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Numerical Investigation of the Fractal Mobile/Immobile Transport Model with Caputo and Caputo-Fabrizio Fractional Derivatives using Finite difference/Spectral Approximations

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This paper discusses a spectral collocation method for numerically solving linear and nonlinear fractal Mobile/Immobile transport model with Caputo and Caputo-Fabrizio fractional derivatives. In the time direction, a finite difference scheme is used to approximate the differential term. Also, for space discretization, we apply the Chebyshev-spectral method. The unconditional stability and convergence of the proposed method are investigated, which provides the theoretical basis of the proposed method for solving the considered equation. Finally, some numerical experiments are considered to examine the efficiency and applicability of it in the sense of accuracy and convergence ratio. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: Caputo Fractional Derivative; Caputo-Fabrizio Fractional Derivative; Fractal Mobile-Immobile Transport Model; Spectral Approximation; stability Analysis; Error Analysis

1. Introduction

Fractional calculus investigates integrals and derivatives of non-integer order and is a classical mathematical field as old as calculus itself [1]. Earlier, fractional calculus was considered as pure mathematics, but now the situation has changed dramatically and fractional calculus become an attractive and important topic among engineers and applied scientists. It is a useful tool for the description of memory and heredity effects [2]. In recent years, the application of fractional-order derivatives has become popular due to its non-locality property, which is an essential property of many complex systems. Various applications have been used in the modeling of different phenomena such as viscoelasticity, nanotechnology, financial modeling, random walk, anomalous transport, control theory of dynamical systems, and biological modeling. This calculus involves different definitions of the fractional operators as well as the Riemann- Liouville fractional derivative (**R-L-FD**), Caputo fractional derivative (**C-FD**), Riesz fractional derivative (**R-FD**), Grunwald-Letnikov fractional derivative (**G-R-FD**), Atangana-Beleanu derivative (**A-BD**) [3, 4]. Recently, the authors of [5] presented a new definition for a fractional derivative without a singular kernel, which is named as Caputo-Fabrizio fractional derivative (**C-F-FD**). Indeed, the fractional models with a singular kernel can not describe as the fluctuations of different scales and material heterogeneities. But, models with **C-F-FD** can describe them. More fully-described work on physical and engineering processes with utilization of fractional order derivatives can be found in [6, 7, 8, 9, 10, 11]. Many phenomena in physics, chemistry, finance, fluid mechanics, and other sciences can be described successfully by fractional models using the fractional calculus [12, 13]. Doungmo Goufo and et al. [4] presented comparative analysis between differential fractional operators for solving the nonlinear Kaup-Kupershmidt equation, so that operators include the **A-BD** and **C-F-FD** which respectively follow the Mittag-Leffler law and the exponential law. In [14], the authors proposed an analytical method to solve systems of the nonlinear fractional differential equations. El-Ajou and et al. in [15] solved the time-fractional nonlinear dispersive partial differential equations in the sense of conformable fractional derivative consisting the time-fractional nonlinear dispersive Boussinesq, time-fractional nonlinear dispersive Klein-Gordon and time-fractional nonlinear dispersive $B(2, 1, 1)$ partial differential equations. Kumar and et al. in [16] presented a comparative study of the modified analytical methods based on of residual power series and auxiliary parameters approaches to solve the

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time-fractional Newell-Whitehead-Segel equations. Nonlinear dynamic and solution of a heat flux model presented by a partial differential equation were analyzed in [17]. Atangana and Baleanu [18] proposed a new fractional derivative with non-local and non-singular kernel for solving fractional heat in material with different scales and also those with heterogeneous media. Tateishi and et al. [19] solved the fractional diffusion equation without external forces and subjected to the free diffusion boundary conditions. In [20] the authors proposed fractal-fractional integrals and derivatives which predict the chaotic behavior of some attractors from applied mathematics. The concept of fractional integrals and derivatives based on of the exponential and Mittag-Leffler laws, are presented in [21, 22]. Furthermore fundamental differences among the exponential decay, power law, Mittag-Leffler law and their possible applications in natural phenomena are discussed. Although many important works have been presented on the theoretical analysis of fractional equations, the obtained solutions for most of them are not explicit. Therefore, many scholars have proposed several numerical investigations based on stability and convergence analysis [23, 24]. A variety of numerical methods have been proposed for fractional differential equations [25, 26, 27].

The spectral methods were initially proposed for computations in fluid dynamics and were promoted generally by meteorologists to investigate global weather modeling, and by fluid dynamicists to study isotropic turbulence. The spectral method also can present an approximation to the solution of a differential equation using a truncated series of smooth basis functions. The critical elements of the spectral methods are the basis functions and the test functions. The basis functions are used to approximate the solution to a finite series of smooth basis functions. One of the aspects which individuates spectral methods compared to finite element method is the choice of basis functions. These functions for spectral methods are infinitely differentiable global functions, while basis functions for finite element method are only local. The widely-used basis functions include Fourier series, Chebychev polynomials and Legendre polynomials. Fourier series are often used in the approximation of periodic functions, while Chebychev polynomials and Legendre polynomials are used for non-periodic functions to avoid the Gibbs phenomenon. The outstanding advantage of the spectral methods is that when the solution is smooth enough, the expansion coefficients decay faster than any polynomial order. Then, only a few terms are enough to reach the acceptable accuracy, which Would be preferable to studying problems with smooth enough solutions. Recently, spectral methods have been a well-known class of approximation methods for the solution of partial differential equations [28]. The aim of [29] is a spectrally formulated finite element approach for solving elastic waves in carbon nanotubes (CNT), where the frequency content of the new signal is at terahertz level. Authors of [30] proposed a Lagrange-Galerkin spectral element method for obtaining the approximate solution for the two-dimensional shallow water equations. Authors of [31] have presented a spectral element method on the basis of Gauss-Lobatto-Legendre quadrature formulas, and finite difference Newmark's explicit time advancing schemes for solving acoustic wave equation. A numerical spectral method for the time-fractional subdiffusion equation with second-order accuracy is presented in [32]. The aim of [33, 34] is to propose the spectral methods for the pricing of European options.

A fractal mobile/immobile transport for solute transport assumes power-law waiting times in the immobile zone, leading to a time-fractional derivative in the mobile/immobile transport model. The **FM/IT** model describes a extensive family of problems, including heat diffusion and ocean acoustic propagation in critical physical phenomena that behave essentially like heat diffusing through a solid [35]. To approximate the **FM/IT** model, significant progress has been made. In this paper, we present a spectral method to compute the approximate solution for **FM/IT** model with **C-FD** and **C-F-FD** [36, 37, 38, 39]:

FM/IT model with C-FD:

Case I: Linear **FM/IT** model with **C-FD**:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^C\partial_t^\alpha \mathcal{U}(x,t) = \gamma_1 \partial_x^2 \mathcal{U}(x,t) - \gamma_2 \mathcal{U}(x,t) + f(x,t), \tag{1}$$

Case II: Nonlinear **FM/IT** model with **C-FD**:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^C\partial_t^\alpha \mathcal{U}(x,t) = \gamma \partial_x^2 \mathcal{U}(x,t) + \mathcal{Q}(\mathcal{U}) + f(x,t), \tag{2}$$

where $(x,t) \in \Omega \times (0,T]$, $\Omega = (-1,1)$, $\mathcal{U} = \mathcal{U}(x,t)$ is a sufficiently differentiable function in $\overline{\Omega} \times [0,T]$ and the term $\mathcal{Q}(\mathcal{U})$ satisfies the following conditions:

- There exists a positive constant c such that $|\mathcal{Q}(\mathcal{U})| \leq c|\mathcal{U}|$,
- There exists a positive constant c such that $|\mathcal{Q}_{\mathcal{U}}(\mathcal{U})| \leq c$.

Also and the time-fractional derivative ${}_0^C\partial_t^\alpha \mathcal{U}(x,t)$ is the **C-FD** defined by

$${}_0^C\partial_t^\alpha \mathcal{U}(x,t) := ({}_0I_t^{1-\alpha} \frac{\partial \mathcal{U}}{\partial t})(x,t), \ 0 < \alpha < 1,$$

in which ${}_0I_t$ being the Riemann-Liouville fractional integral $({}_0I_t^\alpha \mathcal{U})(x,t) = \int_0^t \mathcal{U}(x,s)\nu_\alpha(t-s)ds$ with $\nu_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

FM/IT model with C-F-FD:

Case I: Linear **FM/IT** model with **C-F-FD**:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^{CF}\partial_t^\alpha \mathcal{U}(x,t) = \gamma_1 \partial_x^2 \mathcal{U}(x,t) - \gamma_2 \mathcal{U}(x,t) + f(x,t), \tag{3}$$

Case II: Nonlinear **FM/IT** model with **C-F-FD**:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^{CF}\partial_t^\alpha \mathcal{U}(x,t) = \gamma \partial_x^2 \mathcal{U}(x,t) + \mathcal{Q}(\mathcal{U}) + f(x,t), \tag{4}$$

where the time-fractional derivative ${}_0^{CF}\partial_t^{\alpha}\mathcal{U}(x,t)$ is the **C-F-FD** defined by

$${}_0^{CF}\partial_t^{\alpha}\mathcal{U}(x,t) := \int_0^t \frac{\partial \mathcal{U}(x,s)}{\partial s}\vartheta_\alpha(t-s)ds,\ 0<\alpha<1,$$

in which $\vartheta_\alpha(t) := \frac{\exp\left(-\frac{\alpha}{1-\alpha}(t)\right)}{1-\alpha}$.

The rest of this paper is organized as follows. In Sections 2 and 3, we present computational approaches to construct a numerical solution for fractal Mobile/Immobile transport model with **C-FD** and **C-F-FD**. We prove the convergence and the stability of the method in this section. Some test problems are presented, and the results are shown in Section 4, and we discuss the numerical performance of our method. Finally, in Section 5, some concluding remarks are presented.

2. FM/IT Model With C-FD

2.1. Linear FM/IT Model With C-FD

2.1.1. Discretization of Caputo Derivative and Semi-Discrete Scheme In this subsection, we deal with the linear **FM/IT** model with **C-FD**. For (1), the initial condition:

$$\mathcal{U}(x,t)|_{t=0} = h(x),\ x\in\overline{\Omega}, \tag{5}$$

and the Dirichlet boundary conditions:

$$\mathcal{U}(x,t)|_{x\in\partial\Omega} = 0,\ t>0, \tag{6}$$

is considered.
For discretization of time variable, let $t_k := k\delta t,\ k=0,1,\dots,N$ be an equidistant partition of $[0,T]$, where $\delta t = \frac{T}{N}$. We analogize the time-fractional derivative term by using the finite difference scheme:

$${}_0^{C}\partial_t^{\alpha}\mathcal{U}^{k+1}(x) = \begin{cases} c_{\alpha,\delta t}[(\mathcal{U}^{k+1}(x)-\mathcal{U}^{k}(x))+\sum_{j=1}^{k} d_{\alpha,j}(\mathcal{U}^{k+1-j}(x)-\mathcal{U}^{k-j}(x))], & k\geq 1\\ c_{\alpha,\delta t}(\mathcal{U}^{1}(x)-\mathcal{U}^{0}(x)), & k=0,\end{cases} + r_{1,\mathcal{U}}^{k+1}(x), \tag{7}$$

where $c_{\alpha,\delta t} = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}$ and $d_{\alpha,j} = (j+1)^{1-\alpha} - j^{1-\alpha},\ (j=1,2,\cdots,k)$.

Theorem 1 *([41]) For any $0<\alpha<1$, the coefficients of $d_{\alpha,j}, j=1,2,\cdots$ satisfies the following properties*

- $d_{\alpha,0}=1,\ d_{\alpha,j}, j=0,1,2,\cdots;$
- $d_{\alpha,k}\to 0$ *as* $k\to\infty$;
- $d_{\alpha,j} > d_{\alpha,j+1}, j=0,1,2,\cdots;$
- $d_{\alpha,k}^{-1} \leq \frac{(k+1)^{\alpha}}{1-\alpha}$;
- $\sum_{j=0}^{k-1}(d_{\alpha,j+1}-d_{\alpha,j})+d_{\alpha,k} = d_{\alpha,0}$;
- $\sum_{j=0}^{k} d_{\alpha,j}\delta t^{\alpha} \leq (k+1)^{\alpha}\delta t^{\alpha}$.

Theorem 2 *([41]) For any $0<\alpha<1$, it holds*

$$|r_{1,\mathcal{U}}^{k+1}(x)| \leq \frac{c}{\Gamma(2-\alpha)}\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^{2-\alpha},\ \ -1\leq k\leq N-1,\ \forall x\in\Omega,$$

where c is independent of δt.

Also, the first order temporal derivative can be approximated as follows

$$\frac{\partial \mathcal{U}^{k+1}(x)}{\partial t} = \frac{\mathcal{U}^{k+1}(x)-\mathcal{U}^{k}(x)}{\delta t} + r_{2,\mathcal{U}}^{k+1}(x), \tag{8}$$

where the truncation error $r_{2,\mathcal{U}}^{k+1}(x)$ satisfy $|r_{2,\mathcal{U}}^{k+1}(x)| \leq c\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t$, in which c is independent δt.
Substituting (7) and (8) into (1), we obtain

$$\mathcal{K}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) - c_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2\mathcal{U}^{k+1}(x) = \mathcal{P}_t^{l,\alpha}\mathcal{U}^{k}(x) + F^{k+1}(x) + \mathcal{R}_{\mathcal{U}}^{k+1}(x),\ x\in\overline{\Omega}, \tag{9}$$

where

$$\mathcal{P}_t^{l,\alpha}\mathcal{U}^{k}(x) = \begin{cases} (\mathcal{K}_{2,\alpha,\delta t}+\lambda_2 d_{\alpha,1})\mathcal{U}^{0}(x), & k=0,\\ \mathcal{K}_{2,\alpha,\delta t}\mathcal{U}^{k}(x)+\lambda_2\sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\mathcal{U}^{k-j}(x)+\lambda_2 d_{\alpha,k}\mathcal{U}^{0}(x), & k\geq 1,\end{cases}$$

$$F^{k+1} = c_{\alpha,\delta t}^{-1}f(x,t_{k+1}),\ k=0,1,\cdots,N-2,\ \mathcal{U}^{0}(x) = h(x),$$

and

$$\beta_{\alpha,\delta t}=\frac{\Gamma(2-\alpha)}{\delta t^{1-\alpha}},\ \mathcal{K}_{1,\alpha,\delta t}=\lambda_1\beta_{\alpha,\delta t}+\lambda_2+\gamma_2 c_{\alpha,\delta t}^{-1},\ \mathcal{K}_{2,\alpha,\delta t}=\lambda_1\beta_{\alpha,\delta t}+\lambda_2(1-d_{\alpha,1}).$$

Furthermore the truncation error $\mathcal{R}_{\mathcal{U}}^{k+1}(x)$ satisfy

$$|\mathcal{R}_{\mathcal{U}}^{k+1}(x)|\le|\mathcal{R}_{1,\mathcal{U}}^{k+1}(x)|+|\mathcal{R}_{2,\mathcal{U}}^{k+1}(x)|\le c_\alpha\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^{1+\alpha},$$

in which $\mathcal{R}_{1,\mathcal{U}}^{k+1}(x)=\Gamma(2-\alpha)\delta t^{\alpha}r_{1,\mathcal{U}}^{k+1}(x)$ and $\mathcal{R}_{2,\mathcal{U}}^{k+1}(x)=\Gamma(2-\alpha)\delta t^{\alpha}r_{2,\mathcal{U}}^{k+1}(x)$.
Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (1) and (5)-(6), which is given by:
Scheme L-I: Given $u^0=h(x)$ and find u^{k+1} $(k=0,1,2,\cdots,N-2)$, such that

$$\begin{cases}\mathcal{K}_{1,\alpha,\delta t}u^{k+1}(x)-c_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2u^{k+1}(x)=\mathcal{P}_t^{l,\alpha}u^k(x)+F^{k+1}(x),x\in\overline{\Omega},\\ u^{k+1}|_{x\in\partial\Omega}=0,-1\le k\le N-1,\end{cases}\tag{10}$$

2.1.2. Spectral Approximation to Semi-Discrete Problem (10) Consider the Hilbert space of μ-measurable $L^2((-1,1),d\mu(x))$, where $d\mu(x)=w(x)dx=(1-x^2)^{-\frac12}dx$. The Hilbert space $L^2((-1,1),d\mu(x))$ equipped with inner product

$$\langle u,v\rangle_{0,\omega}=\int_{-1}^{1}u(x)v(x)(1-x^2)^{-\frac12}d(x).$$

Theorem 3 *([40]) Let $\mathbb{P}_M$ denote the set of polynomials of degree$\le M$. If $\mathbf{B}_M$ be a sequence of orthogonal polynomials on $(-1,1)$ of degree $\le M$, i.e.,*

$$\mathrm{B}_M=\{u\in\mathbb{P}_M|\langle u,v\rangle_{0,\omega}=0,\ \forall v\in\mathbb{P}_{M-1}\},$$

then there exists a reproducing kernel $K_M:(-1,1)\times(-1,1)\to\mathbb{R}$ such that

$$u(x)=\langle u,K_M(x,.)\rangle_{0,\omega},\ \forall u\in\mathbb{P}_M,\forall x\in(-1,1),$$

$$0=\langle(x+1)u,K_M(-1,.)\rangle_{0,\omega}=\langle(1-x)u,K_M(1,.)\rangle_{0,\omega},\ \forall u\in\mathbb{P}_{M-1}.$$

Let $\{T_M\}_{M\ge0}$ be the Chebyshev polynomials in $L^2((-1,1),d\mu(x))$ with $degree(P_M)=M$, we consider

$$q_M(x)=\frac{T_{M+2}(x)+c_MT_{M+1}(x)+d_MT_M(x)}{(1-x)(x+1)}\in\mathbb{P}_M,$$

where

$$\begin{aligned}c_M&=-\frac{[T_{M+2}(1)T_M(-1)+T_{M+2}(-1)T_M(1)]}{[T_M(-1)T_{M+1}(1)-T_M(1)T_{M+1}(-1)]},\\ d_M&=-\frac{[T_{M+2}(T_{M+1}(-1)+T_{M+2}(-1)T_{M+1}(1)]}{[T_M(1)T_{M+1}(-1)-T_M(-1)T_{M+1}(1)]}.\end{aligned}$$

Hence, $\{q_M\}_{M\ge0}$ is a sequence of orthogonal polynomials in $L^2((-1,1),d\widetilde{\mu}(x))$ equipped with inner product

$$\langle u,v\rangle_{2,\widetilde{\omega}}=\int_{-1}^{1}u(x)v(x)d\widetilde{\mu}(x),\ d\widetilde{\mu}(x)=\widetilde{\omega}(x)dx=(1-x)(x+1)(1-x^2)^{-\frac12}dx.$$

It is well known [40] that

$$K_{M-2}(x,y)=\sum_{m=0}^{M-2}\frac{q_m(x)q_m(y)}{\|q_m\|_{2,\widetilde{\omega}}^2}=\frac{k_M(q_{M-1}(x)q_{M-2}(y)-q_{M-2}(x)q_{M-1}(y))}{k_{M+1}\|q_{M-2}\|_{0,\widetilde{\omega}}^2(x-y)},\ x\ne y,$$

where $K_{M-2}(.,y)\in\mathbb{P}_{M-2}$ and $-k_{M+1}<0$ is the leading coefficient of x^{M+1} in $(x+1)(1-x)q_{M-1}(x)$.
We also have

$$K_{M-2}(x,x)=\sum_{m=0}^{M-2}\frac{q_m^2(x)}{\|q_m\|_{0,\widetilde{\omega}}^2}=\frac{k_M(q'_{M-1}(x)q_{M-2}(x)-q'_{M-2}(x)q_{M-1}(x))}{k_{M+1}\|q_{M-2}\|_{0,\widetilde{\omega}}^2}.$$

Suppose that $\{z_j\}_{j=1}^{M-1}$ denote the $M-1$ simple zero points of q_{M-1} on $(-1,1)$, then we have

$$K_{M-2}(z_i,z_j)=\sum_{m=0}^{M-2}\frac{q_m(z_i)q_m(z_j)}{\|q_m\|_{0,\widetilde{\omega}}^2}=\begin{cases}0, & i\ne j\\ \widetilde{\omega}_i^{-1}=\frac{k_Mq'_{M-1}(z_i)q_{M-2}(z_i)}{k_{M+1}\|q_{M-2}\|_{0,\widetilde{\omega}}}, & i=j.\end{cases}$$

Let $\{z_j\}_{j=0}^{M}$ denote the $M+1$ simple zero points of $(x+1)(1-x)q_{M-1}$ on $[-1,1]$, it is shown in [40] that there exists a unique set of quadrature weights $\{\omega_j\}_{j=0}^{M}$ such that we have

$$\int_{-1}^{1} u(x)\frac{1}{\sqrt{1-x^2}}dx = \sum_{j=0}^{M}\omega_j u(z_j),\ \forall u\in\mathbb{P}_{2M-1}, z_j=-\cos\frac{\pi j}{M}, j=0,\cdots,M.$$

where

$$\omega_j=\frac{\pi}{\sigma_j M}, j=0,1,\cdots,M,$$

where

$$\sigma_j=\begin{cases}2, & j=0,M,\\ 1, & 1\le j\le M-1,\end{cases}$$

Now, we will give the representation of numerical solution to semi-discrete problem (10) in the space $\mathbb{P}_M$. Given $u_M^0=I_M^c u^0$ and find $u_M^{k+1}\in\mathbb{P}_M$ $(k=0,1,2,\cdots,N-1)$, such that

$$\begin{cases}\mathcal{K}_{1,\alpha,\delta t}u_M^{k+1}(z_i)-c_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2 u_M^{k+1}(z_i)=\mathcal{P}_t^{l,\alpha}u_M^k(z_i)+F^{k+1}(z_i), 1\le i\le M-1,\\ u_M^{k+1}(z_i)=0,\ i=0,M,\ \ -1\le k\le N-1,\end{cases} \tag{11}$$

where

$$\mathcal{P}_t^{l,\alpha}u_M^k(z_i)=\begin{cases}(\mathcal{K}_{2,\alpha,\delta t}+\lambda_2 d_{\alpha,1})u_M^0(z_i), & k=0,\\ \mathcal{K}_{2,\alpha,\delta t}u_M^k(z_i)+\lambda_2\sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})u_M^{k-j}(z_i)+\lambda_2 d_{\alpha,k}u_M^0(z_i), & k\ge 1,\end{cases}$$

and $I_M^c : C[a,b]\to\mathbb{P}_N$ is the interpolation operator associated with $\{z_i,\omega_i\}_{j=0}^{M}$ such that

$$(I_M^c u)(z_i)=u(z_i),\ \ i=0,1,2,\cdots,M.$$

An approximant u_M^k to u^k can be obtained by calculating a truncated series based on

$$\mathbb{P}_M=span\{\phi_j(x), j=0,1,\cdots,M\},\ \phi_j(x)=\frac{(x+1)(1-x)q_{M-1}(x)}{((x+1)(1-x)q_{M-1}(x))'|_{x=z_j}(x-z_j)}$$

as

$$u^k(x)\approx u_M^k(x):=\Phi(x)\{\mathrm{v}\}^k,$$

where

$$\Phi(x)=(\phi_0(x),\phi_1(x),\cdots,\phi_M(x)),$$

and

$$\{\mathrm{v}\}^k=(v_0^k,v_1^k,\cdots,v_M^k)^T.$$

Also $\frac{d^m}{dx^m}\Phi(x)$ can be expressed in the following matrix form

$$\frac{d^m}{dx^m}\Phi(x)=\Phi(x)\mathrm{D}^m,\ \ m\ge 1,$$

where

$$\mathrm{D}=[\mathrm{D}_{ij}]=[\phi_j'(z_i)], i,j=0,1,\cdots,M,\ \ \mathrm{D}^m=\underbrace{\mathrm{D}\mathrm{D}\cdots\mathrm{D}}_{m}.$$

The entries of the first-order differentiation matrix **D** are determined by

$$\mathrm{D}_{ij}\ =\ \begin{cases}\frac{((x-a)(b-x)q_{M-1}(x))'|_{x=z_i}}{((x-a)(b-x)q_{M-1}(x))'|_{x=z_j}(z_i-z_j)}, & i\neq j,\\ \frac{((x-a)(b-x)q_{M-1}(x))''|_{x=z_i}}{2((x-a)(b-x)q_{M-1}(x))''|_{x=z_i}}, & i=j.\end{cases} = \begin{cases}-\frac{2M^2+1}{6}, & i=j=0,\\ \frac{\sigma_i}{\sigma_j}\frac{(-1)^{i+j}}{z_i-z_j}, & i\neq j,\ 0\le i,j\le M,\\ -\frac{z_i}{2(1-z_k^2)}, & i=j,\ 1\le i,j\le M-1,\\ \frac{2M^2+1}{6}, & i=j=M,\end{cases}$$

where

$$\sigma_j=\begin{cases}2, & j=0,M,\\ 1, & 1\le j\le M-1.\end{cases}$$

Then, we approximate $\partial_x^m u_M^k$ by

$$\partial_x^m u_M^k(x):=\Phi(x)\mathrm{D}^m\{\mathrm{v}\}^k,\ \ m\ge 1.$$

Thus, we have:

$$\partial_x^m u_M^k(z_i)=\sum_{j=0}^{M}(\mathbf{D}^m)_{ij}v_j^k, m\ge 1,\ \ 1\le i\le M-1.$$

Therefore, it follows that

$$\begin{cases} \sum_{j=0}^{M}\left[\mathcal{K}_{1,\alpha,\delta t}\delta_{ij}-c_{\alpha,\delta t}^{-1}\gamma_1(\mathrm{D}^2)_{ij}\right]v_j^{k+1}=\digamma^{k+1}(z_i),\ 1\le i\le M-1,\\ \Phi(z_0)\{\mathrm{v}\}^{k+1}=\Phi(z_M)\{\mathrm{v}\}^{k+1}=0. \end{cases} \tag{12}$$

where

$$\digamma^{k+1}(z_i)=\mathcal{P}_t^{l,\alpha}u_M^k(z_i)+F^{k+1}(z_i),1\le i\le M-1.$$

Let us denote

$$\begin{aligned}
&(\mathrm{A})_{ij}=\mathcal{K}_{1,\alpha,\delta t}\delta_{ij}-c_{\alpha,\delta t}^{-1}\gamma_1(\mathrm{D}^2)_{ij},\ 1\le i\le M-1,\ 0\le j\le M,\\
&(\mathrm{A})_{0j}=\delta_{0j},\ (\mathrm{A})_{Mj}=\delta_{Mj},\ 0\le j\le M,\\
&\{\mathrm{b}\}^{k+1}=(0,\digamma^{k+1}(z_1),\digamma^{k+1}(z_2),\cdots,\digamma^{k+1}(z_{M-1}),0)^T,\\
&\{\mathrm{v}\}^{k+1}=(v_0^{k+1},v_1^{k+1},\cdots,v_M^{k+1})^T,
\end{aligned}$$

then, the linear system (12) reduces to

$$\mathbf{A}\{\mathbf{v}\}^{k+1}=\{\mathbf{b}\}^{k+1},\ k=0,1,\cdots,N-2.$$

We define the corresponding discrete inner product as

$$\langle u,v\rangle_M=\sum_{j=0}^{M}\omega_j u(z_j)v(z_j),$$

which induces the norm $\|u\|_M=(\langle u,u\rangle_{M,\omega})^{\frac{1}{2}}$ and satisfies

$$\langle u,v\rangle_M=\langle u,v\rangle_{0,\omega},\ \forall u,v: u.v\in\mathbb{P}_{2M-1}.$$

Consider the weight Sobolov space $H^r((-1,1),d\mu(x))$ as

$$H^r((-1,1),d\mu(x))=\{u\in L^2((-1,1),d\mu(x)):\|u\|_{r,\omega}=(\sum_{j=0}^{r}\|\partial_x u\|_{0,\omega}^2)^{\frac{1}{2}}<\infty\},$$

Moreover, we set $H_0^1((-1,1),d\mu(x))=\{u\in L^2((-1,1),d\mu(x)):\partial_x u\in L^2((-1,1),d\mu(x)),u(-1)=u(1)=0\}$. We also introduce the the bilinear form over $H_0^1((-1,1),d\mu(x))$ as

$$a_\omega\langle u,v\rangle=\langle\partial_x u,\omega^{-1}\partial_x(v\omega)\rangle_{0,\omega}=\int_a^b\partial_x u\partial_x(v\omega)dx,\ \forall u,v\in H_0^1((-1,1),d\mu(x)).$$

Let us denote $\mathbb{X}_M=\{v_M|v_M\in\mathbb{P}_M,\ v_M(z_0)=v_M(z_M)=0.\}$, we can reformulate the scheme (11) as the following:
S-A(L-I): Find the spectral approximation $u_M^{k+1}\in\mathbb{X}_M$ $(k=0,1,2,\cdots,N-1)$, such that for all $v_M\in\mathbb{X}_M$:

$$\mathcal{K}_{1,\alpha,\delta t}\langle u_M^{k+1},v_M\rangle_M+c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle u_M^{k+1},v_M\rangle=\langle\mathcal{P}_t^{l,\alpha}u_M^k,v_M\rangle_M+\langle I_M^c F^{k+1},v_M\rangle_M. \tag{13}$$

2.1.3. Stability Analysis

Lemma 1 *([42])For any $u\in\mathbb{P}_M$, we have*

$$\|u\|_{0,\omega}\le\|u\|_M\le\sqrt{2}\|u\|_{0,\omega}.$$

Lemma 2 *([42]) If $u\in H_0^1((-1,1),d\mu(x))$, then there holds*

$$\|u\|_{0,\omega}\le c\|\partial_x u\|_{0,\omega},$$

where c is positive constant independent of u.

Lemma 3 *([42]) For any $u\in H_0^1((-1,1),d\mu(x))$, we have*

$$\begin{aligned}
&|a_\omega\langle u,u\rangle|\le c\|\partial_x u\|_{0,\omega},\\
&a_\omega\langle u,u\rangle\ge\frac{1}{4}\|\partial_x u\|_{0,\omega}^2,
\end{aligned}$$

where c is positive constant independent of u.

Lemma 4 *([43]) (Discrete Gronwall inequality) Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are nonnegative sequences and c is a nonnegative constant. If*

$$f_i \leq c + \sum_{j=0}^{i-1} g_j f_j, \;\; i \geq 0,$$

then

$$f_i \leq c \prod_{0\leq j\leq i-1} (1+g_j) \leq c e^{\sum_{j=0}^{i-1} g_j}, \;\; i \geq 0.$$

Lemma 5 *Let $u_M^{k+1} \in \mathbb{X}_M,\;\; k = 0, 1, \cdots, N-1$ be the solution of scheme (13). Then the following inequality is holds*

$$\|u_M^{k+1}\|_M^2 + \|\partial_x u_M^{k+1}\|_{0,\omega}^2 \leq C_{1,\alpha,\delta t}\left(\|u_M^0\|_M^2 + \|I_M^c F^{k+1}\|_M^2\right) e^{C_{2,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})},$$

where $\mathbf{C}_{1,\alpha,\delta t}$ and $\mathbf{C}_{2,\alpha,\delta t}$ are positive constants.

Proof 1 *Setting $v_M = u_M^{k+1}$, we get*

$$\mathcal{K}_{1,\alpha,\delta t}\langle u_M^{k+1}, u_M^{k+1}\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle u_M^{k+1}, u_M^{k+1}\rangle = \langle \mathcal{P}_t^{l,\alpha} u_M^k, u_M^{k+1}\rangle_M + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M, \tag{14}$$

where

$$\langle \mathcal{P}_t^{l,\alpha} u_M^k, u_M^{k+1}\rangle_M = \mathcal{K}_{2,\alpha,\delta t}\langle u_M^k, u_M^{k+1}\rangle_M + \lambda_2 \sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\langle u_M^{k-j}, u_M^{k+1}\rangle_M + \lambda_2 d_{\alpha,k}\langle u_M^0, u_M^{k+1}\rangle_M.$$

Using the following inequality

$$ab \leq \frac{1}{2\Theta^2}a^2 + \frac{\Theta^2}{2}b^2, \;\; \forall \Theta \neq 0,$$

we have

$$\begin{aligned}
&\langle \mathcal{P}_t^{l,\alpha} u_M^k, u_M^{k+1}\rangle_M \leq \mathcal{K}_{2,\alpha,\delta t}(\|u_M^k\|_M^2 + \frac{1}{4}\|u_M^{k+1}\|_M^2) + \lambda_2 \sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})(\|u_M^{k-j}\|_M^2 \\
&+\frac{1}{4}\|u_M^{k+1}\|_M^2) + \lambda_2 d_{\alpha,k}(\|u_M^0\|_M^2 + \frac{1}{4}\|u_M^{k+1}\|_M^2) \leq \lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_M^2 + \frac{\mathcal{K}_{2,\alpha,\delta t}}{4}\|u_M^{k+1}\|_M^2 \\
&+\lambda_2 \sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \frac{\lambda_2}{4}(d_{\alpha,1}-d_{\alpha,k})\|u_M^{k+1}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \frac{\lambda_2 d_{\alpha,k}}{4}\|u_M^{k+1}\|_M^2.
\end{aligned} \tag{15}$$

Noting Lemma 3 and using (14) and (15), we have

$$\begin{aligned}
\mathcal{K}_{1,\alpha,\delta t}\|u_M^{k+1}\|_M^2 &+ \frac{c_{\alpha,\delta t}^{-1}\gamma_1}{4}\|\partial_x u_M^{k+1}\|_{0,\omega}^2 \leq \lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_M^2 + \frac{\mathcal{K}_{2,\alpha,\delta t}}{4}\|u_M^{k+1}\|_M^2 \\
&+ \lambda_2 \sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \frac{\lambda_2}{4}(d_{\alpha,1}-d_{\alpha,k})\|u_M^{k+1}\|_M^2 \\
&+ \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \frac{\lambda_2 d_{\alpha,k}}{4}\|u_M^{k+1}\|_M^2 + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M,
\end{aligned}$$

Using Lemmas 1 and 2, it follows that

$$\begin{aligned}
&\left(\mathcal{K}_{1,\alpha,\delta t} - \frac{\mathcal{K}_{2,\alpha,\delta t}}{4} - \frac{\lambda_2}{4}(d_{\alpha,1}-d_{\alpha,k}) - \frac{\lambda_2 d_{\alpha,k}}{4}\right)\|u_M^{k+1}\|_M^2 + \frac{c_{\alpha,\delta t}^{-1}\gamma_1}{4}\|\partial_x u_M^{k+1}\|_{0,\omega}^2 \\
&\leq \lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_M^2 + \lambda_2 \sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 \\
&+\langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M \leq \sqrt{2}\lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_{0,\omega}^2 + \lambda_2 \sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 \\
&+\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M \leq \sqrt{2}c\lambda_1\beta_{\alpha,\delta t}\|\partial_x u_M^k\|_{0,\omega}^2 \\
&+\lambda_2 \sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M
\end{aligned} \tag{16}$$

We know

$$\langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M \leq \frac{1}{3(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2 + \frac{3}{4}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)\|u_M^{k+1}\|_M^2. \tag{17}$$

In view of (16) and (17), we have

$$\gamma_2 c_{\alpha,\delta t}^{-1}\|u_M^{k+1}\|_M^2+\frac{c_{\alpha,\delta t}^{-1}\gamma_1}{4}\|\partial_x u_M^{k+1}\|_{0,\omega}^2\le\sqrt{2}c\lambda_1\beta_{\alpha,\delta t}\|\partial_x u_M^k\|_{0,\omega}^2$$
$$+\lambda_2\sum_{j=1}^{k}(d_{\alpha,k-j}-d_{\alpha,k-j+1})\|u_M^j\|_M^2+\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{1}{3(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2$$
$$\le\frac{\lambda_1\beta_{\alpha,\delta t}\sqrt{2}c}{1-d_{\alpha,1}}\sum_{j=1}^{k}(d_{\alpha,k-j}-d_{\alpha,k-j+1})\|\partial_x u_M^j\|_{0,\omega}^2+\lambda_2\sum_{j=1}^{k}(d_{\alpha,k-j}-d_{\alpha,k-j+1})\|u_M^j\|_M^2$$
$$+\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{1}{3(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2.$$

If

$$C_{1,\alpha,\delta t}=\min\{\gamma_2 c_{\alpha,\delta t}^{-1},\frac{c_{\alpha,\delta t}^{-1}\gamma_1}{4}\},$$
$$C_{2,\alpha,\delta t}=\max\{\lambda_2,\frac{\lambda_1\beta_{\alpha,\delta t}\sqrt{2}c}{1-d_{\alpha,1}}\},$$

we can get the following inequality

$$\begin{aligned}\|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2 &\le \sum_{j=1}^{k}\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(d_{\alpha,k-j}-d_{\alpha,k-j+1})(\|u_M^j\|_M^2+\|\partial_x u_M^j\|_{0,\omega}^2)\\ &+ C_{1,\alpha,\delta t}^{-1}\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2.\end{aligned}$$

Noting Lemma 4, we have

$$\|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2\le\left(C_{1,\alpha,\delta t}^{-1}\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{\|I_M^c F^{k+1}\|_M^2}{3C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\right)e^{\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(d_{\alpha,0}-d_{\alpha,k})},$$

Therefore

$$\|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2\le \mathsf{C}_{1,\alpha,\delta t}\left(\|u_M^0\|_M^2+\|I_M^c F^{k+1}\|_M^2\right)e^{\mathsf{C}_{2,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})},$$

where $\mathsf{C}_{1,\alpha,\delta t}=\max\{C_{1,\alpha,\delta t}^{-1}\lambda_2 d_{\alpha,k},\frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\}$ *and* $\mathsf{C}_{2,\alpha,\delta t}=\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}$.

Theorem 4 *Let* $u_M^{k+1}\in\mathbb{X}_M,\ k=0,1,\cdots,N-1$ *be the solution of scheme (13). Then the scheme (13) is unconditionally stable in the sense that for all* $\delta t>0$.

Proof 2 *Let* $\tilde{u}_M^{k+1}\in\mathbb{X}_M,\ k=0,1,\cdots,N-1$ *is the approximate solution of the scheme (13) with the initial condition* $\tilde{u}_M^0$*. Noting Lemma 8, we obtain*

$$\|u_M^{k+1}-\tilde{u}_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}-\partial_x\tilde{u}_M^{k+1}\|_{0,\omega}^2\le \mathsf{C}_{1,\alpha,\delta t}\|u_M^0-\tilde{u}_M^0\|_M^2 e^{\mathsf{C}_{2,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})}.$$

Therefore the following inequality is holds

$$\|u_M^{k+1}-\tilde{u}_M^{k+1}\|_M^2\le \mathsf{C}_{1,\alpha,\delta t}\|u_M^0-\tilde{u}_M^0\|_M^2 e^{\mathsf{C}_{2,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})}.$$

This completes the proof of Theorem 4.

2.1.4. Error Analysis

Lemma 6 *([44]) For any* $u\in H^r((-1,1),d\mu(x))$*, the following estimate holds*

$$\|u-I_M^c u\|_{s,\omega}\le cM^{2s-r}\|u\|_{r,\omega},\ \ r>\frac{1}{2},0\le s\le r, \tag{18}$$

where c is a positive constant.

Lemma 7 *([44]) Let* Π_M *be the orthogonal projection operator defined by*

$$\Pi_M:=L^2((-1,1),d\mu(x))\to\mathbb{P}_M, \tag{19}$$
$$\langle u-\Pi_M u,v\rangle_{0,\omega}=0,\ \ \forall v\in\mathbb{P}_M. \tag{20}$$

then the following estimate holds

$$\|u-\Pi_M u\|_{s,\omega}\le cM^{\frac{3s}{2}-r}\|u\|_{r,\omega},\ \ r\ge 0,\ \ 0\le s\le 1, \tag{21}$$

where c is a positive constant.

Lemma 8 *([44]) Let $\Pi_M^{1,0}$ be the orthogonal projection operator defined by*

$$\Pi_M^{1,0} := H_0^1((-1,1), d\mu(x)) \to \mathbb{X}_M, \tag{22}$$
$$a_\omega\langle \Pi_M^{1,0} u - u, v\rangle = 0, \ \forall v \in \mathbb{X}_M. \tag{23}$$

then the following estimate holds

$$\|u - \Pi_M^{1,0} u\|_{s,\omega} \leq c M^{s-r} \|u\|_{r,\omega}, \ r \geq 1, \ 0 \leq s \leq 1, \tag{24}$$

where c is a positive constant.

Lemma 9 *([44]) For any $u \in H^r((-1,1), d\mu(x))$, $r > \frac{1}{2}$, the following estimate holds*

$$|\langle E(u), v\rangle_{0,\omega}| = |\langle u, v\rangle_M - \langle u, v\rangle_{0,\omega}| \leq c\{\|u - \Pi_{M-1} u\|_{0,\omega} + \|u - I_M^c u\|_{0,\omega}\}\|v\|_{0,\omega}, \ \forall v \in \mathbb{P}_M, \tag{25}$$

where c is a positive constant.

Lemma 10 *For $0 < \alpha < 1$, $0 \leq k \leq N-1$, it holds*

$$\begin{aligned}\mathcal{K}_{1,\alpha,\delta t}\langle e_M^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle e_M^{k+1}, v_M\rangle &= \mathcal{K}_{2,\alpha,\delta t}\langle e_M^k, v_M\rangle_M + \lambda_2 \sum_{j=1}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\langle e_M^{k-j}, v_M\rangle_M \\ &+ \lambda_2 d_{\alpha,k}\langle e_M^0, v_M\rangle_M + \langle \delta^{k+1}, v_M\rangle_{0,\omega}, \ \forall v_M \in \mathbb{X}_M.\end{aligned} \tag{26}$$

where

$$e_M^{k+1} = u_M^{k+1} - \Pi_M^{1,0}\mathcal{U}^{k+1},$$

and

$$\langle \delta^{k+1}, v_M\rangle_{0,\omega} = \langle \delta_1^{k+1}, v_M\rangle_{0,\omega} + \langle \delta_2^{k+1}, v_M\rangle_{0,\omega} + \langle \delta_3^{k+1}, v_M\rangle_{0,\omega},$$

in which

$$\langle \delta_1^{k+1}, v_M\rangle_{0,\omega} = c_{\alpha,\delta t}^{-1}\langle (I_d - \Pi_M^{1,0})(\lambda_1 \partial_t \mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha \mathcal{U}^{k+1} + \gamma_2 \mathcal{U}^{k+1}), v_M\rangle_{0,\omega},$$

$$\langle \delta_2^{k+1}, v_M\rangle_{0,\omega} = -c_{\alpha,\delta t}^{-1}\langle E(\lambda_1 \partial_t \Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2 \Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega},$$

$$\langle \delta_3^{k+1}, v_M\rangle_{0,\omega} = \langle E(F^{k+1}), v_M\rangle_{0,\omega} + \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M,$$

and $r_{\Pi_M^{1,0}\mathcal{U}}^{k+1} = O(\delta t^{1+\alpha})$.

Proof 3 *We know $a_\omega\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle = a_\omega\langle \mathcal{U}^{k+1}, v_M\rangle$, therefore we can write*

$$c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle = -c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha \mathcal{U}^{k+1} + \gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega} + c_{\alpha,\delta t}^{-1}\langle f^{k+1}, v_M\rangle_{0,\omega}, \tag{27}$$

Furthermore, we have

$$\begin{aligned}&\mathcal{K}_{1,\alpha,\delta t}\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle_M - \langle \mathcal{P}_t^{l,\alpha}\Pi_M^{1,0}\mathcal{U}^k, v_M\rangle_M \\ &= c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_M - \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M,\end{aligned} \tag{28}$$

where $r_{\Pi_M^{1,0}\mathcal{U}}^{k+1} = O(\delta t^{1+\alpha})$.
Now, from (27) and (28), it easily conclude that

$$\begin{aligned}&\mathcal{K}_{1,\alpha,\delta t}\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle = \langle \mathcal{P}_t^{l,\alpha}\Pi_M^{1,0}\mathcal{U}^k, v_M\rangle_M \\ &-c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\mathcal{U}^{k+1} + \gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega} + \langle F^{k+1}, v_M\rangle_{0,\omega} \\ &+c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_M - \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M \\ &= \langle \mathcal{P}_t^{l,\alpha}\Pi_M^{1,0}\mathcal{U}^k, v_M\rangle_M - c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\mathcal{U}^{k+1} + \gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega} \\ &+c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega} \\ &-c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega} \\ &+c_{\alpha,\delta t}^{-1}\langle(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1} + \lambda_2\, {}_0^C\partial_t^\alpha\ \Pi_M^{1,0}\mathcal{U}^{k+1} + \gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_M \\ &+\langle F^{k+1}, v_M\rangle_{0,\omega} - \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M\end{aligned}$$

and therefore

$$\begin{aligned}&\mathcal{K}_{1,\alpha,\delta t}\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle \Pi_M^{1,0}\mathcal{U}^{k+1}, v_M\rangle = \langle \mathcal{P}_t^{l,\alpha}\Pi_M^{1,0}\mathcal{U}^k, v_M\rangle_M\\ &-c_{\alpha,\delta t}^{-1}\langle (I_d-\Pi_M^{1,0})(\lambda_1\partial_t\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\mathcal{U}^{k+1}+\gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega}\\ &+c_{\alpha,\delta t}^{-1}\langle E(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\,\Pi_M^{1,0}\mathcal{U}^{k+1}+\gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega}\\ &+\langle F^{k+1}, v_M\rangle_{0,\omega} - \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M, \end{aligned}\tag{29}$$

Let $e_M^{k+1}=u_M^{k+1}-\Pi_M^{1,0}\mathcal{U}^{k+1}$. *From (13) and (29), we obtain*

$$\begin{aligned}&\mathcal{K}_{1,\alpha,\delta t}\langle e_M^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle e_M^{k+1}, v_M\rangle = \langle \mathcal{P}_t^{l,\alpha}e_M^k, v_M\rangle_M\\ &+c_{\alpha,\delta t}^{-1}\langle (I_d-\Pi_M^{1,0})(\lambda_1\partial_t\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\mathcal{U}^{k+1}+\gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega}\\ &-c_{\alpha,\delta t}^{-1}\langle E(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\,\Pi_M^{1,0}\mathcal{U}^{k+1}+\gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega}\\ &\langle E(F^{k+1}), v_M\rangle_{0,\omega} + \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M, \end{aligned}\tag{30}$$

and therefore we can write

$$\mathcal{K}_{1,\alpha,\delta t}\langle e_M^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle e_M^{k+1}, v_M\rangle = \langle \mathcal{P}_t^{l,\alpha}e_M^k, v_M\rangle_M + \langle \delta^{k+1}, v_M\rangle_{0,\omega},$$

where

$$\langle \delta^{k+1}, v_M\rangle_{0,\omega} = \langle \delta_1^{k+1}, v_M\rangle_{0,\omega} + \langle \delta_2^{k+1}, v_M\rangle_{0,\omega} + \langle \delta_3^{k+1}, v_M\rangle_{0,\omega},$$

in which

$$\langle \delta_1^{k+1}, v_M\rangle_{0,\omega} = c_{\alpha,\delta t}^{-1}\langle (I_d-\Pi_M^{1,0})(\lambda_1\partial\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\mathcal{U}^{k+1}+\gamma_2\mathcal{U}^{k+1}), v_M\rangle_{0,\omega},$$

$$\langle \delta_2^{k+1}, v_M\rangle_{0,\omega} = -c_{\alpha,\delta t}^{-1}\langle E(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^\alpha\,\Pi_M^{1,0}\mathcal{U}^{k+1}+\gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}), v_M\rangle_{0,\omega}.$$

and

$$\langle \delta_3^{k+1}, v_M\rangle_{0,\omega} = \langle E(F^{k+1}), v_M\rangle_{0,\omega} + \langle r_{\Pi_M^{1,0}\mathcal{U}}^{k+1}, v_M\rangle_M.$$

It easily conclude that

$$\begin{aligned}\mathcal{K}_{1,\alpha,\delta t}\langle e_M^{k+1}, v_M\rangle_M + c_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle e_M^{k+1}, v_M\rangle &= \mathcal{K}_{2,\alpha,\delta t}\langle e_M^k, v_M\rangle_M + \lambda_2\sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\langle e_M^{k-j}, v_M\rangle_M\\ &+ \lambda_2 d_{\alpha,k}\langle e_M^0, v_M\rangle_M + \langle \delta^{k+1}, v_M\rangle_{0,\omega},\ \forall v_M\in\mathbb{X}_M.\end{aligned}$$

This completes the proof of Lemma 10.

Theorem 5 *If* $\partial_t^2\mathcal{U}\in L^\infty((0,T];H^r((-1,1),d\mu(x)))$, $r\geq 1$, *then for* $0<\alpha<1$, $0\leq k\leq N-1$, *it holds*

$$\|u_M^{k+1}\|_M^2 \leq \left(B_1^{-1}\lambda_2(1-\alpha)\frac{t_k^{-\alpha}}{\Gamma(2-\alpha)}\|u_M^0\|_M^2 + \widetilde{D}_{1,\alpha,\mathcal{U},f}M^{-2r} + \widetilde{D}_{2,\alpha,\mathcal{U}}\delta t^2\right)e^{\frac{B_{2,\alpha,\delta t}}{B_1}(d_{\alpha,0}-d_{\alpha,k})},$$

where $\widetilde{D}_{1,\alpha,\mathcal{U},f}$ *and* $\widetilde{D}_{2,\alpha,\mathcal{U}}$ *are positive constants.*

Proof 4 *We have*

$$\begin{aligned}&\langle \mathcal{P}_t^{l,\alpha}e_M^k, e_M^{k+1}\rangle_M \leq \mathcal{K}_{2,\alpha,\delta t}(\|e_M^k\|_M^2+\frac{1}{4}\|e_M^{k+1}\|_M^2) + \lambda_2\sum_{j=1}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})(\|e_M^{k-j}\|_M^2\\ &+\frac{1}{4}\|e_M^{k+1}\|_M^2) + \lambda_2 d_{\alpha,k}(\|u_M^0\|_M^2+\frac{1}{4}\|e_M^{k+1}\|_M^2) \leq \lambda_1\beta_{\alpha,\delta t}\|e_M^k\|_M^2 + \frac{\mathcal{K}_{2,\alpha,\delta t}}{4}\|e_M^{k+1}\|_M^2\\ &+\lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|e_M^{k-j}\|_M^2 + \frac{\lambda_2}{4}(d_{\alpha,1}-d_{\alpha,k})\|e_M^{k+1}\|_M^2 + \lambda_2 d_{\alpha,k}\|e_M^0\|_M^2 + \frac{\lambda_2 d_{\alpha,k}}{4}\|e_M^{k+1}\|_M^2.\end{aligned}\tag{31}$$

Noting Lemmas 1, 3 and using (31), we have

$$\begin{aligned}\mathcal{K}_{1,\alpha,\delta t}\|e_M^{k+1}\|_M^2 &+ \frac{c_{\alpha,\delta t}^{-1}\gamma_1}{8}\|\partial_x e_M^{k+1}\|_M^2 \leq \lambda_1\beta_{\alpha,\delta t}\|e_M^k\|_M^2 + \frac{\mathcal{K}_{2,\alpha,\delta t}}{4}\|e_M^{k+1}\|_M^2 + \lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|e_M^{k-j}\|_M^2\\ &+ \frac{\lambda_2}{4}(d_{\alpha,1}-d_{\alpha,k})\|e_M^{k+1}\|_M^2 + \lambda_2 d_{\alpha,k}\|e_M^0\|_M^2 + \frac{\lambda_2 d_{\alpha,k}}{4}\|e_M^{k+1}\|_M^2 + \langle \delta^{k+1}, e_M^{k+1}\rangle_{0,\omega},\end{aligned}$$

It follows that

$$\left(\frac{3}{4}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)+\gamma_2 c_{\alpha,\delta t}^{-1}\right)\|e_M^{k+1}\|_M^2+\frac{c_{\alpha,\delta t}^{-1}\gamma_1}{8}\|\partial_x e_M^{k+1}\|_M^2$$
$$\le \lambda_1\beta_{\alpha,\delta t}\|e_M^k\|_M^2+\lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|e_M^{k-j}\|_M^2+\lambda_2 d_{\alpha,k}\|e_M^0\|_M^2+\langle\delta^{k+1},e_M^{k+1}\rangle_{0,\omega}, \tag{32}$$

Using Lemmas 8 and 9, we obtain

$$\begin{aligned}|\langle\delta_1^{k+1},e_M^{k+1}\rangle_{0,\omega}| &= c_{\alpha,\delta t}^{-1}|\langle(I_d-\Pi_M^{1,0})((\lambda_1\partial_t\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^{\alpha}\mathcal{U}^{k+1}+\gamma_2\mathcal{U}^{k+1})),e_M^{k+1}\rangle_{0,\omega}|\\ &\le D_{1,\alpha,\mathcal{U}}\delta t^{\alpha}M^{-2r}+\frac{1}{4}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)\|e_M^{k+1}\|_M^2,\end{aligned} \tag{33}$$

$$\begin{aligned}|\langle\delta_2^{k+1},e_M^{k+1}\rangle_{0,\omega} &= c_{\alpha,\delta t}^{-1}|\langle E(\lambda_1\partial_t\Pi_M^{1,0}\mathcal{U}^{k+1}+\lambda_2\,{}_0^C\partial_t^{\alpha}\,\Pi_M^{1,0}\mathcal{U}^{k+1}+\gamma_2\Pi_M^{1,0}\mathcal{U}^{k+1}),v_M\rangle_{0,\omega}|\\ &\le D_{2,\alpha,\mathcal{U}}\delta t^{\alpha}M^{-2r}+\frac{1}{4}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)\|e_M^{k+1}\|_M^2,\end{aligned}$$

and

$$\begin{aligned}|\langle\delta_3^{k+1},e_M^{k+1}\rangle_{0,\omega}| &\le |\langle E(F^{k+1}),e_M^{k+1}\rangle_{0,\omega}|+|\langle r^{k+1}_{\Pi_M^{1,0}\mathcal{U}},e_M^{k+1}\rangle_M|\\ &\le D_{3,\alpha,f}\delta t^{\alpha}M^{-2r}+D_{4,\alpha,\Pi_M^{1,0}\mathcal{U}}\delta t^{2+\alpha}+\frac{1}{4}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)\|e_M^{k+1}\|_M^2.\end{aligned} \tag{34}$$

In view of (32), (33) and (34), we have

$$\begin{aligned}\gamma_2\|e_M^{k+1}\|_M^2+\frac{\gamma_1}{8}\|\partial_x e_M^{k+1}\|_M^2 &\le \lambda_1 c_{\alpha,\delta t}\beta_{\alpha,\delta t}\|e_M^k\|_M^2+\lambda_2 c_{\alpha,\delta t}\sum_{j=0}^{k-1}(d_{\alpha,j}-d_{\alpha,j+1})\|e_M^{k-j}\|_M^2\\ &+ \lambda_2 c_{\alpha,\delta t}d_{\alpha,k}\|e_M^0\|_M^2+\widetilde{D}_{1,\alpha,\mathcal{U},f}M^{-2r}+\widetilde{D}_{2,\alpha,\mathcal{U}}\delta t^2.\end{aligned} \tag{35}$$

If

$$B_1=\min\{\gamma_2,\frac{\gamma_1}{8}\},$$
$$B_{2,\alpha,\delta t}=\max\{\lambda_2 c_{\alpha,\delta t},\frac{\lambda_1 c_{\alpha,\delta t}\beta_{\alpha,\delta t}}{1-d_{\alpha,1}}\},$$

Noting $d_{\alpha,k}\le(1-\alpha)k^{-\alpha}$, we can get the following inequality

$$\|e_M^{k+1}\|_M^2+\|\partial_x e_M^{k+1}\|_M^2\le\sum_{j=1}^{k}\frac{B_{2,\alpha,\delta t}}{B_1}(d_{\alpha,k-j}-d_{\alpha,k-j+1})(\|e_M^j\|_M^2+\|\partial_x e_M^j\|_M^2)$$
$$+B_1^{-1}\lambda_2(1-\alpha)\frac{t_k^{-\alpha}}{\Gamma(2-\alpha)}\|u_M^0\|_M^2+\widetilde{D}_{1,\alpha,\mathcal{U},f}M^{-2r}+\widetilde{D}_{2,\alpha,\mathcal{U}}\delta t^2.$$

Noting Lemma 4, we have

$$\begin{aligned}\|u_M^{k+1}\|_M^2 &\le \|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2\\ &\le \left(\frac{B_1^{-1}\lambda_2(1-\alpha)t_k^{-\alpha}}{\Gamma(2-\alpha)}\|u_M^0\|_M^2+\widetilde{D}_{1,\alpha,\mathcal{U},f}M^{-2r}+\widetilde{D}_{2,\alpha,\mathcal{U}}\delta t^2\right)e^{\frac{B_{2,\alpha,\delta t}}{B_1}(d_{\alpha,0}-d_{\alpha,k})},\end{aligned} \tag{36}$$

Corollary 1 *If $\partial_t^2\mathcal{U}\in L^{\infty}((0,T];H^r((-1,1),d\mu(x)))$, $r\ge 1$, then for $0<\alpha<1$, $0\le k\le N-1$, it holds*

$$\|U^{k+1}-u_M^{k+1}\|_M^2\le\widehat{D}_{1,\alpha}\|e_M^0\|_M^2+\widehat{D}_{2,\alpha,\mathcal{U},f}M^{-2r}+\widehat{D}_{2,\alpha,\mathcal{U}}\delta t^2,$$

where $\widehat{D}_{1,\alpha}$, $\widehat{D}_{2,\alpha,\mathcal{U},f}$ and $\widehat{D}_{3,\alpha,\mathcal{U}}$ are positive constants.

Proof 5 *Using of triangle inequality and (36), we have*

$$\|U^{k+1}-u_M^{k+1}\|_M^2\le\|U^{k+1}-\Pi_M^{1,0}U^{k+1}\|_M^2+\|u_M^{k+1}-\Pi_M^{1,0}U^{k+1}\|_M^2\le\|U^{k+1}-\Pi_M^{1,0}U^{k+1}\|_M^2$$
$$+\left(\frac{B_1^{-1}\lambda_2(1-\alpha)t_k^{-\alpha}}{\Gamma(2-\alpha)}\|u_M^0\|_M^2+\widetilde{D}_{1,\alpha,\mathcal{U},f}M^{-2r}+\widetilde{D}_{2,\alpha,\mathcal{U}}\delta t^2\right)e^{\frac{B_{2,\alpha,\delta t}}{B_1}(d_{\alpha,0}-d_{\alpha,k})}. \tag{37}$$

Noting Lemmas 1 and 8 and using (37), we have

$$\|U^{k+1}-u_M^{k+1}\|_M^2\le\widehat{D}_{1,\alpha}\|e_M^0\|_M^2+\widehat{D}_{2,\alpha,\mathcal{U},f}M^{-2r}+\widehat{D}_{2,\alpha,\mathcal{U}}\delta t^2,$$

which completes the proof.

2.2. Nonlinear FM/IT Model With C-FD

2.2.1. Semi-Discrete Scheme and Spectral Approximation In this subsection, we consider the nonlinear **FM/IT** model with **C-FD** with the following conditions

$$\mathcal{U}(x,t)|_{t=0} = h(x),\ x \in \overline{\Omega},, \tag{38}$$

and

$$\mathcal{U}(x,t)|_{x\in\partial\Omega} = 0,\ t>0,. \tag{39}$$

Using Taylor series expansion, we have

$$\begin{cases} \mathcal{Q}(\mathcal{U}^1) = \mathcal{Q}(\mathcal{U}^0) + \mathcal{Q}_{\mathcal{U}}(\mathcal{U}_\vartheta)\partial_t\mathcal{U}(x,t_\tau)\delta t, & k=0 \\ \mathcal{Q}(\mathcal{U}^{k+1}) = 2\mathcal{Q}(\mathcal{U}^k) - \mathcal{Q}(\mathcal{U}^{k-1}) + O(\delta t^2), & k\geq 1. \end{cases} \tag{40}$$

Substituting (7), (8) and (40) into (2), we obtain

$$\begin{aligned} \mathcal{A}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) \quad &- \quad c^{-1}_{\alpha,\delta t}\gamma\partial^2_x\mathcal{U}^{k+1}(x) \\ &= \quad \mathcal{P}^{l,\alpha}_t\mathcal{U}^k(x) + \begin{cases} 2\mathcal{Q}(\mathcal{U}^k) - \mathcal{Q}(\mathcal{U}^{k-1}) + F^{k+1}(x), & k\geq 1, \\ \mathcal{Q}(\mathcal{U}^0) + F^1(x), & k=0, \end{cases} + \mathcal{R}^{k+1}_{\mathcal{U}}(x), \end{aligned} \tag{41}$$

where

$$\mathcal{P}^{l,\alpha}_t\mathcal{U}^k(x) = \begin{cases} \mathcal{A}_{2,\alpha,\delta t}\mathcal{U}^k(x) + \lambda_2\sum_{j=1}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\mathcal{U}^{k-j}(x) + \lambda_2 d_{\alpha,k}\mathcal{U}^0(x), & k\geq 1, \\ (\mathcal{A}_{2,\alpha,\delta t} + \lambda_2 d_{\alpha,1})\mathcal{U}^0(x), & k=0, \end{cases}$$

$$F^{k+1} = c^{-1}_{\alpha,\delta t} f(x,t_{k+1}),\ k=0,1,\cdots,N-2,\ \mathcal{U}^0(x) = h(x),$$

and

$$\beta_{\alpha,\delta t} = \frac{\Gamma(2-\alpha)}{\delta t^{1-\alpha}},\ \mathcal{A}_{1,\alpha,\delta t} = \lambda_1\beta_{\alpha,\delta t} + \lambda_2,\ \mathcal{A}_{2,\alpha,\delta t} = \lambda_1\beta_{\alpha,\delta t} + \lambda_2(1-d_{\alpha,1}).$$

Furthermore the truncation error $\mathcal{R}^{k+1}_{\mathcal{U}}(x)$ satisfy

$$|\mathcal{R}^{k+1}_{\mathcal{U}}(x)| \leq c_\alpha \max_{t\in(0,T]} |\partial^2_t\mathcal{U}(x,t)|\delta t^{1+\alpha}.$$

Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (2) and (38)-(39), which is given by:
Scheme N-I: Given $u^0 = h(x)$ and find u^{k+1} $(k=0,1,2,\cdots,N-1)$, such that

$$\mathcal{A}_{1,\alpha,\delta t}u^{k+1} - c^{-1}_{\alpha,\delta t}\gamma\partial^2_x u^{k+1} = \mathcal{P}^{l,\alpha}_t u^k + \begin{cases} \mathcal{Q}(u^0) + F^1, x\in\overline{\Omega}, & k=0, \\ 2\mathcal{Q}(u^k) - \mathcal{Q}(u^{k-1}) + F^{k+1}, x\in\overline{\Omega}, & k\geq 1, \end{cases} \tag{42}$$

$$u^{k+1}|_{x\in\partial\Omega} = 0,\ -1\leq k\leq N-1, \tag{43}$$

Now, we will give the representation of numerical solution to semi-discrete problem (42)-(43) in the space $\mathbb{X}_M$.
S-A(N-I): Find the spectral approximation $u^{k+1}_M \in \mathbb{X}_M$ $(k=0,1,2,\cdots,N-1)$, such that for all $v_M\in\mathbb{X}_M$:

$$\begin{aligned} &\mathcal{A}_{1,\alpha,\delta t}\langle u^{k+1}_M, v_M\rangle_M + c^{-1}_{\alpha,\delta t}\gamma a_\omega\langle u^{k+1}_M, v_M\rangle \\ &= \langle\mathcal{P}^{l,\alpha}_t u^k_M, v_M\rangle_M + \begin{cases} \langle I^c_M\mathcal{Q}(u^0_M), v_M\rangle_M + \langle I^c_M F^1, v_M\rangle_M, & k=0, \\ \langle 2I^c_M\mathcal{Q}(u^k_M) - I^c_M\mathcal{Q}(u^{k-1}_M), v_M\rangle_M + \langle I^c_M F^{k+1}, v_M\rangle_M, & k\geq 1. \end{cases} \end{aligned} \tag{44}$$

2.2.2. Stability Analysis

Lemma 11 *Let $u^{k+1}_M \in \mathbb{X}_M,\ k=0,1,\cdots,M-1$ be the solution of scheme (44). Then the following inequalities are hold*

$$\|u^1_M\|^2_M + \|\partial_x u^1_M\|^2_{0,\omega} \leq \mathbf{C}_{1,\alpha,\delta t}\|I^c_M F^1\|^2_M,$$

and

$$\|u^{k+1}_M\|^2_M + \|\partial_x u^{k+1}_M\|^2_{0,\omega} \leq \mathbf{C}_{2,\alpha,\delta t}\left(\|u^0_M\|^2_M + \|I^c_M F^{k+1}\|^2_M\right) e^{\mathbf{C}_{3,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})},$$

where $\mathbf{C}_{1,\alpha,\delta t}$, $\mathbf{C}_{2,\alpha,\delta t}$ and $\mathbf{C}_{3,\alpha,\delta t}$ are positive constants.

Proof 6 *Setting $v_M = u_M^{k+1}$, we get*

$$\begin{aligned}&\mathcal{A}_{1,\alpha,\delta t}\langle u_M^{k+1}, u_M^{k+1}\rangle_M + c_{\alpha,\delta t}^{-1}\gamma a_\omega\langle u_M^{k+1}, u_M^{k+1}\rangle\\ &= \langle \mathcal{P}_t^{l,\alpha} u_M^k, u_M^{k+1}\rangle_M + \begin{cases} \langle I_M^c \mathcal{Q}(u_M^0), u_M^1\rangle_M + \langle I_M^c F^1, u_M^1\rangle_M, & k=0,\\ \langle 2I_M^c\mathcal{Q}(u_M^k) - I_M^c\mathcal{Q}(u_M^{k-1}), u_M^{k+1}\rangle_M + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M, & k\geq 1.\end{cases}\end{aligned} \tag{45}$$

We know

$$(\mathcal{A}_{2,\alpha,\delta t} + \lambda_2 d_{\alpha,1})\langle u_M^0, u_M^1\rangle_M \leq (\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\|u_M^0\|_M^2 + \frac{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{4}\|u_M^1\|_M^2, \tag{46}$$

and

$$\begin{aligned}\langle \mathcal{P}_t^{l,\alpha} u_M^k, u_M^{k+1}\rangle_M \quad &\leq \quad \lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_M^2 + \frac{\mathcal{A}_{2,\alpha,\delta t}}{4}\|u_M^{k+1}\|_M^2 + \lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\|u_M^{k-j}\|_M^2\\ &+ \quad \frac{\lambda_2}{4}(d_{\alpha,1} - d_{\alpha,k})\|u_M^{k+1}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \frac{\lambda_2 d_{\alpha,k}}{4}\|u_M^{k+1}\|_M^2,\ k\geq 1.\end{aligned} \tag{47}$$

Using Lemmas 1, 2 and 3 and applying Eqs. (45), (46) and (47), we obtain

$$\begin{aligned}&\frac{3(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{4}\|u_M^1\|_M^2 + \frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\|\partial_x u_M^1\|_{0,\omega}^2 \leq (\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\|u_M^0\|_M^2\\ &+\langle I_M^c\mathcal{Q}(u_M^0), u_M^1\rangle_M + \langle I_M^c F^1, u_M^1\rangle_M \leq \sqrt{2}c_1(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\|\partial_x u_M^0\|_{0,\omega}^2\\ &+\langle I_M^c\mathcal{Q}(u_M^0), u_M^1\rangle_M + \langle I_M^c F^1, u_M^1\rangle_M \leq \sqrt{2}c_1(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\|\partial_x u_M^0\|_{0,\omega}^2\\ &+\frac{1}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}(\|I_M^c F^1\|_M^2 + \|I_M^c\mathcal{Q}(u_M^0)\|_M^2) + \frac{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{2}\|u_M^1\|_M^2,\ k=0,\end{aligned} \tag{48}$$

We know that there exists a positive constants c_2 such that $|\mathcal{Q}(\mathcal{U})| \leq c_2|\mathcal{U}|$. Then, we conclude that

$$\begin{aligned}&\frac{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{4}\|u_M^1\|_M^2 + \frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\|\partial_x u_M^1\|_{0,\omega}^2 \leq \frac{1}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}\|I_M^c F^1\|_M^2\\ &+\frac{c_2^2}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}\|u_M^0\|_M^2 + \sqrt{2}c_1(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\|\partial_x u_M^0\|_{0,\omega}^2,\ k=0,\end{aligned}$$

If

$$\begin{aligned}C_{1,\alpha,\delta t} &= \min\{\frac{\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{4}, \frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\},\\ C_{2,\alpha,\delta t} &= \max\{\frac{c_2^2}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}, \sqrt{2}c_1(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)\},\end{aligned}$$

we can get the following inequality

$$\|u_M^1\|_M^2 + \|\partial_x u_M^1\|_{0,\omega}^2 \leq \frac{1}{C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}\|I_M^c F^1\|_M^2 + \frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(\|u_M^0\|_M^2 + \|\partial_x u_M^0\|_{0,\omega}^2).$$

Now, noting Lemma 4, we obtain

$$\|u_M^1\|_M^2 + \|\partial_x u_M^1\|_{0,\omega}^2 \leq \mathbf{C}_{1,\alpha,\delta t}\|I_M^c F^1\|_M^2,$$

where $\mathbf{C}_{1,\alpha,\delta t} = \frac{e^{\frac{C_{2,\alpha,\delta t}}{C_1}}}{C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}$.
For $k\geq 1$, it can also be shown

$$\begin{aligned}&\frac{3(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{4}\|u_M^{k+1}\|_M^2 + \frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\|\partial_x u_M^{k+1}\|_{0,\omega}^2 \leq \lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_M^2\\ &+\lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \langle 2I_M^c\mathcal{Q}(u_M^k) - I_M^c\mathcal{Q}(u_M^{k-1}), u_M^{k+1}\rangle_M\\ &+\langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M \leq \sqrt{2}\lambda_1\beta_{\alpha,\delta t}\|u_M^k\|_{0,\omega}^2 + \lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\|u_M^{k-j}\|_M^2\\ &+\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2 + \langle 2I_M^c\mathcal{Q}(u_M^k) - I_M^c\mathcal{Q}(u_M^{k-1}), u_M^{k+1}\rangle_M + \langle I_M^c F^{k+1}, u_M^{k+1}\rangle_M\\ &\leq \sqrt{2}c_1\lambda_1\beta_{\alpha,\delta t}\|\partial_x u_M^k\|_{0,\omega}^2 + \lambda_2\sum_{j=0}^{k-1}(d_{\alpha,j} - d_{\alpha,j+1})\|u_M^{k-j}\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2\\ &+\frac{1}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}(\|I_M^c F^{k+1}\|_M^2 + \|2I_M^c\mathcal{Q}(u_M^k) - I_M^c\mathcal{Q}(u_M^{k-1})\|_M^2) + \frac{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{2}\|u_M^{k+}\|_M^2\\ &\leq \sqrt{2}c\lambda_1\beta_{\alpha,\delta t}\|\partial_x u_M^k\|_{0,\omega}^2 + \lambda_2\sum_{j=1}^{k}(d_{\alpha,k-j} - d_{\alpha,k-j+1})\|u_M^j\|_M^2 + \lambda_2 d_{\alpha,k}\|u_M^0\|_M^2\\ &+\frac{1}{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}(\|I_M^c F^{k+1}\|_M^2 + 2c_2^2\|u_M^k\|_M^2 + c_2^2\|u_M^{k-1}\|_M^2) + \frac{(\lambda_1\beta_{\alpha,\delta t} + \lambda_2)}{2}\|u_M^{k+1}\|_M^2.\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}{4}\|u_M^{k+1}\|_M^2+\frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\|\partial_x u_M^{k+1}\|_{0,\omega}^2\\
&\le\left(\lambda_2+\frac{1}{(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\left(\frac{2c_2^2}{(1-d_{\alpha,1})}+\frac{c_2^2}{(d_{\alpha,1}-d_{\alpha,2})}\right)\right)\sum_{j=1}^{k}(d_{\alpha,k-j}-d_{\alpha,k-j+1})\|u_M^j\|_M^2\\
&+\frac{\sqrt{2}c_1\lambda_1\beta_{\alpha,\delta t}}{1-d_{\alpha,1}}\sum_{j=1}^{k}(d_{\alpha,k-j}-d_{\alpha,k-j+1})\|\partial_x u_M^j\|_{0,\omega}^2\\
&+\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{1}{(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2.
\end{aligned}\tag{49}$$

If

$$\begin{aligned}
C_{1,\alpha,\delta t}&=\min\{\frac{(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}{4},\frac{c_{\alpha,\delta t}^{-1}\gamma}{4}\},\\
C_{2,\alpha,\delta t}&=\max\{\left(\lambda_2+\frac{1}{(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\left(\frac{2c_2^2}{(1-d_{\alpha,1})}+\frac{c_2^2}{(d_{\alpha,1}-d_{\alpha,2})}\right)\right),\frac{\sqrt{2}c_1\lambda_1\beta_{\alpha,\delta t}}{1-d_{\alpha,1}}\},
\end{aligned}$$

we can get the following inequality

$$\begin{aligned}
\|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2 &\le \sum_{j=1}^{k}\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(d_{\alpha,k-j}-d_{\alpha,k-j+1})(\|u_M^j\|_M^2+\|\partial_x u_M^j\|_{0,\omega}^2)\\
&+ C_{1,\alpha,\delta t}^{-1}\lambda_2 d_{\alpha,k}\|u_M^0\|_M^2+\frac{1}{C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\|I_M^c F^{k+1}\|_M^2.
\end{aligned}$$

Noting Lemma 4, we have

$$\|u_M^{k+1}\|_M^2+\|\partial_x u_M^{k+1}\|_{0,\omega}^2\le \mathrm{C}_{2,\alpha,\delta t}\left(\|u_M^0\|_M^2+\|I_M^c F^{k+1}\|_M^2\right)e^{C_{3,\alpha,\delta t}(d_{\alpha,0}-d_{\alpha,k})},$$

where $\mathbf{C}_{2,\alpha,\delta t}=\max\{C_{1,\alpha,\delta t}^{-1}\lambda_2 d_{\alpha,k},\frac{1}{C_{1,\alpha,\delta t}(\lambda_1\beta_{\alpha,\delta t}+\lambda_2)}\}$ *and* $\mathbf{C}_{3,\alpha,\delta t}=\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}$.

Theorem 6 *Let $u_M^{k+1}\in\mathbb{X}_M$, $k=0,1,\cdots,M-1$ be the solution of scheme (44). Then the scheme (44) is unconditionally stable in the sense that for all $\delta t>0$.*

Proof 7 *The proof is similar to proof of Theorem 4.*

3. FM/IT Model With C-F-FD

3.1. Linear FM/IT Model with C-F-FD

3.1.1. Discretization of Caputo-Fabrizio Derivative and Semi-Discrete Scheme In this subsection, we deal with the **FM/IT** model with **C-F-FD**. For (3), the initial condition:

$$\mathcal{U}(x,t)|_{t=0}=h(x),\ x\in\overline{\Omega},\tag{50}$$

and the Dirichlet boundary conditions:

$$\mathcal{U}(x,t)|_{x\in\partial\Omega}=0,\ t>0,\tag{51}$$

is considered.
For discretization of time variable, let $t_k:=k\delta t,\ \ k=0,1,\cdots,N$ be an equidistant partition of $[0,T]$, where $\delta t=\frac{T}{N}$. We analogize the time-fractional derivative term by using the finite difference scheme:

$${}_0^{CF}\partial_t^\alpha\mathcal{U}^{k+1}(x)=\left\{\begin{array}{ll}\bar{c}_{\alpha,\delta t}[\mathcal{D}_{\alpha,k+1}^{k+1}(\mathcal{U}^{k+1}(x)-\mathcal{U}^k(x))+\sum_{j=1}^{k}\mathcal{D}_{\alpha,j}^{k+1}(\mathcal{U}^j(x)-\mathcal{U}^{j-1}(x))], & k\ge 1\\ \bar{c}_{\alpha,\delta t}\mathcal{D}_{\alpha,1}^{1}(\mathcal{U}^1(x)-\mathcal{U}^0(x)), & k=0,\end{array}\right.+r_{\mathcal{U}}^{k+1}(x),\tag{52}$$

where $\bar{c}_{\alpha,\delta t}=(\alpha\delta t)^{-1}$ and $\mathcal{D}_{\alpha,j}^{k+1}=\exp(-\frac{\alpha\delta t}{1-\alpha}(k+1-j))-\exp(-\frac{\alpha\delta t}{1-\alpha}(k-j+2)),\ (j=1,2,\cdots,k+1)$.

Theorem 7 *([45]) For any $0<\alpha<1$, the coefficients of $\mathcal{D}_{\alpha,j}^{k+1}$, $j=1,2,\cdots,k+1$ satisfies the following properties*

- $\mathcal{D}_{\alpha,j}^{k+1}>0,\ \forall j\le k+1;$
- $\mathcal{D}_{\alpha,j}^{k+1}\le\mathcal{D}_{\alpha,j+1}^{k+1},\ \forall j\le k;$
- $\mathcal{D}_{\alpha,k+1}^{k+1}=\mathcal{D}_{\alpha,1}^{1},\ \mathcal{D}_{\alpha,k}^{k+1}=\mathcal{D}_{\alpha,1}^{2};$

- $(\mathcal{D}^1_{\alpha,1})^{-1} \le \frac{1-\alpha}{\alpha\delta t} exp(\frac{\alpha\delta t}{1-\alpha})$;
- $\sum_{j=1}^{k-1}(\mathcal{D}^{k+1}_{\alpha,j+1} - \mathcal{D}^{k+1}_{\alpha,j}) + \mathcal{D}^{k+1}_{\alpha,1} = \mathcal{D}^{k+1}_{\alpha,k} = \mathcal{D}^2_{\alpha,1}$.

Theorem 8 *([45]) For any $0 < \alpha < 1$, it holds*

$$r_{\mathcal{U}}^{k+1}(x) = -\frac{1}{1-\alpha}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(s - t_{j-\frac{1}{2}})\frac{\partial^2\mathcal{U}(x,s)}{\partial s^2}|_{s=\varsigma_j}\exp(-\frac{\alpha}{1-\alpha}(t_{k+1}-s))ds,$$

$$|r_{\mathcal{U}}^{k+1}(x)| \le \frac{c}{\alpha}\exp(\frac{2\alpha}{1-\alpha})\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^2, \;\; -1 \le k \le N-1, \; \forall x \in \Omega,$$

where $\varsigma_j \in (t_{j-1}, t_j)$ and c is independent of δt.

Substituting (52) and (8) into (3), we obtain

$$\mathcal{B}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2\mathcal{U}^{k+1}(x) = \mathcal{P}_t^{II,\alpha}\mathcal{U}^k(x) + F^{k+1}(x) + \mathcal{R}_{\mathcal{U}}^{k+1}(x), \; x \in \overline{\Omega} \tag{53}$$

where

$$\mathcal{P}_t^{II,\alpha}\mathcal{U}^k(x) \;\; = \;\; \begin{cases} (\lambda_1\alpha + \lambda_2\mathcal{D}^1_{\alpha,1})\mathcal{U}^0(x), & k=0, \\ \mathcal{B}_{2,\alpha,\delta t}\mathcal{U}^k(x) + \lambda_2\sum_{j=1}^{k-1}(\mathcal{D}^{k+1}_{\alpha,j+1} - \mathcal{D}^{k+1}_{\alpha,j})\mathcal{U}^j(x) + \lambda_2\mathcal{D}^{k+1}_{\alpha,1}\mathcal{U}^0(x), & k \ge 1, \end{cases}$$

$$F^{k+1} = \bar{c}_{\alpha,\delta t}^{-1}f(x, t_{k+1}), \; k = 0, 1, \cdots, N-2, \; \mathcal{U}^0(x) = h(x),$$

and

$$\mathcal{B}_{1,\alpha,\delta t} = \lambda_1\alpha + \lambda_2\mathcal{D}^1_{\alpha,1} + \gamma_2\bar{c}_{\alpha,\delta t}^{-1}, \; \mathcal{B}_{2,\alpha,\delta t} = \lambda_1\alpha + \lambda_2(\mathcal{D}^1_{\alpha,1} - \mathcal{D}^2_{\alpha,1}).$$

Furthermore, we have

$$|\mathcal{R}_{\mathcal{U}}^{k+1}(x)| \le \frac{c}{\alpha}\exp(\frac{2\alpha}{1-\alpha})\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^2, \;\; -1 \le k \le N-1, \; \forall x \in \Omega.$$

Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (3) and (50)-(51), which is given by:

Scheme L-II: Given $u^0 = h(x)$ and find u^{k+1} $(k = 0, 1, 2, \cdots, N-1)$, such that:

$$\begin{cases} \mathcal{B}_{1,\alpha,\delta t}u^{k+1}(x) - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2u^{k+1}(x) = \mathcal{P}_t^{II,\alpha}u^k(x) + F^{k+1}(x), x \in \overline{\Omega}, \\ u^{k+1}|_{x\in\partial\Omega} = 0, -1 \le k \le N-1, \end{cases} \tag{54}$$

3.1.2. Spectral Approximation to Semi-Discrete Problem (54) Now, we will give the representation of numerical solution to semi-discrete problem (54) in the space $\mathbb{P}_M$.

Given $u_M^0 = I_M^c u^0$ and find $u_M^{k+1} \in \mathbb{P}_M$ $(k = 0, 1, 2, \cdots, N-1)$, such that:

$$\begin{cases} \mathcal{B}_{1,\alpha,\delta t}u_M^{k+1}(z_i) - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2u_M^{k+1}(z_i) = \mathcal{P}_t^{II,\alpha}u_M^k(z_i) + F^{k+1}(z_i), 1 \le i \le M-1, \\ u_M^{k+1}(z_i) = 0, \; i = 0, M, \;\; -1 \le k \le N-1, \end{cases} \tag{55}$$

where

$$\mathcal{P}_t^{II,\alpha}u_M^k(z_i) = \begin{cases} (\lambda_1\alpha + \lambda_2\mathcal{D}^1_{\alpha,1})u_M^0(z_i), & k=0, \\ \mathcal{B}_{2,\alpha,\delta t}u_M^k(z_i) + \lambda_2\sum_{j=1}^{k-1}(\mathcal{D}^{k+1}_{\alpha,j+1} - \mathcal{D}^{k+1}_{\alpha,j})u_M^j(z_i) + \lambda_2\mathcal{D}^{k+1}_{\alpha,1}u^0(z_i), & k \ge 1, \end{cases}$$

We can reformulate the scheme (55) as the following:

S-A(L-II): Find the spectral approximation $u_M^{k+1} \in \mathbb{X}_M$ $(k = 0, 1, 2, \cdots, N-2)$, such that for all $v_M \in \mathbb{X}_M$:

$$\mathcal{B}_{1,\alpha,\delta t}\langle u_M^{k+1}, v_M\rangle_M + \bar{c}_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle u_M^{k+1}, v_M\rangle = \langle\mathcal{P}_t^{II,\alpha}u_M^k, v_M\rangle_M + \langle I_M^c F^{k+1}, v_M\rangle_M. \tag{56}$$

Similar to the previous section, the approximate solution u_M^k can be obtained by calculating a truncated series based on $\mathbb{P}_M = span\{\phi_j(x), j = 0, 1, \cdots, M\}$ as the following

$$u^k(x) \approx u_M^k(x) := \Phi(x)\{\mathrm{v}\}^k.$$

Therefore, we get

$$\begin{cases} \sum_{j=0}^M \left[\mathcal{B}_{1,\alpha,\delta t}\delta_{ij} - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1(\mathrm{D}^2)_{ij}\right] v_j^{k+1} = \digamma^{k+1}(z_i), \;\; 1 \le i \le M-1, \\ \Phi(z_0)\{\mathrm{v}\}^{k+1} = \Phi(z_M)\{\mathrm{v}\}^{k+1} = 0. \end{cases} \tag{57}$$

where

$$\digamma^{k+1}(z_i) = \mathcal{P}_t^{II,\alpha}u_M^k(z_i) + F^{k+1}(z_i), 1 \le i \le M-1,$$

Let us denote

$$\begin{aligned}
&(\mathrm{B})_{ij} = \mathcal{B}_{1,\alpha,\delta t}\delta_{ij} - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1(\mathrm{D}^2)_{ij},\ 1 \le i \le M-1,\ 0 \le j \le M,\\
&(\mathrm{B})_{0j} = \delta_{0j},\ (\mathrm{B})_{Mj} = \delta_{Mj},\ 0 \le j \le M,\\
&\{\mathrm{c}\}^{k+1} = (0, F^{k+1}(z_1), F^{k+1}(z_2), \cdots, F^{k+1}(z_{M-1}), 0)^T,\\
&\{\mathrm{v}\}^{k+1} = (v_0^{k+1}, v_1^{k+1}, \cdots, v_M^{k+1})^T,
\end{aligned}$$

then, the linear system (57) reduces to

$$\mathbf{B}\{\mathbf{v}\}^{k+1} = \{\mathbf{c}\}^{k+1},\ k = 0, 1, \cdots, N-2.$$

Theorem 9 *Let $u_M^{k+1} \in \mathbb{X}_M,\ k = 0, 1, \cdots, M-1$ be the solution of scheme (56). Then the scheme (56) is unconditionally stable in the sense that for all $\delta t > 0$.*

Proof 8 *We know $\|u_M^{k+1}\|_{0,\omega} \le c\|\partial_x u_M^{k+1}\|_{0,\omega}$. Set*

$$\begin{aligned}
C_{1,\alpha,\delta t} &= \min\{\gamma_2\bar{c}_{\alpha,\delta t}^{-1}, \frac{\bar{c}_{\alpha,\delta t}^{-1}\gamma_1}{4}\},\\
C_{2,\alpha,\delta t} &= \max\{\lambda_2, \frac{\lambda_1\alpha\sqrt{2}c}{(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^2)}\},
\end{aligned}$$

therefore similar to the proof of Lemma 11, we can get the following inequality

$$\begin{aligned}
\|u_M^{k+1}\|_M^2 + \|\partial_x u_M^{k+1}\|_{0,\omega}^2 &\le \sum_{j=1}^{k}\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1})(\|u_M^j\|_M^2 + \|\partial_x u_M^j\|_{0,\omega}^2)\\
&+ \lambda_2 C_{1,\alpha,\delta t}^{-1}\mathcal{D}_{\alpha,1}^{k+1}\|u_M^0\ |_M^2 + \frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\alpha + \lambda_2\mathcal{D}_{\alpha,1}^1)}\|I_M^c F^{k+1}\|_M^2,
\end{aligned}$$

Noting Lemma 4, we have

$$\|u_M^{k+1}\|_M^2 + \|\partial_x u_M^{k+1}\|_{0,\omega}^2 \le \left(\|u_M^0\|_M^2 + \|I_M^c F^{k+1}\|_M^2\right) e^{\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^{k+1})}. \tag{58}$$

where $\mathbf{C}_{1,\alpha,\delta t} = \max\{\lambda_2 C_{1,\alpha,\delta t}^{-1}\mathcal{D}_{\alpha,1}^{k+1}, \frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\alpha+\lambda_2\mathcal{D}_{\alpha,1}^1)}\}$ and $\mathbf{C}_{2,\alpha,\delta t} = \frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}$.
Using (58), the following inequality is holds

$$\|u_M^{k+1} - \tilde{u}_M^{k+1}\|_M^2 \le \|u_M^{k+1} - \tilde{u}_M^{k+1}\|_M^2 + \||\partial_x u_M^{k+1} - \partial_x\tilde{u}_M^{k+1}\|_M^2 \le \mathrm{C}_{1,\alpha,\delta t}\|u_M^0 - \tilde{u}_M^0\|_M^2 e^{\mathrm{C}_{2,\alpha,\delta t}(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^{k+1})}.$$

This completes the proof of Theorem 9.

3.2. Nonlinear FM/IT Model with C-F-FD

3.2.1. Semi-Discrete Scheme and Spectral Approximation In this subsection, we consider the nonlinear **FM/IT** model with **C-F-FD** with the following conditions

$$\mathcal{U}(x,t)|_{t=0} = h(x),\ x \in \overline{\Omega}, \tag{59}$$

and

$$\mathcal{U}(x,t)|_{x\in\partial\Omega} = 0,\ t > 0, \tag{60}$$

where $(x,t) \in \Omega \times (0,T]$ in which $\Omega = (-1,1)$.
Substituting (8), (40) and (52) into (4), we obtain

$$\begin{aligned}
\mathcal{S}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) &- \bar{c}_{\alpha,\delta t}^{-1}\gamma\partial_x^2\mathcal{U}^{k+1}(x)\\
&= \mathcal{P}_t^{II,\alpha}\mathcal{U}^k(x) + \begin{cases} \mathcal{Q}(\mathcal{U}^0) + F^1(x), & k = 0,\\ 2\mathcal{Q}(\mathcal{U}^k) - \mathcal{Q}(\mathcal{U}^{k-1}) + F^{k+1}(x), & k \ge 1, \end{cases} + \mathcal{R}_{\mathcal{U}}^{k+1}(x),
\end{aligned} \tag{61}$$

where

$$\mathcal{P}_t^{II,\alpha}\mathcal{U}^k(x) = \begin{cases} (\mathcal{S}_{2,\alpha,\delta t} + \lambda_2\mathcal{D}_{\alpha,1}^1)\mathcal{U}^0(x), & k = 0,\\ \mathcal{S}_{2,\alpha,\delta t}\mathcal{U}^k(x) + \lambda_2\sum_{j=1}^{k-1}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1})\mathcal{U}^j(x) + \lambda_2\mathcal{D}_{\alpha,1}^{k+1}\mathcal{U}^0(x), & k \ge 1, \end{cases}$$

$$F^{k+1} = \bar{c}_{\alpha,\delta t}^{-1}f(x, t_{k+1}),\ k = 0, 1, \cdots, N-2,\ \mathcal{U}^0(x) = h(x),$$

and

$$\mathcal{S}_{1,\alpha,\delta t} = \lambda_1 \alpha + \lambda_2 \mathcal{D}^1_{\alpha,1}, \ \mathcal{S}_{2,\alpha,\delta t} = \lambda_1 \alpha + \lambda_2 (\mathcal{D}^1_{\alpha,1} - \mathcal{D}^2_{\alpha,1}).$$

Furthermore the truncation error $\mathcal{R}^{k+1}_{\mathcal{U}}(x)$ satisfy

$$|\mathcal{R}^{k+1}_{\mathcal{U}}(x)| \leq \frac{c}{\alpha} \exp(\frac{2\alpha}{1-\alpha}) \max_{t\in(0,T]} |\partial^2_t \mathcal{U}(x,t)| \delta t^2, \ \ -1 \leq k \leq N-1, \ \forall x \in \Omega.$$

Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (4) and (59)-(60), which is given by:
Scheme N-II: Given $u^0 = h(x)$ and find u^{k+1} $(k = 0, 1, 2, \cdots, N-1)$, such that

$$\mathcal{S}_{1,\alpha,\delta t} u^{k+1}(x) - \bar{c}^{-1}_{\alpha,\delta t} \gamma \partial^2_x u^{k+1}(x) = \mathcal{P}^{II,\alpha}_t u^k(x) + \left\{ \begin{array}{ll} \mathcal{Q}(u^0) + F^1(x), & k=0, \\ 2\mathcal{Q}(u^k) - \mathcal{Q}(u^{k-1}) + F^{k+1}(x), & k \geq 1, \end{array} \right. \tag{62}$$

$$u^{k+1}|_{x\in\partial\Omega} = 0, -1 \leq k \leq N-1, \tag{63}$$

Now, we will give the representation of numerical solution to semi-discrete problem (62)-(63) in the space $\mathbb{X}_M$.
S-A(N-II): Find the spectral approximation $u^{k+1}_M \in \mathbb{X}_M$ $(k = 0, 1, 2, \cdots, N-1)$, such that for all $v_M \in \mathbb{X}_M$:

$$\begin{aligned} &\mathcal{S}_{1,\alpha,\delta t} \langle u^{k+1}_M, v_M \rangle_M + \bar{c}^{-1}_{\alpha,\delta t} \gamma a_\omega \langle u^{k+1}_M, v_M \rangle \\ &= \langle \mathcal{P}^{II,\alpha}_t u^k_M, v_M \rangle_M + \left\{ \begin{array}{ll} \langle \mathcal{Q}(u^0_M), v_M \rangle_M + \langle I^c_M F^1, v_M \rangle_M, & k=0, \\ \langle 2\mathcal{Q}(u^k_M) - \mathcal{Q}(u^{k-1}_M), v_M \rangle_M + \langle I^c_M F^{k+1}, v_M \rangle_M, & k \geq 1. \end{array} \right. \end{aligned} \tag{64}$$

Similar to Theorem 6, we have the following theorem:

Theorem 10 *Let $u^{k+1}_M \in \mathbb{X}_M$, $k = 0, 1, \cdots, N-1$ be the solution of scheme (64). Then the scheme (64) is unconditionally stable in the sense that for all $\delta t > 0$.*

4. Illustrative Test Problems and Discussion

We have studied some numerical examples to test the performance of the proposed methods. We illustrate the accuracy and stability of the proposed methods by performing **S-A(L-I)**, **S-A(L-II)**, **S-A(N-I)** and **S-A(N-II)** for different values of M and N.
1. (Error measurement criterion) As the exact solution is known, the maximum absolute error $\mathrm{e}^{M,N}_\infty$ and the root mean square error $\mathrm{e}^{M,N}_{rms}$ are measured with the following formulas:

$$\mathrm{e}^{M,N}_\infty = \max_{0\leq i\leq M} |\mathcal{U}^N(z_i) - u^N_M(z_i)|,$$

and

$$\mathrm{e}^{M,N}_{rms} = \sqrt{\frac{1}{M+1} \sum_{i=0}^{M} |\mathcal{U}^N(z_i) - u^N_M(z_i)|^2},$$

As the exact solution is unknown, the maximum absolute error $\mathrm{E}^{M,N}_\infty$ and the root mean square error $\mathrm{E}^{M,N}_{rms}$ are measured with the following formulas:

$$\mathrm{E}^{M,N}_\infty = \max_{0\leq i\leq M} |u^N_M(z_i) - u^{2N}_M(z_i)|,$$

and

$$\mathrm{E}^{M,N}_{rms} = \sqrt{\frac{1}{M+1} \sum_{i=0}^{M} |u^{N/2}_M(z_i) - u^N_M(z_i)|^2},$$

2. (Convergence ratio) As the exact solution is known, the convergence ratio is given by

$$\mathrm{Ratio}_1 = \log_2 \left[\frac{\mathrm{e}^{M,N/2}_\infty}{\mathrm{e}^{M,N}_\infty} \right].$$

As the exact solution is unknown, the convergence ratio is given by

$$\mathrm{Ratio}_2 = \log_2 \left[\frac{\mathrm{E}^{M,N/2}_\infty}{\mathrm{E}^{M,N}_\infty} \right].$$

α	N	40	80	160	320
0.2	$e_\infty^{M,N}$	2.8854e-3	1.4547e-3	7.3821e-4	3.7989e-4
	$e_{rms}^{M,N}$	1.2761e-3	6.3907e-4	3.2019e-4	1.6101e-4
	Ratio$_1$	-	0.9880	0.9786	0.9584
0.5	$e_\infty^{M,N}$	3.0444e-3	1.5119e-3	7.5821e-4	3.8662e-4
	$e_{rms}^{M,N}$	1.3485e-3	6.6533e-4	3.2951e-4	1.6422e-4
	Ratio$_1$	-	1.0098	0.9957	0.9717
0.9	$e_\infty^{M,N}$	4.5480e-3	2.2382e-3	1.1056e-3	5.5148e-4
	$e_{rms}^{M,N}$	2.0218e-3	9.9060e-4	4.8502e-4	2.3790e-4
	Ratio$_1$	-	1.0229	1.0175	1.0034

Table 1. **S-A(L-I):** The maximum absolute error $e_\infty^{M,N}$ and the root mean square error $e_{rms}^{M,N}$ for different values of α with $M = 13$ (Example 1-**Case I**).

Example 1 *In this example, we deal with the following time-fractional mobile/immobile transport equation:*

$$\lambda_1 \frac{\partial \mathcal{V}(x,t)}{\partial t} + \lambda_2\, {}_0^C\partial_t^\alpha \mathcal{V}(x,t) = \gamma_1 \partial_x^2 \mathcal{V}(x,t) - \gamma_2 \partial_x \mathcal{V}(x,t) + g(x,t), \tag{65}$$

where $(x,t) \in (-1,1) \times (0,1]$.
We introduce the following transformation:

$$\mathcal{V}(x,t) = e^{f_0(x)} \mathcal{U}(x,t), \;\; f_0(x) = \frac{\gamma_2 x}{2\gamma_1}, \tag{66}$$

Using the transformation (66), the equation (65) becomes

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2\, {}_0^C\partial_t^\alpha \mathcal{U}(x,t) = \gamma_1 \partial_x^2 \mathcal{U}(x,t) - \kappa_\gamma \mathcal{U}(x,t) + f(x,t), \tag{67}$$

where $\kappa_\gamma = \frac{1}{4}\frac{\gamma_1^2}{\gamma_1}$ *and* $f(x,t) = g(x,t)e^{-f_0(x)}$.

Case I: We consider (65) with the following terms

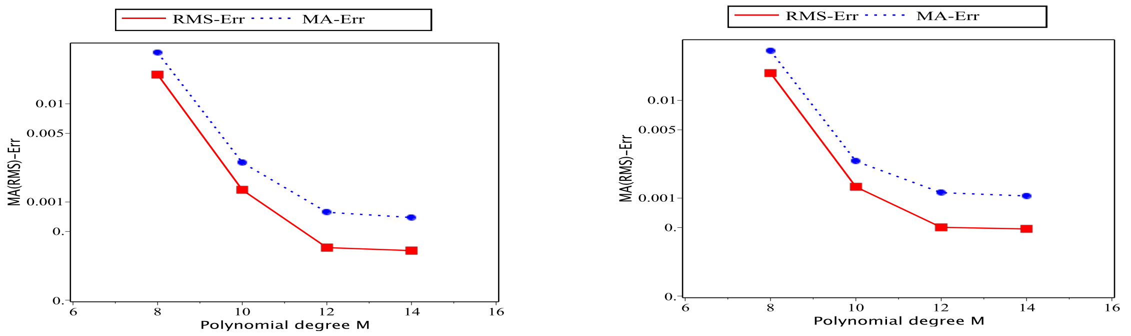
$$\begin{cases} Parameters: \;\; \lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = 1, \\ Force\ term: \;\; f(x,t) = (3t^2 + 4t^3\pi^2)\sin(2\pi x) + 2t^3\pi\cos(2\pi x) + \frac{6t^{3-\alpha}\sin(2\pi x)}{\Gamma(4-\alpha)}, \\ Initial\ condition: \;\; \mathcal{V}(x,0) = 0, \\ Dirichlet\ boundary\ conditions: \;\; \mathcal{V}(-1,t) = \mathcal{V}(1,t) = 0. \end{cases}$$

Then the exact solution $\mathcal{V}$ *is given by* $\mathcal{V}(x,t) = t^3\sin(2\pi x)$.
Using the transformation $\mathcal{V}(x,t) = e^{\frac{x}{2}}\mathcal{U}(x,t)$, *we have* $\kappa_\gamma = \frac{1}{4}$, $f(x,t) = e^{-\frac{x}{2}}g(x,t)$.
In Table 1, we present the maximum absolute error $e_\infty^{M,N}$, *the root mean square error* $e_{rms}^{M,N}$ *and the convergence ratio in the computed solutions of* **S-A(L-I)** *for Example 1-***Case I** *with* $\alpha = 0.2, 0.5, 0.9$. *From the obtained data in Table 1, we can observe that the convergence ratios in temporal direction are close to theoretical convergence order (***TCO***) i.e.* **TCO** $= 1$ *as we expected from Corollary 1. To check the spatial accuracy, we present the maximum absolute error* $e_\infty^{M,N}$ *and the root mean square error* $e_{rms}^{M,N}$ *for* $\alpha = 0.2, 0.9$ *with respect to the polynomial degree* M *for* $N = 160$ *in Figures* **??** *(a1-a2).*

Case II: As another example, we consider (65) with the following terms

$$\begin{cases} Parameters: \;\; \lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = 1, \\ Source\ term: \;\; f(x,t) = 6t^2(x^2-1) - 2t^3 + 2t^3 x, \\ Initial\ condition: \;\; \mathcal{V}(x,0) = 0, \\ Dirichlet\ boundary\ conditions: \;\; \mathcal{V}(-1,t) = \mathcal{V}(1,t) = 0. \end{cases}$$

The exact solution $\mathcal{V}$ *is unknown.*
Using the transformation $\mathcal{V}(x,t) = e^{\frac{x}{2}}\mathcal{U}(x,t)$, *we have* $\kappa_\gamma = \frac{1}{4}$, $f(x,t) = e^{-\frac{x}{2}}g(x,t)$.
Experimental Results of S-A(L-I): *Table 2 presents the experimental results of* **S-A(L-I)** *in temporal direction based on Chebyshev polynomials for Example 1-***Case II** *with* $\alpha = 0.1, 0.3$. *From the obtained results given in Table 2, we observe that, the numerical results agree precisely with the theoretical rate of convergence of Corollary 1. Also, the detailed observation of changes of* $\log_{10}[e_\infty^{M,N}]$ *and* $\log_{10}[e_{rms}^{M,N}]$ *against* N *for* $\alpha = 0.6, 0.9$ *are plotted in Figures* **??** *(a3-a4).*

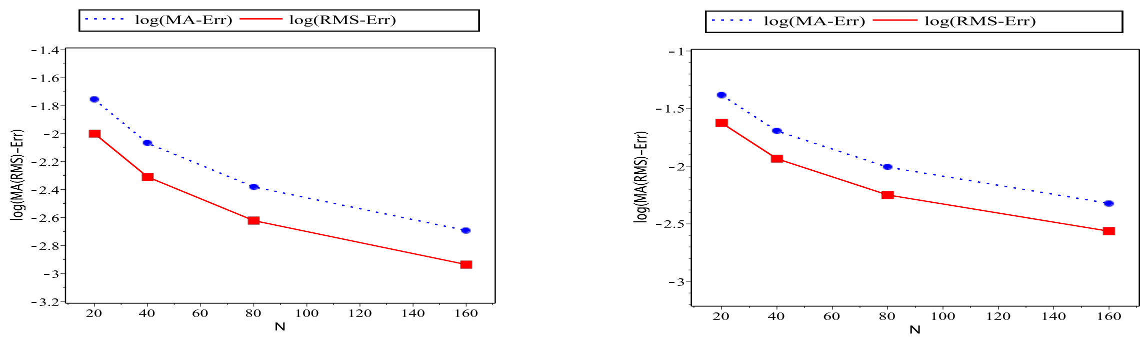


$(a1)$ $\alpha = 0.2$ $(a2)$ $\alpha = 0.9$

Figure 1. S-A(L-I): The changes of $e_{\infty}^{M,N}$ and $e_{rms}^{M,N}$ against M for $\alpha = 0.2, 0.9$ with $N = 160$ (Example 1-Case I).

N	$\alpha = 0.1$ $\mathrm{E}_{\infty}^{M,N}$	$\mathrm{E}_{rms}^{M,N}$	**Ratio**$_2$	$\alpha = 0.3$ $\mathrm{E}_{\infty}^{M,N}$	$\mathrm{E}_{rms}^{M,N}$	**Ratio**$_2$
20	2.3309e-2	1.9121e-2	-	3.3069e-2	1.8981e-2	-
40	1.6993e-2	9.7546e-3	0.9710	1.6605e-2	9.5308e-3	1.0416
80	8.5781e-3	4.9442e-3	0.9862	8.2903e-3	4.7584e-3	0.9544
160	4.3083e-3	2.4731e-3	0.9935	4.1324e-3	2.3719e-3	1.0044

Table 2. **S-A(L-I):** The maximum absolute error $\mathrm{E}_{\infty}^{M,N}$ and the root mean square error $\mathrm{E}_{rms}^{M,N}$ for different values of α with $M = 15$ (Example 1-**Case II**).



$(a3)$ $\alpha = 0.6$ $(a4)$ $\alpha = 0.9$

Figure 2. S-A(L-I): The changes of $\log_{10}(e_{\infty}^{M,N})$ and $\log_{10}(e_{rms}^{M,N})$ against N for different values of α with $M = 15$ (Example 1-Case II).

Example 2 *Consider (2) on* $(-1, 1) \times (0, 1]$ *with the following terms*

$$\begin{cases} \textit{Parameters}: \ \lambda_1 = 1, \lambda_2 = 1, \gamma = 1, \\ \textit{Nonlinear term}: \mathcal{Q}(\mathcal{U}) = -\mathcal{U}^3, \\ \textit{Source term}: \ f(x,t) = (e^{-x}(3t^2 \sin(\pi x) - t^3 \sin(\pi x) + 2t^3 \cos(\pi x)\pi + t^3 \sin(\pi x)\pi^2 \\ +t^9 e^{-2x} \sin(\pi x) - t^9 e^{-2x} \sin(\pi x)\cos^2(\pi x)) + \frac{6t^{-\alpha+3}\sin(\pi x)}{\Gamma(4-\alpha)}), \\ \textit{Initial condition}: \ \mathcal{U}(x,0) = 0, \\ \textit{Dirichlet boundary conditions}: \ \mathcal{U}(-1,t) = \mathcal{U}(1,t) = 0. \end{cases}$$

The exact solution of Example 2 is given by $\mathcal{U}(x,t) = t^3 e^{-x}\sin(\pi x)$.
Experimental Results of S-A(N-I): *Table 3 presents the experimental results of* **S-A(N-I)** *in temporal direction based on Chebyshev polynomials for Example 2 with* $\alpha = 0.2, 0.4, 0.7$.

Example 3 *Consider (3) on* $(-1, 1) \times (0, 1]$ *with the following terms*

$$\begin{cases} \textit{Parameters}: \ \lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = 1, \\ \textit{Source term}: \ f(x,t) = 3e^t \sin(2\pi x) - e^{\frac{\alpha t}{-1+\alpha}} \sin(2\pi x) + 4e^t \sin(2\pi x)\pi^2, \\ \textit{Initial condition}: \ \mathcal{U}(x,0) = \sin(2\pi x), \\ \textit{Dirichlet boundary conditions}: \ \mathcal{U}(-1,t) = \mathcal{U}(1,t) = 0. \end{cases}$$

α	N	80	160	320
0.2	$e_{\infty}^{M,N}$	5.8856e-3	2.5207e-3	1.1484e-3
	$e_{rms}^{M,N}$	2.7491e-3	1.1990e-3	5.5324e-4
	Ratio$_1$	-	1.2234	1.1342
0.4	$e_{\infty}^{M,N}$	5.9068e-3	2.5320e-3	1.1531e-3
	$e_{rms}^{M,N}$	2.7565e-3	1.2032e-3	5.5480e-3
	Ratio$_1$	-	1.2221	1.1348
0.7	$e_{\infty}^{M,N}$	6.4313e-3	2.7753e-3	1.2613e-3
	$e_{rms}^{M,N}$	3.0081e-3	1.3200e-3	6.0670e-4
	Ratio$_1$	-	1.2125	1.1377

Table 3. **S-A(N-I):** The maximum absolute error $e_{\infty}^{M,N}$ and the root mean square error $e_{rms}^{M,N}$ for different values of α with $M = 16$ (Example 2).

	$\alpha = 0.2$			$\alpha = 0.4$		
N	$e_{\infty}^{M,N}$	$e_{rms}^{M,N}$	**Ratio**$_1$	$e_{\infty}^{M,N}$	$e_{rms}^{M,N}$	**Ratio**$_1$
10	3.3029e-3	2.0883e-3	-	4.7579e-3	3.0082e-3	-
20	1.6915e-3	1.0694e-3	0.9654	2.5442e-3	1.6086e-3	0.9031
40	8.5607e-4	5.4126e-4	0.9825	1.3160e-3	8.3204e-4	0.9511
80	4.3066e-4	2.7229e-4	0.9912	6.6929e-4	4.2317e-4	0.9755
160	2.1598e-4	1.3656e-4	0.9956	3.375e-4	2.1340e-4	0.9877
320	1.0816e-4	6.8385e-5	0.9977	1.6947e-4	1.0715e-4	0.9939
	$\alpha = 0.7$			$\alpha = 0.8$		
N	$e_{\infty}^{M,N}$	$e_{rms}^{M,N}$	**Ratio**$_1$	$e_{\infty}^{M,N}$	$e_{rms}^{M,N}$	**Ratio**$_1$
10	2.5040e-2	1.5831e-2	-	6.2126e-2	3.9280e-2	-
20	1.5109e-2	9.5528e-3	0.7288	4.0299e-2	2.5479e-2	0.6244
40	8.2872e-3	5.2396e-3	0.8664	2.2915e-2	1.4488e-2	0.8144
80	4.3385e-3	2.7431e-3	0.9337	1.2213e-2	7.7215e-3	0.9079
160	2.2195e-3	1.4033e-3	0.9670	6.3035e-3	3.9855e-3	0.9542
320	1.1225e-3	7.0973e-3	0.9835	3.2021e-3	2.0246e-3	0.9771

Table 4. **S-A(L-II):** The maximum absolute error $e_{\infty}^{M,N}$ and the root mean square error $e_{rms}^{M,N}$ for different values of α with $M = 17$ (Example 3).

The exact solution of Example 3 is given by $\mathcal{U}(x,t) = e^t \sin(2\pi x)$.

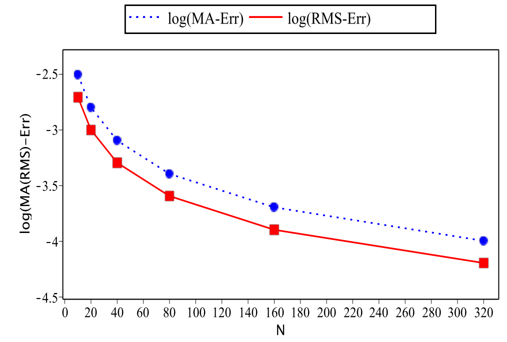
Experimental Results of S-A(L-II): *Table 4 presents the experimental results of* **S-A(L-II)** *in temporal direction based on Chebyshev polynomials for Example 3 with* $\alpha = 0.2, 0.4, 0.7, 0.8$. *From the obtained results given in Table 4, we observe that, the numerical results agree precisely with the theoretical rate of convergence. More detailed observation of changes of* $\log_{10}[e_{\infty}^{M,N}]$ *and* $\log_{10}[e_{rms}^{M,N}]$ *against N for* $\alpha = 0.1, 0.15, 0.6, 0.81$ *are plotted in Figures* **??** *(c1-c4). To check the spatial accuracy, we present the maximum absolute error* $e_{\infty}^{M,N}$ *and the root mean square error* $e_{rms}^{M,N}$ *for* $\alpha = 0.1, 0.15, 0.6, 0.81$ *with respect to the polynomial degree M for* $N = 160$ *in Figures* **??** *(c5-c8).*

Example 4 *Consider (3) on* $(-1, 1) \times (0, 1]$ *with the following terms*

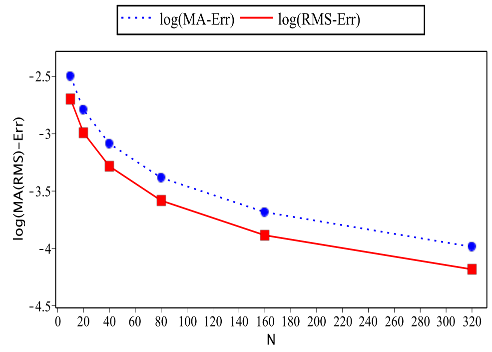
$$\begin{cases} \mathit{Parameters}: \ \lambda_1 = 1, \lambda_2 = 1, \gamma = 1, \\ \mathit{Nonlinear\ term}: \mathcal{Q}(\mathcal{U}) = -\sin(\mathcal{U}), \\ \mathit{Source\ term}: \ f(x,t) = 2e^t \sin(\pi x) - e^{\frac{\alpha t}{-1+alpha}} \sin(\pi x) + e^t \sin(\pi x)\pi^2 + \sin(e^t \sin(\pi x)), \\ \mathit{Initial\ condition}: \ \mathcal{U}(x,0) = \sin(\pi x), \\ \mathit{Dirichlet\ boundary\ conditions}: \ \mathcal{U}(-1,t) = \mathcal{U}(1,t) = 0. \end{cases}$$

The exact solution of Example 4 is given by $\mathcal{U}(x,t) = e^t \sin(\pi x)$.

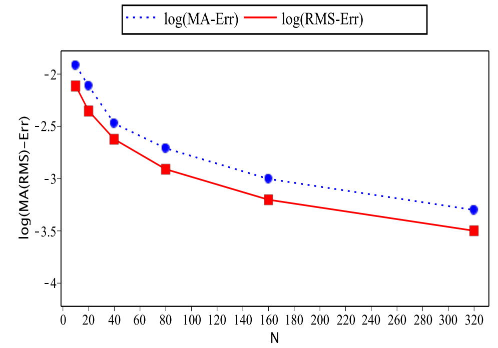
Experimental Results of S-A(N-II): *Table 5 presents the experimental results of* **S-A(N-II)** *in temporal direction based on Chebyshev polynomials for Example 4 with* $\alpha = 0.1, 0.15, 0.6$.



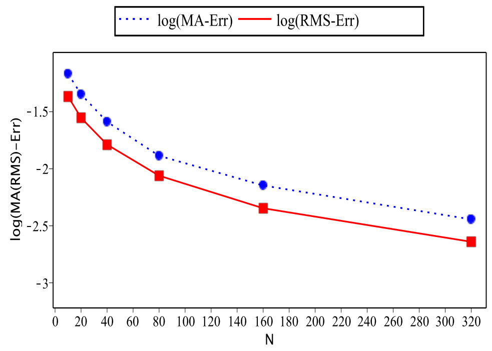
$(c1)\ \alpha = 0.1$



$(c2)\ \alpha = 0.15$

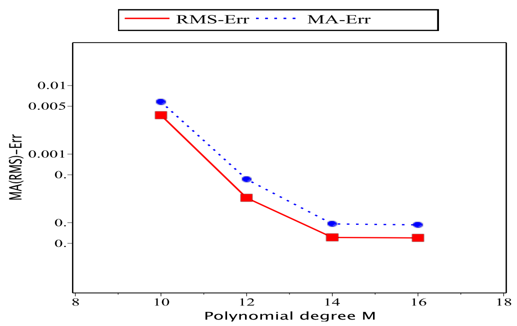


$(c3)\ \alpha = 0.6$

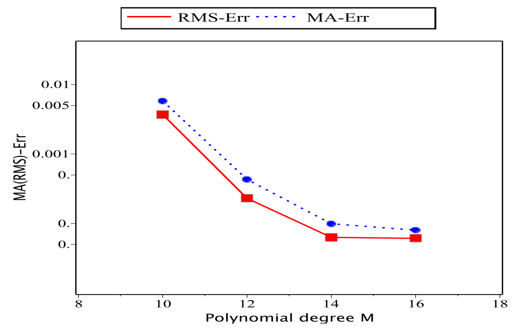


$(c4)\ \alpha = 0.81$

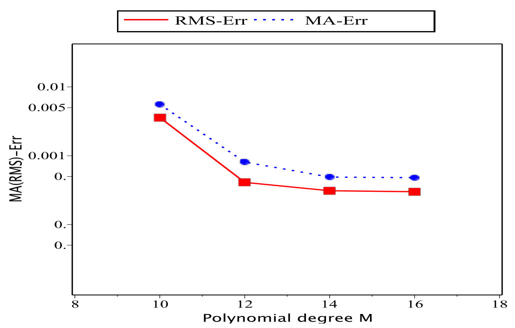
Figure 3. S-A(L-II): The changes of $\log_{10}(e_{\infty}^{M,N})$ and $\log_{10}(e_{rms}^{M,N})$ against N for different values of α with $M = 17$ (Example 3).



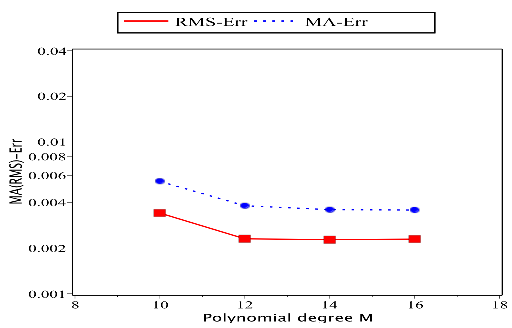
$(c5)\ \alpha = 0.1$



$(c6)\ \alpha = 0.15$



$(c7)\ \alpha = 0.6$



$(c8)\ \alpha = 0.81$

Figure 4. S-A(L-II): The changes of $e_{\infty}^{M,N}$ and $e_{rms}^{M,N}$ against M for $\alpha = 0.1, 0.15, 0.6, 0.81$ with $N = 320$ (Example 3).

α	N	80	160	320
0.1	$e_{\infty}^{M,N}$	1.4117e-3	7.2585e-3	3.6786e-3
	$e_{rms}^{M,N}$	8.6583e-3	4.4498e-3	2.2547e-4
	Ratio$_1$	-	0.9597	0.9805
0.15	$e_{\infty}^{M,N}$	1.4445e-3	7.4247e-4	3.7635e-4
	$e_{rms}^{M,N}$	8.8597e-4	4.5517e-4	2.3067e-4
	Ratio$_1$	-	0.9602	0.9803
0.6	$e_{\infty}^{M,N}$	6.8372e-3	3.4971e-3	1.7683e-3
	$e_{rms}^{M,N}$	4.1907e-3	2.1433e-3	1.0837e-3
	Ratio$_1$	-	0.9672	0.9838

Table 5. **S-A(N-II):** The maximum absolute error $e_{\infty}^{M,N}$ and the root mean square error $e_{rms}^{M,N}$ for different values of α with $M = 16$ (Example 4).

5. Conclusion

In this paper, a spectral method is developed to solve **FM/IT** model with **C-FD** and **C-F-FD**. Furthermore, the unconditional stability and convergence of the numerical method are discussed, which provides the theoretical basis of the proposed method. The proposed method is computationally capable due to its simple implementation but with reasonable accuracy. It can be easily viewed from obtained numerical solutions and error norms that this is an excellent method to achieve a numerical solution of the time-fractional Mobile/Immobile transport model.

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