260}.14+\sqrt{391}.23,\sqrt{529}.15+\sqrt{338}.24,\sqrt{430}.17+\sqrt{559}.26+\sqrt{138}.35)$	$10<13<22<25<32<33<40<50$	1	E	1	2	4
2	$(0,\ 0,\ 0,\ 1.13,\ 1.23,\ 1.14,\ 1.16{+}1.25{+}1.34,\ 1.27{+}1.45\ )$	$\frac{256}{2159}<\frac{53}{298}<\frac{106}{447}<\frac{461}{1296}<\frac{371}{894}<\frac{599}{1263}<\frac{265}{447}<\frac{689}{894}$	1	VPA	1	2	5
3	$(0,\ 0,\ 0,\ 1.13,\ 1.14,\ 1.34,\ 1.15{+}1.23,\ 1.17{+}1.24{+}1.35)$	$4<42<53<57<96<105<120<164$	1	VPA	1	2	5
4	$(0,0,0,1.13,\ 1.14,\ 1.15,\ 1.16{+}1.23,1.27{+}\ 1.36{+}1.45)$	$1<4<5<6<7<8<9<13$	1	VPA	1	2	6
5	$(0,0,0,\sqrt{162}.13,\sqrt{33}.12,\sqrt{99}.15+\sqrt{135}.23,\sqrt{77}.14+\sqrt{110}.25,\sqrt{168}.16+\sqrt{140}.24)$	$32<43<54<66<75<87<88<119$	1	E	1	4	4
6	$(0,0,0,\sqrt{45}.13,\sqrt{6}.12,\sqrt{40}.14+\sqrt{19}.23,\sqrt{42}.25+\sqrt{22}.34,\sqrt{45}.16+\sqrt{19}.24)$	$46<60<82<86<106<121<125<167$	1	E	1	2	4
7	$(0,0,0,\sqrt{31}.13,2.12,3\sqrt{2}.14+5.23,\sqrt{14}.15+2\sqrt{5}.34,\sqrt{31}.16+5.24)$	$32<44<53<65<76<87<88<119$	1	E	1	4	4
8	$(0,\ 0,\ 0,\ 1.12,\ 1.13{,}1.23,\ 1.15{+}1.24,\ 1.27{+}1.36{+}1.45)$	$2<3<4<5<6<7<8<11$	1	VPA	1	2	4
9	$(0,0,0,1.12,\ 1.13,\ 1.14{+}1.23,\ 1.24,\ 1.27{+}1.36{+}1.45)$	$2<3<4<5<6<7<8<11$	1	VPA	1	2	4
10	$(0,\ 0,\ 0,\ 1.12,\ 1.13,\ 1.14{+}1.23,\ 1.15{+}1.24,\ 1.47{+}1.56)$	$2<3<4<5<6<7<8<13$	1	VPA	1	2	4
11	$(0,\ 0,\ 0,\ 1.12,\ 1.13,\ 1.14{+}1.23,\ 1.15{+}1.24,\ 1.37{+}1.46)$	$2<3<4<5<6<7<8<12$	1	VPA	1	2	4
12	$(0,0,0,\sqrt{755}.12,\sqrt{750}.13,\sqrt{1701}.14+\sqrt{258}.23,\sqrt{1782}.15+\sqrt{1075}.24,\sqrt{1911}.27+\sqrt{2002}.36)$	$2<3<4<5<6<7<8<11$	1	E	1	2	4
13	$(0,0,0,\sqrt{168}.12,\sqrt{162}.13,\sqrt{99}.14,\sqrt{7}.15+\sqrt{110}.24,\sqrt{33}.16+\sqrt{140}.25+\sqrt{135}.34)$	$2<3<4<5<6<7<8<9$	1	E	1	2	4
14	$(0,\ 0,\ 1.12,\ 0,\ 1.14,\ 1.15,\ 1.16{+}1.23,\ 1.27{+}1.36{+}1.45)$	$44<203<247<318<371<406<468<680$	1	VPA	1	2	5
15	$(0,0,\sqrt{11685}.12,0,\sqrt{13596}.13,\sqrt{16480}.23,\sqrt{20188}.15+\sqrt{7885}.24,\sqrt{13653}.17+\sqrt{12566}.26+\sqrt{9213}.34)$	$2<3<5<6<7<8<9<11$	1	E	1	2	5
16	$(0,\ 0,\ 1.12,\ 0,\ 1.13,\ 1.14{+}1.23,\ 1.15{+}1.24,\ 1.47{+}1.56)$	$\frac{53}{475}<\frac{285}{1703}<\frac{793}{2843}<\frac{65}{1945}<\frac{5068}{12979}<\frac{1175}{2633}<\frac{737}{1468}<\frac{3080}{3681}$	1	VPA	1	2	5
17	$(0,0,\sqrt{14820}.12,0,\sqrt{15774}.13,\sqrt{7885}.14+\sqrt{19120}.23,\sqrt{23422}.15,\sqrt{6692}.17+\sqrt{22464}.26+\sqrt{11952}.34)$	$2<3<5<6<7<8<9<11$	1	E	1	2	5
18	$(0,0,\sqrt{1020}.12,0,\sqrt{5610}.13,\sqrt{3071}.14+\sqrt{6800}.23,\sqrt{8330}.15+\sqrt{3984}.24,\sqrt{6364}.17+\sqrt{8256}.26)$	$2<3<5<6<7<8<9<11$	1	E	1	4	5
19	$(0,0,\sqrt{1533}.12,0,\sqrt{3120}.13,\sqrt{1022}.14+\sqrt{2720}.23,\sqrt{1533}.15+\sqrt{1460}.24,\sqrt{2574}.16+\sqrt{2244}.25)$	$2<3<5<6<7<8<9<10$	1	E	1	4	5
20	$(0,0,\sqrt{53}.12,\sqrt{82}.13,0,2\sqrt{3}.15,\sqrt{53}.14+\sqrt{34}.25,\sqrt{55}.27+\sqrt{55}.34+\sqrt{38}.56)$	$10<39<49<78<87<116<126<184$	1	E	1	2	5
21	$(0,0,\sqrt{11448}.12,\sqrt{12935}.13,\ 0,\ \sqrt{5771}.15,\ \sqrt{9646}.14+\sqrt{6766}.23,\ \sqrt{10044}.27{+}\sqrt{8463}.34{+}\sqrt{10945}.56)$	$3<61<64<122<174<232<235<351$	1	E	1	2	5
22	$(0,\ 0,\ 1.12,\ 1.13,\ 1.23,\ 1.14,\ 1.16{+}1.25,\ 1.37{+}1.46)$	$\frac{58}{579}<\frac{241}{1604}<\frac{445}{1777}<\frac{1104}{3149}<\frac{359}{896}<\frac{1176}{2609}<\frac{979}{1777}<\frac{3029}{3780}$	1	VPA	1	2	6
23	$(0,\ 0,\ 1.12,\ 1.13,\ 1.23,\ 1.14,\ 1.16{+}1.25,\ 1.27{+}1.36)$	$\frac{291}{2708}<\frac{59}{366}<\frac{817}{3041}<\frac{1415}{3762}<\frac{521}{1212}<\frac{457}{945}<\frac{172}{291}<\frac{1163}{1546}$	1	VPA	1	2	6
24	$(0,0,\sqrt{\frac{9381359}{51474290}}.12,\ \sqrt{\frac{55}{201}}.13,\ \sqrt{\frac{58}{497}}.23,\ \sqrt{\frac{6716831}{26617490}}.14,\ \sqrt{\frac{31}{497}}.16{+}\ \sqrt{\frac{50}{251}}.25,\ \sqrt{\frac{8121971}{51474290}}.26{+}\sqrt{\frac{22627}{103570}}.34)$	$\frac{231}{2014}<\frac{754}{4383}<\frac{162}{565}<\frac{2852}{7105}<\frac{506}{1103}<\frac{465}{901}<\frac{504}{799}<\frac{342}{497}$	1	E	1	2	5
25	$(0,0,1.12,\ 1.13,\ 1.14,\ 1.15,\ 1.23,\ 1.27{+}\ 1.36{+}1.45)$	$1<4<5<6<7<8<9<13$	1	VPA	1	2	6
*26	$(0,\ 0,\ 1.12,\ 1.13,\ 1.14,\ 1.15,\ 1.16{+}1.23,\ 1.47{+}1.56)$	$1<4<5<6<7<8<9<15$	1	VPA	1	2	7
*27	$(0,\ 0,\ 1.12,\ 1.13,\ 1.14,\ 1.15,\ 1.16{+}1.23,\ 1.37{+}1.46)$	$1<4<5<6<7<8<9<14$	1	VPA	1	2	7
*28	$(0,0,\sqrt{621}.12,\sqrt{5736}.13,\sqrt{6270}.14,\sqrt{3889}.15,\sqrt{4824}.16+\sqrt{4565}.23,\sqrt{7409}.27+\sqrt{6231}.45)$	$1<4<5<6<7<8<9<13$	1	E	1	2	7
29	$(0,0,\sqrt{26}.12,\sqrt{31}.13,\sqrt{30}.14,4.15,\sqrt{11}.16+\sqrt{11}.23,\sqrt{26}.25+\sqrt{30}.34)$	$1<4<5<6<7<8<9<11$	1	E	1	2	6
**30	$(0,0,\sqrt{603}.12,\ \sqrt{1995}.13,\ \sqrt{1943}.14,\ \sqrt{2680}.15,\ \sqrt{1742}.16{+}\ \sqrt{2695}.23,\ \sqrt{1824}.17{+}\ \sqrt{2464}.24)$	$1<4<5<6<7<8<9<10$	1	E	1	4	7
**31	$(0,0,0,0,\sqrt{238}.14,\ \sqrt{253}.23,\ \sqrt{253}.15{+}\ \sqrt{84}.34,\ \sqrt{153}.17{+}\ \sqrt{230}.26{+}\ \sqrt{54}.35)$	$11<18<22<25<36<40<47<58$	1	E	2	2	4
32	$(0,0,0,0,1.13,\ 1.24,\ 1.15{+}1.34,\ 1.27{+}1.36)$	$11<13<16<22<27<35<38<51$	1	VPA	3	2	4
33	$(0,0,0,0,1.13,\ 1.23,\ 1.15{+}1.34,\ 1.27{+}1.46)$	$11<15<16<22<27<31<38<53$	1	VPA	3	2	4
34	$(0,0,0,0,6.13,\sqrt{30}.23,\sqrt{22}.15+\sqrt{22}.24,\sqrt{30}.16+5.25)$	$5<6<7<11<12<13<17<18$	1	E	3	4	4
35	$(0,0,0,0,\sqrt{2}.13,\ \sqrt{2}.15+1.23,\ \sqrt{2}.24\ ,\ \sqrt{2}.16+1.25$	$2<4<5<6<7<9<10<11$	1	E	3	4	4
36	$(0,0,0,0,3.13,2\sqrt{2}.15+2.23,2\sqrt{2}.24+2.35,3.16+2.25)$	$6<12<19<24<25<35<40<49$	1	E	2	2	4
37	$(0,0,0,0,\sqrt{7}.13,0,2\sqrt{2}.14+\sqrt{7}.23,3.17+3.25+\sqrt{2}.36)$	$5<8<10<13<15<17<20<27$	1	E	3	2	3
38	$(0,0,0,0,2.13,\ \sqrt{5}.23,\ 2.14,\ 2.16{+}\sqrt{5}.25)$	$8<10<11<16<19<21<24<29$	1	E	4	4	3
39	$(0,0,0,0,\sqrt{30}.13,\sqrt{22}.14+5.23,\sqrt{22}.24,6.16+\sqrt{30}.25)$	$5<6<8<9<13<14<15<19$	1	E	3	4	3
40	$(0,0,0,0,\sqrt{21}.13,2\sqrt{5}.14+\sqrt{14}.23,\sqrt{11}.24+\sqrt{21}.35,3\sqrt{3}.16+3\sqrt{2}.25)$	$17<21<50<54<67<92<96<151$	1	E	2	2	3
41	$(0,0,0,0,3.13,\sqrt{14}.14,\sqrt{14}.16+2\sqrt{3}.23,2\sqrt{3}.17+4.25)$	$2<5<6<7<8<9<11<13$	1	E	3	4	4
42	$(0,0,0,0,\sqrt{21}.13,4\sqrt{2}.14,\sqrt{22}.16+\sqrt{29}.23,\sqrt{21}.17+\sqrt{29}.25+\sqrt{22}.46)$	$10<18<33<42<43<63<73<94$	1	E	2	2	4
43	$(0,0,0,0,1.13,\ 1.14{+}1.23,\ 1.15{+}1.34,\ 1.27{+}1.46)$	$34<49<53<68<87<102<121<170$	1	VPA	2	2	4
44	$(0,0,0,0,3\sqrt{2}.13,\ \sqrt{11}.14+\sqrt{22}.23,\ 2\sqrt{7}.15{+}\ \sqrt{14}.24,\ \sqrt{22}.17{+}\ 2\sqrt{7}.26)$	$10<15<23<28<33<38<43<53$	1	E	2	4	4
45	$(0,0,0,0,3\sqrt{3}.13,\sqrt{14}.14+\sqrt{23}.23,\sqrt{14}.15+4.24,3\sqrt{3}.16+\sqrt{23}.25)$	$10<15<18<23<28<33<38<43$	1	E	2	4	3
46	$(0,0,0,13,\ 0,\ 15,\ 14{+}23,\ 17{+}24{+}56)$	$3<6<12<15<21<32<34<45$	1	E	2	2	4
47	$(0,0,0,1.13,\ 1.23,\ 1.14,\ 1.25{+}1.34,\ 1.17{+}1.36)$	$72<83<94<166<177<238<260<332$	1	VPA	2	2	4
48	$(0,0,0,\sqrt{1794}.13,\sqrt{1353}.14,\sqrt{1640}.34,\sqrt{2009}.15+\sqrt{1404}.23,\sqrt{1978}.17+\sqrt{1548}.24)$	$11<73<107<118<169<185<220<311$	1	E	2	4	5
49	$(0,0,0,\sqrt{819}.13,\ \sqrt{828}.14,\sqrt{975}.23,\ \sqrt{805}.15{+}\sqrt{253}.34,\ 3\sqrt{70}.16{+}\ 5\sqrt{30}.24$	$34<82<90<124<169<183<203<239$	1	E	2	4	4
50	$(0,0,0,\sqrt{22}.13,\ 2\sqrt{7}.14,2\sqrt{7}.23,\ 3\sqrt{2}.15{+}\sqrt{22}.26,\ \sqrt{11}.16{+}\ \sqrt{14}.24$	$8<23<37<45<97<104<105<134$	1	E	2	4	4
51	$(0,0,0,1.13{,}1.14,\ 1.15,\ 1.16{,}1.27{+}1.36{+}\ 1.45)$	$10<123<133<143<153<163<173<296$	1	VPA	2	2	6
52	$(0,0,0,\sqrt{21}.13,\sqrt{22}.14,4\sqrt{2}.15,\sqrt{22}.16+\sqrt{29}.23,\sqrt{21}.17+\sqrt{29}.24)$	$5<20<22<27<32<37<42<47$	1	E	2	4	6
53	$(0,0,0,0,\sqrt{19}.12,\ \sqrt{22}.23,\ \sqrt{24}.15{+}\ \sqrt{13}.24,\ \sqrt{19}.17{+}\ \sqrt{22}.36+\ \sqrt{19}.45)$	$18<22<27<36<40<49<58<76$	1	E	2	2	4
54	$(0,0,0,0,\ 1.12,\ 1.14{+}1.23,\ 1.25{+}1.34,\ 1.37{+}1.56)$	$19<20<29<30<39<49<59<88$	1	VPA	2	2	4
55	$(0,0,0,0,1.12,\ 1.14,\ 1.25{+}1.34,\ 1.17{+}1.36)$	$32<45<59<63<77<95<122<154$	1	VPA	3	2	4

Table 1: 8-dimensional nilsoliton metric Lie algebras-Nullity 1

	Lie Bracket	Derivation	AoM	E/VPA	Rank	Index	NI
56	(0,0,0,0,1.12, 1.14+1.23, 1.25+1.34, 1.17+1.36)	$71 < 90 < 116 < 135 < 161 < 206 < 251 < 322$	1	VPA	2	2	4
57	$(0, 0, 0, 0, \sqrt{2}.12, \sqrt{2}.14 + \sqrt{2}.23, \sqrt{2}.24, \sqrt{3}.16 + \sqrt{3}.35)$	$5 < 6 < 7 < 8 < 11 < 13 < 14 < 18$	1	E	3	2	3
58	(0,0,0,0, 1.12, 1.14, 1.16+1.25+1.34, 1.17+1.36)	$29 < 45 < 58 < 61 < 74 < 90 < 119 < 148$	1	VPA	2	2	4
59	$(0, 0, 0, 0, 2.12, \sqrt{3}.14 + 2.23, \sqrt{2}.15 + \sqrt{3}.24, \sqrt{5}.16 + \sqrt{5}.35)$	$8 < 11 < 13 < 16 < 19 < 24 < 27 < 32$	1	E	2	2	3
60	(0,0,0,0, 1.12, 1.13, 1.16+1.25+1.34, 1.27+1.56)	$71 < 79 < 87 < 142 < 150 < 158 < 229 < 308$	1	VPA	2	2	4
61	(0,0,0,0,$3\sqrt{3}$.12, $3\sqrt{3}$.13, $\sqrt{14}$.16+ 4.25+$\sqrt{14}$.34, $\sqrt{23}$.26+ $\sqrt{23}$.35)	$27 < 28 < 29 < 54 < 55 < 56 < 83 < 84$	1	E	2	2	3
62	(0,0,0,0, 1.12, 1.13, 1.15+1.24, 1.37+1.56)	$5 < 6 < 7 < 10 < 11 < 12 < 16 < 23$	1	VPA	3	2	4
63	(0,0,0,0, 1.12, 1.13, 1.15+1.24, 1.16+1.25+1.34)	$7 < 10 < 13 < 14 < 17 < 20 < 24 < 27$	0	-	2	2	3
64	$(0, 0, 0, 0, 3\sqrt{2}.12, 3\sqrt{2}.13, 3\sqrt{2}.14 + \sqrt{10}.23, \sqrt{21}.27 + 4.36 + \sqrt{21}.45)$	$46 < 51 < 56 < 61 < 97 < 102 < 107 < 158$	1	E	2	2	3
65	$(0, 0, 0, \sqrt{2}.13, \sqrt{2}.12, 0, 2.15 + \sqrt{2}.23, \sqrt{3}.17 + \sqrt{3}.24 + \sqrt{2}.36)$	$7 < 9 < 14 < 15 < 16 < 18 < 21 < 26$	1	E	2	2	4
66	$(0, 0, 0, 2\sqrt{5}.13, 3.12, 3\sqrt{2}.25, 3\sqrt{2}.15 + 4.23, 5.17 + 2\sqrt{5}.24)$	$6 < 7 < 10 < 12 < 13 < 14 < 15 < 21$	1	E	2	4	4
67	$(0, 0, 0, 1.12, 0, 2\sqrt{3}.15 + 1.23, \sqrt{15}.16 + \sqrt{15}.34, \sqrt{22}.27)$	$5 < 7 < 11 < 12 < 13 < 18 < 23 < 30$	2	E	3	2	4
68	$(0, 0, 0, 3.12, 0, 3.23, 2\sqrt{2}.14, \sqrt{7}.16 + \sqrt{2}.25 + \sqrt{7}.34)$	$7 < 9 < 11 < 16 < 18 < 20 < 23 < 27$	1	E	3	2	3
69	(0,0,0,1.12, 1.23, 1.14, 1.16+1.35, 1.27+1.46)	$30 < 32 < 45 < 62 < 77 < 92 < 122 < 154$	1	VPA	2	2	5
70	$(0, 0, 0, \sqrt{3}.12, 0, 2.14, 2.16 + \sqrt{3}.23, \sqrt{3}.17 + \sqrt{2}.25 + \sqrt{3}.34)$	$2 < 5 < 6 < 7 < 8 < 9 < 11 < 13$	1	E	2	2	5
71	(0,0,0,1.12, 1.23, 1.14, 1.16+1.25, 1.17+1.45)	$74 < 77 < 145 < 151 < 222 < 225 < 299 < 373$	1	VPA	2	2	5
72	$(0, 0, 0, 1.12, 0, 1.14 + 1.23, 1.15 + 1.24, 1.37 + 1.46)$	$51 < 70 < 102 < 121 < 140 < 172 < 191 < 293$	1	VPA	2	2	4
73	$(0, 0, 0, \sqrt{5}.12, 0, \sqrt{3}.14 + 2.23, \sqrt{2}.15 + \sqrt{3}.24, \sqrt{5}.16 + 2.34)$	$7 < 10 < 14 < 17 < 20 < 24 < 27 < 31$	1	E	2	4	4
74	$(0, 0, 0, 6\sqrt{10}.12, \sqrt{624}.23, \sqrt{574}.14, \sqrt{255}.15 + \sqrt{442}.34, \sqrt{451}.16 + \sqrt{651}.25)$	$19 < 24 < 33 < 43 < 57 < 62 < 76 < 81$	1	E	2	2	4
****75-11	$(0, 0, 0, 2\sqrt{3}.12, \sqrt{21}.14, \sqrt{14}.23, 4.15, 3.27 + \sqrt{14}.36 + 2\sqrt{3}.45)$	$2 < 4 < 5 < 6 < 8 < 9 < 10 < 14$	1	E	2	2	5
76	(0,0,0,1.12, 1.14, 1.23, 1.15+1.24, 1.37+1.46)	$45 < 90 < 98 < 135 < 180 < 188 < 225 < 323$	1	VPA	2	2	5
77	$(0, 0, 0, \sqrt{15}.12, 2\sqrt{5}.14, \sqrt{15}.15, 2\sqrt{6}.26 + 2\sqrt{6}.45, 6.37)$	$10 < 21 < 24 < 31 < 41 < 51 < 72 < 96$	2	E	3	2	6
78	(0,0,0,1.12, 0, 1.13, 1.16+1.25, 1.27+1.46)	$10 < 14 < 19 < 24 < 25 < 29 < 39 < 53$	1	VPA	3	2	4
79	$(0, 0, 0, \sqrt{5}.12, \sqrt{5}.13, 2.24, \sqrt{13}.25 + \sqrt{13}.34, 3\sqrt{2}.17)$	$5 < 8 < 10 < 13 < 15 < 21 < 23 < 28$	2	E	3	2	4
80	(0,0,0,1.12, 1.13,0, 1.15+ 1.24,1.27+1.36 + 1.45)	$8 < 13 < 18 < 21 < 26 < 29 < 34 < 47$	1	VPA	2	2	4
81	(0,0,0,1.12, 1.13, 1.23, 1.15+1.24,1.37 + 1.46)	$69 < 77 < 85 < 146 < 154 < 162 < 223 < 308$	1	VPA	2	2	4
82	(0,0,0,1.12, 1.13, 1.23, 1.15+1.24, 1.27+1.45)	$25 < 41 < 57 < 66 < 82 < 98 < 107 < 148$	1	VPA	2	2	4
83	$(0, 0, 0, \sqrt{5}.12, \sqrt{2}.13, \sqrt{5}.23, 2.15 + 2.24, \sqrt{5}.16 + \sqrt{5}.34)$	$5 < 6 < 7 < 11 < 12 < 13 < 17 < 18$	1	E	2	4	3
84	(0,0,0,1.12, 1.13, 1.15, 1.24, 1.26+1.45)	$21 < 47 < 61 < 68 < 82 < 103 < 115 < 150$	1	VPA	3	2	4
85	(0,0,0,1.12, 1.13, 1.15+1.24, 1.35, 1.26+1.45)	$13 < 15 < 17 < 28 < 30 < 43 < 47 < 58$	1	VPA	2	2	4
86	(0,0,0,1.12, 1.13, 1.15+ 1.23, 1.24+ 1.37+ 1.46)	$11 < 22 < 31 < 33 < 42 < 53 < 55 < 86$	1	VPA	2	2	4
87	(0,0,0,1.12, 1.13, 1.15+1.23, 1.24, 1.26+1.45)	$22 < 44 < 57 < 66 < 79 < 101 < 110 < 145$	1	VPA	2	2	4
88	(0,0,0,$\sqrt{6}$.12, 3.13, $\sqrt{10}$.15, $\sqrt{10}$.24, $\sqrt{5}$.16+ 2.25+ $\sqrt{6}$.34)	$2 < 4 < 5 < 6 < 7 < 9 < 10 < 11$	1	E	2	2	3
89	$(0, 0, 0, \sqrt{6}.12, \sqrt{5}.13, \sqrt{10}.15 + 2.23, \sqrt{10}.24, 3.16 + \sqrt{6}.34)$	$2 < 4 < 5 < 6 < 7 < 9 < 10 < 11$	1	E	2	4	4
90	(0,0,0,1.12, 1.13, 1.15, 1.16+1.24, 1.26+1.45)	$3 < 8 < 10 < 11 < 13 < 16 < 19 < 24$	1	VPA	2	2	4
91	$(0, 0, 0, \sqrt{21}.12, \sqrt{29}.13, 4\sqrt{2}.15, \sqrt{22}.16 + \sqrt{22}.24, \sqrt{21}.25 + \sqrt{29}.34)$	$17 < 37 < 40 < 54 < 57 < 74 < 91 < 94$	1	E	2	2	4
92	$(0, 0, 0, 3.12, \sqrt{7}.13, \sqrt{2}.23, 2\sqrt{2}.14, 3.25 + \sqrt{7}.34)$	$7 < 9 < 11 < 16 < 18 < 20 < 23 < 27$	1	E	3	4	3
93	$(0, 0, 0, \sqrt{2}.12, \sqrt{7}.13, 3.23, 2\sqrt{2}.14, \sqrt{7}.16 + 3.25)$	$7 < 9 < 11 < 16 < 18 < 20 < 23 < 27$	1	E	3	4	3
94	(0,0,0,1.12, 1.13, 1.14, 1.24, 1.36+1.45)	$25 < 45 < 59 < 70 < 84 < 95 < 115 < 154$	1	VPA	3	2	4
95	(0,0,0,1.12, 1.13, 1.14, 1.24, 1.27+1.36+1.45)	$8 < 13 < 18 < 21 < 26 < 29 < 34 < 47$	1	VPA	2	2	4
96	(0,0,0,1.12, 1.13, 1.14, 1.23, 1.36+1.45)	$22 < 49 < 50 < 71 < 72 < 93 < 99 < 143$	1	VPA	3	2	4
97	(0,0,0,1.12, 1.13, 1.14, 1.23, 1.27+1.36+1.45)	$8 < 16 < 17 < 24 < 25 < 32 < 33 < 49$	1	VPA	2	2	4
98	(0,0,0,1.12,1.13,1.14+1.23,1.24,1.36+1.45)	$29 < 45 < 58 < 74 < 87 < 103 < 119 < 161$	1	VPA	2	2	4
99	$(0, 0, 0, 5.12, 2\sqrt{5}.13, 3\sqrt{2}.14, 3\sqrt{2}.24, 3.16 + 2\sqrt{5}.25 + 4.34)$	$4 < 5 < 8 < 9 < 12 < 13 < 14 < 17$	1	E	2	2	4
100	(0,0,0,1.12, 1.13, 1.14, 1.16, 1.36+1.45)	$14 < 57 < 58 < 71 < 72 < 85 < 99 < 143$	1	VPA	3	2	4
101	(0,0,0,1.12, 1.13, 1.14, 1.16+1.35,1.27+ 1.46)	$64 < 110 < 119 < 174 < 183 < 238 < 302 < 412$	1	VPA	2	2	5
102	(0,0,0,2$\sqrt{3}$.12, 3.13, $\sqrt{14}$.14, $\sqrt{14}$.16, 4.25+2$\sqrt{3}$.34)	$2 < 5 < 6 < 7 < 8 < 9 < 11 < 13$	1	E	3	2	4
103	(0,0,0,1.12,1.13,1.14,1.16+1.24,1.37+1.56)	$5 < 10 < 13 < 15 < 18 < 20 < 25 < 38$	1	VPA	2	2	5
104	(0,0,0,1.12, 1.13, 1.14, 1.16+1.24,1.36+ 1.45)	$45 < 90 < 116 < 135 < 161 < 180 < 225 < 226$	1	VPA	2	2	4
105	(0,0,0,$\sqrt{819}$.12, $\sqrt{630}$.13, 6$\sqrt{23}$.14, $\sqrt{805}$.16 + $\sqrt{253}$.24, $\sqrt{975}$.25+$\sqrt{750}$.34)	$45 < 90 < 119 < 135 < 164 < 180 < 225 < 254$	1	E	2	2	4

Table 2: 8-dimensional nilsoliton metric Lie algebras-Nullity 1-Continued

	Lie Bracket	Derivation	AoM	E/VPA	Rank	Index	NI
106	(0,0,0,1.12, 1.13, 1.14, 1.16+1.23,1.36+ 1.45)	$10 < 29 < 30 < 39 < 40 < 49 < 59 < 79$	1	VPA	2	2	4
107	(0,0,0,$\sqrt{7}$.12, $\sqrt{7}$.13, $\sqrt{15}$.14, $\sqrt{15}$.15 + $\sqrt{10}$.24, $\sqrt{17}$.27+$\sqrt{17}$.36)	$8 < 13 < 18 < 21 < 26 < 29 < 34 < 47$	1	E	2	2	4
108	$(0, 0, 0, 3\sqrt{3}.12, 3\sqrt{3}.13, \sqrt{14}.14, \sqrt{14}.15 + 4.24, \sqrt{23}.25 + \sqrt{23}.34)$	$5 < 7 < 9 < 12 < 14 < 17 < 19 < 21$	1	E	2	4	3
109	(0,0,0,1.12, 1.13, 1.14, 1.15+1.23, 1.47+1.56)	$5 < 10 < 12 < 15 < 17 < 20 < 22 < 37$	1	VPA	2	2	4
110	(0,0,0,1.12,1.13,1.14,1.15+1.23,1.27+1.45)	$8 < 16 < 17 < 24 < 25 < 32 < 33 < 49$	1	VPA	2	2	4
111	(0,0,0,$\sqrt{5}$.12, $\sqrt{5}$.13, $\sqrt{19}$.14, $\sqrt{17}$.15 + $\sqrt{6}$.23, $\sqrt{19}$.27+$\sqrt{17}$.36)	$8 < 16 < 17 < 24 < 25 < 32 < 33 < 49$	1	E	2	2	4
112	(0,0,0,12,13,14,15+23,16+24)	$5 < 10 < 14 < 15 < 19 < 20 < 24 < 25$	0	-	2	4	4
113	$(0, 0, 3.12, 0, \sqrt{21}.23, \sqrt{14}.14, \sqrt{14}.16 + 2\sqrt{3}.25, 2\sqrt{3}.17 + 4.35)$	$2 < 3 < 5 < 7 < 8 < 9 < 11 < 13$	1	E	2	4	5
114	(0,0,1.12, 0 ,1.14, 1.15, 1.16, 1.27+1.36)	$2 < 22 < 24 < 25 < 27 < 29 < 31 < 53$	1	VPA	3	2	5
115	(0,0,1.12, 0 ,1.14, 1.15, 1.16+1.23, 1.27+1.36)	$3 < 13 < 16 < 20 < 23 < 26 < 29 < 42$	1	VPA	2	2	5
116	(0,0,1.12, 0 ,1.13, 1.23, 1.15+1.24, 1.37+1.46)	$13 < 22 < 35 < 39 < 48 < 57 < 61 < 96$	1	VPA	2	2	5
117	(0,0,$\sqrt{287}$.12,0, $\sqrt{528}$.13, $\sqrt{456}$.23, $\sqrt{287}$.15 + $\sqrt{246}$.24, $\sqrt{374}$.16+$\sqrt{323}$.25)	$15 < 19 < 34 < 45 < 49 < 53 < 64 < 68$	1	E	2	4	4
118-25	(0,0,2$\sqrt{3}$.12,0, 2$\sqrt{5}$.13, $\sqrt{15}$.15, 2$\sqrt{3}$.24, 2$\sqrt{3}$.26+$\sqrt{15}$.35)	$13 < 24 < 37 < 48 < 50 < 63 < 72 < 87$	1	E	3	4	5
119	$(0, 0, \sqrt{5}.12, 0, 4.13, \sqrt{5}.15 + 2\sqrt{7}.24, \sqrt{35}.26 + \sqrt{35}.35, \sqrt{46}.17)$	$11 < 13 < 24 < 33 < 35 < 46 < 59 < 70$	2	E	2	2	6
120	$(0, 0, \sqrt{46}.12, 0, 6\sqrt{5}.13, \sqrt{161}.15 + \sqrt{96}.23, \sqrt{161}.24, \sqrt{165}.16 + \sqrt{88}.25)$	$7 < 14 < 21 < 24 < 28 < 35 < 38 < 42$	1	E	2	4	5
121	(0,0,1.12, 0 ,1.13, 1.15, 1.16+1.24, 1.37+1.56)	$5 < 13 < 18 < 20 < 23 < 28 < 33 < 51$	1	VPA	2	2	6
122	(0,0,1.12, 0 ,1.13, 1.15, 1.16+1.24, 1.27+1.36)	$5 < 11 < 16 < 20 < 21 < 26 < 31 < 42$	2	VPA	2	2	6
123	$(0, 0, 2\sqrt{6}.12, 0, 2\sqrt{10}.13, \sqrt{31}.15, 2.16 + 2\sqrt{6}.24, 2\sqrt{6}.26 + \sqrt{31}.35)$	$25 < 52 < 77 < 100 < 102 < 127 < 152 < 179$	1	E	2	2	5
124	$(0, 0, \sqrt{3}.12, 0, 2\sqrt{2}.13, 2\sqrt{7}.15 + \sqrt{3}.24, \sqrt{33}.16 + \sqrt{33}.34, 5\sqrt{2}.27)$	$9 < 17 < 26 < 27 < 35 < 44 < 53 < 70$	1	E	2	2	6
125	$(0, 0, \sqrt{22}.12, 0, \sqrt{30}.13, 6.23, \sqrt{22}.14, 5.16 + \sqrt{30}.25)$	$4 < 5 < 9 < 11 < 13 < 14 < 15 < 18$	1	E	3	4	4
126	$(0, 0, \sqrt{14}.12, 0, 3\sqrt{3}.13, \sqrt{14}.14 + \sqrt{23}.23, 4.24, 3\sqrt{3}.16 + \sqrt{23}.25)$	$20 < 23 < 43 < 46 < 63 < 66 < 69 < 86$	1	E	2	4	4
127	(0, 0, 1.12, 0 , 1.13, 1.14, 1.16+1.23, 1.37+1.56)	$2 < 5 < 7 < 8 < 9 < 10 < 12 < 19$	1	VPA		2	4
128	$(0, 0, \sqrt{253}.12, 0, \sqrt{630}.13, \sqrt{828}.14, \sqrt{805}.16 + \sqrt{819}.23, 5\sqrt{30}.17 + \sqrt{975}.25)$	$37 < 90 < 127 < 143 < 164 < 180 < 217 < 254$	1	E	2	4	4
129	(0,0,2$\sqrt{6}$.12, 0, 2$\sqrt{10}$.13, 2.14, 2$\sqrt{6}$.15 + 2$\sqrt{6}$.24, $\sqrt{31}$.27+$\sqrt{31}$.35)	$32 < 45 < 77 < 96 < 109 < 128 < 141 < 186$	1	E	2	2	5
130	$(0, 0, \sqrt{442}.12, 0, \sqrt{451}.13, \sqrt{255}.14 + \sqrt{451}.23, \sqrt{574}.15, \sqrt{624}.26 + 6\sqrt{10}.34)$	$17 < 22 < 39 < 44 < 56 < 61 < 73 < 83$	1	E	2	2	4
131	$(0, 0, 4.12, 0, 3\sqrt{3}.13, \sqrt{14}.14, \sqrt{14}.15 + 3\sqrt{3}.23, \sqrt{23}.17 + \sqrt{23}.25)$	$7 < 14 < 21 < 27 < 28 < 34 < 35 < 42$	1	E	2	4	5
**132	$(0, 0, 3.12, \sqrt{14}.13, 0, 0, 3.14 + \sqrt{6}.25, 3.27 + 3.34 + \sqrt{6}.56)$	$2 < 6 < 8 < 13 < 15 < 18 < 21 < 30$	1	E	2	2	5
**133	(0,0,$\sqrt{10925}$.12, $\sqrt{16709}$.13, $\sqrt{5642}$.23, $\sqrt{14030}$.14, $\sqrt{11718}$.25, $\sqrt{9690}$.26+$\sqrt{12444}$.34)	$65 < 82 < 147 < 212 < 229 < 277 < 311 < 359$	1	E	2	4	5
134	$(0, 0, 2\sqrt{3}.12, 4.13, 0, 2\sqrt{3}.14, 2.15, \sqrt{13}.26 + \sqrt{13}.34)$	$8 < 27 < 35 < 43 < 48 < 51 < 56 < 78$	1	E	3	2	5
135	$(0, 0, 2.12, \sqrt{5}.13, 0, 2.14, 1.15 + 1.23, 2.26 + 2.34)$	$3 < 8 < 11 < 14 < 16 < 17 < 19 < 25$	1	E	2	2	4
136	(0,0,6$\sqrt{2}$.12, $\sqrt{85}$.13, 0, 8.14+ $\sqrt{34}$.23, $\sqrt{34}$.15, 9.26+ 6$\sqrt{2}$.34)	$4 < 8 < 12 < 16 < 17 < 20 < 21 < 28$	1	E	2	2	5
137	$(0, 0, \sqrt{209}.12, \sqrt{126}.13, \sqrt{209}.14, \sqrt{88}.23, \sqrt{259}.35, \sqrt{72}.17 + \sqrt{148}.45)$	$3 < 32 < 35 < 38 < 41 < 67 < 76 < 79$	2	E	2	2	6
138	(0,0,$\sqrt{13}$.12, 4.13, $\sqrt{13}$.14, 2.23, $\sqrt{29}$.25+ $\sqrt{29}$.34, $\sqrt{34}$.17)	$1 < 4 < 5 < 6 < 7 < 9 < 11 < 12$	2	E	2	2	6
*139	(0,0,$\sqrt{3}$.12, 2.13, $\sqrt{3}$.14, 3$\sqrt{3}$.25+ 3$\sqrt{3}$.34, $\sqrt{38}$.16, 2$\sqrt{5}$.17)	$1 < 20 < 21 < 22 < 23 < 43 < 44 < 45$	2	E	2	2	7
140	(0,0,$\sqrt{11}$.12, $\sqrt{21}$.13, $\sqrt{21}$.14, 3$\sqrt{3}$.23, 2$\sqrt{5}$.15, $\sqrt{14}$.16+ 3$\sqrt{2}$.24)	$19 < 50 < 69 < 88 < 107 < 119 < 126 < 138$	1	E	2	4	5
141	(0, 0, 1.12, 1.13, 1.14, 1.15, 0, 1.27+1.36+1.45)	$7 < 14 < 21 < 27 < 28 < 34 < 35 < 42$	1	VPA	2	2	6
142	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.23, 1.36+1.45)	$23 < 139 < 162 < 185 < 208 < 231 < 301 < 393$	1	VPA	2	2	6
143	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.23, 1.26+1.35)	$1 < 4 < 5 < 6 < 7 < 8 < 9 < 12$	1	VPA	2	2	6
*144	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16, 1.37+1.46)	$2 < 71 < 73 < 75 < 77 < 79 < 81 < 154$	1	VPA	2	2	7
*145	$(0, 0, \sqrt{184}.12, \sqrt{3876}.13, \sqrt{3195}.14, \sqrt{1804}.15, \sqrt{3366}.16, \sqrt{4218}.27 + \sqrt{3663}.45)$	$10 < 123 < 133 < 143 < 153 < 163 < 173 < 296$	1	E	2	2	7
146	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16, 1.27+1.36)	$10 < 123 < 133 < 143 < 153 < 163 < 173 < 296$	1	VPA	2	2	7
147	(0, 0, 1.12, 1.13, 1.14, 1.15, 1.16, 1.26+1.35)	$4 < 29 < 33 < 37 < 41 < 45 < 49 < 74$	1	VPA	2	2	6
148	$(0, 0, 8.12, \sqrt{85}.13, 6\sqrt{2}.14, \sqrt{34}.15, \sqrt{34}.16, 6\sqrt{2}.25 + 9.34)$	$2 < 11 < 13 < 15 < 17 < 19 < 21 < 28$	1	E	2	2	6

Table 3: 8-dimensional nilsoliton metric Lie algebras-Nullity 1-Continued 2

	Lie Bracket	Derivation	AoM	E/VPA	Rank	Nullity	Index	NI
1	$(0,0,0,\sqrt{6}.13,\sqrt{5}.14+\sqrt{5}.23,\sqrt{5}.34,\sqrt{6}.15+\sqrt{5}.24,\sqrt{6}.17+\sqrt{6}.25)$	$2<4<14<16<23<25<35<37)$	2	E	2	2	2	5
2	(0,0,0,1.13,1.14, 1.15, 1.26+1.35,1.17+1.36+1.45),	$1<11<12<13<14<15<26<27$	2	VPA	1	2	2	6
3	$(0,0,0,\sqrt{10}.13,\sqrt{11}.14,\sqrt{5}.15+\sqrt{10}.23,\sqrt{11}.16+\sqrt{11}.24,\sqrt{10}.17+\sqrt{10}.25)$	$8<24<35<43<51<59<67<75$	2	E	2	2	4	6
4	$(0,0,0,1.13,1.14,1.15+1.23,1.16+1.24,1.17+1.25+1.34)$	$34<111<172<206<249<292<344<369$	2	VPA	1	2	2	6
5	$(0,0,0,1.12,1.14+1.23,1.24,1.15+1.34,1.27+1.36)$	$22<25<44<47<69<72<91<116$	2	VPA	2	2	2	5
6	$(0,0,0,1.12,1.14+1.23,1.24,1.15+1.34,1.17+1.26+1.35)$	$18914<28370<37828<47284<66199<75657<85113<104027$	2	VPA	1	2	2	5
7	$(0,0,0,1.12,1.13,1.24,1.25+1.34,1.17+1.26+1.45)$	$20<29<38<49<58<78<87<107$	2	VPA	2	2	2	4
8	$(0,0,0,5\sqrt{2}.12,\sqrt{65}.13,\sqrt{70}.25+\sqrt{91}.34,\sqrt{65}.16+\sqrt{91}.45,6\sqrt{2}.17+6\sqrt{2}.26)$	$5<10<13<15<18<28<33<38$	3	E	2	2	2	5
9	$(0,0,0,1.12,1.13,1.15+1.24,1.26+1.45,1.27+1.56)$	$18<29<40<47<58<76<105<134$	2	VPA	2	2	2	5
10	$(0,0,0,1.12,1.13,1.15+1.24,1.25+1.34,1.26+1.45)$	$20<29<38<49<58<78<87<107$	2	VPA	2	2	2	4
11	$(0,0,0,1.12,1.13,1.15+1.23,1.25+1.34,1.27+1.46)$	$5<10<13<15<18<23<28<38$	2	VPA	2	2	2	4
12	(0,0,0,1.12, 1.13, 1.15+ 1.24,1.25+1.34, 1.17+ 1.26),	$20<29<38<49<58<78<87<107$	2	VPA	2	2	2	4
13	$(0,0,0,1.12,1.13,1.15+1.23,1.26+1.45,1.17+1.46)$	$5<10<14<15<19<24<34<39$	2	VPA	2	2	2	5
14	$(0,0,0,1.12,1.13,1.15,1.16+1.25+1.34,1.27+1.46)$	$5<10<13<15<18<23<28<38$	2	VPA	2	2	2	5
15	$(0,0,0,1.12,1.13,1.15+1.23,1.16+1.34,1.27+1.46)$	$5<10<13<15<18<23<28<38$	2	VPA	2	2	2	5
16	$(0,0,1.12,0,1.23,1.24,1.16+1.25+1.34,1.17+1.35)$	$7<8<15<16<23<24<31<38$	2	VPA	2	2	2	5
17	$(0,0,1.12,0,1.14+1.23,0,1.16+1.25+1.34,1.17+1.35)$	$7<8<15<16<23<24<31<38$	2	VPA	2	2	2	5
18	$(0,0,1.12,0,1.14+1.23,1.24,1.16+1.25,1.17+1.35)$	$7<8<15<16<23<24<31<38$	2	VPA	2	2	2	5
19	$(0,0,1.12,0,1.14,1.15+1.23,1.26+1.35,1.17+1.36)$	$5<19<24<33<38<43<62<67$	2	VPA	2	2	2	5
20	$(0,0,1.12,0,1.14,1.15+1.23,1.26+1.35,1.17+1.36+1.45)$	$31<125<156<235<266<313<470<485$	2	VPA	1	2	2	5
21	$(0,0,1.12,0,1.13+1.24,0,1.15+1.34,1.27+1.35+1.46)$	$18<21<39<40<65<72<91<108$	2	VPA	2	2	2	5
22	$(0,0,1.12,0,1.13,1.15+1.24,1.26+1.35,1.17+1.36)$	$11<13<24<33<35<46<59<70$	2	VPA	2	2	2	6
23	$(0,0,1.12,0,1.13,1.15+1.24,1.16+1.34,1.27+1.36)$	$9<17<26<27<35<44<53<70$	2	VPA	2	2	2	6
24	$(0,0,\sqrt{37}.12,0,3\sqrt{5}.13,\sqrt{37}.15+\sqrt{37}.24,5.16+5.34,2\sqrt{10}.26+2\sqrt{10}.35)$	$41<71<112<123<153<194<235<265$	2	E	2	2	2	5
25	$(0,0,1.12,1.13,0,1.15+1.23,1.16+1.24,1.17+1.34)$	$5<14<19<24<28<33<38<43$	2	EA	2	2	4	5
26	$(0,0,1.12,1.13,1.14,0,1.26+1.35,1.17+1.36+1.45)$	$1<11<12<13<14<15<26<27$	2	VPA	2	2	2	6
27	$(0,0,3\sqrt{2}.12,4.13,\sqrt{19}.14,0,3\sqrt{2}.25+\sqrt{19}.34,\sqrt{14}.17+2.26+\sqrt{14}.35)$	$1<4<5<6<7<8<11<12$	2	E	2	2	2	6
28	$(0,0,1.12,1.13,1.14,1.23,1.26+1.35,1.17+1.36+1.45)$	$1<3<4<5<6<7<10<11$	2	VPA	1	2	2	6
29	$(0,0,1.12,1.13,1.14,1.23,1.25+1.34,1.17+1.35)$	$1<4<5<6<7<9<11<12$	2	VPA	2	2	2	6
30	(0,0,1.12, 1.13, 1.14, 1.23,1.25+1.34, 1.17+1.26+1.35),	$1<3<4<5<6<7<9<10$	2	VPA	1	2	2	6
31	$(0,0,\sqrt{10}.12,2\sqrt{2}.13,\sqrt{19}.14,\sqrt{10}.25+\sqrt{19}.34,+\sqrt{26}.16+\sqrt{26}.35,6.27)$	$13<20<33<46<59<79<92<112$	3	E	2	2	2	7
32	$(0,0,1.12,1.13,1.14,1.15,1.25+1.34,1.26+1.35)$	$1<4<5<6<7<8<11<12$	2	VPA	2	2	2	6
33	$(0,0,1.12,1.13,1.14,1.15+1.23,1.26+1.35,1.27+1.56)$	$1<3<4<5<6<7<10<13$	2	VPA	1	2	2	7
34	$(0,0,1.12,1.13,1.14,1.15+1.23,1.25+1.34,1.27+1.46)$	$1<3<4<5<6<7<9<12$	2	VPA	1	2	2	6
35	$(0,0,1.12,1.13,1.14,1.15+1.23,1.25+1.34,1.26+1.35)$	$1<3<4<5<6<7<9<10$	2	VPA	1	2	2	6
36	(0,0,1.12, 1.13, 1.14, 1.15,1.26+ 1.35, 1.17+ 1.45),	$1<11<12<13<14<15<26<27$	2	VPA	2	2	2	7
37	$(0,0,1.12,1.13,1.14,1.15,1.25+1.34,1.17+1.26)$	$1<4<5<6<7<8<11<12$	2	VPA	2	2	2	6
38	$(0,0,1.12,1.13,1.14,1.15+1.23,1.36+1.45,1.17+1.46)$	$\frac{49}{802}<\frac{520}{2837}<\frac{1579}{6461}<\frac{1232}{4033}<\frac{136}{371}<\frac{473}{1106}<\frac{539}{802}<\frac{272}{371}$	2	VPA	1	2	2	7
39	(0,0,1.12, 1.13, 1.14, 1.15+1.23,1.26+ 1.35, 1.17+ 1.45),	$1<3<4<5<6<7<10<11$	2	VPA	1	2	2	7
40	(0,0,1.12, 1.13, 1.14, 1.15+1.23,1.25+ 1.34, 1.17+ 1.26),	$1<3<4<5<6<7<9<10$	2	VPA	1	2	2	6
41	$(0,0,1.12,1.13,1.14,1.15,1.16+1.25+1.34,1.27+1.45)$	$1<2<3<4<5<6<7<9$	2	VPA	1	2	2	7
42	$(0,0,1.12,1.13,1.14,1.15,1.16+1.25+1.34,1.27+1.36)$	$1<2<3<4<5<6<7<9$	2	VPA	1	2	2	7
43	(0,0,1.12, 1.13, 1.14, 1.15+1.23,1.16+ 1.24,1.17+ 1.34),	$1<3<4<5<6<7<8<9$	2	EA	1	2	4	7
44	$(0,0,0,1.12,0,1.15+1.23,1.16+1.34,1.27+1.46)$	$5<7<11<12<13<18<23<30$	2	VPA	3	2	2	4
45	(0,0,0,$\sqrt{14}$.12, $\sqrt{14}$.13, $\sqrt{6}$.24, $\sqrt{13}$.25+$\sqrt{13}$.34,$\sqrt{14}$.17+ $\sqrt{13}$.45),	$5<8<10<13<15<21<23<28$	2	E	3	2	2	4
46	$(0,0,1.12,0,1.14,1.15,1.26+1.35,1.17+1.36)$	1¡ $11<12<13<14<15<26<27$	2	VPA	3	2	2	5
47	$(0,0,0,\sqrt{5}.13,\sqrt{5}.14+\sqrt{5}.23,\sqrt{6}.15+\sqrt{6}.24,\sqrt{5}.16+\sqrt{6}.25,\sqrt{5}.17+\sqrt{5}.26)$	$1<2<5<6<7<8<9<10$	3	E	2	3	4	6
48	$(0,0,0,1.13,1.14+1.23,1.15+1.24,1.16+1.25,1.17+1.26+1.34)$	$4<8<28<32<39<49<53<57$	3	EA	1	3	2	6
49	$(0,0,0,1.12,1.14+1.23,1.15+1.34,1.16+1.35,1.27+1.46)$	$9<17<18<26<35<44<53<70$	3	VPA	2	3	2	6
50	$(0,0,1.12,1.13,1.14+1.23,1.25+1.34,1.351.17+1.26+1.45)$,	$1<2<3<4<5<7<8<9$	3	VPA	1	3	2	6
51	$(0,0,\sqrt{15}.12,\sqrt{14}.13,\sqrt{15}.14,\sqrt{15}.25+\sqrt{15}.34,\sqrt{14}.16+\sqrt{14}.35,\sqrt{10}.17+\sqrt{10}.45)$	$1<20<21<22<23<43<44<45$	3	E	2	3	2	7
52	$(0,0,1.12,1.13,1.14+1.23,1.25+1.34,1.16+1.35,1.17+1.26)$	$1<2<3<4<5<7<8<9$	3	VPA	1	3	2	7
53	$(0,0,1.12,1.13,1.14+1.23,1.15+1.24,1.16+1.25+1.34,1.17)$	$1<2<3<4<5<6<7<8$	3	VPA	1	3	2	7

Table 4: 8-dimensional nilsoliton metric Lie algebras-Nullity 2-3