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RESEARCH ARTICLE

Zero dissipation limit to rarefaction wave with vacuum for the micropolar compressible flow with temperature-dependent transport coefficients

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ABSTRACT

In this article, we consider the zero dissipation limit of the micropolar equations with temperature-dependent viscosity and heat-conduction coefficient. If the given rarefaction wave to the corresponding conservation systems connects to vacuum at one side, we can construct a sequence of solutions to the micropolar equations which converge to the given rarefaction wave with vacuum as the transport coefficients tend to zero. And the uniform convergence rate can be obtained. The key point in our analysis is how to control the degeneracies of the density, the temperature and the temperature-dependent viscosities at the vacuum region in the zero dissipation limit process.

KEYWORDS:
micropolar equations, temperature-dependent transport coefficients, zero dissipation limit, vacuum.

1 | INTRODUCTION AND MAIN RESULT

In the present paper, we consider the zero dissipation limit of the one-dimensional micropolar equations

$$\begin{cases} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= (\epsilon\mu(\theta)u_x)_x, \\ (\rho w)_t + (\rho u w)_x + \epsilon\xi(\theta)w &= (\epsilon\lambda(\theta)w_x)_x, \\ (\rho E)_t + (\rho u E + up)_x &= (\epsilon\kappa(\theta)\theta_x)_x + (\epsilon\mu(\theta)uu_x)_x + \epsilon\lambda(\theta)(w_x)^2 + \epsilon\xi(\theta)w^2, \end{cases} \tag{1.1}$$

here, $x \in \mathbf{R}$, $t > 0$, the unknowns are $\rho(x,t) \geq 0$, $u(x,t)$, $w(x,t)$ and $\theta(x,t) \geq 0$ which denote the mass density, fluid velocity, microrotational velocity and absolute temperature respectively. Moreover, $p = p(\rho,\theta)$ is pressure and $E = e + \frac{u^2}{2}$ is the specific total energy, where $e = e(\rho,\theta)$ is the specific internal energy. The transport coefficients are $\epsilon\mu(\theta)$, $\epsilon\xi(\theta)$, $\epsilon\lambda(\theta)$ and $\epsilon\kappa(\theta)$ which represent the shear viscosity, microrotation viscosity, angular viscosity and heat conductivity coefficient, respectively.

The micropolar equations was firstly introduced by Eringen[11] to deal with a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements. Which enables us to consider some physical phenomena that cannot be explained by the classical Navier-Stokes equations. Due to its importance in mathematics and physics, a lot of attentions have been payed to the micropolar fluid systems:

- For one dimensional case, Mujaković studied the local existence and global existence to an initial-boundary value problem in[22,23]. Later, Cui and Yin[5] studied the stability of the composite wave for the inflow problem on the micropolar fluid model, Jin and Duan[16] verified the stability of rarefaction waves for 1-D compressible viscous micropolar fluid model.

Recently, Duan[10] proved the global solutions for the one-dimensional with zero heat conductivity case, and Su verified the global existence and low Mach number limit of a compressible micropolar fluid in[26,27].

- For multidimensional case, Mujaković et al.[6,7,8,9,24] considered the three dimensional spherical symmetry solution and derived its local existence, global existence, uniqueness and large time behavior. Recently, Gong and Zhang[13] obtained the nonlinear stability of planar rarefaction wave to 3D micropolar equations.

Its worth to note that all the results obtained above are made in the absence of vacuum. When vacuum is under consideration, due to the difficulties caused by the vacuum, the corresponding results are much less, the global well-posedness of strong solutions was verified in[2,28] for the one dimensional case. Chen et al.[3,4] derived the global weak solutions with discontinuous initial data and vacuum and the blow up criterion with vacuum. Very recently, Gong[12] studied the zero dissipation limit to rarefaction wave with one-side vacuum state for the 1D micropolar equations with constant transport coefficient.

The most results which we mentioned before concern the case that the transport coefficients are constants. While according to the Chapman-Enskog expansion theory for rarefied gas dynamics (cf.[1]), the transport coefficients should be temperature-dependent. In this paper, in the base of the former study[12], we further concerns the zero dissipation limit to the rarefaction wave with vacuum for the system (1.1) with temperature-dependent transport coefficients. We assume the transport coefficients in system (1.1) satisfy $\epsilon > 0$ and

$$\mu(\theta) = \mu_1\theta^\alpha, \quad \xi(\theta) = \xi_1\theta^\alpha, \quad \lambda(\theta) = \lambda_1\theta^\alpha, \quad \kappa(\theta) = \kappa_1\theta^\alpha \tag{1.2}$$

for positive constants $\mu_1, \xi_1, \lambda_1, \kappa_1$ and $\alpha > 0$. For simplicity, we take $\mu_1 = \xi_1 = \lambda_1 = \kappa_1 = 1$ in the rest of this paper. In this article, we consider the ideal polytropic gas, that is, the pressure $p(\rho,\theta)$ and internal energy $e(\rho,\theta)$ are given respectively by the relations

$$p = R\rho\theta = A\rho^\gamma \exp\frac{R}{\gamma-1}S, \quad e = \frac{R}{\gamma-1}\theta + const, \tag{1.3}$$

where $\gamma > 1$ is the adiabatic exponent, A and R are positive constants, and S is the entropy. We take $A = R = \gamma - 1$ for simplicity. The second law of the thermodynamics shows that

$$de = \theta dS + \frac{p}{\rho^2}d\rho. \tag{1.4}$$

As we all know, if $w(x,t) = Const$ and $\xi_1 = 0$, the system (1.1) reduces to the compressible Navier-Stokes equations, and further more, tends to the inviscid Euler equations in formally if we take $\epsilon \to 0$:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + (\rho u E + up)_x = 0. \end{cases} \tag{1.5}$$

In fact, the zero dissipation limit of viscous flows with basic wave patterns is one of the important problems in the theory of the compressible fluid. For the Navier-Stokes equations, there have respectable results been obtained in this aspect: For the researches without vaccum, the readrs may please refer to[15,20,29,31] and references therein. When vacuum appears, Jiu, Wang and Xin[17] verified the large time asymptotic behavior toward rarefaction wave for solutions to the one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity for general initial data. The zero dissipation limit of the full compressible Navier-Stokes equations to a rarefaction wave with vacuum was proved by Li and Wang[19] for the constant transport coefficients case and by Li, Wang and Wang[18] for the temperature-dependent transport coefficients case. While, so far, the related researches on zero dissipation limit of micropolar equations are few.

The Euler system (1.5) is a strict hyperbolic system of conservation laws with three distinct eigenvalues:

$$\lambda_1(\rho,u,S) = u - \sqrt{p_\rho(\rho,S)}, \quad \lambda_2(\rho,u,S) = u, \quad \lambda_3(\rho,u,S) = u + \sqrt{p_\rho(\rho,S)},$$

the corresponding right eigenvectors to λ_1 and λ_3 are:

$$r_1(\rho,u,S) = \left(-\rho, \sqrt{p_\rho(\rho,S)}, 0\right)^t, \quad r_3(\rho,u,S) = \left(\rho, \sqrt{p_\rho(\rho,S)}, 0\right)^t,$$

and $\lambda_i, r_i (i = 1,3)$ satisfy the following relation

$$r_i(\rho,u,S)\cdot\nabla_{(\rho,u,S)}\lambda_i(\rho,u,S) \neq 0, \quad i = 1,3, \quad \forall\theta > 0.$$

Define two 1-Riemann invariants by

$$\Sigma_1^{(1)} = u + \int^{\rho} \frac{\sqrt{p_z(z,S)}}{z} dz, \quad \Sigma_1^{(2)} = S, \tag{1.6}$$

and two 3-Riemann invariants by

$$\Sigma_3^{(1)} = u - \int^{\rho} \frac{\sqrt{p_z(z,S)}}{z} dz, \quad \Sigma_3^{(2)} = S, \tag{1.7}$$

such that

$$\nabla_{(\rho,u,S)}\Sigma_i^{(j)}(\rho,u,S)\cdot r_i(\rho,u,S) \equiv 0, \quad i = 1,3, \quad j = 1,2, \quad \forall \rho > 0.$$

Now we give a description of the rarefaction wave connected to the vacuum to the compressible Euler equations (1.5)(cf.[25]). For definiteness, 3-rarefaction wave will be considered. If we study the compressible Euler system (1.5) with the Riemann initial data

$$\begin{cases} \rho(x,0) = 0, & x < 0, \\ (\rho,u,\theta)(x,0) = (\rho_+,u_+,\theta_+), & x > 0, \end{cases} \tag{1.8}$$

where the left side is the vacuum state and $\rho_+ > 0, u_+, \theta_+ > 0$ are prescribed constants on the right state, and then the Riemann problem (1.5), (1.8) admits a 3-rarefaction wave which connected to the vacuum on the left side. In fact, the 3-Riemann invariant $\Sigma_3^{(i)}(\rho,u,\theta), (i = 1,2)$ is constant in (x,t) along the 3-rarefaction wave curve, then we can obtain the velocity $u_- = \Sigma_3^{(1)}(\rho_+,u_+,\theta_+)$ being the speed of the fluid coming into the vacuum from the 3-rarefaction wave. This 3-rarefaction wave $(\rho^{r_3},u^{r_3},\theta^{r_3})(\xi), (\xi = \frac{x}{t})$ connecting the vacuum $\rho = 0$ to (ρ_+,u_+,θ_+) is the self-similar solution of (1.5) defined by

$$\lambda_3(\rho^{r_3}(\xi),u^{r_3}(\xi),\theta^{r_3}(\xi)) = \begin{cases} \rho^{r_3}(\xi) \equiv 0, & \xi < \lambda_3(0,u_-,0) = u_-, \\ \xi, & u_- \le \xi \le \lambda_3(\rho_+,u_+,\theta_+), \\ \lambda_3(\rho_+,u_+,\theta_+), & \xi > \lambda_3(\rho_+,u_+,\theta_+), \end{cases} \tag{1.9}$$

and

$$\Sigma_3^{(1)}(\rho^{r_3}(\xi),u^{r_3}(\xi),\theta^{r_3}(\xi)) = \Sigma_3^{(1)}(0,u_-,0) = \Sigma_3^{(1)}(\rho_+,u_+,\theta_+). \tag{1.10}$$

Thus we define the momentum $m^{r_3} = m^{r_3}(\xi)$ and the total internal energy $\tilde{e^{r_3}} = \tilde{e^{r_3}}(\xi)$ of a 3-rarefaction wave by

$$m^{r_3}(\xi) := \begin{cases} \rho^{r_3}(\xi)u^{r_3}(\xi), & (\rho^{r_3} > 0), \\ 0, & (\rho^{r_3} = 0), \end{cases} \tag{1.11}$$

and

$$\tilde{e^{r_3}}(\xi) := \begin{cases} \rho^{r_3}(\xi)\theta^{r_3}(\xi), & (\rho^{r_3} > 0), \\ 0, & (\rho^{r_3} = 0). \end{cases} \tag{1.12}$$

Motivated by[14,18,19], we want to construct a sequence of solutions $(\rho^\epsilon, m^\epsilon, \tilde{e^\epsilon} := \rho^\epsilon\theta^\epsilon, w^\epsilon)(x,t)$ to the micropolar equations (1.1) which converge to the 3-rarefaction wave $(\rho^{r_3}, m^{r_3}, \tilde{e^{r_3}} = \rho^{r_3}\theta^{r_3}, 0)\left(\frac{x}{t}\right)$ defined above as ϵ tends to zero. By selecting the well-prepared initial data depending on the viscosity of the micropolar equations, the influence of the initial layer can be ignored.

Now we state our main result as follows:

Theorem 1. Let $(\rho^{r_3}, m^{r_3}, \tilde{e^{r_3}})(\frac{x}{t})$ be the 3-rarefaction wave defined by (1.9)-(1.12) with left side is vacuum state. Then there exists a constant $\epsilon_0 > 0$ small enough such that for $\forall\epsilon$, $0 < \epsilon < \epsilon_0$, we can construct a family of global smooth solutions $(\rho^\epsilon, m^\epsilon, \tilde{e^\epsilon} := \rho^\epsilon\theta^\epsilon, w^\epsilon)(x,t)$ to the micropolar equations (1.1) satisfying

(i)
$$(\rho^\epsilon - \rho^{r_3}, m_\epsilon - m^{r_3}, \tilde{e^\epsilon} - \tilde{e^{r_3}}, w^\epsilon), (\rho^\epsilon_x, m^\epsilon_x, \tilde{e^\epsilon}_x, w^\epsilon_x) \in C^0(0,+\infty; L^2(\mathbf{R})),$$
$$(u^\epsilon_{xx}, \theta^\epsilon_{xx}, w^\epsilon_{xx}) \in L^2(0,+\infty; L^2(\mathbf{R})).$$

(ii) As ϵ tends to zero, $(\rho^\epsilon, m^\epsilon, \tilde{e^\epsilon}, w^\epsilon)(x,t)$ converges to $(\rho^{r_3}, m^{r_3}, \tilde{e^{r_3}}, 0)(\frac{x}{t})$ pointwise except at $(0,0)$. Furthermore, for any given positive constant l, there exists a constant $C_l > 0$, independent of ϵ, such that

$$\begin{aligned} \sup_{t\ge l}\left\|\rho^\epsilon(\cdot,t) - \rho^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} &\le C_l\epsilon^a|\ln\epsilon|, \quad \sup_{t\ge l}\left\|m^\epsilon(\cdot,t) - m^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} \le C_l\epsilon^a|\ln\epsilon|, \\ \sup_{t\ge l}\left\|\tilde{e^\epsilon}(\cdot,t) - \tilde{e^{r_3}}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} &\le C_l\epsilon^a|\ln\epsilon|, \quad \sup_{t\ge l}\|w^\epsilon(\cdot,t)\|_{L^\infty} \le C_l\epsilon^a, \end{aligned} \tag{1.13}$$

with the positive constant a is given by

$$a = \frac{1}{18\gamma + 12\alpha(\gamma - 1)}. \tag{1.14}$$

Remark 1. Some remarks can be summarized as follows:

- In the vacuum region, according to the state equation (1.3) and the fact that the entropy S is constant along the 3-rarefaction wave, the absolute temperature also becomes zero. Thus, not only the density but also the temperature cause the degeneracies in the vacuum region and so the temperature-dependent viscosities do, which causes the terms for the temperature-dependent viscosities becomes very complicated.

- Its seen that, the decay rate a given in (1.14) is monotone decrease as the parameter α grows. Which is consistent with observation that the viscosity effect becomes weaker as α grows due to the vacuum.

In this paper, the main difficulty is how to control the degeneracies caused by the vacuum in the rarefaction wave. To overcome this difficulty, firstly, because of the invalidation of Lagrangian transformation when the fluid involves vacuum, we have to deal with the micropolar equations in Eulerian coordinates. Then, we cut off the 3-rarefaction wave with vacuum along the rarefaction wave curve to control the convection terms, which couples by density and velocity. More precisely, for any $\nu > 0$ to be determined, the states $(\rho_\nu, u_\nu, \theta_\nu) = (\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1})$ and (ρ_+, u_+, θ_+) can be connected by the cut-off rarefaction wave, where u_ν can be determined uniquely by the definition of the 3-rarefaction wave curve. Finally, the desired solution sequences of the micropolar system (1.1) can be established around the approximate rarefaction wave.

On the other hand, compared to the previous works [12] for the micropolar equations with the constant viscosity case, some new difficulties occur for the full micropolar equations (1.1) with temperature-dependent viscosities considered in the present paper. For example, as we mentioned in the remark, the terms for temperature-dependent transport coefficients become very complicated, see (3.26). Actually, the derivative estimates of the perturbation of the density depend on the second-order derivative estimates of velocity with some degenerate coefficients is quite different from the constant viscosity case in [12]. So, we choose the convergence rate a suitably as in (1.14) and choose the parameters ν, δ as in (3.7) to close the a priori estimates and yields the desired result.

The rest of this paper is arranged as follows: In section 2, we construct a smooth 3-rarefaction wave which approximates the cut-off rarefaction wave based on the inviscid Burgers equation, and then we give some properties of this approximate rarefaction wave. In section 3, we show the proof of the a priori estimates. Finally, the Theorem 1 is proved in section 4.

Notations. Throughout this article, $L^q(\Omega)(1 \le q \le \infty)$ denotes the usual Lebesgue space with norm $\|\cdot\|_{L^q}$, and $H^k(\Omega)(k \in \mathbb{N})$ represents the usual Sobolev space with norm $\|\cdot\|_k$. We take $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ for simplicity. And $C(I; H^p(\Omega))$ denotes the space of continuous functions on the interval I with values in $H^p(\Omega)$ and $L^2(I; H^p(\Omega))$ represents the space of L^2-functions on I with values in $H^p(\Omega)$.

2 | RAREFACTION WAVE

As we all know, there is no exact rarefaction wave profile for the micropolar equations (1.1), and the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Matsumura-Nishihara [21] and Xin [30].

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} W_t + WW_x = 0, \\ W(x,0) = \begin{cases} W_-, & x < 0, \\ W_+, & x > 0. \end{cases} \end{cases} \tag{2.1}$$

If $W_- < W_+$, the Riemann problem (2.1) admits a rarefaction wave solution $W^r(x,t) = W^r(\frac{x}{t})$ given by

$$W^r\left(\frac{x}{t}\right) = \begin{cases} W_-, & \frac{x}{t} \le W_-, \\ \frac{x}{t}, & W_- \le \frac{x}{t} \le W_+, \\ W_+, & \frac{x}{t} \ge W_+. \end{cases} \tag{2.2}$$

However, $W^r\left(\frac{x}{t}\right)$ is only Lipchitz continuous. We construct the smooth approximate rarefaction wave $W^r_\delta(x,t)$ by the solution of following Burgers equation (see[30])

$$\begin{cases} W_t + WW_x = 0, \\ W(x,0) = W_\delta(x) = W(\frac{x}{\delta}) = \frac{W_+ + W_-}{2} + \frac{W_+ + W_-}{2}\tanh\frac{x}{\delta}, \end{cases} \tag{2.3}$$

here $\delta > 0$ is a small parameter to be determined. We choose $\delta = \epsilon^a$ in (3.7) with a given by (1.14), then the solution $W^r_\delta(x,t)$ of the problem (2.3) is given by

$$W^r_\delta(x,t) = W_\delta(x_0(x,t)), \quad x = x_0(x,t) + W_\delta(x_0(x,t)). \tag{2.4}$$

And $W_\delta(x_0(x,t))$ has the following properties.

Lemma 1. (See[14,30]) The problem (2.3) has a unique global smooth solution $W^r_\delta(t,x)$ for each $\delta > 0$ such that

1. $W_- < W^r_\delta(x,t) < W_+$, $\partial_x W^r_\delta(x,t) > 0$, for $x \in \mathbf{R}$, $t \geq 0$, $\delta > 0$.
2. For any $t > 0$, $\delta > 0$ and $p \in [1,\infty]$, the following inequalities hold:

$$\begin{aligned} \|\partial_x W^r_\delta(\cdot,t)\|_{L^p} &\leq C(W_+ - W_-)^{\frac{1}{p}}(\delta+t)^{-1+\frac{1}{p}}, \\ \|\partial_x^2 W^r_\delta(\cdot,t)\|_{L^p} &\leq C(\delta+t)^{-1}(\delta+t)^{-1+\frac{1}{p}}, \\ \left|\frac{\partial^2 W^r_\delta(\cdot,t)}{\partial x^2}\right| &\leq \frac{4}{\delta}\frac{\partial W^r_\delta(\cdot,t)}{\partial x}. \end{aligned} \tag{2.5}$$

3. There exist a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0,\delta_0]$, $t > 0$,

$$\|W^r_\delta(\cdot,t) - W^r(\cdot/t)\|_{L^\infty} \leq C\delta t^{-1}[\ln(1+t) + |\ln\delta|].$$

The proof of Lemma 1 can be found in Xin[30], so we omit the details here for brevity.

As mentioned in the introduction, in order to overcome the degeneracies caused by the vacuum, we will cut off the 3-rarefaction wave with vacuum along the rarefaction wave curve. In detail, for any $\nu > 0$ to be determined, we can obtain a state $(\rho_\nu, u_\nu, \theta_\nu) = (\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1})$ which belongs to the 3-rarefaction wave curve, where $\bar{S} = S_+ = -(\gamma-1)\log\rho_+ + \log\theta_+$. In fact, the 3-Riemann invariant $\Sigma_3^{(i)}(\rho,u,\theta), (i = 1,2)$ is constant along the 3-rarefaction wave curve, and u_ν can be computed by $u_\nu = \Sigma_3^{(1)}(\rho_+,u_+,\theta_+) + 2\sqrt{\frac{\gamma}{\gamma-1}\nu^{\gamma-1}e^{S_+}}$. Then we can construct a new 3-rarefaction wave $(\rho_\nu^{r_3}, u_\nu^{r_3}, \theta_\nu^{r_3})(\xi), (\xi = \frac{x}{t})$ that connect the states $(\rho_\nu, u_\nu, \theta_\nu) = (\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1})$ and (ρ_+, u_+, θ_+) and which can be expressed exactly as

$$\lambda_3(\rho_\nu^{r_3}, u_\nu^{r_3}, \theta_\nu^{r_3})(\xi) = \begin{cases} \lambda_3(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}), & \xi < \lambda_3(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}), \\ \xi, & \lambda_3(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}) \leq \xi \leq \lambda_3(\rho_+, u_+, \theta_+), \\ \lambda_3(\rho_+, u_+, \theta_+), & \xi > \lambda_3(\rho_+, u_+, \theta_+), \end{cases} \tag{2.6}$$

and

$$\Sigma_3^{(1)}(\rho_\nu^{r_3}, u_\nu^{r_3}, \theta_\nu^{r_3}) = \Sigma_3^{(1)}(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}) = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+). \tag{2.7}$$

Correspondingly, the momentum and total internal energy can be defined by $m_\nu^{r_3} := \rho_\nu^{r_3}u_\nu^{r_3}$ and $e_\nu^{\tilde{r}_3} := \rho_\nu^{r_3}\theta_\nu^{r_3}$ respectively. It is easy to see that the cut-off 3-rarefaction wave $(\rho_\nu^{r_3}, m_\nu^{r_3}, e_\nu^{\tilde{r}_3})(\frac{x}{t})$ converges to the original 3-rarefaction wave with vacuum $(\rho^{r_3}, m^{r_3}, e^{\tilde{r}_3})(\frac{x}{t})$ in the sup-norm with the convergence rate ν as ν tends to zero. In fact, we have

Lemma 2. There exist a constant $\nu_0 \in (0,1)$ such that for $\nu \in (0,\nu_0]$, $t > 0$,

$$\|(\rho_\nu^{r_3}, m_\nu^{r_3}, e_\nu^{\tilde{r}_3})(\cdot/t) - (\rho^{r_3}, m^{r_3}, e^{\tilde{r}_3})(\cdot/t)\|_{L^\infty} \leq C\nu.$$

Now, with the above preparation, we can construct the approximate rarefaction wave $(\bar{\rho}_{\nu,\delta}, \bar{u}_{\nu,\delta}, \bar{\theta}_{\nu,\delta})(x,t)$ of the cut-off 3-rarefaction wave $(\rho_\nu^{r_3}, m_\nu^{r_3}, \theta_\nu^{r_3})(\frac{x}{t})$ to the compressible Euler equations (1.5) by

$$\begin{cases} W_+ = \lambda_3(\rho_+, u_+, \theta_+), \quad W_- = \lambda_3(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}), \\ W^r_\delta(t,x) = \lambda_3(\bar{\rho}_{\nu,\delta}, \bar{u}_{\nu,\delta}, \bar{\theta}_{\nu,\delta})(t,x), \\ \Sigma_3^{(1)}(\bar{\rho}_{\nu,\delta}, \bar{u}_{\nu,\delta}, \bar{\theta}_{\nu,\delta})(t,x) = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+) = \Sigma_3^{(1)}(\nu, u_\nu, e^{\bar{S}}\nu^{\gamma-1}). \end{cases} \tag{2.8}$$

Here W_δ^r defined in (2.4) is the solution of Burgers equation (2.3). From here on, for simplicity, we abbreviate $(\bar\rho_{\nu,\delta}, \bar u_{\nu,\delta}, \bar\theta_{\nu,\delta})(x,t)$ to $(\bar\rho, \bar u, \bar\theta)(x,t)$. Then, it's easy to check that the approximate cut-off 3-rarefaction wave $(\bar\rho, \bar u, \bar\theta)(x,t)$ defined above satisfies

$$\begin{cases} \bar\rho_t + (\bar\rho\bar u)_x = 0, \\ (\bar\rho\bar u)_t + (\bar\rho\bar u^2 + \bar p)_x = 0, \\ (\bar\rho\bar E)_t + (\bar\rho\bar u\bar E + \bar u\bar p)_x = 0, \end{cases} \tag{2.9}$$

where $\bar E = \bar e + \frac{\bar u^2}{2}$, $\bar p = R\bar\rho\bar\theta$. The properties of the approximate rarefaction wave $(\bar\rho, \bar u, \bar\theta)(x,t)$ are listed in Lemma 3 as below.

Lemma 3. (See [14,18,19]) The approximate cut-off 3-rarefaction wave $(\bar\rho, \bar u, \bar\theta)(x,t)$ defined in (2.8) satisfies:

1. $\bar u_x(x,t) = \frac{2}{\gamma+1}(W_\delta^r)_x > 0$, for $x \in \mathbf{R}$, $t \geq 0$,
   $\bar\rho_x = \frac{1}{\sqrt{\gamma(\gamma-1)e^{S_+}}}\bar\rho^{\frac{3-\gamma}{2}}\bar u_x$, and $\bar\rho_{xx} = \frac{1}{\sqrt{\gamma(\gamma-1)e^{S_+}}}\bar\rho^{\frac{3-\gamma}{2}}\bar u_{xx} + \frac{3-\gamma}{2\gamma(\gamma-1)e^{S_+}}\bar\rho^{2-\gamma}(\bar u_x)^2$,
   $\bar\theta_x = \sqrt{\frac{\gamma-1}{\gamma}}\bar\theta^{\frac12}\bar u_x$, and $\bar\theta_{xx} = \sqrt{\frac{\gamma-1}{\gamma}}\bar\theta^{\frac12}\bar u_{xx} + \frac{\gamma-1}{2\gamma}(\bar u_x)^2$.
2. For any $t > 0$, $\delta > 0$ and $p \in [1, \infty]$, there exists a positive constant C such that
$$\begin{aligned} \|\bar u_x(\cdot,t)\|_{L^p} \leq& C(W_+ - W_-)^{-\frac1p}(\delta+t)^{-1+\frac1p}, \\ \|\bar u_{xx}(\cdot,t)\|_{L^p} \leq& C(\delta+t)^{-1}\delta^{-1+\frac1p}. \end{aligned}$$
3. There exist a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0,\delta_0]$, $t > 0$,
$$\left\| \left(\bar\rho - \rho_\nu^{r_3}, \bar u - u_\nu^{r_3}, \bar\theta - \theta_\nu^{r_3}\right)(\cdot,t) \right\|_{L^\infty} \leq C\delta t^{-1}\left[\ln(1+t) + |\ln\delta|\right].$$

3 | A PRIORI ESTIMATES

3.1 | Reformulation of the problem

In order to prove the Theorem 1, we consider the Cauchy problem (1.1) with the smooth initial data

$$(\rho^\epsilon, u^\epsilon, \theta^\epsilon, w^\epsilon)(x,0), \tag{3.1}$$

and treat the global smooth solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon, w^\epsilon)$ of system (1.1),(3.1) as the perturbation around the approximate rarefaction wave $(\bar\rho, \bar u, \bar\theta, 0)$. For convenience, we reformulate the system by introducing a scaling for the independent variables

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}. \tag{3.2}$$

Then we denote the perturbation

$$(\phi, \psi, \chi, w)(y,\tau) = (\rho^\epsilon - \bar\rho, u^\epsilon - \bar u, \theta^\epsilon - \bar\theta, w^\epsilon - 0)(x,t),$$

and assume

$$\begin{aligned} (\phi_0, \psi_0, \chi_0, w_0)(y) :=& (\rho^\epsilon, u^\epsilon, \theta^\epsilon, w^\epsilon)(x,0) - (\bar\rho, \bar u, \bar\theta, 0)(x,0), \\ \|(\phi_0, \psi_0, \chi_0, w_0)\|^2_{H^1(\mathbf{R},dy)} =& O(1)\epsilon^{\frac13+\gamma a}. \end{aligned} \tag{3.3}$$

For simplicity of notation, without causing confusion, the superscription of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon, w^\epsilon)$ will be omitted as (ρ, u, θ, w) from now on. Then the system (1.1),(3.1) can be rewritten in form of $(\phi, \psi, \chi, w)(y,\tau)$ as follows

$$\begin{cases} \phi_\tau + \rho\psi_y + u\phi_y = -f, \\ \rho\psi_\tau + \rho u\psi_y + (\gamma-1)(\theta\phi_y + \rho\chi_y) - \mu(\bar\theta)\psi_{yy} \\ \qquad\qquad = -g + \mu(\bar\theta)_y u_y + \big((\mu(\theta) - \mu(\bar\theta))u_y\big)_y, \\ \rho w_\tau + \rho u w_y + \epsilon^2\xi(\theta)w - \lambda(\theta)w_{yy} = 0, \\ \rho\chi_\tau + \rho u\chi_y + (\gamma-1)\rho\theta\psi_y - \kappa(\bar\theta)\chi_{yy} \\ \qquad\qquad = -h + \kappa(\bar\theta)_y\theta_y + \big((\kappa(\theta) - \kappa(\bar\theta))\theta_y\big)_y + \mu(\theta)u_y^2, \\ (\phi, \psi, \chi, w)(y,0) = 0, \end{cases} \tag{3.4}$$

where

$$\begin{cases} f &= \bar{u}_y\phi + \bar{\rho}_y\psi, \\ g &= -\mu(\bar{\theta})\bar{u}_{yy} + \rho\psi\bar{u}_y + (\gamma-1)\left(\bar{\rho}_y\chi - \frac{\bar{\rho}_y\bar{\theta}\phi}{\bar{\rho}}\right), \\ h &= (\gamma-1)\rho\chi\bar{u}_y + \rho\psi\bar{\theta}_y - \kappa(\bar{\theta})\bar{\theta}_{yy} + \lambda(\theta)(w_y)^2 + \epsilon^2\xi(\theta)w^2. \end{cases} \tag{3.5}$$

Next, we seek a global (in time) and bounded (in L^∞ norm) solution $(\phi,\psi,\chi,w)(y,\tau)$ to the Cauchy problem (3.4). Firstly, we define the functional space

$$X(0,\tau_1(\epsilon)) = \left\{ (\phi,\psi,\chi,w) \,\middle|\, \begin{array}{l} (\phi,\psi,\chi,w) \in C^0([0,\tau_1(\epsilon)];H^1(\mathbf{R})), \quad \phi_y \in L^2(0,\tau_1(\epsilon);L^2(\mathbf{R})), \\ (\psi_y,\chi_y,w_y) \in L^2(0,\tau_1(\epsilon);H^1(\mathbf{R})) \end{array} \right\}$$

with $0<\tau_1(\epsilon)\le\infty$.

Our main result can be stated as following:

Theorem 2. There exist positive constants ϵ_1 and C independent of ϵ, such that if $0<\epsilon<\epsilon_1$, then the Cauchy problem (3.4),(3.5),(3.3) admits a unique global-in-time solution $(\phi,\psi,\chi,w)\in X(0,+\infty)$ satisfying

$$\begin{aligned} &\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}} \left(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}w^2 + \bar{\rho}^{2-\gamma}\chi^2\right)dy + \int_0^{+\infty}\int_{\mathbf{R}} \left(\bar{\theta}^\alpha\psi_y^2 + \bar{\theta}^{\alpha-1}\chi_y^2 + \bar{\theta}^\alpha w_y^2\right)dyd\tau \\ &+ \int_0^{+\infty}\int_{\mathbf{R}} \left[\bar{u}_y(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}^{2-\gamma}\chi^2) + \epsilon^2\bar{\theta}^\alpha w^2\right]dyd\tau \le C\epsilon^{\frac{1}{3}}, \\ &\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}} \frac{\bar{\theta}^{2\alpha}}{\bar{\rho}^3}\phi_y^2 dy + \int_0^{+\infty}\int_{\mathbf{R}} \frac{\bar{\theta}^{1+\alpha}}{\bar{\rho}^2}\phi_y^2 dyd\tau \le C\epsilon^{\frac{1}{3}-3\gamma a}|\ln\epsilon|^{-3\gamma}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}} (\psi_y^2+\chi_y^2+w_y^2)dy + \int_0^{+\infty}\int_{\mathbf{R}} \left[(\psi_y^2+\chi_y^2+w_y^2)\bar{u}_y + \frac{\epsilon^2\bar{\theta}^\alpha}{\bar{\rho}}w_y^2\right]dyd\tau \\ &+ \int_0^{+\infty}\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha}{\bar{\rho}}(\psi_{yy}^2+\chi_{yy}^2+w_{yy}^2)dyd\tau \le C\epsilon^{\frac{1}{9}}, \end{aligned}$$

where a is given by (1.14).

Suppose the solution is exists in time interval $[0,\tau_1(\epsilon)]$. Some a priori assumptions can be listed as follows, which will be used in the proof of Theorem 2.

$$\begin{aligned} &\sup_{0\le\tau\le\tau_1(\epsilon)} \|\phi(\cdot,\tau)\|_{L^\infty} \le \epsilon^a, \qquad \sup_{0\le\tau\le\tau_1(\epsilon)} \|\psi(\cdot,\tau)\|_{L^\infty} \le \epsilon^a, \\ &\sup_{0\le\tau\le\tau_1(\epsilon)} \|\chi(\cdot,\tau)\|_{L^\infty} \le \epsilon^{(\gamma-1)a}, \quad \sup_{0\le\tau\le\tau_1(\epsilon)} \|w(\cdot,\tau)\|_{L^\infty} \le \epsilon^a, \\ &\sup_{0\le\tau\le\tau_1(\epsilon)} \|(\psi_y,\chi_y)(\cdot,\tau)\| \le 1, \end{aligned} \tag{3.6}$$

where the positive constant a is given in (1.14). We take

$$\nu = \epsilon^a|\ln\epsilon|, \quad \delta = \epsilon^a \tag{3.7}$$

in what follows. Then we have $\nu \ge \bar{C}\epsilon^a$, where $\bar{C} = \max\{2,(2e^{-\bar{S}})^{\frac{1}{\gamma-1}}\}$ if $\epsilon \ll 1$. From the a priori assumptions (3.6), it follows that

$$\frac{\bar{\rho}}{2} \le \rho \le \frac{3\bar{\rho}}{2}, \quad \frac{\bar{\theta}}{2} \le \theta \le \frac{3\bar{\theta}}{2}. \tag{3.8}$$

Actually, if $\epsilon \ll 1$, we have

$$\begin{cases} \rho = \bar{\rho}+\phi \ge \bar{\rho} - \|\phi\|_{L^\infty} \ge \bar{\rho} - \epsilon^a \ge \bar{\rho} - \frac{\nu}{2} \ge \frac{\bar{\rho}}{2}, \\ \rho = \bar{\rho}+\phi \le \bar{\rho} + \|\phi\|_{L^\infty} \le \bar{\rho} + \epsilon^a \le \bar{\rho} + \frac{\nu}{2} \le \frac{3\bar{\rho}}{2}, \end{cases}$$

and noticed that $\bar{\theta} = \bar{\rho}^{\gamma-1}e^{\bar{S}} \geq v^{\gamma-1}e^{\bar{S}}$, which can be obtained by the definition of rarefaction wave profile in (2.8), we have

$$\begin{cases} \theta = \bar{\theta} + \phi \geq \bar{\theta} - \|\phi\|_{L^\infty} \geq \bar{\theta} - \epsilon^{a(\gamma-1)} \geq \bar{\theta} - \dfrac{e^{\bar{S}}}{2}v^{\gamma-1} \geq \dfrac{\bar{\theta}}{2}, \\ \theta = \bar{\theta} + \phi \leq \bar{\theta} + \|\phi\|_{L^\infty} \leq \bar{\theta} + \epsilon^{a(\gamma-1)} \leq \bar{\theta} + \dfrac{e^{\bar{S}}}{2}v^{\gamma-1} \leq \dfrac{3\bar{\theta}}{2}. \end{cases}$$

Theorem 2 can be proved by the standard method which is based on the local existence result combined with the a priori estimates stated in proposition 1. Since the proof of the local existence of the solution to (3.4) is standard, we omit the details here for brevity. Now we will devote ourselves to prove the following a priori estimates.

Proposition 1 (*A priori estimates*)**.** Let $\gamma > 1$ and $(\phi, \psi, \chi, w) \in X(0, \tau_1(\epsilon))$ be a solution to the system (3.4), where $\tau_1(\epsilon)$ is the maximum existence time of the solution satisfying the a priori assumptions (3.6). Then there exists a positive constants ϵ_2 such that if $0 < \epsilon \leq \epsilon_2$, then

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbf{R}} \left(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}w^2 + \bar{\rho}^{2-\gamma}\chi^2\right) dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left(\bar{\theta}^\alpha \psi_y^2 + \bar{\theta}^{\alpha-1}\chi_y^2 + \bar{\theta}^\alpha w_y^2\right) dyd\tau \\ &+ \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left[\bar{u}_y(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}^{2-\gamma}\chi^2) + \epsilon^2\bar{\theta}^\alpha w^2\right] dyd\tau \leq \epsilon^{\frac{1}{3}}, \end{aligned} \tag{3.9}$$

$$\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbf{R}} \frac{\bar{\theta}^{2\alpha}}{\bar{\rho}^3}\phi_y^2 dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \frac{\bar{\theta}^{1+\alpha}}{\bar{\rho}^2}\phi_y^2 dyd\tau \leq \epsilon^{\frac{1}{3}-3\gamma a}|\ln\epsilon|^{-3\gamma}, \tag{3.10}$$

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbf{R}} (\psi_y^2 + \chi_y^2 + w_y^2) dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left[(\psi_y^2 + \chi_y^2 + w_y^2)\bar{u}_y + \frac{\epsilon^2\bar{\theta}^\alpha}{\bar{\rho}} w_y^2\right] dyd\tau \\ &+ \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left[\frac{\bar{\theta}^\alpha}{\bar{\rho}}(\psi_{yy}^2 + \chi_{yy}^2 + w_{yy}^2)\right] dyd\tau \leq \epsilon^{\frac{1}{9}}, \end{aligned} \tag{3.11}$$

where the positive constant a is given by (1.14).

The proof of proposition 1 consists of several lemmas, the first one is the following basic energy estimates.

3.2 | Basic energy estimates

Lemma 4. Under the conditions of Proposition 1, for $0 \leq \tau \leq \tau_1(\epsilon)$ and $\epsilon \ll 1$ is small enough, we have

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbf{R}} \left(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}w^2 + \bar{\rho}^{2-\gamma}\chi^2\right) dy \\ &+ \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left(\bar{u}_y(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}^{2-\gamma}\chi^2) + \bar{\theta}^\alpha\psi_y^2 + \bar{\theta}^{\alpha-1}\chi_y^2 + \bar{\theta}^\alpha w_y^2 + \epsilon^2\bar{\theta}^\alpha w^2\right) dyd\tau \leq \epsilon^{\frac{1}{3}}. \end{aligned} \tag{3.12}$$

Proof. Firstly, we define the relative entropy-entropy flux pair (η, q) as

$$\begin{cases} \eta = -\bar{\theta}\left\{\rho S - \bar{\rho}\bar{S} - \nabla_{\mathbf{X}}(\rho S)|_{\mathbf{X}=\bar{\mathbf{X}}} \cdot (\mathbf{X} - \bar{\mathbf{X}})\right\}, \\ q = -\bar{\theta}\left\{\rho u S - \bar{\rho}\bar{u}\bar{S} - \nabla_{\mathbf{X}}(\rho S)|_{\mathbf{X}=\bar{\mathbf{X}}} \cdot (\mathbf{Y} - \bar{\mathbf{Y}})\right\}, \end{cases} \tag{3.13}$$

where

$$\begin{aligned} \mathbf{X} &= \left(\rho, \rho u, \rho w, \rho\left(\theta + \frac{u^2}{2}\right)\right), \\ \mathbf{Y} &= \left(\rho u, \rho u^2 + (\gamma-1)\rho\theta, \rho u w, \rho u\left(\gamma\theta + \frac{u^2}{2}\right)\right). \end{aligned} \tag{3.14}$$

Then we can obtain

$$\begin{cases} \eta = \rho\theta - \bar{\theta}\rho S + \rho\left[(\bar{S}-\gamma)\bar{\theta} + \frac{1}{2}|u-\bar{u}|^2\right] + (\gamma-1)\bar{\rho}\bar{\theta} \\ \quad = (\gamma-1)\rho\bar{\theta}\Phi\left(\frac{\bar{\rho}}{\rho}\right) + \frac{\rho\psi^2}{2} + \frac{\rho w^2}{2} + \rho\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right), \\ q = u\eta + (\gamma-1)(u-\bar{u})(\rho\theta - \bar{\rho}\bar{\theta}) - \lambda(\theta)ww_y, \end{cases} \tag{3.15}$$

with

$$\Phi(\eta) := \eta - \ln\eta - 1.$$

Through direct calculation, we get

$$\begin{aligned} &\eta_\tau + q_y + H\bar{u}_y + \lambda(\theta)w_y^2 + \epsilon^2\xi(\theta)w^2 \\ =&\psi\left(\mu(\theta)u_y\right)_y + \frac{\chi}{\theta}\mu(\theta)u_y^2 + \frac{\chi}{\theta}\left(\kappa(\theta)\theta_y\right)_y - \lambda(\theta)_y w_y w, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} H =&\rho(u-\bar{u})^2 + (\gamma-1)\rho\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right) + (\gamma-1)^2\rho\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right) \\ &+ \sqrt{\frac{\gamma-1}{\gamma}}\bar{\theta}^{\frac{1}{2}}\rho(u-\bar{u})\left((\gamma-1)\log\frac{\bar{\rho}}{\rho} + \log\frac{\theta}{\bar{\theta}}\right) \\ \geq&(1-\epsilon)\rho(u-\bar{u})^2 + (\gamma-1)\rho\bar{\theta}\left[\Phi\left(\frac{\theta}{\bar{\theta}}\right) + (\gamma-1)\Phi\left(\frac{\theta}{\bar{\theta}}\right) - \frac{1}{4\epsilon\gamma}\left((\gamma-1)\log\frac{\bar{\rho}}{\rho} + \log\frac{\theta}{\bar{\theta}}\right)^2\right], \end{aligned} \tag{3.17}$$

with $\epsilon > 0$ to be determined later. Then we can derive that (see [18])

$$H \geq C\left[\bar{\rho}\psi^2 + \frac{\bar{\theta}}{\bar{\rho}}\phi^2 + \frac{\bar{\rho}}{\bar{\theta}}\chi^2\right]. \tag{3.18}$$

Then, integrating (3.16) over $\mathbf{R}\times[0,\tau]$ and using (3.8) yields

$$\begin{aligned} &\int_{\mathbf{R}}\left(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}w^2 + \bar{\rho}^{2-\gamma}\chi^2\right)(\tau,y)dy \\ &+ \int_0^\tau\int_{\mathbf{R}}\left[\bar{u}_y\left(\bar{\rho}^{\gamma-2}\phi^2 + \bar{\rho}\psi^2 + \bar{\rho}^{2-\gamma}\chi^2\right) + \bar{\theta}^\alpha\psi_y^2 + \bar{\theta}^{\alpha-1}\chi_y^2 + \bar{\theta}^\alpha w_y^2 + \epsilon^2\bar{\theta}^\alpha w^2\right]dyd\tau \\ \leq&\epsilon^{\frac{1}{3}} + \int_0^\tau\int_{\mathbf{R}}\left[\left|\bar{\theta}^{\alpha-1}\chi\right|\left|(\bar{\theta}_{yy},\bar{u}_y^2)\right| + \left|\bar{\theta}^\alpha\psi\bar{u}_{yy}\right| + \left|\bar{\theta}^{\alpha-1/2}\psi\bar{u}_y^2\right| + \left|\bar{\theta}^{\alpha-1}\chi_y w_y w\right|\right]dyd\tau \\ &+ \int_0^\tau\int_{\mathbf{R}}\left|\bar{u}_y\left(\bar{\theta}^{\alpha-1}\chi\psi_y, \bar{\theta}^{\alpha-3/2}\chi\chi_y, \bar{\theta}^{\alpha-1}\psi\chi_y, \bar{\theta}^{\alpha-1/2}w_y w\right)\right|dyd\tau \\ :=&\epsilon^{\frac{1}{3}} + \sum_{i=1}^5 I_i. \end{aligned} \tag{3.19}$$

Where we used the fact that, for all $\gamma > 1$, from (3.3), we can get

$$\int_{\mathbf{R}}\left(\bar{\rho}_0^{\gamma-2}\phi_0^2 + \bar{\rho}_0\psi_0^2 + \bar{\rho}_0 w_0^2 + \bar{\rho}_0^{2-\gamma}\chi_0^2\right)dy \leq \epsilon^{1/3}. \tag{3.20}$$

By Lemma 3 and Sobolev's inequality, we have

$$\begin{aligned} I_1 =&\int_0^\tau\int_{\mathbf{R}}|\bar{\theta}^{\alpha-1}|\left|(\bar{\theta}_{yy},\bar{u}_y^2)\right|dyd\tau \leq C\nu^{1-\gamma}\int_0^\tau\|(\bar{\theta}_{yy},\bar{u}_y^2))\|_{L^1}\|\chi\|^{1/2}\|\chi_y\|^{1/2}d\tau \\ \leq&C\nu^{-\frac{(3+\alpha)(1-\gamma)}{4}}\int_0^\tau\frac{1}{\tau+\delta/\epsilon}\|\chi\|^{1/2}\left\|\sqrt{\bar{\theta}^{\alpha-1}}\chi_y\right\|^{1/2}d\tau \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 d\tau + C\nu^{-\frac{(3+\alpha)(1-\gamma)+2}{3}} \sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho^{2-\gamma}}\chi\right\|^{2/3} \int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3} d\tau \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 d\tau + \frac{1}{24}\sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho^{2-\gamma}}\chi\right\|^2 + C\nu^{-\frac{(3+\alpha)(1-\gamma)+2}{2}} \left(\frac{\epsilon}{\delta}\right)^{1/2} \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 d\tau + \frac{1}{24}\sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho^{2-\gamma}}\chi\right\|^2 + \epsilon^{\frac{1}{3}},
\end{aligned}
$$

where we have used the following fact

$$
C\nu^{-\frac{(3+\alpha)(1-\gamma)+2}{2}} \left(\frac{\epsilon}{\delta}\right)^{1/2} = C\epsilon^{\frac{1-a(3\gamma-1+\alpha(\gamma-1))}{2}} |\ln\epsilon|^{-\frac{(3+\alpha)(1-\gamma)+2}{2}} \le \epsilon^{\frac{1}{3}}, \quad \text{if } \epsilon \ll 1.
$$

For I_2, by Cauchy's inequality and Sobolev's inequality, it holds that

$$
\begin{aligned}
I_2 = \int_0^\tau \int_{\mathbf{R}} \left|\bar\theta^\alpha \psi \bar u_{yy}\right| dy d\tau &\le C\int_0^\tau \|\bar u_{yy}\|_{L^1} \|\psi\|^{1/2} \|\psi_y\|^{1/2} d\tau \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha}\psi_y\right\|^2 d\tau + C\nu^{-\frac{\alpha(\gamma-1)}{3}} \int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3} \|\psi\|^{2/3} d\tau \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha}\psi_y\right\|^2 d\tau + C\nu^{-\frac{\alpha(\gamma-1)+1}{3}} \sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho}\psi\right\|^{2/3} \int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3} d\tau \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha}\psi_y\right\|^2 d\tau + \frac{1}{24}\sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho}\psi\right\|^2 + C\nu^{-\frac{\alpha(\gamma-1)+1}{2}} \left(\frac{\epsilon}{\delta}\right)^{1/2} \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha}\psi_y\right\|^2 d\tau + \frac{1}{24}\sup_{0\le\tau\le\tau_1(\epsilon)} \left\|\sqrt{\bar\rho}\psi\right\|^2 + \epsilon^{\frac{1}{3}},
\end{aligned}
$$

where we used the following fact

$$
C\nu^{-\frac{\alpha(\gamma-1)+1}{2}} \left(\frac{\epsilon}{\delta}\right)^{1/2} = C\epsilon^{\frac{1-a(2+\alpha(\gamma-1))}{2}} |\ln\epsilon|^{-\frac{(\alpha(1-\gamma)+1}{2}} \le \epsilon^{\frac{1}{3}}, \quad \text{if } \epsilon \ll 1.
$$

For I_3, we have

$$
\begin{aligned}
I_3 = \int_0^\tau \int_{\mathbf{R}} \left|\bar u_y^2 \bar\theta^{\alpha-\frac{1}{2}}\psi\right| dy d\tau &\le \frac{1}{24}\int_0^\tau \int_{\mathbf{R}} \bar u_y \bar\rho \psi^2 dy d\tau + C\nu^{-\gamma} \int_0^\tau \int_{\mathbf{R}} \bar u_y^3 dy d\tau \\
&\le \frac{1}{24}\int_0^\tau \int_{\mathbf{R}} \bar u_y \bar\rho \psi^2 dy d\tau + C\nu^{-\gamma} \int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^2 d\tau \\
&\le \frac{1}{24}\int_0^\tau \int_{\mathbf{R}} \bar u_y \bar\rho \psi^2 dy d\tau + C\nu^{-\gamma} \left(\frac{\epsilon}{\delta}\right).
\end{aligned}
$$

For I_4,

$$
\begin{aligned}
I_4 = \int_0^\tau \int_{\mathbf{R}} \left|\bar\theta^{\alpha-1}\chi_y w_y w\right| dy d\tau &\le C\nu^{1-\gamma} \|w\|_{L^\infty} \int_0^\tau \int_{\mathbf{R}} \chi_y w_y dy d\tau \\
&\le \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 d\tau + C\nu^{(1-\gamma)} \|w\|_{L^\infty}^2 \int_0^\tau \left\|\sqrt{\bar\theta^\alpha} w_y\right\|^2 d\tau
\end{aligned}
$$

$$\leq \frac{1}{24}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 d\tau + C\nu^{(2\alpha-3)(1-\gamma)}\epsilon^{2a}\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha}}w_y\right\|^2 d\tau$$
$$\leq \frac{1}{24}\int_0^\tau \left(\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2 + \left\|\sqrt{\bar\theta^{\alpha}}w_y\right\|^2\right) d\tau,$$

where we used the following fact

$$C\nu^{(2\alpha-3)(1-\gamma)}\epsilon^{2a} = C\epsilon^{a(5-3\gamma-2\alpha(\gamma-1))}|\ln\epsilon|^{-(2\alpha-3)(\gamma-1)} \leq \frac{1}{24}, \quad \text{if } \epsilon \ll 1.$$

And for I_5,

$$I_5 = \int_0^\tau\int_{\mathbf{R}} \left|\bar u_y\left(\bar\theta^{\alpha-1}\chi\psi_y, \bar\theta^{\alpha-3/2}\chi\chi_y, \bar\theta^{\alpha-1}\psi\chi_y, \bar\theta^{\alpha-1/2}w_y w\right)\right| dyd\tau$$
$$\leq \int_0^\tau\int_{\mathbf{R}} \bar\rho^{-\frac12}\bar\theta^{\frac{\alpha-1}{2}}|\bar u_y|\left|\sqrt{\bar\rho^{2-\gamma}}\chi\right|\left(\left|\bar\theta^{\frac{\alpha}{2}}\psi_y\right| + \left|\bar\theta^{\frac{\alpha-1}{2}}\chi_y\right|\right) dyd\tau$$
$$+ \int_0^\tau\int_{\mathbf{R}} \bar\rho^{-\frac12}\bar\theta^{\frac{\alpha-1}{2}}|\bar u_y|\left(\left|\sqrt{\bar\rho}\psi\right|\left|\bar\theta^{\frac{\alpha-1}{2}}\chi_y\right| + \left|\sqrt{\bar\rho}w\right|\left|\bar\theta^{\frac{\alpha}{2}}w_y\right|\right) dyd\tau$$
$$\leq \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \left(\bar\theta^\alpha\psi_y^2 + \bar\theta^{\alpha-1}\chi_y^2 + \bar\theta^\alpha w_y^2\right) dyd\tau + C\frac{\epsilon}{\nu^\gamma\delta}\int_0^\tau\int_{\mathbf{R}} \bar u_y\left(\bar\rho^{2-\gamma}\chi^2 + \bar\rho\psi^2 + \bar\rho w^2\right) dyd\tau$$
$$\leq \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \left(\bar\theta^\alpha\psi_y^2 + \bar\theta^{\alpha-1}\chi_y^2 + \bar\theta^\alpha w_y^2 + \bar u_y\bar\rho^{2-\gamma}\chi^2 + \bar u_y\bar\rho\psi^2 + \bar u_y\bar\rho w^2\right) dyd\tau,$$

where we have used the fact that

$$C\frac{\epsilon}{\nu^\gamma\delta} = C\epsilon^{1-a-a\gamma}|\ln\epsilon|^{-\gamma} \leq \epsilon^{1/2}|\ln\epsilon|^{-\gamma} \leq \frac{1}{24}, \quad \text{if } \epsilon \ll 1.$$

Substituting all the above estimates into (3.19), we can obtain (3.12), that's to say we have completed the proof of Lemma 4. □

3.3 | Estimates of higher order derivatives

In this subsection, we deduce some estimates on the higher order derivatives.

Lemma 5. Under the conditions of Proposition 1, for $0 \leq \tau \leq \tau_1(\epsilon)$ and $\epsilon \ll 1$ is small enough, we have

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}} \frac{\bar\theta^{2\alpha}}{\bar\rho^3}\phi_y^2 dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \frac{\bar\theta^{1+\alpha}}{\bar\rho^2}\phi_y^2 dyd\tau \leq \epsilon^{1/3-3a\gamma}|\ln\epsilon|^{-3\gamma}, \tag{3.21}$$

and

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}} \psi_y^2 dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left(\bar u_y\psi_y^2 + \frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2\right) dyd\tau \leq \epsilon^{1/6}. \tag{3.22}$$

Proof. First, differentiating the equation $(3.4)_1$ with respect to y and then multiplying the result by $\mu^2(\bar\theta)\frac{\phi_y}{\rho^3}$ gives

$$\begin{aligned}&\left(\mu^2(\bar\theta)\frac{\phi_y^2}{2\rho^3}\right)_\tau + \left(\mu^2(\bar\theta)\frac{u\phi_y^2}{2\rho^3}\right)_y + \mu^2(\bar\theta)\frac{\psi_{yy}\phi_y}{\rho^2}\\ &= -\mu^2(\bar\theta)\frac{\phi_y}{\rho^3}\left(\bar u_{yy}\phi + \bar\rho_{yy}\psi + 2\bar\rho_y\psi_y\right) + \frac{\phi_y^2}{2\rho^3}\left[\mu^2(\bar\theta)_\tau + \mu^2(\bar\theta)_y u\right].\end{aligned} \tag{3.23}$$

Multiplying the equation $(3.4)_2$ by $\mu(\bar\theta)\frac{\phi_y}{\rho^2}$ yields

$$\begin{aligned}
&\left(\mu(\bar\theta)\frac{\phi_y\psi}{2\rho}\right)_\tau-\left(\mu(\bar\theta)\frac{\psi\phi_\tau}{\rho}\right)_y-\mu(\bar\theta)\psi_y^2+(\gamma-1)\mu(\bar\theta)\frac{\theta\phi_y^2}{\rho^2}+(\gamma-1)\mu(\bar\theta)\frac{\chi_y\phi_y}{\rho}\\
&-\mu^2(\bar\theta)\frac{\psi_{yy}\phi_y}{\rho^2}+\mu(\bar\theta)\left(-\bar u_y\frac{\psi_y\phi}{\rho}+\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}-\bar\rho\bar u_y\frac{\psi\phi_y}{\rho^2}\right)+\mu(\bar\theta)g\frac{\phi_y}{\rho^2}\\
=&\mu(\bar\theta)\mu(\bar\theta)_y\frac{u_y\phi_y}{\rho^2}\mu(\bar\theta)_\tau\frac{\psi\phi_y}{\rho}-\mu(\bar\theta)_y\frac{\psi\phi_\tau}{\rho}-\mu(\bar\theta)\frac{\psi^2}{\rho^2}\bar\rho_y(\phi_y+\bar\rho_y)\\
&+\left[(\mu(\theta)-\mu(\bar\theta))u_y\right]_y\mu(\bar\theta)\frac{\phi_y}{\rho^2}.
\end{aligned}\tag{3.24}$$

Adding the equation (3.23) and (3.24) together, and then integrating the result over $\mathbf{R}\times[0,\tau]$ we obtain

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\mu^2(\bar\theta)\frac{\phi_y^2}{2\rho^3}+\mu(\bar\theta)\frac{\phi_y\psi}{2\rho}\right)dy+\int_0^\tau\int_{\mathbf{R}}\left[(\gamma-1)\mu(\bar\theta)\frac{\phi_y^2}{\rho^{3-\gamma}}+(\gamma-1)\mu(\bar\theta)\frac{\phi_y\chi_y}{\rho}\right]dyd\tau\\
=&\int_{\mathbf{R}}\left(\mu^2(\bar\theta_0)\frac{\phi_{0y}^2}{2\rho_0^3}+\mu(\bar\theta_0)\frac{\phi_{0y}\psi_0}{2\rho_0}\right)dy+\int_0^\tau\int_{\mathbf{R}}\left(\mu(\bar\theta)\psi_y^2-\mu(\bar\theta)g\frac{\phi_y}{\rho^2}\right)dyd\tau\\
&+\int_0^\tau\int_{\mathbf{R}}\left\{\mu(\bar\theta)\left(\bar u_y\frac{\psi_y\phi}{\rho}-\bar\rho\bar u_y\frac{\psi\phi}{\rho^2}+\bar\rho\bar u_y\frac{\psi\phi_y}{\rho^2}\right)-\mu^2(\bar\theta)\frac{\phi_y}{\rho^3}\left(\bar u_{yy}\phi+\bar\rho_{yy}\psi+2\bar\rho_y\psi_y\right)\right\}dyd\tau\\
&+\int_0^\tau\int_{\mathbf{R}}\left\{\frac{\phi_y^2}{2\rho^3}\left[\mu^2(\bar\theta)_\tau+\mu^2(\bar\theta)_yu\right]+\mu(\bar\theta)\mu(\bar\theta)_y\frac{u_y\phi_y}{\rho^2}+\mu(\bar\theta)_\tau\frac{\psi\phi_y}{\rho}-\mu(\bar\theta)_y\frac{\psi\phi_\tau}{\rho}\right\}dyd\tau\\
&+\int_0^\tau\int_{\mathbf{R}}\left\{\mu(\bar\theta)\bar\rho_y\frac{\psi^2}{\rho^2}(\phi_y+\bar\rho_y)+\left[(\mu(\theta)-\mu(\bar\theta))u_y\right]_y\mu(\bar\theta)\frac{\phi_y}{\rho^2}\right\}dyd\tau.
\end{aligned}\tag{3.25}$$

Then combining the Lemma 4, $(3.5)_1$, (3.25), and the fact that

$$\begin{aligned}
\left[(\mu(\theta)-\mu(\bar\theta))u_y\right]_y=&((\mu(\theta)-\mu(\bar\theta))(\psi_{yy}+\bar u_{yy})+\alpha(\theta^{\alpha-1}\theta_y-\bar\theta^{\alpha-1}\bar\theta_y)(\psi_y+\bar u_y)\\
=&((\mu(\theta)-\mu(\bar\theta))(\psi_{yy}+\bar u_{yy})+\alpha\left(\theta^{\alpha-1}\chi_y\psi_y+\theta^{\alpha-1}\chi_y\bar u_y\right.\\
&\left.+(\theta^{\alpha-1}-\bar\theta^{\alpha-1})\bar\theta_y\psi_y+(\theta^{\alpha-1}-\bar\theta^{\alpha-1})\bar\theta_y\bar u_y\right),
\end{aligned}\tag{3.26}$$

then we can obtain

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\mu^2(\bar\theta)\frac{\phi_y^2}{2\rho^3}+\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho w^2+\bar\rho^{2-\gamma}\chi^2\right)dy\\
&+\int_0^\tau\int_{\mathbf{R}}\left(\bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho^{2-\gamma}\chi^2)+\bar\theta^\alpha\psi_y^2+\bar\theta^{\alpha-1}\chi_y^2+\bar\theta^\alpha w_y^2+\epsilon^2\bar\theta^\alpha w^2+\frac{\bar\theta^{\alpha+1}}{\rho^2}\phi_y^2\right)dyd\tau\\
\le&Cv^{-3}\int_{\mathbf{R}}\phi_{0y}^2dy+\left|\int_0^\tau\int_{\mathbf{R}}\left[\mu(\bar\theta)\bar u_y\frac{\psi_y\phi}{\rho}+\mu(\bar\theta)\bar u_y\bar\rho\frac{\psi\phi_y}{\rho^2}-2\mu^2(\bar\theta)\frac{\phi_y}{\rho^3}\bar\rho\psi_y\right]dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left[-\mu^2(\bar\theta)\frac{\phi_y}{\rho^3}(u_{yy}\phi+\bar\rho_{yy}\psi)+\mu(\theta)\frac{\phi_y}{\rho^2}(\mu(\theta)-\mu(\bar\theta))\bar u_{yy}\right]dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\mu(\bar\theta)g\frac{\phi_y}{\rho^2}dyd\tau\right|+C\left|\int_0^\tau\int_{\mathbf{R}}\left[\frac{\phi_y^2}{2\rho^3}\left(\mu^2(\bar\theta)_\tau+\mu^2(\bar\theta)_yu\right)\right]dyd\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left\{-\mu(\bar\theta)\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}+\mu(\bar\theta)\mu(\bar\theta)_y\frac{\psi_y\phi_y}{\rho^2}-\mu(\bar\theta)_y\frac{\psi\left(\rho\psi_y+u\phi_y+\bar u_y\phi+\bar\rho_y\psi\right)}{\rho}\right\}dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left\{-\mu(\bar\theta)\bar\rho_y^2\frac{\psi^2}{\rho^2}+\alpha\left[\theta^{\alpha-1}\chi_y\bar u_y+\left(\theta^{\alpha-1}-\bar\theta^{\alpha-1}\right)\bar\theta_y\psi_y\right]\mu(\bar\theta)\frac{\phi_y}{\rho^2}\right\}dyd\tau\right| \qquad (3.27)\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left(\mu(\bar\theta)_\tau\frac{\psi\phi_y}{\rho}-\mu(\bar\theta)\bar\rho_y\frac{\psi^2}{\rho^2}\phi_y\right)dyd\tau\right|+C\left|\int_0^\tau\int_{\mathbf{R}}\left(\mu(\bar\theta)\frac{\phi_y}{\rho^2}(\mu(\theta)-\mu(\bar\theta))\psi_{yy}\right)dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left(\mu(\bar\theta)\mu(\bar\theta)_y\frac{\bar u_y\phi_y}{\rho^2}+\alpha\left(\theta^{\alpha-1}-\bar\theta^{\alpha-1}\right)\bar\theta_y\bar u_y\mu(\bar\theta)\frac{\phi_y}{\rho^2}\right)dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\theta^{\alpha-1}\chi_y\psi_y\mu(\bar\theta)\frac{\phi_y}{\rho^2}dyd\tau\right|+C\epsilon^{\frac13}\\
:=&\sum_{i=1}^9 J_i+C\epsilon^{\frac13+(\gamma-3)a}|\ln\epsilon|^{-3}.
\end{aligned}
$$

The terms on the right-hand side of (3.27) will be estimates one by one as follows. From Lemma 2.3, (3.6) and the Cauchy's inequality, it holds that

$$
\begin{aligned}
J_1\le&C\left|\int_0^\tau\int_{\mathbf{R}}\left\{\bar\rho^{\frac{\alpha(\gamma-1)-\gamma}{2}}\bar u_y\left(\left|\sqrt{\bar\theta^\alpha}\psi_y\right|\left|\sqrt{\bar\rho^{\gamma-2}}\phi\right|,\left|\sqrt{\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}}\phi_y\right|\left|\sqrt{\bar\rho}\psi\right|\right)\right\}dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left(\bar\rho^{\alpha(\gamma-1)-\gamma}\bar u_y\left|\sqrt{\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}}\phi_y\right|\left|\sqrt{\bar\theta^\alpha}\psi_y\right|\right)dyd\tau\right|\\
\le&\frac{1}{24}\int_0^\tau\int_{\mathbf{R}}\left(\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2+\bar\theta^\alpha\psi_y^2\right)dyd\tau+C\frac{\epsilon}{\nu^\gamma\delta}\int_0^\tau\int_{\mathbf{R}}\bar u_y\left(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2\right)dyd\tau.
\end{aligned}
$$

It from Lemma 2.1 and Lemma 2.3 holds that

$$
\left|\bar\rho_{xx}\right|\le C\left(\bar\rho^{\frac{3-\gamma}{2}}\frac{\bar u_x}{\delta}+\bar\rho^{2-\gamma}\bar u_x^2\right).
$$

Therefore, we have

$$
\begin{aligned}
J_2\le&C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\frac{3\alpha(\gamma-1)-3\gamma}{2}}\frac{\epsilon}{\delta}\bar u_y\left|\sqrt{\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}}\phi_y\right|\left(\left|\sqrt{\bar\rho^{\gamma-2}}\phi\right|+\left|\sqrt{p}\psi\right|+\left|\sqrt{\bar\rho^{2-\gamma}}\chi\right|\right)dyd\tau\\
\le&\frac{1}{24}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2dyd\tau+C\left(\frac{\epsilon}{\nu\sqrt{\delta}}\right)^3\int_0^\tau\int_{\mathbf{R}}\bar u_y\left(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho^{2-\gamma}\chi^2\right)dyd\tau,
\end{aligned}
$$

and

$$
J_3=C\left|\int_0^\tau\int_{\mathbf{R}}\mu(\bar\theta)g\frac{\phi_y}{\rho^2}dyd\tau\right|\le\frac{1}{24}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\bar\rho^{-2}g^2dyd\tau.
$$

From (3.5),(3.6) and Lemma 2.3, it holds that

$$
\begin{aligned}
|g|\le&C\left(\bar\theta^\alpha\left|\bar u_{yy}\right|+\left|\bar\rho\bar u_y\psi\right|+\left|\bar\rho_y\chi\right|+\left|\bar\rho_y\bar\rho^{\gamma-2}\phi\right|\right)\\
\le&C\left(\bar\theta^\alpha\left|\bar u_{yy}\right|+\bar u_y\left(\left|\bar\rho\psi\right|+\left|\bar\rho^{\frac{3-\gamma}{2}}\chi\right|+\left|\bar\rho^{\frac{\gamma-1}{2}}\phi\right|\right)\right),
\end{aligned}
$$

So, the last term in J_3 can be estimated by

$$
\begin{aligned}
&\left|\int_0^\tau\int_{\mathbf{R}} \bar\theta^{\alpha-1}\bar\rho^{-2}g^2\,dy\,d\tau\right|\\
\le& C\nu^{-1-\gamma}\int_0^\tau \left\|\bar u_{yy}\right\|^2 d\tau + C\int_0^\tau\int_{\mathbf{R}} \bar\theta^{\alpha-1}\bar\rho^{-1}\bar u_y^2\left(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho^{2-\gamma}\chi^2\right)dy\,d\tau\\
\le& C\nu^{-1-\gamma}\left(\frac{\epsilon}{\delta}\right)^2 + C\frac{\epsilon}{\delta\nu^\gamma}\int_0^\tau\int_{\mathbf{R}} \bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho^{2-\gamma}\chi^2)dy\,d\tau.
\end{aligned}
$$

For J_4, we have

$$
\begin{aligned}
J_4 \le& C\left|\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2\bar\theta^{\alpha-3/2}\bar\rho^{-1}\bar u_y\,dy\,d\tau\right| \le C\nu^{\frac{1-3\gamma}{2}}\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2\,dy\,d\tau\\
\le& \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2\,dy\,d\tau, \quad \text{if } \epsilon \ll 1.
\end{aligned}
$$

And

$$
\begin{aligned}
J_5 \le& C\int_0^\tau\int_{\mathbf{R}} \bar\rho^{\alpha(\gamma-1)-\gamma}\bar u_y\left|\sqrt{\bar\theta^{\alpha+1}\bar\rho^{-2}}\phi_y\right|\left|\sqrt{\bar\theta^\alpha}\psi_y\right|dy\,d\tau\\
&+C\int_0^\tau\int_{\mathbf{R}}\left\{\bar\rho^{\alpha(\gamma-1)-\gamma}\bar u_y\left|\sqrt{\bar\theta^{\alpha+1}\bar\rho^{-2}}\phi_y\right|\left|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right| + \bar\rho^{\frac{\alpha(\gamma-1)-\gamma}{2}}\bar u_y\left|\sqrt{\bar\rho}\psi\right|\left|\sqrt{\bar\theta^\alpha}\psi_y\right|\right\}dy\,d\tau\\
&+C\int_0^\tau\int_{\mathbf{R}} \bar\rho^{\frac{\alpha(\gamma-1)}{2}-\gamma}\bar u_y\left|\sqrt{\bar\theta^{\alpha+1}\bar\rho^{-2}}\phi_y\right||\sqrt{\bar\rho}\psi|\,dy\,d\tau\\
&+C\int_0^\tau\int_{\mathbf{R}} \bar\rho^{\alpha(\gamma-1)-\gamma}\bar u_y\left(\left|\sqrt{\bar u_y\bar\rho}\psi\right|\left|\sqrt{\bar u_y\bar\rho^{\gamma-2}}\phi\right| + \bar u_y\bar\rho\psi^2\right)dy\,d\tau\\
\le& \left(\frac{1}{48}+C\frac{\epsilon}{\nu^\gamma\delta}\right)\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2\,dy\,d\tau\\
&+C\frac{\epsilon}{\nu^{2\gamma}\delta}\int_0^\tau\int_{\mathbf{R}}\left(\bar\theta^\alpha\psi_y^2+\bar\theta^{\alpha-1}\chi_y^2+\bar u_y\bar\rho\psi^2+\bar u_y\bar\rho^{\gamma-2}\phi^2\right)dy\,d\tau.
\end{aligned}
$$

Similarly, J_6 and J_7 can be estimated as follows

$$
\begin{aligned}
J_6 \le& C\int_0^\tau\int_{\mathbf{R}} \bar\theta^{\alpha/2-1}\bar\rho^{-1/2}|\bar u_y|\left|\sqrt{\bar\theta^{\alpha+1}\bar\rho^{-2}}\phi_y\right|\left|\sqrt{\bar\rho}\psi\right|dy\,d\tau\\
\le& \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2\,dy\,d\tau + C\nu^{1-2\gamma}\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbf{R}} \bar u_y\bar\rho\psi^2\,dy\,d\tau,
\end{aligned}
$$

and

$$
J_7 \le C\left|\int_0^\tau\int_{\mathbf{R}} \mu(\bar\theta)\frac{\phi_y}{\rho^2}(\mu(\theta)-\mu(\bar\theta))\psi_{yy}\,dy\,d\tau\right|
$$

$$
\begin{aligned}
&\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2\chi^2\bar\theta^{2\alpha-3}\bar\rho^{-1}dyd\tau\\
&\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\nu^{1-3\gamma}\sup_{[0,\tau_1(\epsilon)]}\|\sqrt{\rho^{2-\gamma}}\chi\|\sup_{[0,\tau_1(\epsilon)]}\|\chi_y\|\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2 dyd\tau\\
&\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\nu^{1-3\gamma}\epsilon^{1/6}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2 dyd\tau.
\end{aligned}
$$

Recalling Lemma 2.3 and (3.8), we can get

$$
\begin{aligned}
J_8 &\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\int_0^\tau\int_{\mathbf{R}} \bar\theta^{3\alpha-2}\bar\rho^{-2}\bar u_y^4 dyd\tau\\
&\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\nu^{-2\gamma}\int_0^\tau \left\|\bar u_y\right\|_{L^4}^4 d\tau\\
&\le \frac{1}{24}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2 dyd\tau + C\left(\frac{\epsilon}{\delta\nu^\gamma}\right)^2.
\end{aligned}
$$

And

$$
\begin{aligned}
J_9 =&C\left|\alpha\int_0^\tau\int_{\mathbf{R}} \theta^{\alpha-1}\chi_y\psi_y\mu(\bar\theta)\frac{\phi_y}{\rho^2}dyd\tau\right|\\
\le& C\nu^{-\frac{\gamma}{2}}\int_0^t \left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|\left\|\psi_y\right\|^{1/2}\left\|\psi_{yy}\right\|^{1/2}d\tau\\
\le& C\nu^{-\frac{2\gamma+\alpha(\gamma-1)}{4}}\int_0^t \left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|\left\|\psi_y\right\|^{1/2}\left\|\sqrt{\bar\theta^{\alpha}\bar\rho^{-1}}\psi_{yy}\right\|^{1/2}d\tau\\
\le& \nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2 dyd\tau\\
&+C\nu^{-\frac{2\gamma+3\alpha(\gamma-1)}{3}}|\ln\epsilon|^{\frac{1}{3}}\int_0^\tau \left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|^{4/3}\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^{4/3}\left\|\psi_y\right\|^{2/3}d\tau\\
\le& \nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2 dyd\tau + \frac{1}{24}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|^2\\
&+C\nu^{-2\gamma-3\alpha(\gamma-1)}|\ln\epsilon|\left(\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^{4/3}\left\|\psi_y\right\|^{2/3}d\tau\right)^3\\
\le& \nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}} \frac{\bar\theta^{\alpha}}{\bar\rho}\psi_{yy}^2 dyd\tau + \frac{1}{24}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|^2\\
&+C\nu^{-2\gamma-4\alpha(\gamma-1)}|\ln\epsilon|\left(\int_0^\tau \left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|^2+\left\|\sqrt{\bar\theta^{\alpha}}\psi_y\right\|^2d\tau\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&\le \nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+\frac{1}{24}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|^2\\
&\quad + C\nu^{-2\gamma-4\alpha(\gamma-1)}|\ln\epsilon|\epsilon\\
&\le \nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+\frac{1}{24}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\theta^{2\alpha}\bar\rho^{-3}}\phi_y\right\|^2+\epsilon^{1/3},
\end{aligned}
$$

where we have used the fact that

$$
C\nu^{-2\gamma-4\alpha(\gamma-1)}|\ln\epsilon|\epsilon=C\epsilon^{1-(2\gamma+4\alpha(\gamma-1))a}|\ln\epsilon|^{1-2\gamma-4\alpha(\gamma-1)}\le\epsilon^{1/3},\quad \text{if } \epsilon\ll 1.
$$

Plugging J_1-J_9 into (3.27), we can deduce

$$
\begin{aligned}
&\int_{\mathbf{R}}\left(\frac{\bar\theta^{2\alpha}}{\bar\rho^3}\phi_y^2+\bar\rho\psi^2+\bar\rho w^2+\bar\rho^{2-\gamma}\chi^2\right)(\tau,y)dy\\
&+\int_0^\tau\int_{\mathbf{R}}\left(\bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2+\bar\rho^{2-\gamma}\chi^2)+\bar\theta^\alpha\psi_y^2+\bar\theta^{\alpha-1}\chi_y^2+\bar\theta^\alpha w_y^2+\epsilon^2\bar\theta^\alpha w^2+\frac{\bar\theta^{\alpha+1}}{\rho^2}\phi_y^2\right)dyd\tau\\
&\le C\left(\nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}+\nu^{1-3\gamma}\epsilon^{1/6}\right)\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\epsilon^{1/3+(\gamma-3)a}|\ln\epsilon|^{-3}.
\end{aligned}
\tag{3.28}
$$

In particular, it holds that

$$
\begin{aligned}
&\sup_{\tau\in[0,\tau_(\epsilon)]}\int_{\mathbf{R}}\frac{\bar\theta^{2\alpha}}{\bar\rho^3}\phi_y^2dy+\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^{1+\alpha}}{\bar\rho^2}\phi_y^2dyd\tau\\
&\le C\left(\nu^{2\alpha(\gamma-1)}|\ln\epsilon|^{-1}+\nu^{1-3\gamma}\epsilon^{1/6}\right)\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\epsilon^{1/3+(\gamma-3)a}|\ln\epsilon|^{-3}.
\end{aligned}
\tag{3.29}
$$

Next, we estimate the term $\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau$, firstly we note that

$$
\left[(\mu(\theta)-\mu(\bar\theta))u_y\right]_y=\left[\mu(\theta)-\mu(\bar\theta)\right](\psi_{yy}+\bar u_{yy})+\mu(\theta)_yu_y-\mu(\bar\theta)_yu_y,
$$

and take

$$
\bar g=-\mu(\theta)\bar u_{yy}+\rho\psi\bar u_y+(\gamma-1)\left(\bar\rho_y\chi-\frac{\bar\rho_y\bar\theta\phi}{\bar\rho}\right),
$$

then $(3.4)_2$ can be rewrite as

$$
\rho\psi_\tau+\rho u\psi_y+(\gamma-1)(\theta\phi_y+\rho\chi_y)-\mu(\theta)\psi_{yy}=-\bar g+\mu(\theta)_yu_y,
\tag{3.30}
$$

multiplying (3.30) by $\frac{-\psi_{yy}}{\rho}$ yields

$$
\begin{aligned}
&\left(\frac{\psi_y^2}{2}\right)_\tau-\left(\psi_y\psi_\tau+u\frac{\psi_y^2}{2}\right)_y+\bar u_y\frac{\psi_y^2}{2}+\frac{\mu(\theta)\psi_{yy}^2}{\rho}\\
&=\bar g\frac{\psi_{yy}}{\rho}-\frac{\psi_y^3}{2}+(\gamma-1)(\theta\phi_y+\rho\chi_y)\frac{\psi_{yy}}{\rho}-\mu(\theta)_yu_y\frac{\psi_{yy}}{\rho}.
\end{aligned}
\tag{3.31}
$$

Then integrating the last equation over $\mathbf{R}\times[0,\tau]$, we have

$$
\begin{aligned}
&\int_{\mathbf{R}}\frac{\psi_y^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\left(\frac{\bar u_y\psi_y^2}{2}+\frac{\mu(\theta)\psi_{yy}^2}{\rho}\right)dyd\tau\\
&=\int_{\mathbf{R}}\frac{\psi_{0y}^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\left[\bar g\frac{\psi_{yy}}{\rho}-\frac{\psi_y^3}{2}+(\gamma-1)\left(\theta\phi_y+\rho\chi_y\right)\frac{\psi_{yy}}{\rho}-\mu(\theta)_yu_y\frac{\psi_{yy}}{\rho}\right]dyd\tau.
\end{aligned}
\tag{3.32}
$$

Similar to J_3, it holds that

$$
\begin{aligned}
\left|\int_0^\tau\int_{\mathbf{R}} \bar{g}\frac{\psi_{yy}}{\rho}dyd\tau\right| \le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-2\alpha(\gamma-1)}\int_0^\tau\int_{\mathbf{R}} \bar{\theta}^{\alpha-1}\bar{\rho}^{-1}g^2dyd\tau \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-2\alpha(\gamma-1)-\gamma}\frac{\epsilon}{\delta}\epsilon^{1/3} \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + \epsilon^{1/3},
\end{aligned}
\tag{3.33}
$$

where in the last inequality we used the fact that

$$
C\nu^{-2\alpha(\gamma-1)-\gamma}\frac{\epsilon}{\delta} = C\epsilon^{1-a(\gamma+1+2\alpha(\gamma-1))}|\ln\epsilon|^{-2\alpha(\gamma-1)-\gamma} \le C\epsilon^{\frac{1}{2}}|\ln\epsilon|^{-2\alpha(\gamma-1)-\gamma} \le 1, \quad \text{if } \epsilon \ll 1.
$$

And

$$
\begin{aligned}
\left|\int_0^\tau\int_{\mathbf{R}} -\frac{\psi_y^3}{2}dyd\tau\right| \le & C\int_0^\tau \|\psi_{yy}\|^{1/2}\|\psi_y\|^{5/2}dyd\tau \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-\frac{4\alpha(\gamma-1)}{3}}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\psi_y\|^{\frac{4}{3}}\int_0^\tau\int_{\mathbf{R}} \bar{\theta}^\alpha\psi_y^2dyd\tau \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-\frac{4\alpha(\gamma-1)}{3}}\epsilon^{\frac{1}{3}},
\end{aligned}
\tag{3.34}
$$

where we have used the a priori assumptions (3.6).

By direct calculation, we can get

$$
\begin{aligned}
& \left|\int_0^\tau\int_{\mathbf{R}} (\gamma-1)\left(\theta\phi_y+\rho\chi_y\right)\frac{\psi_{yy}}{\rho}dyd\tau\right| \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\int_0^\tau\int_{\mathbf{R}} \bar{\rho}^{-\gamma-2\alpha((\gamma-1)}\left(\frac{\bar{\theta}^{\alpha+1}}{\bar{\rho}^2}\phi_y^2+\bar{\theta}^{\alpha-1}\chi_y^2\right)dyd\tau \\
\le & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\psi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-2\alpha(\gamma-1)}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^{\alpha+1}}{\bar{\rho}^2}\phi_y^2dyd\tau + C\nu^{-2\alpha(\gamma-1)}\epsilon^{1/3}.
\end{aligned}
\tag{3.35}
$$

As for the last term on the right-hand of (3.32), we have

$$
\begin{aligned}
& \left|\int_0^\tau\int_{\mathbf{R}} -\mu(\theta)_y u_y\frac{\psi_{yy}}{\rho}dyd\tau\right| \\
\le & C\int_0^\tau\int_{\mathbf{R}} \bar{\theta}^{\alpha-1}\bar{\rho}^{-1}|\psi_{yy}|\left(|\psi_y\chi_y| + \bar{u}_y\left(\bar{\theta}^{1/2}|\psi_y| + |\chi_y|\right) + \bar{\theta}^{1/2}\bar{u}_y^2\right)dyd\tau \\
:= & \sum_{i=1}^3 H_i
\end{aligned}
\tag{3.36}
$$

By the Lemma 2.3, Cauchy inequality, the terms H_i $(i = 1, 2, 3)$ will be estimates as follows:

$$
H_1 = C\int_0^\tau\int_{\mathbf{R}} \bar{\theta}^{\alpha-1}\bar{\rho}^{-1}\left|\psi_{yy}\psi_y\chi_y\right|dyd\tau
$$

$$
\begin{aligned}
&\leq C\nu^{-\gamma/2}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\psi_{yy}\right\|\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|\|\psi_y\|^{1/2}\|\psi_{yy}\|^{1/2}d\tau\\
&\leq C\nu^{-\frac{2\gamma+\alpha(\gamma-1)}{4}}\int_0^\tau \left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\psi_{yy}\right\|^{3/2}\left\|\sqrt{\bar\theta^{\alpha-1}}\chi_y\right\|\|\psi_y\|^{1/2}d\tau\\
&\leq \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau + C\nu^{1-3\gamma-\alpha(\gamma-1)}\sup_{[0,\tau_1(\epsilon)]}\|\chi_y\|^2\sup_{[0,\tau_1(\epsilon)]}\|\psi_y\|^2\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\chi_y^2dyd\tau\\
&\leq \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau + C\nu^{1-3\gamma-\alpha(\gamma-1)}\epsilon^{1/3}\\
&\leq \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau + \epsilon^{\frac{1}{6}+a},
\end{aligned}
$$

where we used the fact that

$$
C\nu^{1-3\gamma-\alpha(\gamma-1)}\epsilon^{1/6}\leq \nu^{1-3\gamma-2\alpha(\gamma-1)}\epsilon^{1/6}=\epsilon^a|\ln\epsilon|^{1-3\gamma-2\alpha(\gamma-1)}\leq\epsilon^a,\quad \text{if } \epsilon\ll 1,
$$

and also, obviously, the last term in (3.34) and (3.35) can be controlled by $\epsilon^{\frac{1}{6}+a}$ due to the above inequality. Similarly,

$$
\begin{aligned}
H_2 =& C\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\bar\rho^{-1}|\psi_{yy}|\bar u_y\left(\bar\theta^{1/2}|\psi_y|+|\chi_y|\right)dyd\tau\\
\leq& \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+\int_0^\tau\int_{\mathbf{R}}(\bar\theta^\alpha\psi_y^2+\bar\theta^{\alpha-1}\chi_y^2)(\bar\theta\bar\rho)^{-1}\bar u_y^2dyd\tau\\
\leq& \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\nu^{-\gamma}\left(\frac{\epsilon}{\delta}\right)^2\epsilon^{1/3}\\
\leq& \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+\epsilon^{\frac{1}{3}}.
\end{aligned}
$$

And likewise we have

$$
\begin{aligned}
H_3 =& C\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\bar\rho^{-1}|\psi_{yy}|\bar u_y^2dyd\tau\leq\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\bar\rho^{-1}\bar u_y^4dyd\tau\\
\leq& \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\nu^{-\gamma}\int_0^\tau\|\bar u_y\|_{L^4}^4d\tau\leq\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+C\nu^{-\gamma}\left(\frac{\epsilon}{\delta}\right)^2\\
\leq& \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2dyd\tau+\epsilon^{\frac{1}{3}}.
\end{aligned}
$$

Hence, we can obtain

$$
\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\psi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left(\bar u_y\psi_y^2+\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2\right)dyd\tau\leq\nu^{-2\alpha(\gamma-1)}\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\frac{\bar\theta^{\alpha+1}}{\bar\rho^2}\phi_y^2dyd\tau+\epsilon^{\frac{1}{6}+a}. \tag{3.37}
$$

Then, plugging (3.37) into (3.29), we can derive that

$$\sup_{\tau\in[0,\tau_(\epsilon)]}\int_{\mathbf{R}}\frac{\bar\theta^{2\alpha}}{\bar\rho^3}\phi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\frac{\bar\theta^{1+\alpha}}{\bar\rho^2}\phi_y^2dyd\tau\le\epsilon^{1/3-3\gamma a}|\ln\epsilon|^{-3\gamma},\quad \text{if } \epsilon\ll 1. \tag{3.38}$$

Meanwhile, it holds that

$$\begin{aligned}&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\psi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left(\bar u_y\psi_y^2+\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2\right)dyd\tau\le C\nu^{-2\alpha(\gamma-1)}\epsilon^{1/3-3\gamma a}|\ln\epsilon|^{-3\gamma}+\epsilon^{1/6+a}\\&=\epsilon^{\frac16}|\ln\epsilon|^{-3\gamma-2\alpha(\gamma-1)}+\epsilon^{1/6+a}\le\epsilon^{\frac16}.\end{aligned} \tag{3.39}$$

So we have completed the proof of Lemma 5. □

Lemma 6. Under the conditions of Proposition 1, for $0\le\tau\le\tau_1(\epsilon)$ and $\epsilon\ll 1$ small enough, we have

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}w_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left(\bar u_yw_y^2+\frac{\epsilon^2\bar\theta^\alpha}{\bar\rho}w_y^2+\frac{\bar\theta^\alpha}{\bar\rho}w_{yy}^2\right)dyd\tau\le\epsilon^{1/3}. \tag{3.40}$$

Proof. Multiplying $(3.4)_3$ by $\frac{-w_{yy}}{\rho}$ yields

$$\begin{aligned}&\left(\frac{w_y^2}{2}\right)_\tau-\left(w_yw_\tau+u\frac{w_y^2}{2}+\frac{\epsilon^2ww_y}{\rho}\right)_y+\bar u_y\frac{w_y^2}{2}+\frac{\lambda(\theta)w_{yy}^2}{\rho}+\frac{\epsilon^2\xi(\theta)w_y^2}{\rho}\\&=-\frac{\epsilon^2\xi(\theta)_yww_y}{\rho}+\frac{\epsilon^2\xi(\theta)ww_y\phi_y}{\rho}+\frac{\epsilon^2\xi(\theta)ww_y\bar\rho_y}{\rho}.\end{aligned} \tag{3.41}$$

Integrating the last equation over $\mathbf{R}\times[0,\tau]$ gives

$$\begin{aligned}&\int_{\mathbf{R}}\frac{w_y^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\left(\frac{\bar u_y}{2}w_y^2+\frac{\lambda(\theta)w_{yy}^2}{\rho}+\frac{\epsilon^2\xi(\theta)w_y^2}{\rho}\right)dyd\tau\\&=\int_{\mathbf{R}}\frac{w_{0y}^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\left(-\frac{\epsilon^2\xi(\theta)_yww_y}{\rho}+\frac{\epsilon^2\xi(\theta)ww_y\phi_y}{\rho}+\frac{\epsilon^2\xi(\theta)ww_y\bar\rho_y}{\rho}\right)dyd\tau\\&:=\frac12\|w_{0y}\|^2+\sum_{i=1}^3M_i.\end{aligned} \tag{3.42}$$

For the first term on the right hand side of the equation (3.42), we have

$$\begin{aligned}M_1=&\left|\int_0^\tau\int_{\mathbf{R}}-\frac{\epsilon^2\xi(\theta)_yww_y}{\rho}dyd\tau\right|\le C\epsilon^2\nu^{-1}\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\left(\bar\theta_y+\chi_y\right)ww_ydyd\tau\\\le&\frac1{24}\int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha w_y^2dyd\tau+C\epsilon^2\nu^{-(2\alpha+1)(\gamma-1)-2}\left(\frac{\epsilon}{\delta}\right)^{-1}\int_0^\tau\int_{\mathbf{R}}\epsilon^2\bar\theta^\alpha w^2dyd\tau\\&+C\epsilon^4\nu^{-(\alpha+2)(\gamma-1)-2}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|w(\cdot,\tau)\|_{L^\infty}^2\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\chi_y^2dyd\tau\\\le&\frac1{24}\int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha w_y^2dyd\tau+C\epsilon^2\nu^{-(2\alpha+1)(\gamma-1)-2}\epsilon^{2a}\left(\frac{\epsilon}{\delta}\right)^{-1}\epsilon^{1/3}\\&+C\epsilon^4\nu^{-(\alpha+2)(\gamma-1)-2}\epsilon^{2a}\epsilon^{1/3}\end{aligned}$$

$$\leq \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \bar{\theta}^{\alpha} w_y^2 dy d\tau + \epsilon^{1/3},$$

where we have used the following two facts

$$C\epsilon^2 \nu^{-(2\alpha+1)(\gamma-1)-2}\epsilon^{2a}\left(\frac{\epsilon}{\delta}\right)^{-1} = C\epsilon^{1-a((2\alpha+1)(\gamma-1)+1)}|\ln\epsilon|^{-((2\alpha+1)(\gamma-1)+2)} \leq 1, \quad \text{if } \epsilon \ll 1,$$

and

$$C\epsilon^4 \nu^{-(2\alpha+1)(\gamma-1)-2}\epsilon^{2a} = C\epsilon^{4-a(2\alpha+1)(\gamma-1)}|\ln\epsilon|^{-((2\alpha+1)(\gamma-1)+2)} \leq 1, \quad \text{if } \epsilon \ll 1.$$

Similarity,

$$\begin{aligned}
M_2 =& \left|\int_0^{\tau}\int_{\mathbf{R}} \frac{\epsilon^2 \xi(\theta) w w_y \phi_y}{\rho} dy d\tau\right| \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \frac{\bar{\theta}^{\alpha}}{\bar{\rho}^2}\phi_y^2 dy d\tau + C\epsilon^4 \nu^{-(2\alpha+1)(\gamma-1)-2} \sup_{\tau\in[0,\tau_1(\epsilon)]} \|w(\cdot,\tau)\|_{L^\infty}^2 \int_0^{\tau}\int_{\mathbf{R}} \bar{\theta}^{\alpha} w_y^2 dy d\tau \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \frac{\bar{\theta}^{\alpha}}{\bar{\rho}^2}\phi_y^2 dy d\tau + C\epsilon^4 \nu^{-(2\alpha+1)(\gamma-1)-2}\epsilon^{2a}\epsilon^{1/3} \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \frac{\bar{\theta}^{\alpha}}{\bar{\rho}^2}\phi_y^2 dy d\tau + \epsilon^{1/3},
\end{aligned}$$

where we have used the fact that

$$C\epsilon^4 \nu^{-(2\alpha+1)(\gamma-1)-2}\epsilon^{2a} = C\epsilon^{4-a(2\alpha+1)(\gamma-1)}|\ln\epsilon|^{-((2\alpha+1)(\gamma-1)+2)} \leq 1, \quad \text{if } \epsilon \ll 1.$$

For M_3, we have

$$\begin{aligned}
M_3 =& \left|\int_0^{\tau}\int_{\mathbf{R}} \frac{\epsilon^2 \xi(\theta) w w_y \bar{\rho}_y}{\rho} dy d\tau\right| \\
\leq& C\epsilon^2 \nu^{-\frac{(\gamma+1)}{2}} \sup_{\tau\in[0,\tau_1(\epsilon)]} \|w(\cdot,\tau)\|_{L^\infty} \int_0^{\tau}\int_{\mathbf{R}} \bar{u}_y w_y dy d\tau \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \bar{\theta}^{\alpha} w_y^2 dy d\tau + C\epsilon^4 \nu^{-((\gamma+1)+\alpha(\gamma-1))} \sup_{\tau\in[0,\tau_1(\epsilon)]} \|w(\cdot,\tau)\|_{L^\infty}^2 \int_0^{\tau}\int_{\mathbf{R}} \bar{u}_y^2 dy d\tau \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \bar{\theta}^{\alpha} w_y^2 dy d\tau + C\epsilon^4 \nu^{-((\gamma+1)+\alpha(\gamma-1))}\epsilon^{2a}\left(\frac{\epsilon}{\delta}\right)^{-1} \\
\leq& \frac{1}{24}\int_0^{\tau}\int_{\mathbf{R}} \bar{\theta}^{\alpha} w_y^2 dy d\tau + \epsilon^{1/3},
\end{aligned}$$

where we have used the fact that

$$\epsilon^4 \nu^{-((\gamma+1)+\alpha(\gamma-1))}\epsilon^{2a}\left(\frac{\epsilon}{\delta}\right)^{-1} = C\epsilon^{3-a(\gamma+\alpha(\gamma-1))}|\ln\epsilon|^{-(\alpha(\gamma-1)+(\gamma+1))} \leq \epsilon^{1/3}, \quad \text{if } \epsilon \ll 1.$$

Substituting $M_1 - M_3$ into (3.42), then we can derive (3.40). In other words, we have completed the proof of Lemma 6. □

Finally, we give the estimate of $\sup_\tau \|\chi_y\|$ in the following lemma.

Lemma 7. Under the conditions of Proposition 1, for $0 \le \tau \le \tau_1(\epsilon)$ and $\epsilon \ll 1$ small enough, we have

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}} \chi_y^2 dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}} \left(\bar{u}_y\chi_y^2 + \frac{\bar{\theta}^\alpha}{\bar{\rho}}\chi_{yy}^2\right) dyd\tau \le \epsilon^{\frac{1}{9}}. \tag{3.43}$$

Proof. Firstly, note that

$$\left[\left(\kappa(\theta)-\kappa(\bar{\theta})\right)\theta_y\right]_y = \left[\kappa(\theta)-\kappa(\bar{\theta})\right](\chi_{yy}+\bar{\theta}_{yy}) + \left[\kappa(\theta)_y - \kappa(\bar{\theta})_y\right]\theta_y,$$

and take

$$\bar{h} = -\kappa(\theta)\bar{\theta}_{yy} + (\gamma-1)\rho\chi\bar{u}_y + \rho\psi\bar{\theta}_y + \lambda(\theta)(w_y)^2 + \epsilon^2\xi(\theta)w^2,$$

then $(3.4)_4$ can be rewrote as

$$\rho\chi_\tau + \rho u\chi_y + (\gamma-1)\rho\theta\psi_y - \kappa(\theta)\chi_{yy} = -\bar{h} + \kappa(\theta)_y\theta_y + \mu(\theta)u_y^2, \tag{3.44}$$

multiplying (3.44) by $\frac{-\chi_{yy}}{\rho}$ yields

$$\begin{aligned}
&\left(\frac{\chi_y^2}{2}\right)_\tau - \left(\chi_y\chi_\tau + u\frac{\chi_y^2}{2}\right)_y + \bar{u}_y\frac{\chi_y^2}{2} + \frac{\kappa(\theta)}{\rho}\chi_{yy}^2\\
&= \bar{h}\frac{\chi_{yy}}{\rho} + (\gamma-1)\theta\psi_y\chi_{yy} - \frac{\psi_y\chi_y^2}{2} + \left(\kappa(\theta)_y\theta_y + \mu(\theta)u_y^2\right)\frac{\chi_{yy}}{\rho}.
\end{aligned} \tag{3.45}$$

Then integrating the last equation over $\mathbf{R}\times[0,\tau]$, it holds that

$$\begin{aligned}
&\int_{\mathbf{R}} \frac{\chi_y^2}{2}dy + \int_0^\tau\int_{\mathbf{R}} \left(\frac{\bar{u}_y\chi_y^2}{2} + \frac{\kappa(\theta)\chi_{yy}^2}{\rho}\right) dyd\tau\\
=&\int_{\mathbf{R}} \frac{\chi_{0y}^2}{2}dy + \int_0^\tau\int_{\mathbf{R}} \left(\bar{h}\frac{\chi_{yy}}{\rho} + (\gamma-1)\theta\psi_y\chi_{yy} - \frac{\psi_y\chi_y^2}{2} + \left[\kappa(\theta)_y\theta_y + \mu(\theta)u_y^2\right]\frac{\chi_{yy}}{\rho}\right) dyd\tau\\
:=&\frac{1}{2}\|\chi_{0y}\|^2 + \sum_{i=1}^4 N_i.
\end{aligned} \tag{3.46}$$

By virtue of (3.5), (3.8) and Lemma 3, we get that

$$\begin{aligned}
|\bar{h}| \le& C\left(\bar{\theta}^\alpha\left|\bar{\theta}_{yy}\right| + \left|\bar{\rho}\bar{\theta}_y\psi\right| + \left|\bar{u}_y\bar{\rho}\chi\right| + \bar{\theta}^\alpha w_y^2 + \epsilon^2\bar{\theta}^\alpha w^2\right)\\
\le& C\left\{\bar{\theta}^\alpha\left(\bar{\theta}^{1/2}|\bar{u}_{yy}| + \bar{u}_y^2\right) + \bar{u}_y\left(\left|\bar{\theta}^{1/2}\bar{\rho}\psi\right| + |\bar{\rho}\psi_y|\right) + \bar{\theta}^\alpha w_y^2 + \epsilon^2\bar{\theta}^\alpha w^2\right\}.
\end{aligned}$$

From Cauchy's inequality, it holds that

$$\begin{aligned}
N_1 =& \int_0^\tau\int_{\mathbf{R}} \bar{h}\frac{\chi_{yy}}{\rho}dyd\tau \le \frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\chi_{yy}^2}{\bar{\rho}}dyd\tau + C\int_0^\tau\int_{\mathbf{R}} \frac{\bar{h}^2}{\bar{\theta}^\alpha\bar{\rho}}dyd\tau\\
\le&\frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\chi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-1}\int_0^\tau (\|\bar{u}_{yy}\|^2 + \|\bar{u}_y\|_{L^4}^4)d\tau\\
&+ C\nu^{-1}\int_0^\tau\int_{\mathbf{R}} \left(\bar{\theta}^\alpha w_y^2 + \epsilon^2\bar{\theta}^\alpha w^2\right)dyd\tau + C\int_0^\tau\int_{\mathbf{R}} \bar{u}_y(\bar{\rho}\psi^2 + \bar{\rho}^{2-\gamma}\chi^2)\bar{\theta}^{1-\alpha}\bar{u}_ydyd\tau\\
\le&\frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\chi_{yy}^2}{\bar{\rho}}dyd\tau + C\nu^{-1}\left(\frac{\epsilon}{\delta}\right)^2 + C\nu^{-1}\epsilon^{1/3} + C\nu^{-1}\left(\frac{\epsilon}{\delta}\right)\epsilon^{1/3}\\
\le&\frac{1}{8}\int_0^\tau\int_{\mathbf{R}} \frac{\bar{\theta}^\alpha\chi_{yy}^2}{\bar{\rho}}dyd\tau + \epsilon^{1/3}.
\end{aligned} \tag{3.47}$$

Similarly, we have

$$\begin{aligned} N_2 =& \left| \int_0^\tau \int_{\mathbf{R}} (\gamma-1)\theta\psi_y \chi_{yy} dy d\tau \right| \le \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + \int_0^\tau \int_{\mathbf{R}} \bar\rho \psi_y^2 \bar\theta^{2-\alpha} dy d\tau \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + C\nu^{-2\alpha(\gamma-1)} \epsilon^{1/3}. \end{aligned} \tag{3.48}$$

By sobolev's inequality, we obtain

$$\begin{aligned} N_3 =& \left| \int_0^\tau \int_{\mathbf{R}} -\frac{\psi_y \chi_y^2}{2} dy d\tau \right| \le C \int_0^\tau \|\psi_y\|^{1/2} \|\chi_{yy}\|^{1/2} \|\chi_y\|^{3/2} d\tau \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + C\nu^{-\alpha(\gamma-1)/3} \int_0^\tau \|\psi_y\|^{2/3} \|\chi_y\|^2 d\tau \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + C\nu^{-4\alpha(\gamma-1)/3} \sup_{\tau\in[0,\tau_1]} \|\psi_y\|^{4/3} \int_0^\tau \int_{\mathbf{R}} \bar\theta^{\alpha-1} \chi_y^2 dy d\tau \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + C\nu^{-4\alpha(\gamma-1)/3} \epsilon^{1/3} \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + \epsilon^{1/9}, \end{aligned} \tag{3.49}$$

where we have used the a priori assumptions (3.6), and in the last inequality we have used the fact that

$$C\nu^{-4\alpha(\gamma-1)/3}\epsilon^{1/3} = C\nu^{\frac{1-4\alpha(\gamma-1)a}{3}} |\ln\epsilon|^{-\frac{4\alpha((\gamma-1)}{3}} \le \epsilon^{1/9}, \quad \text{if } \epsilon \ll 1.$$

By the Cauchy's inequality, we have

$$\begin{aligned} N_4 =& \left| \int_0^\tau \int_{\mathbf{R}} \left(\kappa(\theta)_y \theta_y + \mu(\theta) u_y^2 \right) \frac{\chi_{yy}}{\rho} dy d\tau \right| \\ \le& C \int_0^\tau \int_{\mathbf{R}} \bar\theta^\alpha \bar\rho^{-1} |\chi_{yy}| \left(\bar\theta^{-1} \chi_y^2 + \psi_y^2 + \bar u_y^2 \right) dy d\tau := \sum_i^3 O_i. \end{aligned} \tag{3.50}$$

Now we estimate the terms on the right-hand side of (3.50) one by one. By the Sobolev inequality, it holds that

$$\begin{aligned} O_1 =& \int_0^\tau \int_{\mathbf{R}} \bar\theta^{\alpha-1} \bar\rho^{-1} \left| \chi_{yy} \chi_y^2 \right| dy d\tau \le C\nu^{1/2-\gamma} \int_0^\tau \left\| \sqrt{\bar\theta^\alpha \bar\rho^{-1}} \chi_{yy} \right\| \left\| \chi_y \right\|_{L^4}^2 d\tau \\ \le& C\nu^{1/2-\gamma} \int_0^\tau \left\| \sqrt{\bar\theta^\alpha \bar\rho^{-1}} \chi_{yy} \right\| \left\| \chi_{yy} \right\|^{\frac{1}{2}} \left\| \chi_y \right\|^{\frac{3}{2}} d\tau \\ \le& C\nu^{\frac{2-4\gamma-\alpha(\gamma-1)}{4}} \int_0^\tau \left\| \sqrt{\bar\theta^\alpha \bar\rho^{-1}} \chi_{yy} \right\|^{\frac{3}{2}} \left\| \chi_y \right\|^{\frac{3}{2}} d\tau \\ \le& \frac{1}{8} \int_0^\tau \int_{\mathbf{R}} \frac{\bar\theta^\alpha}{\bar\rho} \chi_{yy}^2 dy d\tau + C\nu^{2-4\gamma-\alpha(\gamma-1)} \int_0^\tau \left\| \chi_y \right\|^6 d\tau \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+Cv^{2-4\gamma-2\alpha(\gamma-1)}\sup_{\tau\in[0,\tau_1(\epsilon)]}\left\|\chi_y\right\|^4\int_0^\tau\int_{\mathbf{R}}\bar\theta^{\alpha-1}\chi_y^2dyd\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+Cv^{2-4\gamma-2\alpha(\gamma-1)}\epsilon^{\frac{1}{3}}\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+\epsilon^{\frac{1}{9}},
\end{aligned}
$$

where in the last inequality we have used the fact that

$$
Cv^{2-4\gamma-2\alpha(\gamma-1)}\epsilon^{\frac{1}{3}}=C\epsilon^{\frac{1}{3}+a(2-4\gamma-2\alpha(\gamma-1))}|\ln\epsilon|^{2-4\gamma-2\alpha(\gamma-1)}\leq\epsilon^{\frac{1}{9}},\quad \text{if } \epsilon\ll 1.
$$

Similarity, we have

$$
\begin{aligned}
O_2= & \int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha\bar\rho^{-1}\left|\chi_{yy}\psi_y^2\right|dyd\tau\leq Cv^{-\frac{1}{2}}\int_0^\tau\left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\chi_{yy}\right\|\left\|\psi_y\right\|_{L^4}^2d\tau\\
\leq & Cv^{-\frac{1}{2}}\int_0^\tau\left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\chi_{yy}\right\|\left\|\psi_{yy}\right\|^{\frac{1}{2}}\left\|\psi_y\right\|^{\frac{3}{2}}d\tau\\
\leq & Cv^{-\frac{2+\alpha(\gamma-1)}{4}}\int_0^\tau\left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\chi_{yy}\right\|\left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\psi_{yy}\right\|^{\frac{1}{2}}\left\|\psi_y\right\|^{\frac{3}{2}}d\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+Cv^{-\frac{2+\alpha(\gamma-1)}{2}}\int_0^\tau\left\|\sqrt{\bar\theta^\alpha\bar\rho^{-1}}\psi_{yy}\right\|\left\|\psi_y\right\|^3d\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\left(\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2+\frac{\bar\theta^\alpha}{\bar\rho}\psi_{yy}^2\right)dyd\tau+Cv^{-2-\alpha(\gamma-1)}\int_0^\tau\left\|\psi_y\right\|^6d\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+\frac{1}{8}\epsilon^{\frac{1}{9}}+Cv^{-2-2\alpha(\gamma-1)}\sup_{[0,r_1(c)]}\left\|\psi_y\right\|^4\int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha\psi_ydyd\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+\frac{1}{8}\epsilon^{\frac{1}{9}}+Cv^{-2-2\alpha(\gamma-1)}\epsilon^{\frac{5}{9}}\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+\epsilon^{1/9}.
\end{aligned}
$$

Recalling Lemma 2.3 and using the Cauchy inequality, it holds that

$$
\begin{aligned}
O_3= & \int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha\bar\rho^{-1}\left|\chi_{yy}\right|\bar u_y^2dtd\tau\leq\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\theta^\alpha\bar\rho^{-1}\bar u_y^4dyd\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+Cv^{-1}\int_0^\tau\|\bar u_y\|_{L^4}^4d\tau\\
\leq & \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\frac{\bar\theta^\alpha}{\bar\rho}\chi_{yy}^2dyd\tau+\epsilon^{1/9}.
\end{aligned}
$$

Substituting N_1-N_4 into (3.46) we can derive (3.43) immediately. So, we have proved the Lemma 7. □

Therefore, combining Lemma 4 - Lemma 7 together, we obtain (3.9), (3.10), (3.11). It follows from (3.9) − (3.11) that if ϵ is suitably small, then

$$\begin{aligned}
\sup_{0\le\tau\le\tau_1(\epsilon)} \|\phi(\cdot,\tau)\|_{L^\infty} &\le \sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)} \|\phi(\cdot,\tau)\|^{1/2}\left\|\phi_y(\cdot,\tau)\right\|^{1/2}\\
&\le C\sup_{0\le\tau\le\tau_1(\epsilon)}\left(\nu^{-\gamma-2\alpha(\gamma-1)}\int_{\mathbf{R}}\bar{\rho}^{\gamma-2}\phi^2dy\int_{\mathbf{R}}\frac{\bar{\theta}^{2\alpha}}{\bar{\rho}^3}\phi_y^2dy\right)^{\frac14}\\
&\le C\left(\nu^{-\gamma-2\alpha(\gamma-1)}\epsilon^{\frac13}\cdot\epsilon^{\frac13-3a\gamma}|\ln\epsilon|^{-3\gamma}\right)^{\frac14}\\
&= C\epsilon^{\frac{4\gamma+3\alpha(\gamma-1)}{2(18\gamma+12\alpha(\gamma-1))}}|\ln\epsilon|^{-\frac{2\gamma+\alpha(\gamma-1)}{2}}\le\epsilon^{\frac{1}{18\gamma+12\alpha(\gamma-1)}}=\epsilon^a,\\
\sup_{0\le\tau\le\tau_1(\epsilon)} \|\psi(\cdot,\tau)\|_{L^\infty} &\le \sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)} \|\psi(\cdot,\tau)\|^{1/2}\left\|\psi_y(\cdot,\tau)\right\|^{1/2}\\
&\le C\sup_{0\le\tau\le\tau_1(\epsilon)}\left(\nu^{-1}\int_{\mathbf{R}}\bar{\rho}\psi^2dy\int_{\mathbf{R}}\psi_y^2dy\right)^{\frac14}\\
&\le C\left(\nu^{-1}\epsilon^{\frac13}\cdot\epsilon^{\frac19}\right)^{\frac14}=C\epsilon^{\frac19-\frac{a}{4}}|\ln\epsilon|^{-\frac14}\\
&= C\epsilon^{\frac{24\gamma+16\alpha(\gamma-1)-3}{12(18\gamma+12\alpha(\gamma-1))}}|\ln\epsilon|^{-\frac14}\le\epsilon^{\frac{1}{18\gamma+12\alpha(\gamma-1)}}=\epsilon^a,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{0\le\tau\le\tau_1(\epsilon)} \|w(\cdot,\tau)\|_{L^\infty} &\le \sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)} \|w(\cdot,\tau)\|^{1/2}\left\|w_y(\cdot,\tau)\right\|^{1/2}\\
&\le C\sup_{0\le\tau\le\tau_1(\epsilon)}\left(\nu^{-1}\int_{\mathbf{R}}\bar{\rho}w^2dy\int_{\mathbf{R}}w_y^2dy\right)^{\frac14}\\
&\le C\left(\nu^{-1}\epsilon^{\frac13}\cdot\epsilon^{\frac19}\right)^{\frac14}=C\epsilon^{\frac19-\frac{a}{4}}|\ln\epsilon|^{-\frac14}\\
&= C\epsilon^{\frac{24\gamma+16\alpha(\gamma-1)-3}{12(18\gamma+12\alpha(\gamma-1))}}|\ln\epsilon|^{-\frac14}\le\epsilon^{\frac{1}{18\gamma+12\alpha(\gamma-1)}}=\epsilon^a,\\
\sup_{0\le\tau\le\tau_1(\epsilon)} \|\chi(\cdot,\tau)\|_{L^\infty} &\le \sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)} \|\chi(\cdot,\tau)\|^{1/2}\left\|\chi_y(\cdot,\tau)\right\|^{1/2}\\
&\le C\sup_{0\le\tau\le\pi(\epsilon)}\left(\nu^{-2}\int_{\mathbf{R}}\bar{\rho}^{2-\gamma}\chi^2dy\int_{\mathbf{R}}\chi_y^2dy\right)^{\frac14}\\
&\le C\left(\nu^{-2}\epsilon^{\frac13}\cdot\epsilon^{\frac19}\right)^{\frac14}=C\epsilon^{\frac19-\frac{a}{2}}|\ln\epsilon|^{-\frac12}\\
&= C\epsilon^{\frac{12\gamma+8\alpha(\gamma-1)-3}{6(18\gamma+12\alpha(\gamma-1))}}|\ln\epsilon|^{-\frac12}\le\epsilon^{\frac{\gamma-1}{18\gamma+12\alpha(\gamma-1)}}=\epsilon^{(\gamma-1)a}.
\end{aligned}$$

Therefore, the a priori assumptions (3.6) are proved if $\epsilon \ll 1$. We have completed the proof of the Proposition 1.

It is easy to see that, in the maximum time interval $[0,\tau_1(\epsilon)]$, the a priori assumption (3.6) are worse than that we have obtained the a priori estimates (3.9)-(3.11). On the basis of these a priori estimates, we claim that $\tau_1(\epsilon)=\infty$. In fact, if $\tau_1(\epsilon)<\infty$, we can using the local existence result at time $\tau=\tau_1(\epsilon)$ again, and then we can find another time $\tau_2(\epsilon)>\tau_1(\epsilon)$ so that the solution satisfies the assumptions (3.6) in the time interval $[0,\tau_2(\epsilon)]$ which contradicts the assumption that $\tau_1(\epsilon)$ is the maximum time. So, we can extend the local solution to the global solution in $[0,\infty)$ for small but fixed ϵ.

4 | PROOF OF THEOREM 1

In this section, we prove Theorem 1. Actually, there only left (1.13) with a given in (1.14) need to be proved. By Lemma 2, Lemma 3, Proposition 1 and $\nu=\epsilon^a|\ln\epsilon|$ and $\delta=\epsilon^a$, it holds that for any given positive constant l there exists a positive constant

C_l which is independent of ϵ such that

$$\begin{aligned}\sup_{t\geq l}\left\|\rho(\cdot,t)-\rho^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\leq&\sup_{\tau\in[0,+\infty)}\|\phi(\cdot,t)\|_{L^\infty}+\sup_{t\geq l}\left\|\bar{\rho}(\cdot,t)-\rho_\nu^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\&+\sup_{t\geq l}\left\|\rho_\nu^{r_3}\left(\frac{\cdot}{t}\right)-\rho^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\\leq&C_l\left(\epsilon^a+\delta|\ln\epsilon|+\nu\right)\leq C_l\epsilon^a|\ln\epsilon|,\end{aligned}$$

similarly we have

$$\begin{aligned}\sup_{t\geq l}\left\|m(\cdot,t)-m^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\leq&C\sup_{\tau\in[0,+\infty)}\left(\|\phi\|_{L^\infty}+\|\psi\|_{L^\infty}\right)+\sup_{t\geq l}\left\|\bar{m}(\cdot,t)-m_\nu^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\&+\sup_{t\geq l}\left\|m_\nu^{r_3}\left(\frac{\cdot}{t}\right)-m^{r_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\\leq&C_l\left(\epsilon^a+\delta|\ln\epsilon|+\nu\right)\leq C_l\epsilon^a|\ln\epsilon|,\\\sup_{t\geq l}\left\|\tilde{e}(\cdot,t)-e^{\tilde{r}_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\leq&C\sup_{\tau\in[0,+\infty)}\left(\|\chi\|_{L^\infty}+\|\phi\|_{L^\infty}\right)+\sup_{t\geq l}\left\|\bar{\rho}\bar{\theta}(\cdot,t)-e_\nu^{\tilde{r}_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\&+\sup_{t\geq l}\left\|e_\nu^{\tilde{r}_3}\left(\frac{\cdot}{t}\right)-e^{\tilde{r}_3}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\\leq&C_l\left(\epsilon^a+\delta|\ln\epsilon|+\nu\right)\leq C_l\epsilon^a|\ln\epsilon|,\end{aligned}$$

where we have used the fact that

$$\sup_{\tau\in[0,+\infty)}\|\chi(\cdot,\tau)\|_{L^\infty}\leq C\epsilon^{1/9-a/2}|\ln\epsilon|^{-1/2}\leq\epsilon^a,$$

and lastly,

$$\sup_{t\geq l}\|w(\cdot,\tau)\|\leq C_l\epsilon^a.$$

Thus the proof of Theorem 1 is completed.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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References

1. S. Chapman, T.G. Cowling, The mathematical theory of nonuniform gases, 3rd edn, (Cambridge University Press, 1990).

2. M. Chen, Global strong solutions for the viscous, micropolar, compressible flow, J. Partial Differ. Equ., 24 (2011), pp. 158–164.

3. M. Chen, B. Huang, and J. Zhang, Blowup criterion for the three-dimensional equations of compressible viscous micropolar fluids with vacuum, Nonlinear Anal., 79 (2013), pp. 1–11.

4. M. CHEN, X. XU, AND J. ZHANG, Global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum, Commun. Math. Sci., 13 (2015), pp. 225–247.

5. H. CUI AND H. YIN, Stability of the composite wave for the inflow problem on the micropolar fluid model, Commun. Pure Appl. Anal., 16 (2017), pp. 1265–1292.

6. I. DRAŽIĆ AND N. MUJAKOVIĆ, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: a local existence theorem, Bound. Value Probl., (2012), pp. 2012:69, 25.

7. ——, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: a global existence theorem, Bound. Value Probl., (2015), pp. 2015:98, 21.

8. ——, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: large time behavior of the solution, J. Math. Anal. Appl., 431 (2015), pp. 545–568.

9. I. DRAŽIĆ, L. SIMČIĆ, AND N. MUJAKOVIĆ, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: regularity of the solution, J. Math. Anal. Appl., 438 (2016), pp. 162–183.

10. R. DUAN, Global solutions for a one-dimensional compressible micropolar fluid model with zero heat conductivity, J. Math. Anal. Appl., 463 (2018), pp. 477–495.

11. A. C. ERINGEN, Theory of micropolar fluids, J. Math. Mech., 16 (1966), pp. 1–18.

12. G. GONG,Zero dissipation limit to rarefaction wave with vacuum for the one-dimensional nonisentropic micropolar equations, preprint.

13. G. GONG, L. ZHANG, Asymptotic stability of planar rarefaction wave to 3D micropolar equations, J. Math. Anal. Appl. 485 (2020), no. 2, 123819, 19 pp.

14. F. HUANG, M. LI, AND Y. WANG, Zero dissipation limit to rarefaction wave with vacuum for one-dimensional compressible Navier-Stokes equations, SIAM J. Math. Anal., 44 (2012), pp. 1742–1759.

15. S. JIANG, G. NI, W. SUN, Vanishing viscosity limit to rarefaction waves for the Navier-Stokes equations of one-dimensional compressible heat-conducting fluids, SIAM J. Math. Anal. 38 (2006), no. 2, pp. 368–384.

16. J. JIN AND R. DUAN, Stability of rarefaction waves for 1-D compressible viscous micropolar fluid model, J. Math. Anal. Appl., 450 (2017), pp. 1123–1143.

17. Q. JIU, Y. WANG, Z. XIN, Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity, SIAM J. Math. Anal. 45 (2013), no. 5, pp. 3194–3228.

18. M. LI, T. WANG, AND Y. WANG, The limit to rarefaction wave with vacuum for 1D compressible fluids with temperature-dependent transport coefficients, Anal. Appl. (Singap.), 13 (2015), pp. 555–589.

19. M.-J. LI AND T. WANG, Zero dissipation limit to rarefaction wave with vacuum for one-dimensional full compressible Navier-Stokes equations, Commun. Math. Sci., 12 (2014), pp. 1135–1154.

20. S. MA,Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier-Stokes equations, J. Differential Equations 248 (2010), no. 1, pp. 95–110.

21. A. MATSUMURA AND K. NISHIHARA, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math., 3 (1986), pp. 1–13.

22. N. MUJAKOVIĆ, One-dimensional flow of a compressible viscous micropolar fluid: a global existence theorem, Glas. Mat. Ser. III, 33(53) (1998), pp. 199–208.

23. ——, One-dimensional flow of a compressible viscous micropolar fluid: a local existence theorem, Glas. Mat. Ser. III, 33(53) (1998), pp. 71–91.

24. N. MUJAKOVIĆ AND I. DRAŽIĆ, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: uniqueness of a generalized solution, Bound. Value Probl., (2014), pp. 2014:226, 17.

25. J. SMOLLER, Shock waves and reaction-diffusion equations, vol. 258 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, New York, second ed., 1994.

26. J. SU, Global existence and low Mach number limit to a 3D compressible micropolar fluids model in a bounded domain, Discrete Contin. Dyn. Syst., 37 (2017), pp. 3423–3434.

27. ——, Low Mach number limit of a compressible micropolar fluid model, Nonlinear Anal. Real World Appl., 38 (2017), pp. 21–34.

28. L. WAN AND L. ZHANG, Global solutions to the micropolar compressible flow with constant coefficients and vacuum, Nonlinear Anal. Real World Appl. 51 (2020), pp. 76N10 (35Q35)

29. Y. WANG, Zero dissipation limit of the compressible heat-conducting Navier-Stokes equations in the presence of the shock, Acta Math. Sci. Ser. B (Engl. Ed.) 28 (2008), no. 4, pp. 727–748.

30. Z. P. XIN, Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases, Comm. Pure Appl. Math., 46 (1993), pp. 621–665.

31. Z. XIN, H. ZENG, Convergence to rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations, J. Differential Equations 249 (2010), no. 4, pp. 827–871.

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