POST-QUANTUM OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES

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ABSTRACT. In this study, we give the notions about some new post-quantum partial derivatives and then use these derivatives to prove an integral equality via post-quantum double integrals. We establish some new post-quantum Ostrowski type inequalities for differentiable coordinated functions using the newly established equality. We also show that the results presented in this paper are the extensions of some existing results.

1. INTRODUCTION

A. M. Ostowski established the following intriguing integral inequality in 1938, which is known in the literature as the Ostrowski inequality.

Theorem 1. *[42] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a differentiable function on (π_1, π_2) whose derivative is bounded on (π_1, π_2), i.e., $\left\| F'(\tau) \right\|_\infty := \sup \left| F'(\tau) \right| < \infty$, for all $\tau \in (\pi_1, \pi_2)$. Then we have the following integral inequality:*

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(x)dx \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{\pi_1 + \pi_2}{2})}{(\pi_2 - \pi_1)^2} \right] (\pi_2 - \pi_1) \left\| F' \right\|_\infty, \tag{1.1}$$

for all $x \in [\pi_1, \pi_2]$. The $\frac{1}{4}$ is the best possible.

Inequality (1.1) can be rewritten in the following way:

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(x)dx \right| \leq \left[\frac{(x - \pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left\| F' \right\|_\infty. \tag{1.2}$$

Since 1938, numerous mathematicians have worked on and around the Ostrowski inequality, in a variety of ways and with a variety of applications in Numerical Analysis and Probability, etc.

Many authors investigate several versions of the Ostrowski integral inequality for bounded variation mappings, Lipschitzian mappings, monotonic mappings, absolutely continuous mappings, convex mappings, and n-times differentiable mappings with error estimates for various particular means and numerical quadrature techniques. For recent results and generalizations concerning Ostrowski's inequality, one can consult [7, 8, 13, 21, 23, 24, 36, 37, 43–45, 47] and the references therein.

The following is a formal definition of co-ordinated convex (concave) functions:

Definition 1. *A function $F : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $\tau, s \in [0, 1]$, if it satisfies the following inequality:*

$$\begin{aligned} &F(\tau x + (1 - \tau)\, y, su + (1 - s)\, v) \\ \leq\ &\tau s\, F(x, u) + \tau(1 - s)F(x, v) + s(1 - \tau)F(y, u) + (1 - \tau)(1 - s)F(y, v). \end{aligned} \tag{1.3}$$

The mapping F is a co-ordinated concave on Δ if the inequality (1.3) holds in reversed direction for all $\tau, s \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

For co-ordinated convex functions, M. A. Latif et al. established the following Ostrowski type inequalities in [35]:

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Theorem 2. *[35] Let $F : \Delta := [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \pi_1, \pi_3 \geq 0$ such that $\frac{\partial^2 F}{\partial s \partial \tau} \in L(\Delta)$. If $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right|$ is co-ordinated convex on Δ and $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right| \leq M,\ (x, y) \in \Delta$, then the following inequality holds:*

$$\begin{aligned} &\left| F(x,y) + \frac{1}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \int_{\pi_1}^{\pi_2} \int_{\pi_3}^{\pi_4} F(u,v)dvdu - \pi_{11} \right| \\ \leq\ & M \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left[\frac{(y-\pi_3)^2 + (\pi_4 - y)^2}{2(\pi_4 - \pi_3)} \right], \end{aligned} \tag{1.4}$$

where

$$\pi_{11} = \frac{1}{\pi_4 - \pi_3} \int_{\pi_3}^{\pi_4} F(x,v)dv + \frac{1}{\pi_2 - \pi_1} \int_{\pi_3}^{\pi_4} F(u,y)dy.$$

Theorem 3. *[35] Let $F : \Delta := [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \pi_1, \pi_3 \geq 0$ such that $\frac{\partial^2 F}{\partial s \partial \tau} \in L(\Delta)$. If $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right|^p$ is co-ordinated convex on Δ, $s > 1,\ \frac{1}{s} + \frac{1}{r} = 1$ and $\left|\frac{\partial^2 F}{\partial s \partial \tau}(x,y)\right| \leq M,\ (x, y) \in \Delta$, then the following inequality holds:*

$$\begin{aligned} &\left| F(x,y) + \frac{1}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \int_{\pi_1}^{\pi_2} \int_{\pi_3}^{\pi_4} F(u,v)dvdu - \pi_{11} \right| \\ \leq\ & \frac{M}{(1+r)^{\frac{2}{r}}} \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left[\frac{(y-\pi_3)^2 + (\pi_4 - y)^2}{2(\pi_4 - \pi_3)} \right], \end{aligned} \tag{1.5}$$

where π_{11} is defined in Theorem 2.

Theorem 4. *[35] Let $F : \Delta := [\pi_1, \pi_2] \times [\pi_3, \pi_4] \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \pi_1, \pi_3 \geq 0$ such that $\frac{\partial^2 F}{\partial s \partial \tau} \in L(\Delta)$. If $\left|\frac{\partial^2 F}{\partial s \partial \tau}\right|^p$ is co-ordinated convex on Δ, $p \geq 1$ and $\left|\frac{\partial^2 F}{\partial s \partial \tau}(x,y)\right| \leq M,\ (x, y) \in \Delta$, then the following inequality holds:*

$$\begin{aligned} &\left| F(x,y) + \frac{1}{(\pi_2 - \pi_1)(\pi_4 - \pi_3)} \int_{\pi_1}^{\pi_2} \int_{\pi_3}^{\pi_4} F(u,v)dvdu - \pi_{11} \right| \\ \leq\ & \frac{M}{4} \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2(\pi_2 - \pi_1)} \right] \left[\frac{(y-\pi_3)^2 + (\pi_4 - y)^2}{2(\pi_4 - \pi_3)} \right], \end{aligned} \tag{1.6}$$

where π_{11} is defined in Theorem 2.

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [26, 29]. Quantum calculus also has many applications in quantum information theory which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [14, 15]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by F. H. Jackson [25, 27]. In 1966, W. Al-Salam [11] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, J. Tariboon and S. K. Ntouyas introduced ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral in [49]. In 2020, S. Bermudo et al. introduced the notion of ${}^{\pi_2}D_q$ derivative and q^{π_2}-integral in [12]. P. N. Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or shortly (p, q)-calculus in [46]. In [48], M. Tunç and E. Göv gave the post-quantum variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Y.-M. Chu et al. introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p, q)^{\pi_2}$-integral in [22].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [2, 5, 9, 10, 12, 16, 17, 28, 38, 39], the authors used ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [40], M. A. Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, E. R. Nwaeze et al. proved certain parameterized quantum integral inequalities in [41]. M. A. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [31]. H. Budak et al. [18], M. A. Ali et al. [1, 3] and M. Vivas-Cortez et al. [50] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions on can consult [4, 6, 20]. M. Kunt et al. [32] generalized the results of [9] and proved Hermite-Hadamard type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, M. A. Latif et al. [33] found the right estimates of Hermite-Hadamard type inequalities proved by M. Kunt et al. [32]. To prove Ostrowski's inequalities, Y.-M. Chu et al. [22] used the concepts of ${}^{\pi_2}D_{p,q}$-difference operator and $(p,q)^{\pi_2}$-integral.

The following quantum variants of inequalities (1.4)-(1.6) proved my H. Budak et al. in [19].

Theorem 5. *[19] Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° and partial q_1q_2-derivatives $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ be continuous and integrable on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$. If $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right| \le M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the following quantum Ostrowski's type inequality:*

$$\begin{aligned}(1.7)\qquad &\left|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_x^{\pi_2}\int_y^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_x^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right.\right.\\
&+\left.\int_{\pi_1}^{x}\int_y^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_4-\pi_3}\left[\int_y^{\pi_4}F(x,s)\ {}^{\pi_4}d_{q_2}s+\int_{\pi_3}^{y}F(x,s)\ {}_{\pi_3}d_{q_2}s\right]\\
&\left.-\frac{1}{\pi_2-\pi_1}\left[\int_x^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{q_1}\tau+\int_{\pi_1}^{x}F(\tau,y)\ {}_{\pi_1}d_{q_1}\tau\right]+F(x,y)\right|\\
\le\ &\frac{M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\frac{q_1q_2\left(1+[2]_{q_1}\right)\left(1+[2]_{q_2}\right)}{[3]_{q_1}[3]_{q_2}}\\
&\left[\frac{(\pi_2-x)^2+(x-\pi_1)^2}{[2]_{q_1}}\right]\left[\frac{(\pi_4-y)^2+(y-\pi_3)^2}{[2]_{q_2}}\right]\end{aligned}$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ where $q_1,q_2\in(0,1)$.

Theorem 6. *[19] Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° and partial q_1q_2-derivatives $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ be continuous and integrable on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$. If $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right| \le M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the following quantum Ostrowski's type inequality:*

$$\begin{aligned}(1.8)\qquad &\left|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_x^{\pi_2}\int_y^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_x^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right.\right.\\
&+\left.\int_{\pi_1}^{x}\int_y^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_4-\pi_3}\left[\int_y^{\pi_4}F(x,s)\ {}^{\pi_4}d_{q_2}s+\int_{\pi_3}^{y}F(x,s)\ {}_{\pi_3}d_{q_2}s\right]\end{aligned}$$

$$-\frac{1}{\pi_2-\pi_1}\left[\int_x^{\pi_2} F(\tau,y)\ {}^{\pi_2}d_{q_1}\tau+\int_{\pi_1}^{x} F(\tau,y)\ {}_{\pi_1}d_{q_1}\tau\right]+F(x,y)\Bigg|$$
$$\leq \frac{q_1q_2M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left(\frac{1}{[r+1]_{q_1}}\frac{1}{[r+1]_{q_2}}\right)^{\frac{1}{r}}\left[(\pi_2-x)^2+(x-\pi_1)^2\right]\left[(\pi_4-y)^2+(y-\pi_3)^2\right]$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *where* $q_1,q_2\in(0,1)$ *and* $\frac{1}{r}+\frac{1}{s}=1,\ s>1$.

Theorem 7. *[19] Let* $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* q_1q_2*-differentiable function on* Δ° *and partial* q_1q_2*-derivatives* $\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}$, $\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$, $\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}$ *be continuous and integrable on* $[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\Delta^\circ$. *If* $\left|\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}^{\pi_4}\partial_{q_2}s}\right|$, $\left|\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|$, $\left|\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{q_1}\tau\ {}_{\pi_3}\partial_{q_2}s}\right|\leq M$ *for all* $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, *then we have the following quantum Ostrowski's type inequality:*

$$\begin{aligned}(1.9)\qquad &\Bigg|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_x^{\pi_2}\int_y^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_x^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\ {}^{\pi_2}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right.\\
&\left.+\int_{\pi_1}^{x}\int_y^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}^{\pi_4}d_{q_2}s+\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\ {}_{\pi_1}d_{q_1}\tau\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_4-\pi_3}\left[\int_y^{\pi_4}F(x,s)\ {}^{\pi_4}d_{q_2}s+\int_{\pi_3}^{y}F(x,s)\ {}_{\pi_3}d_{q_2}s\right]\\
&-\frac{1}{\pi_2-\pi_1}\left[\int_x^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{q_1}\tau+\int_{\pi_1}^{x}F(\tau,y)\ {}_{\pi_1}d_{q_1}\tau\right]+F(x,y)\Bigg|\\
\leq\ &\frac{Mq_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left(\frac{\left(1+[2]_{q_1}\right)\left(1+[2]_{q_2}\right)}{[3]_{q_1}[3]_{q_2}}\right)^{\frac{1}{s}}\\
&\times\left[\frac{(\pi_2-x)^2+(x-\pi_1)^2}{[2]_{q_1}}\right]\left[\frac{(\pi_4-y)^2+(y-\pi_3)^2}{[2]_{q_2}}\right]\end{aligned}$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *where* $q_1,q_2\in(0,1)$ *and* $s\geq 1$.

Inspired by this ongoing studies, we introduce some new notions of post-quantum partial derivatives and prove some new ostrowski type inequalities for the functions of two variables by using the post-quantum double integrals and newly introduced post-quantum partial derivatives. Moreover, we show that the results presented in this paper are the extensions of results proved in [19] and [34].

The following is the structure of this paper: A brief overview of the concepts of q-calculus, as well as some related works, is given in Section 2. In Section 3, we recall the notions of (p,q)-calculus and give some realted works. In Section 4, we show the relationship between the results presented here and comparable results in the literature by proving some new post-quantum Ostrowski type inequalities for the functions of two variables. Section 5 concludes with some recommendations for future studies.

2. Quantum Calculus and Some Inequalities

In this section, we present some required definitions and inequalities.

In [27], F. H. Jackson gave the q-Jackson integral from 0 to π_2 for $0<q<1$ as follows:

$$(2.1)\qquad \int_0^{\pi_2}F(x)\ d_qx=(1-q)\pi_2\sum_{n=0}^{\infty}q^nF(\pi_2q^n)$$

provided the sum converge absolutely. Moreover, he gave the q-Jackson integral in an arbitrary interval $[\pi_1,\pi_2]$ as:

$$\int_{\pi_1}^{\pi_2}F(x)\ d_qx=\int_0^{\pi_2}F(x)\ d_qx-\int_0^{\pi_1}F(x)\ d_qx\ .$$

Definition 2. *[49] For a continuous function $F:[\pi_1,\pi_2]\to\mathbb{R}$, then q_{π_1}-derivative of F at $x\in[\pi_1,\pi_2]$ is characterized by the expression:*

$$ {}_{\pi_1}D_qF(x) = \frac{F(x)-F(qx+(1-q)\pi_1)}{(1-q)(x-\pi_1)},\ x\neq\pi_1. \tag{2.2}$$

For $x=\pi_1$, we state ${}_{\pi_1}D_qF(\pi_1)=\lim_{x\to\pi_1}\ {}_{\pi_1}D_qF(x)$ if it exists and it is finite.

Definition 3. *[12] For a continuous function $F:[\pi_1,\pi_2]\to\mathbb{R}$, then q^{π_2}-derivative of F at $x\in[\pi_1,\pi_2]$ is characterized by the expression:*

$$ {}^{\pi_2}D_qF(x) = \frac{F(qx+(1-q)\pi_2)-F(x)}{(1-q)(\pi_2-x)},\ x\neq\pi_2. \tag{2.3}$$

For $x=\pi_2$, we state ${}^{\pi_2}D_qF(\pi_2)=\lim_{x\to\pi_2}\ {}^{\pi_2}D_qF(x)$ if it exists and it is finite.

Definition 4. *[49] Let $F:[\pi_1,\pi_2]\to\mathbb{R}$ be a continuous function. Then, the q_{π_1}-definite integral on $[\pi_1,\pi_2]$ is defined as:*

$$\begin{aligned}\int_{\pi_1}^{\pi_2}F(x)\ {}_{\pi_1}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty}q^nF(q^n\pi_2+(1-q^n)\pi_1) \\ &= (\pi_2-\pi_1)\int_0^1 F((1-\tau)\pi_1+\tau\pi_2)\ d_q\tau\ .\end{aligned}\tag{2.4}$$

On the other hand, S. Bermudo et al. gave the following new definition:

Definition 5. *[12] Let $F:[\pi_1,\pi_2]\to\mathbb{R}$ be a continuous function. Then, the q^{π_2}-definite integral on $[\pi_1,\pi_2]$ is defined as:*

$$\begin{aligned}\int_{\pi_1}^{\pi_2}F(x)\ {}^{\pi_2}d_qx &= (1-q)(\pi_2-\pi_1)\sum_{n=0}^{\infty}q^nF(q^n\pi_1+(1-q^n)\pi_2) \\ &= (\pi_2-\pi_1)\int_0^1 F(\tau\pi_1+(1-\tau)\pi_2)\ d_q\tau\ .\end{aligned}\tag{2.5}$$

For more details about q^{π_2}-integrals and corresponding inequalities one can see [12].

Now, let's give the following notation which will be used many times in the next sections (see, [29]):

$$[n]_q=\frac{q^n-1}{q-1}.$$

Moreover, we give the following Lemma for our main results:

Lemma 1. *[49] We have the equality*

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\alpha}\ {}_{\pi_1}d_qx = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_q}$$

for $\alpha\in\mathbb{R}\backslash\{-1\}$.

In [20], H. Budak et al. proved the following variant of quantum Ostrowski inequality using the q_{π_1} and q^{π_2}-integrals:

Theorem 8. *[20] Let $F:[\pi_1,\pi_2]\subset\mathbb{R}\to\mathbb{R}$ be a function and ${}^{\pi_2}D_qF$, ${}_{\pi_1}D_qF$ be two continuous and integrable functions on $[\pi_1,\pi_2]$. If $\left|{}^{\pi_2}D_q\ F(\tau)\right|,\left|{}_{\pi_1}D_q\ F(\tau)\right|\leq M$ for all $\tau\in[\pi_1,\pi_2]$, then we have the following quantum Ostrowski type inequality:*

$$\left|F(x)-\frac{1}{\pi_2-\pi_1}\left[\int_{\pi_1}^{x}F(\tau)\ {}_{\pi_1}d_q\tau\ +\int_{x}^{\pi_2}F(\tau)\ {}^{\pi_2}d_q\tau\ \right]\right| \tag{2.6}$$

$$\leq \frac{qM}{(\pi_2-\pi_1)}\left[\frac{(x-\pi_1)^2+(\pi_2-x)^2}{[2]_q}\right]$$

for all $x\in[\pi_1,\pi_2]$ *where* $0<q<1$.

On the other hand, the authors gave the following definitions of $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_2}^{\pi_3}$ and $q^{\pi_2\pi_4}$ integrals and related inequalities of Hermite-Hadamard type:

Definition 6. *[17, 34] Suppose that* $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subset\mathbb{R}^2\to\mathbb{R}$ *is a continuous function. Then, the following* $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_3}^{\pi_2}$ *and* $q^{\pi_2\pi_4}$ *integrals on* $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *are defined by*

$$\int_{\pi_1}^{x}\int_{\pi_3}^{y}F(\tau,s)\;{}_{\pi_3}d_{q_2}s\;{}_{\pi_1}d_{q_1}\tau = (1-q_1)(1-q_2)(x-\pi_1)(y-\pi_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF(q_1^nx+(1-q_1^n)\pi_1,q_2^my+(1-q_2^m)\pi_3)$$

$$\int_{\pi_1}^{x}\int_{y}^{\pi_4}F(\tau,s)\;{}^{\pi_4}d_{q_2}s\;{}_{\pi_1}d_{q_1}\tau = (1-q_1)(1-q_2)(x-\pi_1)(\pi_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF(q_1^nx+(1-q_1^n)\pi_1,q_2^my+(1-q_2^m)\pi_4) \tag{2.7}$$

$$\int_{x}^{\pi_2}\int_{\pi_3}^{y}F(\tau,s)\;{}_{\pi_3}d_{q_2}s\;{}^{\pi_2}d_{q_1}\tau = (1-q_1)(1-q_2)(\pi_2-x)(y-\pi_3)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF(q_1^nx+(1-q_1^n)\pi_2,q_2^my+(1-q_2^m)\pi_3) \tag{2.8}$$

and

$$\int_{x}^{\pi_2}\int_{y}^{\pi_4}F(\tau,s)\;{}^{\pi_4}d_{q_2}s\;{}^{\pi_2}d_{q_1}\tau = (1-q_1)(1-q_2)(\pi_2-x)(\pi_4-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF(q_1^nx+(1-q_1^n)\pi_2,q_2^my+(1-q_2^m)\pi_4) \tag{2.9}$$

respectively, for $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$.

Definition 7. *[34, 52] Let* $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a continuous function of two variables. Then, the partial* q_1*-derivatives,* q_2*-derivatives and* q_1q_2*-derivatives at* $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *can be given as follows:*

$$\frac{{}_{\pi_1}\partial_{q_1}F(x,y)}{{}_{\pi_1}\partial_{q_1}x} = \frac{F(q_1x+(1-q_1)\pi_1,y)-F(x,y)}{(1-q_1)(x-\pi_1)},\; x\neq\pi_1$$

$$\frac{{}_{\pi_3}\partial_{q_2}F(x,y)}{{}_{\pi_3}\partial_{q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_3)-F(x,y)}{(1-q_2)(y-\pi_3)},\; y\neq\pi_3$$

$$\frac{{}_{\pi_1,\,\pi_3}\partial^2_{q_1,q_2}F(x,y)}{{}_{\pi_1}\partial_{q_1}x\;{}_{\pi_3}\partial_{q_2}y} = \frac{1}{(x-\pi_1)(y-\pi_3)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_3)\right.$$
$$\left.-F(q_1x+(1-q_1)\pi_1,y)-F(x,q_2y+(1-q_2)\pi_3)+F(x,y)\right],\; x\neq\pi_1,\; y\neq\pi_3$$

$$\frac{{}^{\pi_2}\partial_{q_1}F(x,y)}{{}^{\pi_2}\partial_{q_1}x} = \frac{F(q_1x+(1-q_1)\pi_2,y)-F(x,y)}{(1-q_1)(\pi_2-x)},\; x\neq\pi_2$$

$$\begin{aligned}
\frac{{}^{\pi_4}\partial_{q_2}F(x,y)}{{}^{\pi_4}\partial_{q_2}y} &= \frac{F(x,q_2y+(1-q_2)\pi_4)-F(x,y)}{(1-q_2)(\pi_4-y)},\ y\neq\pi_4\\
\frac{{}_{\pi_1}^{\pi_4}\partial^2_{q_1,q_2}F(x,y)}{{}_{\pi_1}\partial_{q_1}x\ {}^{\pi_4}\partial_{q_2}y} &= \frac{1}{(x-\pi_1)(\pi_4-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_4)\right.\\
&\quad \left.-F(q_1x+(1-q_1)\pi_1,y)-F(x,q_2y+(1-q_2)\pi_4)+F(x,y)\right],\ x\neq\pi_1,\ y\neq\pi_4,\\
\frac{{}_{\pi_3}^{\pi_2}\partial^2_{q_1,q_2}F(x,y)}{{}^{\pi_2}\partial_{q_1}x\ {}_{\pi_3}\partial_{q_2}y} &= \frac{1}{(\pi_2-x)(y-\pi_3)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_3)\right.\\
&\quad \left.-F(q_1x+(1-q_1)\pi_2,y)-F(x,q_2y+(1-q_2)\pi_3)+F(x,y)\right],\ x\neq\pi_2,\ y\neq\pi_3,\\
\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2}F(x,y)}{{}^{\pi_2}\partial_{q_1}x\ {}^{\pi_4}\partial_{q_2}y} &= \frac{1}{(\pi_2-x)(\pi_4-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)\pi_2,q_2y+(1-q_2)\pi_4)\right.\\
&\quad \left.-F(q_1x+(1-q_1)\pi_2,y)-F(x,q_2y+(1-q_2)\pi_4)+F(x,y)\right],\ x\neq\pi_2,\ y\neq\pi_4.
\end{aligned}$$

3. Post-quantum calculus and some inequalities

In this section, we review some fundamental notions and notations of (p,q)-calculus.

The $[n]_{p,q}$ is said to be (p,q)-integers and expressed as:

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}$$

with $0<q<p\leq 1$. The $[n]_{p,q}!$ and $\begin{bmatrix} n\\ k\end{bmatrix}!$ are called (p,q)-factorial and (p,q)-binomial, respectively, and expressed as:

$$\begin{aligned}
[n]_{p,q}! &= \prod_{k=1}^{n}[k]_{p,q},\ n\geq 1,\ [0]_{p,q}!=1,\\
\begin{bmatrix} n\\ k\end{bmatrix}! &= \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.
\end{aligned}$$

Definition 8. *[46] The (p,q)-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$D_{p,q}F(x)=\frac{F(px)-F(qx)}{(p-q)x},\ x\neq 0$$

with $0<q<p\leq 1$.

Definition 9. *[48] The $(p,q)_{\pi_1}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$ {}_{\pi_1}D_{p,q}F(x)=\frac{F(px+(1-p)\pi_1)-F(qx+(1-q)\pi_1)}{(p-q)(x-\pi_1)},\ x\neq\pi_1 \tag{3.1}$$

with $0<q<p\leq 1$. For $x=\pi_1$, we state ${}_{\pi_1}D_{p,q}F(\pi_1)=\lim_{x\to\pi_1}\ {}_{\pi_1}D_{p,q}F(x)$ if it exists and it is finite.

Definition 10. *[22] The $(p,q)^{\pi_2}$-derivative of mapping $F:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$ {}^{\pi_2}D_{p,q}F(x)=\frac{F(qx+(1-q)\pi_2)-F(px+(1-p)\pi_2)}{(p-q)(\pi_2-x)},\ x\neq\pi_2. \tag{3.2}$$

with $0<q<p\leq 1$. For $x=\pi_2$, we state ${}^{\pi_2}D_{p,q}F(\pi_2)=\lim_{x\to\pi_2}\ {}^{\pi_2}D_{p,q}F(x)$ if it exists and it is finite.

Remark 1. *It is clear that if we use $p=1$ in (3.1) and (3.2), then the equalities (3.1) and (3.2) reduce to (2.2) and (2.3), respectively.*

Definition 11. *[48] The definite $(p,q)_{\pi_1}$-integral of mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{\pi_1}^{x}\digamma(\tau)\ {}_{\pi_1}d_{p,q}\tau=(p-q)(x-\pi_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_1\right)\tag{3.3}$$

with $0<q<p\le 1$.

Definition 12. *[22] The definite $(p,q)^{\pi_2}$-integral of mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{x}^{\pi_2}\digamma(\tau)\ {}^{\pi_2}d_{p,q}\tau=(p-q)(\pi_2-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_2\right)\tag{3.4}$$

with $0<q<p\le 1$.

Remark 2. *It is evident that if we pick $p=1$ in (3.3) and (3.4), then the equalities (3.3) and (3.4) change into (2.4) and (2.5), respectively.*

Remark 3. *If we take $\pi_1=0$ and $x=\pi_2=1$ in (3.3), then we have*

$$\int_0^1\digamma(\tau)\ {}_0d_{p,q}\tau=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}\right).$$

Similarly, by taking $x=\pi_1=0$ and $\pi_2=1$ in (3.4), then we obtain that

$$\int_0^1\digamma(\tau)\ {}^1d_{p,q}\tau=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(1-\frac{q^n}{p^{n+1}}\right).$$

Lemma 2. *[51] We have the following equalities*

$$\int_{\pi_1}^{\pi_2}(\pi_2-x)^{\alpha}\ {}^{\pi_2}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\alpha}\ {}_{\pi_1}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}},$$

where $\alpha\in\mathbb{R}-\{-1\}$.

In [32], M. Kunt et al. proved the following HH type inequalities for convex functions via $(p,q)_{\pi_1}$-integral:

Theorem 9. *[32] For a convex mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)_{\pi_1}$-integral:*

$$\digamma\left(\frac{q\pi_1+p\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\digamma(x)\ {}_{\pi_1}d_{p,q}x\le\frac{q\digamma(\pi_1)+p\digamma(\pi_2)}{[2]_{p,q}},\tag{3.5}$$

where $0<q<p\le 1$.

Recently, M. Vivas-Cortez et al. [51] proved the following HH type inequalities for convex functions using the $(p,q)^{\pi_2}$-integral:

Theorem 10. *[51] For a convex mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)^{\pi_2}$-integral:*

$$\digamma\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}\digamma(x)\ {}^{\pi_2}d_{p,q}x\le\frac{p\digamma(\pi_1)+q\digamma(\pi_2)}{[2]_{p,q}},\tag{3.6}$$

where $0<q<p\le 1$.

In [30] and [53], the authors gave the following notions of post-quantum integrals for the functions of two variables.

Definition 13. *[30, 53] For a function $\digamma:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\to\mathbb{R}$,*

(1) *the* $(p,q)_{\pi_1}^{\pi_4}$ *integral of* F *is given as:*

$$\int_{\pi_1}^{x}\int_{y}^{\pi_4} F(\tau,s)\ {}^{\pi_4}d_{p_2,q_2}s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(x-\pi_1)(\pi_4-y)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_1,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x,y\in[\pi_1,p_1\pi_2+(1-p_1)\pi_1]\times[p_2\pi_3+(1-p_2)\pi_4,\pi_4]$.

(2) *the* $(p,q)_{\pi_3}^{\pi_2}$ *integral of* F *is given as:*

$$\int_{x}^{\pi_2}\int_{\pi_3}^{y} F(\tau,s)\ {}_{\pi_3}d_{p_2,q_2}s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(\pi_2-x)(y-\pi_3)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_2,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x,y\in[p_1\pi_1+(1-p_1)\pi_1,\pi_2]\times[\pi_3,p_2\pi_4+(1-p_2)\pi_3]$.

(3) *the* $(p,q)^{\pi_2\pi_4}$ *integral of* F *is given as:*

$$\int_{x}^{\pi_2}\int_{y}^{\pi_4} F(\tau,s)\ {}^{\pi_4}d_{p_2,q_2}s\ {}^{\pi_2}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(\pi_2-x)(\pi_4-y)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_2,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_4\right),$$

where $x,y\in[p_1\pi_1+(1-p_1)\pi_2,\pi_2]\times[p_2\pi_3+(1-p_2)\pi_4,\pi_4]$.

(4) *the* $(p,q)_{\pi_1\pi_3}$ *integral of* F *is given as:*

$$\int_{\pi_1}^{x}\int_{\pi_3}^{y} F(\tau,s)\ {}_{\pi_3}d_{p_2,q_2}s\ {}_{\pi_1}d_{p_1,q_1}\tau = (p_1-q_1)(p_2-q_2)(x-\pi_1)(y-\pi_3)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\pi_1,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\pi_3\right)$$

where $x,y\in[\pi_1,p_1\pi_2+(1-p_1)\pi_3]\times[\pi_3,p_2\pi_4+(1-p_2)\pi_3]$.

Remark 4. *It is obvious that if we use* $p_1=p_2=1$*, then Definition 13 transforms into Definition 6.*

In [30], H. Kalsoom et al. introduced the following notions of post-quantum partial derivatives.

Definition 14. *[30] Let* $F:[\pi_1,\pi_2]\times[\pi_3,\pi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a continuous function of two variables. Then the partial* p_1q_1*-derivatives,* p_2q_2*-derivatives and* $p_1q_1p_2q_2$*-derivatives at* $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$ *can be given as follows:*

$$\frac{{}_{\pi_1}\partial_{p_1,q_1}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x} = \frac{F(q_1x+(1-q_1)\pi_1,y)-F(p_1x+(1-p_1)\pi_1,y)}{(p_1-q_1)(x-\pi_1)},\ x\neq\pi_1$$

$$\frac{{}_{\pi_3}\partial_{p_2,q_2}F(x,y)}{{}_{\pi_3}\partial_{p_2,q_2}y} = \frac{F(x,q_2y+(1-q_2)\pi_3)-F(x,p_2y+(1-p_2)\pi_3)}{(p_2-q_2)(y-\pi_3)},\ y\neq\pi_3$$

$$\begin{aligned}\frac{{}_{\pi_1,\ \pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1}x\ {}_{\pi_3}\partial_{p_2,q_2}y} &= \frac{1}{(x-\pi_1)(y-\pi_3)(p_1-q_1)(p_2-q_2)}\left[F(q_1x+(1-q_1)\pi_1,q_2y+(1-q_2)\pi_3)\right.\\ &\quad -F(q_1x+(1-q_1)\pi_1,p_2y+(1-p_2)\pi_3)-F(p_1x+(1-p_1)\pi_1,q_2y+(1-q_2)\pi_3)\\ &\quad \left.+F(p_1x+(1-p_1)\pi_1,p_2y+(1-p_2)\pi_3)\right],\ x\neq\pi_1,\ y\neq\pi_3.\end{aligned}$$

Now, from the above given concepts, we give the following new post-quantum partial derivatives.

Definition 15. *Let $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then the partial p_1q_1-derivatives, p_2q_2-derivatives and $p_1q_1p_2q_2$-derivatives at $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ can be given as follows:*

$$\begin{aligned}
\frac{{}^{\pi_2}\partial_{p_1,q_1} F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1} x} &= \frac{F(q_1x+(1-q_1)\pi_2, y) - F(p_1x+(1-p_1)\pi_2, y)}{(p_1-q_1)(\pi_2-x)}, \ x \neq \pi_2 \\
\frac{{}^{\pi_4}\partial_{p_2,q_2} F(x,y)}{{}^{\pi_4}\partial_{p_2,q_2} y} &= \frac{F(x, q_2y+(1-q_2)\pi_4) - F(x, p_2y+(1-p_2)\pi_4)}{(p_2-q_2)(\pi_4-y)}, \ y \neq \pi_4 \\
\frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(x,y)}{{}_{\pi_1}\partial_{p_1,q_1} x \ {}^{\pi_4}\partial_{p_2,q_2} y} &= \frac{1}{(x-\pi_1)(\pi_4-y)(p_1-q_1)(p_2-q_2)} [F(q_1x+(1-q_1)\pi_1, q_2y+(1-q_2)\pi_4) \\
&\quad - F(q_1x+(1-q_1)\pi_1, p_2y+(1-p_2)\pi_4) - F(p_1x+(1-p_1)\pi_1, q_2y+(1-q_2)\pi_4) \\
&\quad + F(p_1x+(1-p_1)\pi_1, p_2y+(1-p_2)\pi_4)], \ x \neq \pi_1, \ y \neq \pi_4, \\
\frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2} F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1} x \ {}_{\pi_3}\partial_{p_2,q_2} y} &= \frac{1}{(\pi_2-x)(y-\pi_3)(p_1-q_1)(p_2-q_2)} [F(q_1x+(1-q_1)\pi_2, q_2y+(1-q_2)\pi_3) \\
&\quad - F(q_1x+(1-q_1)\pi_2, p_2y+(1-p_2)\pi_3) - F(p_1x+(1-p_1)\pi_2, q_2y+(1-q_2)\pi_3) \\
&\quad + F(p_1x+(1-p_1)\pi_2, p_2y+(1-p_2)\pi_3)], \ x \neq \pi_2, \ y \neq \pi_3, \\
\frac{{}^{\pi_2, \ \pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(x,y)}{{}^{\pi_2}\partial_{p_1,q_1} x \ {}^{\pi_4}\partial_{p_2,q_2} y} &= \frac{1}{(\pi_2-x)(\pi_4-y)(p_1-q_1)(p_2-q_2)} [F(q_1x+(1-q_1)\pi_2, q_2y+(1-q_2)\pi_4) \\
&\quad - F(q_1x+(1-q_1)\pi_2, p_2y+(1-p_2)\pi_4) - F(p_1x+(1-p_1)\pi_2, q_2y+(1-q_2)\pi_4) \\
&\quad + F(p_1x+(1-p_1)\pi_2, p_2y+(1-p_2)\pi_4)], \ x \neq \pi_2, \ y \neq \pi_4.
\end{aligned}$$

Remark 5. *It is obvious that if we set $p_1 = p_2 = 1$ in Definitions 14 and 15, then we obtain the Definition 7.*

4. Quantum Ostrowski Type Inequalities for Function of Two variables

In this section, we prove some new post-quantum Ostrowski type inequalities for the functions of two variables.

Lemma 3. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially $p_1q_1p_2q_2$-differentiable function on Δ°. If partial $p_1q_1p_2q_2$-derivatives $\frac{{}^{\pi_2, \ \pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau \ {}^{\pi_4}\partial_{p_2,q_2} s}$, $\frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau \ {}^{\pi_4}\partial_{p_2,q_2} s}$, $\frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2} F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau \ {}_{\pi_3}\partial_{p_2,q_2} s}$ and $\frac{{}_{\pi_1, \ \pi_3}\partial^2_{p_1,q_1,p_2,q_2} F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau \ {}_{\pi_3}\partial_{p_2,q_2} s}$ are continuous and integrable on $[\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \Delta^\circ$, then following identity holds for $p_1q_1p_2q_2$-integrals:*

$$\begin{aligned}
& {}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s)) \\
= \ & \frac{q_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)} \\
& \times \left[(\pi_2-x)^2(\pi_4-y)^2 \int_0^1 \int_0^1 \tau s \frac{{}^{\pi_2, \ \pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4)}{{}^{\pi_2}\partial_{p_1,q_1}\tau \ {}^{\pi_4}\partial_{p_2,q_2} s} \ d_{p_1,q_1}\tau \ d_{p_2,q_2}s \right. \\
& + (\pi_2-x)^2(y-\pi_3)^2 \int_0^1 \int_0^1 \tau s \frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2} F(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3)}{{}^{\pi_2}\partial_{p_1,q_1}\tau \ {}_{\pi_3}\partial_{p_2,q_2} s} \ d_{p_1,q_1}\tau \ d_{p_2,q_2}s \\
& + (x-\pi_1)^2(\pi_4-y)^2 \int_0^1 \int_0^1 \tau s \frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2} F(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4)}{{}_{\pi_1}\partial_{p_1,q_1}\tau \ {}^{\pi_4}\partial_{p_2,q_2} s} \ d_{p_1,q_1}\tau \ d_{p_2,q_2}s
\end{aligned}$$

$$+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2\int_0^1\int_0^1 \tau s \frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2} F\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2} s}\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\Bigg]$$

where

$$\begin{aligned}(4.1)\quad & {}^{\pi_3\pi_4}_{\pi_1\pi_2}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right)\\
= & \frac{1}{p_1p_2\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right.\\
& +\int_{p_1x+(1-p_1\pi_2)}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\\
& +\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\\
& \left.+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right]\\
& -\frac{1}{p_2\left(\pi_4-\pi_3\right)}\left[\int_{p_2y+(1-p_2)\pi_4}^{\pi_4} F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s+\int_{\pi_3}^{p_2y+(1-p_2)\pi_3} F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s\right]\\
& -\frac{1}{\pi_2-\pi_1}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2} F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1} F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau\right]+F(x,y)\end{aligned}$$

for all $(x,y)\in[\pi_1,\pi_2]\times[\pi_3,d]$ *and* $0<q_i<p_i\le 1$.

Proof. From Definitions 14 and 15, we have

$$\begin{aligned}(4.2)\quad & \frac{{}^{\pi_2,\ \pi_4}\partial^2_{p_1,q_1,p_2,q_2} F\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}\\
= & \frac{1}{\left(p_1-q_1\right)\left(p_2-q_2\right)\left(\pi_2-x\right)\left(\pi_4-y\right)\tau s}\left[F\left(\tau q_1x+(1-\tau q_1)\pi_2, sq_2y+(1-sq_2)\pi_4\right)\right.\\
& -F\left(\tau q_1x+(1-\tau q_1)\pi_2, sp_2y+(1-sp_2)\pi_4\right)-F\left(\tau p_1x+(1-\tau p_1)\pi_2, sq_2y+(1-sq_2)\pi_4\right)\\
& \left.+F\left(\tau p_1x+(1-\tau p_1)\pi_2, sp_2y+(1-sp_2)\pi_4\right)\right],\end{aligned}$$

$$\begin{aligned}(4.3)\quad & \frac{{}^{\pi_2}_{\pi_3}\partial^2_{p_1,q_1,p_2,q_2} F\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\\
= & \frac{1}{\left(p_1-q_1\right)\left(p_2-q_2\right)\left(\pi_2-x\right)\left(y-\pi_3\right)\tau s}\left[F\left(\tau q_1x+(1-\tau q_1)\pi_2, sq_2y+(1-sq_2)\pi_3\right)\right.\\
& -F\left(\tau q_1x+(1-\tau q_1)\pi_2, sp_2y+(1-sp_2)\pi_4\right)-F\left(\tau p_1x+(1-\tau p_1)\pi_2, sq_2y+(1-sq_2)\pi_3\right)\\
& \left.+F\left(\tau p_1x+(1-\tau p_1)\pi_2, sp_2y+(1-sp_2)\pi_4\right)\right],\end{aligned}$$

$$\begin{aligned}(4.4)\quad & \frac{{}^{\pi_4}_{\pi_1}\partial^2_{p_1,q_1,p_2,q_2} F\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4\right)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}\\
= & \frac{1}{\left(p_1-q_1\right)\left(p_2-q_2\right)\left(x-\pi_1\right)\left(\pi_4-y\right)\tau s}\left[F\left(\tau q_1x+(1-\tau q_1)\pi_1, sq_2y+(1-sq_2)\pi_4\right)\right.\\
& -F\left(\tau q_1x+(1-\tau q_1)\pi_1, sp_2y+(1-sp_2)\pi_4\right)-F\left(\tau p_1x+(1-\tau p_1)\pi_1, sq_2y+(1-sq_2)\pi_4\right)\\
& \left.+F\left(\tau p_1x+(1-\tau p_1)\pi_1, sp_2y+(1-sp_2)\pi_4\right)\right],\end{aligned}$$

$$\begin{aligned}
(4.5)\quad & \frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s} \\
= \quad & \frac{1}{(p_1-q_1)(p_2-q_2)(x-\pi_1)(y-\pi_3)\tau s}\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sq_2 y+(1-sq_2)\pi_3\right)\right. \\
& -F\left(\tau q_1 x+(1-\tau q_1)\pi_1, sp_2 y+(1-sp_2)\pi_3\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sq_2 y+(1-sq_2)\pi_3\right) \\
& \left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_1, sp_2 y+(1-sp_2)\pi_3\right)\right].
\end{aligned}$$

By the equality (4.2) and Definition 13, we have

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 \tau s\frac{{}^{\pi_2,\ \pi_4}\partial^2_{p_1,q_1,p_2,q_2}F\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(\pi_2-x)(\pi_4-y)}\int_0^1\int_0^1\left[F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sq_2 y+(1-sq_2)\pi_4\right)\right. \\
&\quad -F\left(\tau q_1 x+(1-\tau q_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)-F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sq_2 y+(1-sq_2)\pi_4\right) \\
&\quad \left.+F\left(\tau p_1 x+(1-\tau p_1)\pi_2, sp_2 y+(1-sp_2)\pi_4\right)\right]\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
&= \frac{1}{(\pi_2-x)(\pi_4-y)}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n+1}}{p_1^{n+1}}x+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\pi_2, \frac{q_2^{m+1}}{p_2^{m+1}}y+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\pi_4\right)\right. \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n+1}}{p_1^{n+1}}x+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\pi_2, \frac{q_2^{m}}{p_2^{m}}y+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\pi_4\right) \\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}x+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\pi_2, \frac{q_2^{m+1}}{p_2^{m+1}}y+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\pi_4\right) \\
&\quad \left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n}}x+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\pi_2, \frac{q_2^{m}}{p_2^{m}}y+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\pi_4\right)\right]
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{1}{(\pi_2-x)(\pi_4-y)} \\
& \times\left[\frac{p_1p_2}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^{n}}{p_1^{n+1}}p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^{m}}{p_2^{m+1}}p_2\right)\pi_4\right)\right. \\
& -\frac{p_2}{q_1q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(x, \frac{q_2^{m}}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^{m}}{p_2^{m+1}}p_2\right)\pi_4\right) \\
& -\frac{p_1}{q_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^{n}}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^{n}}{p_1^{n+1}}p_1\right)\pi_2, y\right)+\frac{1}{q_1q_2}F(x,y) \\
& -\frac{p_1}{q_1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^{n}}{p_1^{n+1}}p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^{m}}{p_2^{m+1}}p_2\right)\pi_4\right) \\
& +\frac{1}{q_1}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(x, \frac{q_2^{m}}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^{m}}{p_2^{m+1}}p_2\right)\pi_4\right) \\
& -\frac{p_2}{q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^{n}}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^{n}}{p_1^{n+1}}p_1\right)\pi_2, \frac{q_2^{m}}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^{m}}{p_2^{m+1}}p_2\right)\pi_4\right) \\
& +\frac{1}{q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^{n}}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^{n}}{p_1^{n+1}}p_1\right)\pi_2, y\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}F\left(\frac{q_1^n}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)\pi_2,\frac{q_2^m}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)\pi_4\right)\Bigg] \\
= \quad & \frac{1}{(\pi_2-x)(\pi_4-y)} \\
&\times\left[\frac{(p_1-q_1)(p_2-q_2)}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^mF\left(\frac{q_1^n}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)\pi_2,\frac{q_2^m}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)\pi_4\right)\right. \\
&-\frac{(p_2-q_2)}{q_1q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}F\left(x,\frac{q_2^m}{p_2^{m+1}}p_2y+\left(1-\frac{q_2^m}{p_2^{m+1}}p_2\right)\pi_4\right) \\
&\left.-\frac{(p_1-q_1)}{q_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}F\left(\frac{q_1^n}{p_1^{n+1}}p_1x+\left(1-\frac{q_1^n}{p_1^{n+1}}p_1\right)\pi_2,y\right)+\frac{1}{q_1q_2}F(x,y)\right] \\
= \quad & \frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(\pi_2-x)^2(\pi_4-y)^2}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right. \\
&-\frac{1}{p_2(\pi_2-x)(\pi_4-y)^2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s \\
&\left.-\frac{1}{p_1(\pi_2-x)^2(\pi_4-y)}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\frac{1}{(\pi_2-x)(d-y)}F(x,y)\right].
\end{aligned}
$$

Similarly, by the equality (4.3), (4.4) and (4.5) we obtain the identities

$$
I_2=\int_0^1\int_0^1\tau s\frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_2,sy+(1-s)\pi_3)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.6}
$$

$$
\begin{aligned}
= \quad & \frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(\pi_2-x)^2(y-\pi_3)^2}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right. \\
&-\frac{1}{p_2(\pi_2-x)(y-\pi_3)^2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s \\
&\left.-\frac{1}{p_1(\pi_2-x)^2(y-\pi_3)}\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\frac{1}{(\pi_2-x)(y-\pi_3)}F(x,y)\right],
\end{aligned}
$$

(4.7)

$$
\begin{aligned}
I_3 \quad = \quad & \int_0^1\int_0^1\tau s\frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_4)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s}\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
= \quad & \frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(x-\pi_1)^2(\pi_4-y)^2}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right. \\
&-\frac{1}{p_2(x-\pi_1)(\pi_4-y)^2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s \\
&\left.-\frac{1}{p_1(x-\pi_1)^2(\pi_4-y)}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau+\frac{1}{(x-\pi_1)(\pi_4-y)}F(x,y)\right],
\end{aligned}
$$

and

(4.8)

$$
I_4 \quad = \quad \int_0^1\int_0^1\tau s\frac{{}_{\pi_1,\pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_3)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s
$$

$$
\begin{aligned}
&= \frac{1}{q_1q_2}\left[\frac{1}{p_1p_2(x-\pi_1)^2(y-\pi_3)^2}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right.\\
&\quad -\frac{1}{p_2(x-\pi_1)(y-\pi_3)^2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s\\
&\quad \left.-\frac{1}{p_1(x-\pi_1)^2(y-\pi_3)}\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau+\frac{1}{(x-\pi_1)(y-\pi_3)}F(x,y)\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{q_1q_2(\pi_2-x)^2(\pi_4-y)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_1+\frac{q_1q_2(\pi_2-x)^2(y-\pi_3)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_2\\
&+\frac{q_1q_2(x-\pi_1)^2(\pi_4-y)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_3+\frac{q_1q_2(x-\pi_1)^2(y-\pi_3)^2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}I_4\\
= &\ \frac{1}{p_1p_2(\pi_2-\pi_1)(\pi_4-\pi_3)}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\right.\\
&+\int_{p_1x+(1-p_1\pi_2)}^{\pi_2}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}^{\pi_2}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\\
&+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}^{\pi_4}d_{p_2,q_2}s\\
&\left.+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(\tau,s)\ {}_{\pi_1}d_{p_1,q_1}\tau\ {}_{\pi_3}d_{p_2,q_2}s\right]\\
&-\frac{1}{p_2(\pi_4-\pi_3)}\left[\int_{p_2y+(1-p_2)\pi_4}^{\pi_4}F(x,s)\ {}^{\pi_4}d_{p_2,q_2}s+\int_{\pi_3}^{p_2y+(1-p_2)\pi_3}F(x,s)\ {}_{\pi_3}d_{p_2,q_2}s\right]\\
&-\frac{1}{\pi_2-\pi_1}\left[\int_{p_1x+(1-p_1)\pi_2}^{\pi_2}F(\tau,y)\ {}^{\pi_2}d_{p_1,q_1}\tau+\int_{\pi_1}^{p_1x+(1-p_1)\pi_1}F(\tau,y)\ {}_{\pi_1}d_{p_1,q_1}\tau\right]+F(x,y)\\
= &\ {}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))
\end{aligned}
$$

which completes the proof. □

Remark 6. *In Lemma 3, if we set $p_1 = p_2 = 1$, then the Lemma 3 reduces to [19, Lemma 2].*

Remark 7. *In Lemma 3, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Lemma 3 reduces to [35, Lemma 1].*

In terms of brevity, we will use the following notations

$$
\begin{aligned}
\Phi(\tau,s) &= \frac{{}^{\pi_2,\ \pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s},\ \Psi(\tau,s)=\frac{{}_{\pi_1}^{\pi_4}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}^{\pi_4}\partial_{p_2,q_2}s},\\
\Theta(\tau,s) &= \frac{{}_{\pi_3}^{\pi_2}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}^{\pi_2}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}\text{ and }\Omega(\tau,s)=\frac{{}_{\pi_1,\ \pi_3}\partial^2_{p_1,q_1,p_2,q_2}F(\tau,s)}{{}_{\pi_1}\partial_{p_1,q_1}\tau\ {}_{\pi_3}\partial_{p_2,q_2}s}.
\end{aligned}
$$

Theorem 11. *Suppose that the assumptions of Lemma 3 hold. If $|\Phi(\tau,s)|$, $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$ and $|\Omega(\tau,s)|$ are co-ordinated convex on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the inequality*

$$
\begin{aligned}
&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))\right|\\
\leq &\ \frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\frac{q_1q_2}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\\
&\times\Big[(\pi_2-x)^2(\pi_4-y)^2
\end{aligned}
$$

$$\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}|\Phi(x,y)|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)|\Phi(x,\pi_4)|$$

$$+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)|\Phi(\pi_2,y)|+|\Phi(\pi_2,\pi_4)|\Big)$$

$$+(\pi_2-x)^2(y-\pi_3)^2$$

$$\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}|\Theta(x,y)|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)|\Theta(x,\pi_3)|$$

$$+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)|\Theta(\pi_2,y)|+|\Theta(\pi_2,\pi_3)|\Big)$$

$$(x-\pi_1)^2(\pi_4-y)^2$$

$$\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}|\Psi(x,y)|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)|\Psi(x,\pi_4)|$$

$$+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)|\Psi(\pi_1,y)|+|\Psi(\pi_1,\pi_4)|\Big)$$

$$+(x-\pi_1)^2(y-\pi_3)^2$$

$$\times\Big([2]_{p_1,q_1}[2]_{p_2,q_2}|\Omega(x,y)|+[2]_{p_1,q_1}\Big([3]_{p_2,q_2}-[2]_{p_2,q_2}\Big)|\Omega(x,\pi_3)|$$

$$+[2]_{p_2,q_2}\Big([3]_{p_1,q_1}-[2]_{p_1,q_1}\Big)|\Omega(\pi_1,y)|+|\Omega(\pi_1,\pi_3)|\Big)\Big].$$

Proof. Taking modulus in (4.1), we have

$$\begin{aligned}
&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))\right| \\
\leq\ & \frac{q_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)} \\
&\times\Big[(\pi_2-x)^2(\pi_4-y)^2\int_0^1\int_0^1\tau s\,|\Phi(\tau x+(1-\tau)\pi_2,sy+(1-s)\pi_4)|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+(\pi_2-x)^2(y-\pi_3)^2\int_0^1\int_0^1\tau s\,|\Theta(\tau x+(1-\tau)\pi_2,sy+(1-s)\pi_3)|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+(x-\pi_1)^2(\pi_4-y)^2\int_0^1\int_0^1\tau s\,|\Psi(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_4)|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
&+(x-\pi_1)^2(y-\pi_3)^2\int_0^1\int_0^1\tau s\,|\Omega(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_3)|\ d_{p_1,q_1}\tau d_{p_2,q_2}s\Big].
\end{aligned}\tag{4.9}$$

Since $|\Phi(\tau,s)|$ is co-ordinated convex, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\tau s\,|\Phi(\tau x+(1-\tau)\pi_2,sy+(1-s)\pi_4)|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \\
\leq\ & \int_0^1\int_0^1\tau s\left[\begin{array}{c}\tau s\,|\Phi(x,y)|+\tau(1-s)|\Phi(x,\pi_4)|+(1-\tau)s\,|\Phi(\pi_2,y)| \\ +(1-\tau)(1-s)|\Phi(\pi_2,\pi_4)|\end{array}\right]d_{p_1,q_1}\tau d_{p_2,q_2}s
\end{aligned}\tag{4.10}$$

$$= \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Phi(x,y)|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)|\Phi(x,\pi_4)|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)|\Phi(\pi_2,y)|+|\Phi(\pi_2,\pi_4)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}.$$

By the similar way, as $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$ and $|\Omega(\tau,s)|$ are co-ordinated convex, we establish

$$\int_0^1\int_0^1 \tau s\left|\Theta\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.11}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Theta(x,y)|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)|\Theta(x,\pi_3)|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)|\Theta(\pi_2,y)|+|\Theta(\pi_2,\pi_3)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}},$$

$$\int_0^1\int_0^1 \tau s\left|\Psi\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.12}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Psi(x,y)|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)|\Psi(x,\pi_4)|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)|\Psi(\pi_1,y)|+|\Psi(\pi_1,\pi_4)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}$$

and

$$\int_0^1\int_0^1 \tau s\left|\Omega\left(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.13}$$

$$\leq \frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}|\Omega(x,y)|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)|\Omega(x,\pi_3)|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)|\Omega(\pi_1,y)|+|\Omega(\pi_1,\pi_3)|\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}.$$

If we substitute the inequalities (4.10)-(4.13) in (4.9), then we obtain the desired result. □

Remark 8. *In Theorem 11, if we set $p_1 = p_2 = 1$, then Theorem 11 reduces to [19, Theorem 5].*

Corollary 1. *In Theorem 11, if we choose $|\Phi(\tau,s)|$, $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$, $|\Omega(\tau,s)| \leq M$ for all $(\tau,s) \in [\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right)\right|\\ \leq\ & \frac{M}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\frac{q_1q_2[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)+[2]_{p_2,q_2}[3]_{p_1,q_1}+1}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\\ &\times\left[\frac{(\pi_2-x)^2+(x-\pi_1)^2}{[2]_{p_1,q_1}}\right]\left[\frac{(\pi_4-y)^2+(y-\pi_3)^2}{[2]_{p_2,q_2}}\right].\end{aligned}$$

Remark 9. *In Corollary 1, if we put $p_1 = p_2 = 1$, then we recapture the inequality (1.7).*

Remark 10. *In Corollary 1, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Corollary 1 reduces to Theorem 2.*

Theorem 12. *Suppose that the assumptions of Lemma 3 are hold. If $|\Phi(\tau,s)|^s$, $|\Theta(\tau,s)|^s$, $|\Psi(\tau,s)|^s$ and $|\Omega(\tau,s)|^s$ are co-ordinated convex on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F(\tau,s)\right)\right|\\ \leq\ & \frac{q_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}\end{aligned}$$

$$\times\left[(\pi_2-x)^2(\pi_4-y)^2\left(\frac{\begin{array}{c}|\Phi(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Phi(x,\pi_4)|^s+\left([2]_{p_1,q_1}-1\right)|\Phi(\pi_2,y)|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Phi(\pi_2,\pi_4)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}\right.$$

$$+(\pi_2-x)^2(y-\pi_3)^2\left(\frac{\begin{array}{c}|\Theta(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Theta(x,\pi_3)|^s+\left([2]_{p_1,q_1}-1\right)|\Theta(\pi_2,y)|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Theta(\pi_2,\pi_3)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

$$+(x-\pi_1)^2(\pi_4-y)^2\left(\frac{\begin{array}{c}|\Psi(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Psi(x,\pi_4)|^s+\left([2]_{p_1,q_1}-1\right)|\Psi(\pi_1,y)|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Psi(\pi_1,\pi_4)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

$$\left.+(x-\pi_1)^2(y-\pi_3)^2\left(\frac{\begin{array}{c}|\Omega(x,y)|^s+\left([2]_{p_2,q_2}-1\right)|\Omega(x,\pi_3)|^s+\left([2]_{p_1,q_1}-1\right)|\Omega(\pi_1,y)|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)|\Omega(\pi_1,\pi_3)|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}\right]$$

where $\frac{1}{r}+\frac{1}{s}=1,\ s>1.$

Proof. From the Lemma 3, we have

$$\begin{aligned}
(4.14)\qquad & \left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}(F(\tau,s))\right| \\
\leq\ & \frac{q_1q_2}{(\pi_2-\pi_1)(\pi_4-\pi_3)} \\
& \times\left[(\pi_2-x)^2(\pi_4-y)^2\int_0^1\int_0^1 \tau s\left|\Phi(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right. \\
& +(\pi_2-x)^2(y-\pi_3)^2\int_0^1\int_0^1 \tau s\left|\Theta(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
& +(x-\pi_1)^2(\pi_4-y)^2\int_0^1\int_0^1 \tau s\left|\Psi(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_4)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
& \left.+(x-\pi_1)^2(y-\pi_3)^2\int_0^1\int_0^1 \tau s\left|\Omega(\tau x+(1-\tau)\pi_1, sy+(1-s)\pi_3)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right].
\end{aligned}$$

By using the well-known Hölder inequality and the co-ordinated convexity of $|\Phi(\tau,s)|^s$, we obtain

$$\begin{aligned}
(4.15)\ & \int_0^1\int_0^1 \tau s\left|\Phi(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \\
\leq\ & \left(\int_0^1\int_0^1 \tau^r s^r\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{r}}\left(\int_0^1\int_0^1\left|\Phi(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4)\right|^s\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right)^{\frac{1}{s}} \\
\leq\ & \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}
\end{aligned}$$

$$\times\left(\int_0^1\int_0^1\left[\tau s\left|\Phi\left(x,y\right)\right|^s+\tau\left(1-s\right)\left|\Phi\left(x,\pi_4\right)\right|^s\right.\right.$$
$$\left.\left.+\left(1-\tau\right)s\left|\Phi\left(\pi_2,y\right)\right|^s+\left(1-\tau\right)\left(1-s\right)\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\right]d_{p_1,q_1}\tau\ d_{p_2,q_2}s\right)^{\frac{1}{s}}$$

$$= \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$
$$\times\left(\frac{\begin{array}{c}\left|\Phi\left(x,y\right)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Phi\left(x,\pi_4\right)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Phi\left(\pi_2,y\right)\right|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}.$$

Similarly, we have

$$\int_0^1\int_0^1\tau s\left|\Theta\left(\tau x+\left(1-\tau\right)\pi_2,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.16}$$
$$\leq \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$
$$\times\left(\frac{\begin{array}{c}\left|\Theta\left(x,y\right)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Theta\left(x,\pi_3\right)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Theta\left(\pi_2,y\right)\right|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Theta\left(\pi_2,\pi_3\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}},$$

$$\int_0^1\int_0^1\tau s\left|\Psi\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_4\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.17}$$
$$\leq \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$
$$\times\left(\frac{\begin{array}{c}\left|\Psi\left(x,y\right)\right|^s+\left([2]_{p_2,q_2}-1\right)\left|\Psi\left(x,\pi_4\right)\right|^s+\left([2]_{p_1,q_1}-1\right)\left|\Psi\left(\pi_1,y\right)\right|^s\\+\left([2]_{p_2,q_2}-1\right)\left([2]_{p_1,q_1}-1\right)\left|\Psi\left(\pi_1,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

and

$$\int_0^1\int_0^1\tau s\left|\Omega\left(\tau x+\left(1-\tau\right)\pi_1,sy+\left(1-s\right)\pi_3\right)\right|\ d_{p_1,q_1}\tau\ d_{p_2,q_2}s \tag{4.18}$$
$$\leq \left(\frac{1}{[r+1]_{p_1,q_1}}\frac{1}{[r+1]_{p_2,q_2}}\right)^{\frac{1}{r}}$$

$$\left(\frac{\begin{array}{c}\left|\Omega\left(x,y\right)\right|^{s}+\left(\left[2\right]_{p_{2},q_{2}}-1\right)\left|\Omega\left(x,\pi_{3}\right)\right|^{s}+\left(\left[2\right]_{p_{1},q_{1}}-1\right)\left|\Omega\left(\pi_{1},y\right)\right|^{s}\\+\left(\left[2\right]_{p_{2},q_{2}}-1\right)\left(\left[2\right]_{p_{1},q_{1}}-1\right)\left|\Omega\left(\pi_{1},\pi_{3}\right)\right|^{s}\end{array}}{\left[2\right]_{p_{1},q_{1}}\left[2\right]_{p_{2},q_{2}}}\right)^{\frac{1}{s}}.$$

By substituting the inequalities (4.15)-(4.18) in (4.14), then we obtain the required result. □

Remark 11. *In Theorem 12, if we use $p_1 = p_2 = 1$, then Theorem 12 reduces to [19, Theorem 6].*

Corollary 2. *In Theorem 12, if we choose $|\Phi(\tau,s)|$, $|\Theta(\tau,s)|$, $|\Psi(\tau,s)|$, $|\Omega(\tau,s)| \leq M$ for all $(\tau,s) \in [\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right|\\ \leq\;&\frac{q_1q_2M}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\left(\frac{1}{\left[r+1\right]_{p_1,q_1}}\frac{1}{\left[r+1\right]_{p_2,q_2}}\right)^{\frac{1}{r}}\left[\left(\pi_2-x\right)^2+\left(x-\pi_1\right)^2\right]\left[\left(\pi_4-y\right)^2+\left(y-\pi_3\right)^2\right].\end{aligned}$$

Remark 12. *In Corollary 2, if we set $p_1 = p_2 = 1$, then we recpature the inequality (1.8).*

Remark 13. *In Corollary 2, if we set $p_1 = p_2 = 1$ and $q_1, q_2 \to 1^-$, then Corollary 2 reduces to Theorem 3.*

Theorem 13. *Suppose that the assumptions of Lemma 3 hold. If $|\Phi(\tau,s)|^s$, $|\Theta(\tau,s)|^s$, $|\Psi(\tau,s)|^s$ and $|\Omega(\tau,s)|^s$, $s \geq 1$ are co-ordinated convex on $[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we have the inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right|\\ \leq\;&\frac{1}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\frac{q_1q_2}{\left[2\right]_{p_1,q_1}\left[2\right]_{p_2,q_2}}\\ &\times\Bigg[\left(\pi_2-x\right)^2\left(\pi_4-y\right)^2\\ &\times\left(\frac{\begin{array}{c}\left[2\right]_{p_1,q_1}\left[2\right]_{p_2,q_2}\left|\Phi\left(x,y\right)\right|^s+\left[2\right]_{p_1,q_1}\left(\left[3\right]_{p_2,q_2}-\left[2\right]_{p_2,q_2}\right)\left|\Phi\left(x,\pi_4\right)\right|^s\\+\left[2\right]_{p_2,q_2}\left(\left[3\right]_{p_1,q_1}-\left[2\right]_{p_1,q_1}\right)\left|\Phi\left(\pi_2,y\right)\right|^s+\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\end{array}}{\left[3\right]_{p_1,q_1}\left[3\right]_{p_2,q_2}}\right)^{\frac{1}{s}}\\ &+\left(\pi_2-x\right)^2\left(y-\pi_3\right)^2\\ &\times\left(\frac{\begin{array}{c}\left[2\right]_{p_1,q_1}\left[2\right]_{p_2,q_2}\left|\Theta\left(x,y\right)\right|^s+\left[2\right]_{p_1,q_1}\left(\left[3\right]_{p_2,q_2}-\left[2\right]_{p_2,q_2}\right)\left|\Theta\left(x,\pi_3\right)\right|^s\\+\left[2\right]_{p_2,q_2}\left(\left[3\right]_{p_1,q_1}-\left[2\right]_{p_1,q_1}\right)\left|\Theta\left(\pi_2,y\right)\right|^s+\left|\Theta\left(\pi_2,\pi_3\right)\right|^s\end{array}}{\left[3\right]_{p_1,q_1}\left[3\right]_{p_2,q_2}}\right)^{\frac{1}{s}}\\ &+\left(x-\pi_1\right)^2\left(\pi_4-y\right)^2\end{aligned}$$

$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Psi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Psi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Psi\left(\pi_1,y\right)\right|^s+\left|\Psi\left(\pi_1,\pi_4\right)\right|^s\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

$$+\left(x-\pi_1\right)^2\left(y-\pi_3\right)^2$$

$$\left.\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Omega\left(x,y\right)\right|+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Omega\left(x,\pi_3\right)\right|\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Omega\left(\pi_1,y\right)\right|+\left|\Omega\left(\pi_1,\pi_3\right)\right|\end{array}}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)\right].$$

Proof. By using the power mean inequality and the co-ordinated convexity of $|\Phi(\tau,s)|^s$, we obtain

$$\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.19}$$

$$\begin{aligned}
&\le \left(\int_0^1\int_0^1 \tau s\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{1-\frac{1}{s}}\\
&\quad\times\left(\int_0^1\int_0^1 \tau s\left|\Phi\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_4\right)\right|^s\ d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{s}}\\
&\le \left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}\\
&\quad\times\left(\int_0^1\int_0^1 \tau s\left[\tau s\left|\Phi\left(x,y\right)\right|^s+\tau(1-s)\left|\Phi\left(x,\pi_4\right)\right|^s\right.\right.\\
&\quad\left.\left.+(1-\tau)s\left|\Phi\left(\pi_2,y\right)\right|^s+(1-\tau)(1-s)\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\right]d_{p_1,q_1}\tau d_{p_2,q_2}s\right)^{\frac{1}{s}}\\
&= \left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}\\
&\quad\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Phi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Phi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Phi\left(\pi_2,y\right)\right|^s+\left|\Phi\left(\pi_2,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}.
\end{aligned}$$

Similarly, since $|\Theta(\tau,s)|^s$, $|\Psi(\tau,s)|^s$ and $|\Omega(\tau,s)|^s$ are co-ordinated convex, we establish

$$\int_0^1\int_0^1 \tau s\left|\Theta\left(\tau x+(1-\tau)\pi_2, sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s \tag{4.20}$$

$$\le \left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}$$

$$\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Theta\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Theta\left(x,\pi_3\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Theta\left(\pi_2,y\right)\right|^s+\left|\Theta\left(\pi_2,\pi_3\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}$$

(4.21)
$$\begin{aligned}&\int_0^1\int_0^1\tau s\left|\Psi\left(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_4\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s\\ \leq\ &\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}\\ &\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Psi\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Psi\left(x,\pi_4\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Psi\left(\pi_1,y\right)\right|^s+\left|\Psi\left(\pi_1,\pi_4\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}\end{aligned}$$

and

(4.22)
$$\begin{aligned}&\int_0^1\int_0^1\tau s\left|\Omega\left(\tau x+(1-\tau)\pi_1,sy+(1-s)\pi_3\right)\right|\ d_{p_1,q_1}\tau d_{p_2,q_2}s\\ \leq\ &\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\right)^{1-\frac{1}{s}}\\ &\times\left(\frac{\begin{array}{c}[2]_{p_1,q_1}[2]_{p_2,q_2}\left|\Omega\left(x,y\right)\right|^s+[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)\left|\Omega\left(x,\pi_3\right)\right|^s\\+[2]_{p_2,q_2}\left([3]_{p_1,q_1}-[2]_{p_1,q_1}\right)\left|\Omega\left(\pi_1,y\right)\right|^s+\left|\Omega\left(\pi_1,\pi_3\right)\right|^s\end{array}}{[2]_{p_1,q_1}[2]_{p_2,q_2}[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}.\end{aligned}$$

If we substitute the inequalities (4.19)-(4.22) in (4.14), then we obtain the desired result. □

Remark 14. *In Theorem 13, if we assume $p_1=p_2=1$, then Theorem 13 becomes [19, Theorem 7].*

Corollary 3. *In Theorem 13, if we choose $\left|\Phi(\tau,s)\right|,\left|\Theta(\tau,s)\right|,\left|\Psi(\tau,s)\right|,\left|\Omega(\tau,s)\right|\leq M$ for all $(\tau,s)\in[\pi_1,\pi_2]\times[\pi_3,\pi_4]$, then we obtain the following post-quantum Ostrowski type inequality*

$$\begin{aligned}&\left|{}_{\pi_1\pi_2}^{\pi_3\pi_4}\mathcal{J}_{p_1,q_1,p_2,q_2}\left(F\left(\tau,s\right)\right)\right|\\ \leq\ &\frac{Mq_1q_2}{\left(\pi_2-\pi_1\right)\left(\pi_4-\pi_3\right)}\left(\frac{[2]_{p_1,q_1}\left([3]_{p_2,q_2}-[2]_{p_2,q_2}\right)+[2]_{p_2,q_2}[3]_{p_1,q_1}+1}{[3]_{p_1,q_1}[3]_{p_2,q_2}}\right)^{\frac{1}{s}}\\ &\times\left[\frac{\left(\pi_2-x\right)^2+\left(x-\pi_1\right)^2}{[2]_{p_1,q_1}}\right]\left[\frac{\left(\pi_4-y\right)^2+\left(y-\pi_3\right)^2}{[2]_{p_2,q_2}}\right].\end{aligned}$$

Remark 15. *In Corollary 3, if we consider $p_1=p_2=1$, then we recapture the inequality (1.9).*

Remark 16. *In Corollary 3, if we consider $p_1=p_2=1$ and $q_1,q_2\to1^-$, then Corollary 3 reduces to Theorem 4.*

5. Conclusion

In this study, we proved some new post-quantum variants of Ostrowski type inequalities for the differentiable functions of two variables. We also proved that the results proved in this study are the refinements of some existing results in the field of integral inequalities. It is an interesting and new problem that the upcoming researchers can obtain the similar inequalities for the other kind of convexity in their future work.

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