Multiple solutions for a class of quasilinear Choquard equations *

Xian Wu †

Department of Mathematics, Kunming University, Kunming, Yunnan 650214, P.R.China

Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650092, P.R.China

email: wuxian2042@163.com

ABSTRACT. In this paper, we study the following quasilinear Choquard equations of the form

$$-\Delta u+V(x)u-\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u=(|x|^{-\mu}*G(u))g(u),\ x\ \in R^N,$$

where $1\geq\alpha>\frac{1}{2}$, $V\in C(\mathbb{R}^N,\mathbb{R})$, $g\in C(\mathbb{R}^N,\ \mathbb{R})$. Distinguished from two situations $\lim\limits_{|x|\to\infty}V(x)=+\infty$ or $\lim\limits_{|x|\to\infty}V(x)<+\infty$, we research the existence of nontrivial solutions and a sequence of high energy solutions.

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1. Introduction and Preliminaries

Consider the following quasilinear Choquard equations of the form

$$-\Delta u+V(x)u-\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u=(|x|^{-\mu}*G(u))g(u),\ x\ \in\mathbb{R}^N,\tag{1.1}$$

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†Corresponding Author: wuxian2042@163.com

where $1 \geq \alpha > \frac{1}{2}$, $V \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}^N,\ \mathbb{R})$.

As $\alpha = 1$ and $\mu = 0$, the equation (1.1) degenerate into the form

$$-\Delta u + V(x)u - \Delta(|u|^2)u = g(u),\ x\ \in \mathbb{R}^N. \tag{1.2}$$

Solutions of the equation (1.2) are standing waves the following quasilinear Schrödinger equation of the form

$$i\psi_t + \triangle\psi - V(x)\psi + k\triangle(h(|\psi|^2))h'(|\psi|^2)\psi + g(\psi) = 0, \quad x \in R^N. \tag{1.3}$$

The quasilinear Schrödinger equations (1.3) are derived as models of several physical phenomena, such as see [9, 12, 13, 20, 23]. It begins with [22] for the studies on Mathematics. In the resent years, greater important attention has been paid to the equation (1.2), for example, see [5, 6, 7, 11, 14, 15, 16, 19, 24, 26]. Especially, in [14], the ground state solutions for the following problems

$$-\Delta u + V(x)u - \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda|u|^{p-1}u,\ x\ \in \mathbb{R}^N$$

was studied via the Lagrange multiplier method; in [3], the uniqueness of the ground state solutions for the following the problems

$$-\Delta u + \lambda u - \kappa\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = |u|^{p-1}u,\ x\ \in \mathbb{R}^N$$

was studied via a dual approach.

As $\alpha = \frac{1}{2}$, the equation (1.1) degenerate into the form

$$-\Delta u + V(x)u = (|x|^{-\mu} * G(u))g(u),\ x\ \in \mathbb{R}^N. \tag{1.4}$$

(1.4) first appeared in [21], it was used to describe the quantum mechanics of a polaron. Next, Choquard used (1.4) to describe an electron trapped in its own hole (see [10]). In [17], (1.4) was used as a model of self-gravitating matter. Recently, greater important attention has been paid to the equation (1.4), for example, see [1, 2, 4, 18, 27, 30, 31].

In [29], the author had studied the existence of positive solutions, negative solutions and sequence of high energy solutions for the equation (1.1) with $\frac{1}{2} < \alpha \leq 1$, $\mu = 0$ and

$\lim\limits_{|x|\to\infty} V(x) = +\infty$. In the present paper, we study the equation (1.1) with $\frac{1}{2} < \alpha \leq 1$, $0 < \mu < 2 < N$. Distinguished from two situations $\lim\limits_{|x|\to\infty} V(x) = +\infty$ or $\lim\limits_{|x|\to\infty} V(x) < +\infty$, we research the existence of nontrivial solutions and a sequence of high energy solutions.

In order to reduce the statements for main results, we list the assumptions as follows:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf\limits_{x\in\mathbb{R}^N} V(x)$ and $V_\infty := \lim\limits_{|x|\to\infty} V(x) \leq +\infty$.

(g_1) $g \in C(\mathbb{R}, \mathbb{R})$, $tg(t) > 0$ for all $t \neq 0$, and there exist $C_1 > 0, 2\alpha(2 - \frac{\mu}{N}) \leq q_1 \leq q_2 < 2\alpha(\frac{2N-\mu}{N-2})$ such that

$$|g(t)| \leq C_1(|t|^{q_1-1} + |t|^{q_2-1})$$

for all $t \in \mathbb{R}$.

(g_2)

$$tg(t) - 4\alpha G(t) \geq 0, \ \forall t \in \mathbb{R},$$

where $G(t) = \int_0^t g(s)ds$.

Set

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)dx$$

and the norm

$$\|u\|_{H^1} = [\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx]^{1/2}.$$

When $V_\infty = +\infty$, set

$$E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2]dx < +\infty\}$$

with the inner product

$$\langle u, v\rangle_E = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + V(x)uv]dx$$

and the norm

$$\|u\|_E = \langle u, u\rangle_E^{1/2}.$$

When $V_\infty < +\infty$, set $E = H^1(\mathbb{R}^N)$. Then E is a Hilbert space. By the continuity of the embedding

$$E \hookrightarrow L^s(\mathbb{R}^N), s \in [2, 2^*],$$

there exist constants $a_s > 0$, $2 \le s \le 2^*$, such that

$$\|u\|_s \le a_s \|u\|_E, \quad \forall u \in E,$$

where we denote by $\|\cdot\|_s$ the norm of $L^s(\mathbb{R}^N)$. Moreover, by Lemma 3.4 in [33] we know that as $V_\infty = +\infty$, the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for each $2 \le s < 2^*$.

We observe that formally equation (1.1) is the Euler-Lagrange equation associated of the natural energy functional $J : E \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \frac{1}{4\alpha}\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * G(u))G(u)dx.$$

Clearly, $\frac{1}{4\alpha}\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx = \alpha \int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 dx$. Hence

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \alpha\int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * G(u))G(u)dx.$$

According to [3], we can define f by

$$f'(t) = \frac{1}{\sqrt{1 + 2\alpha |f(t)|^{2(2\alpha-1)}}} \quad on \quad t \in [0, +\infty)$$

and

$$f(-t) = -f(t) \quad on \quad t \in (-\infty, 0].$$

After the change of variables, we obtain the following functional

$$I(v) := J(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v)dx - \frac{1}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(v)))G(f(v))dx,$$

which is well defined in E under the assumptions (V) and (g_1). Moreover, a critical point v of the functional I corresponds to a weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2\alpha |f(v)|^{2(2\alpha-1)}}} [(|x|^{-\mu} * G(f(v)))g(f(v)) - V(x)f(v)] \ in \ \mathbb{R}^N \tag{1.5}$$

and $u = f(v)$ is a weak solution (1.1).

The following lemma appeared in [29].

Lemma 2.1 The function $f(t)$ enjoys the following properties:

(f_1) f is uniquely defined C^∞ function and invertible.

(f_2) $0 < f'(t) \leq 1,\ \forall t \in R.$

(f_3) $|f(t)| \leq |t|,\ \forall t \in R.$

(f_4) $\lim\limits_{t\to 0} \dfrac{f(t)}{t} = 1.$

(f_5) $\lim\limits_{t\to +\infty} \dfrac{f^{2\alpha}(t)}{t} = \sqrt{2\alpha}, \qquad \lim\limits_{t\to -\infty} \dfrac{f^{2\alpha}(t)}{t} = -\sqrt{2\alpha}.$

(f_6) $\frac{f(t)}{2} \leq \alpha t f'(t) \leq \alpha f(t),\ \ \forall t \geq 0; \quad \alpha f(t) \leq \alpha t f'(t) \leq \frac{f(t)}{2},\ \ \forall t \leq 0.$

(f_7) $f^{2\alpha}(t) \leq \sqrt{2\alpha}|t|,\ \ \forall t \in R.$

(f_8) The function $f^2(t)$is strictly convex.

(f_9) There exists a positive constant $\theta > 0$ such that

$$|f(t)| \geq \begin{cases} \theta|t|, & |t| \leq 1, \\ \theta|t|^{\frac{1}{2\alpha}}, & |t| \geq 1. \end{cases}$$

(f_{10}) There exist positive constants C_1 and C_2 such that

$$|t| \leq C_1|f(t)| + C_2|f(t)|^{2\alpha},\ \forall t \in \mathbb{R}.$$

(f_{11}) $|f^{2\alpha-1}(t)f'(t)| < \frac{1}{\sqrt{2\alpha}},\ \forall t \in \mathbb{R}.$

(f_{12}) $f(t)$ is odd, $f^2(t)$ is even.

(f_{13}) For each $\xi > 0$, there exists a positive constant C(ξ) such that

$$f^{2\alpha}(\xi t) \leq C(\xi) f^{2\alpha}(t).$$

(f_{14}) The function $f(t)f'(t)t^{-1}$ is strictly decreasing for $t > 0$.

(f_{15}) The function $f^p(t)f'(t)t^{-1}$ is strictly increasing for $p \geq 4\alpha - 1$ and $t > 0$.

By the Hardy-Littlewood-Sobolev inequality (For example, see Proposition 1.1 in [4]) and Lemma 2.1 we can prove the following Lemma 2.2.

Lemma 2.2 If assumptions (V) and (g_1) hold, then the functionals I is well defined on E, and $I \in C^1(E, \mathbb{R})$.

Through out the paper, C and C_i are used in various places to denote positive constants.

3. Main results

Theorem 3.1 Assume the conditions (V) and (g_1)-(g_2) hold. Then the equation (1.1) has a nontrivial solution. Furthermore, if g is odd, then the equation (1.1) has a sequence of solutions $\{u_n\} \subset E$ such that $\|u_n\|_E \to \infty$ and $J(u_n) \to +\infty$.

Proof. First, we prove that I satisfies the Cerami condition. Let $\{v_n\} \subset E$ be any Cerami sequence of I, i.e. $\{I(v_n)\}$ is bounded and $(1 + \|v_n\|_E)I'(v_n) \to 0$ in E^*. Set $A_n^2 := \int_{\mathbb{R}^N}[|\nabla v_n|^2 + V(x)f^2(v_n)]dx$. By Lemma 2.1 (f_{11}) and (g_2), there exists a constant $C > 0$ such that

$$\begin{aligned}
C &\geq I(v_n) - \frac{1}{8\alpha}\langle I'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{4\alpha - 1}{8\alpha}\int_{\mathbb{R}^N} V(x)f^2(v_n)dx \\
&\quad + \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(v_n)))[\frac{1}{8\alpha}f(v_n)g(f(v_n)) - \frac{1}{2}G(f(v_n))]dx \\
&\geq \frac{1}{4}A_n^2.
\end{aligned}$$

This shows that the sequence $\{A_n\}$ is bounded. Moreover, from Step 2 of proof of Theorem 3.1 in [29] we know that there exists a constant $C > 0$ such that $A_n^2 \geq C\|v_n\|_E^2$, and hence the sequence $\{v_n\}$ is bounded in E. Note that, for each $t \in \mathbb{R}$, by Lemma 2.1 (f_{11}) one has

$$\frac{d}{dt}[f(t)f'(t)] = (f'(t))^2 + f(t)f''(t) > 2(1-\alpha)|f'(t)|^2 = \frac{2(1-\alpha)}{1 + 2\alpha|f(t)|^{2(2\alpha-1)}} \geq 0.$$

Hence, as the proof of Lemma 3.11 in [8], we may prove that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x)[f(v_n)f'(v_n) - f(v)f'(v)](v_n - v)dx \geq C\|v_n - v\|_E^2. \quad (3.1)$$

Moreover, by (g_1) and Lemma 2.1 (f_7), (f_{11}), we have

$$|g(f(t))f'(t)| \leq C_2(|t|^{\frac{q_1}{2\alpha}-1} + |t|^{\frac{q_2}{2\alpha}-1}), \quad \forall\ t \in \mathbb{R}, \quad (3.2)$$

and

$$|G(f(t))| \leq C_2(|t|^{\frac{q_1}{2\alpha}} + |t|^{\frac{q_2}{2\alpha}}), \quad \forall\ t \in \mathbb{R}. \quad (3.3)$$

Set $s = \frac{2N}{2N-\mu}$. Then $s \in (1, \frac{N}{\mu})$ and $G(f(w))$, $g(f(w))f'(w)w \in L^s(\mathbb{R}^N)$ for all $w \in E$. Notice that

$$2 \leq \frac{sq_1}{2\alpha} \leq \frac{sq_2}{2\alpha} < 2^*.$$

(1^0) For the case $V_\infty = +\infty$, then by the compactness of embedding $E \hookrightarrow L^s(\mathbb{R}^N)(2 \leq s < 2^*)$, up to a subsequence, one has $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(\mathbb{R}^N)$ for all $2 \leq s < 2^*$ and $v_n(x) \to v(x)$ *a.e. on* $\mathbb{R}^N$. By the Hardy-Littlewood-Sobolev inequality we know that

$$\begin{aligned}
&|\int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(v_n)))g(f(v_n))f'(v_n) - (|x|^{-\mu} * G(f(v)))g(f(v))f'(v)](v_n - v)dx| \\
\leq& C[\|G(f(v_n))\|_s \cdot \|g(f(v_n))f'(v_n)(v_n - v)\|_s + \|G(f(v))\|_s \cdot \|g(f(v))f'(v)(v_n - v)\|_s] \\
\leq& C[\|g(f(v_n))f'(v_n)(v_n - v)\|_s + \|g(f(v))f'(v)(v_n - v)\|_s] \\
\leq& C(\|v_n - v\|_{\frac{sq_1}{2\alpha}} + \|v_n - v\|_{\frac{sq_2}{2\alpha}}) \to 0. \qquad (3.4)
\end{aligned}$$

(2^0) For the case $V_\infty < +\infty$, then by the boundedness of $\{v_n\}$ in E, up to a subsequence, we can assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{loc}(\mathbb{R}^N)$ for each $s \in [1, 2^*)$ and $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$. By (3.2) and (3.4) we know that

$$\begin{aligned}
&|\int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(v_n)))g(f(v_n))f'(v_n) - (|x|^{-\mu} * G(f(v)))g(f(v))f'(v)](v_n - v)dx| \\
\leq& C[\|g(f(v_n))f'(v_n)(v_n - v)\|_s + \|g(f(v))f'(v)(v_n - v)\|_s] \qquad (3.5)
\end{aligned}$$

and for any $\varepsilon > 0$ there exist $0 < \delta_0 < \rho_0$ such that

$$|g(f(t))f'(t)| \leq \varepsilon(|t|^{\frac{N-\mu}{N}} + |t|^{\frac{2+N-\mu}{N-2}}) + \chi_{[\delta_0,\rho_0]}(|t|)|g(f(t))f'(t)|, \quad \forall\ t \in \mathbb{R},$$

where $\chi_{[\delta_0,\rho_0]}$ is the characteristic function on $[\delta_0,\rho_0]$. Hence

$$\begin{aligned}
&\|g(f(v_n))f'(v_n)(v_n-v)\|_s^s\\
\leq& C\varepsilon\int_{\mathbb{R}^N}(|v_n|^{\frac{s(N-\mu)}{N}}+|v_n|^{\frac{s(2+N-\mu)}{N-2}})|v_n-v|^s dx\\
&+C\int_{\mathbb{R}^N}\chi_{[\delta_0,\rho_0]}(|v_n|)|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
&C\varepsilon\int_{\mathbb{R}^N}(|v_n|^{\frac{s(N-\mu)}{N}}+|v_n|^{\frac{s(2+N-\mu)}{N-2}})|v_n-v|^s dx\\
\leq& C\varepsilon[\|v_n\|_2^{\frac{2(N-\mu)}{2N-\mu}}\|v_n-v\|_2^s+\|v_n-v\|_{2^*}^s\|v_n\|_{2^*}^{\frac{2^*(2+N-\mu)}{2N-\mu}}]\leq C\varepsilon.
\end{aligned}\tag{3.7}$$

For any $r>0$ and $n\in\mathbb{N}$, set $B_r:=\{x\in\mathbb{R}^N:|x|<r\}$, $B_r^c:=\mathbb{R}^N\backslash B_r$ and

$$A_n:=\{x\in\mathbb{R}^N:\delta_0\leq|v_n(x)|\leq\rho_0\}.$$

Then

$$\begin{aligned}
&C\int_{\mathbb{R}^N}\chi_{[\delta_0,\rho_0]}(|v_n|)|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx\\
=&C\int_{A_n}|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx\\
=&C[\int_{A_n\cap B_r}|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx+\int_{A_n\cap B_r^c}|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx]\\
\leq& C\int_{A_n\cap B_r}|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx+C|A_n\cap B_r^c|,
\end{aligned}\tag{3.8}$$

where $|A_n\cap B_r^c|$ denotes the Lebesgue measure of $A_n\cap B_r^c$. Similar to (3.4), we have

$$C\int_{A_n\cap B_r}|g(f(v_n))f'(v_n)|^s|v_n-v|^s dx=o_n(1).\tag{3.9}$$

Now, we prove

$$\lim_{r\to+\infty}|A_n\cap B_r^c|=0.\tag{3.10}$$

Indeed, if this is not true, then there exist $\delta>0$ and $r_k\uparrow+\infty$ such that

$$|A_n\cap B_{r_k}^c|\geq\delta,\quad\forall\ k\in\mathbb{N}.$$

Obviously,

$$|A_n\cap B_{r_k}^c|\leq|A_n|:=\beta_n<+\infty,\quad\forall\ k\in\mathbb{N}.$$

On the other hand, set $\Omega_k := B^c_{r_k} \backslash B^c_{r_{k+1}}$. We have $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$, and

$$B^c_{r_k} = \bigcup_{i=k}^{\infty} \Omega_i, \quad \forall\ k \in \mathbb{N}.$$

Hence

$$\delta \leq |A_n \cap B^c_{r_k}| = \sum_{i=k}^{\infty} |A_n \cap \Omega_i|, \quad \forall\ k \in \mathbb{N},$$

and hence

$$\beta_n \geq |A_n \cap B^c_{r_1}| = \sum_{i=1}^{\infty} |A_n \cap \Omega_i| = +\infty.$$

This is a contradiction. Therefore, (3.10) holds. Now, we prove

$$\lim_{r \to +\infty} |A_n \cap B^c_r| = 0 \ \ uniformly \ \ in \ \ n \in \mathbb{N}. \tag{3.11}$$

In fact, for any $\varepsilon > 0$ there exists a $r_0 > 1$ such that

$$\int_{B^c_r} |v|^2 dx < \varepsilon \ \ whenever \ \ r \geq r_0.$$

Take $t_1 = r_0, t_j \uparrow +\infty$ be such that $D_j := B^c_{t_j} \backslash B^c_{t_{j+1}}$, $B^c_{r_0} = \bigcup\limits_{j=1}^{\infty} D_j$ and

$$\int_{D_j} |v|^2 dx < \frac{\varepsilon}{2^j}, \quad \forall\ j \in \mathbb{N}.$$

By Faou Lemma, we get

$$\limsup_{n \to \infty} \int_{A_n \cap D_j} |v_n|^2 dx \leq \int_{D_j} |v|^2 dx < \frac{\varepsilon}{2^j}, \quad \forall\ j \in \mathbb{N}.$$

Hence, as $r \geq r_0$, one has

$$\begin{aligned}
\delta_0^2 \limsup_{n \to \infty} |A_n \cap B^c_r| \leq & \delta_0^2 \limsup_{n \to \infty} |A_n \cap B^c_{r_0}| \\
\leq & \limsup_{n \to \infty} \int_{A_n \cap B^c_{r_0}} |v_n|^2 dx \\
= & \limsup_{n \to \infty} \sum_{j=1}^{\infty} \int_{A_n \cap D_j} |v_n|^2 dx \\
< & \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.
\end{aligned}$$

Together with (3.10), we know that (3.11) holds.

For any $\varepsilon > 0$, by (3.11) there exists a large $r > 0$ such that $|A_n \cap B_r^c| < \varepsilon$. Using (3.6) – (3.9) we know

$$\|g(f(v_n))f'(v_n)(v_n - v)\|_s^s = o_n(1). \tag{3.12}$$

Similarly, we have

$$\|g(f(v))f'(v)(v_n - v)\|_s. \tag{3.13}$$

Consequently, from (3.5) we get

$$\int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(v_n)))g(f(v_n))f'(v_n) - (|x|^{-\mu} * G(f(v)))g(f(v))f'(v)](v_n - v)dx = o_n(1).$$

Summing up (1^0) and (2^0), together with (3.1) we obtain

$$\begin{aligned}
o_n(1) =& \langle I'(v_n) - I'(v), v_n - v\rangle \\
=& \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 dx + \int_{\mathbb{R}^N} V(x)[f(v_n)f'(v_n) - f(v)f'(v)](v_n - v)dx \\
& - \int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(v_n)))g(f(v_n))f'(v_n) - (|x|^{-\mu} * G(f(v)))g(f(v))f'(v)](v_n - v)dx \\
\geq& C\|v_n - v\|_E^2 + o_n(1).
\end{aligned}$$

Hence $v_n \to v$ in E. This shows that I satisfies the Cerami condition.

Next, we prove that I has a mountain pass geometry. Indeed, set $A(v) := [\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)f^2(v))dx]^{\frac{1}{2}}$. As paragraph 1 of proof of Lemma 3.3 in [8] we can prove that there exist $C, \rho_1 > 0$ such that

$$A^2(v) \geq C\|v\|_E^2, \quad whenever \quad \|v\|_E \leq \rho_1. \tag{3.14}$$

Similar to (3.4) we know that

$$|\int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(v)))G(f(v))dx| \leq C\|G(f(v))\|_s^2 \leq C(\|v\|_E^{\frac{q_1}{\alpha}} + \|v\|_E^{\frac{q_2}{\alpha}})$$

Consequently, for small $0 < \rho < \min\{1, \rho_1\}$, we have

$$I(v) \geq C[\|v\|_E^2 - \|v\|_E^{\frac{q_1}{\alpha}} - \|v\|_E^{\frac{q_2}{\alpha}}] = C(\rho^2 - \rho^{\frac{q_1}{\alpha}} - \rho^{\frac{q_2}{\alpha}}) := \delta > 0 \; whenever \; \|v\|_E = \rho.$$

Take an $e \in E$ with $\|e\|_E = 1$. Set $B(e) := \frac{1}{2}\int_{\mathbb{R}^N}(|x|^{-\mu} * G(f(e)))G(f(e))dx$ and $\beta(t) := B(te)$ for $t > 0$. Then, by (g_2) and Lemma 2.1 (f_6), one has

$$\frac{\beta'(t)}{\beta(t)} \geq \frac{4}{t}, \quad \forall\, t > 0. \tag{3.15}$$

For $s > 1$, integrating (3.15) over $[1, s]$ we obtain $B(se) \geq s^4 B(e)$. Hence

$$I(se) \leq \frac{1}{2}s^2 - \frac{1}{2}s^4 \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(e)))G(f(e))dx \to -\infty \tag{3.16}$$

as $s \to +\infty$. Consequently, there is a $s_0 > \max\{1, \rho\}$ such that $I(v_0) := I(s_0 e) < 0$. This shows that I has a mountain pass geometry.

Moreover, for any finite-dimensional subspace $\tilde{E} \subset E$, we assert that there exists a constant $R > \rho$ such that $I < 0$ on $\tilde{E} \backslash B_R$. Otherwise, there is a sequence $\{v_n\} \subset \tilde{E}$ such that $s_n := \|v_n\|_E \to \infty$ and $I(v_n) \geq 0$. Set $e_n := \frac{v_n}{\|v_n\|_E}$. By (3.16), one has

$$0 \leq I(v_n) = I(s_n e_n) \leq \frac{1}{2}s_n^2 - \frac{1}{2}s_n^4 \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(e_n)))G(f(e_n))dx,$$

and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(e_n)))G(f(e_n))dx = 0.$$

By the Fatou Lemma we know that

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(e)))G(f(e))dx = 0.$$

Hence $e(x) = 0$ *a.e.* $x \in \mathbb{R}^N$. By the equivalency of all norms in $\tilde{E}$, there is a constant $C_2 > 0$ such that

$$\|v\|_2^2 \geq C_2 \|v\|_E^2, \quad \forall\, v \in \tilde{E}.$$

Hence

$$0 = \lim_{n\to\infty} \|e_n\|_2^2 \geq \lim_{n\to\infty} b\|e_n\|_E^2 = C_2,$$

a contradiction. This shows that there exists a constant $R > 0$ such that $I < 0$ on $\tilde{E} \backslash B_R$.

Since $E \hookrightarrow L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_j\}$. Set $E_k := span\{e_1, ..., e_k\}$ and $Z_k := E_k^{\perp}$, then $E = E_k \oplus Z_k$. Hence

$$I|_{S_\rho \cap Z_k} \geq \delta > 0.$$

Notice that the Deformation Theorem still hold under the Cerami condition (see [32]). Hence Theorem 2.2 and Theorem 9.12 in [25] hold under the Cerami condition. Therefore, Theorem 3.1 follows from Theorem 2.2 and Theorem 9.12 in [25]. This completes the proof. $\square$

Remark 3.2 From the proof of Theorem 3.1 we know that when $V_\infty = +\infty$, the limit of q_1 can relax as $2\alpha(2-\frac{\mu}{N}) \le q_1$.

Theorem 3.3 Assume the conditions (V) with $V_\infty < +\infty$ and (g_1) hold. If $V \in C^1(\mathbb{R}^N, \mathbb{R})$ with $x \cdot \nabla V(x) \in L^\infty(\mathbb{R}^N)$ and $V(rx)$ is non-increasing in $r \in \mathbb{R}$. Then the equation (1.1) has a nontrivial solution. Especially, if $V \equiv 1$, then the equation (1.1) has a ground state solution.

Proof. Set

$$\begin{aligned} P(v) := & \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x) f^2(v) dx + \frac{1}{2}\int_{\mathbb{R}^N} (x\cdot\nabla V(x)) f^2(v) dx \\ & - \frac{2N-\mu}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(v)))G(f(v)) dx. \end{aligned}$$

Define a mapping $\Phi : \mathbb{R}\times E \to E$ by

$$\Phi(r,v)(x) := v(e^{-r}x), \quad \forall\ (r,v) \in\ \mathbb{R}\times E \to E.$$

Then

$$\begin{aligned} I(\Phi(r,v)) = & \frac{e^{(N-2)r}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{e^{Nr}}{2}\int_{\mathbb{R}^N} V(e^r x) f^2(v) dx \\ & - \frac{e^{(2N-\mu)r}}{2}\int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(v)))G(f(v)) dx. \end{aligned}$$

Set

$$\widetilde{\Gamma} := \{\tilde\gamma \in C([0,1], \mathbb{R}\times E) : \tilde\gamma(0) = (0,0),\ I\circ\Phi(\tilde\gamma(1)) < 0\}.$$

Then

$$\Gamma := \{\Phi\circ\tilde\gamma : \tilde\gamma \in \widetilde{\Gamma}\} = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}$$

and

$$\inf_{\tilde\gamma\in\widetilde{\Gamma}} \sup_{t\in[0,1]} I\circ\Phi(\tilde\gamma(t)) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) := c.$$

From the proof of Theorem 3.1 we know that for small $\rho > 0$, we have

$$I(v) \ge \delta > 0 \ \textit{whenever}\ \|v\|_E = \rho.$$

Moreover, for any $v \in E\backslash\{0\}$ and $t>0$, set $v_t(x)=v(\frac{x}{t})$. Then

$$\begin{aligned}
I(v_t)=&\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{t^N}{2}\int_{\mathbb{R}^N}V(tx)f^2(v)dx-\frac{t^{2N-\mu}}{2}\int_{\mathbb{R}^N}(|x|^{-\mu}*G(f(v)))G(f(v))dx\\
\leq&\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{t^N}{2}C\int_{\mathbb{R}^N}f^2(v)dx-\frac{t^{2N-\mu}}{2}\int_{\mathbb{R}^N}(|x|^{-\mu}*G(f(v)))G(f(v))dx\\
&\to-\infty
\end{aligned}$$

as $t\to+\infty$. Hence, I has a mountain pass geometry, and hence $c>0$. Using Theorem 2.9 in [28], for $M=[0,1]$, $M_0=\{0,1\}$, $X=\mathbb{R}\times E$ and $\varphi=I\circ\Phi$, we know that there exists a sequence $\{(r_n,v_n)\}\subset\mathbb{R}\times E$ such that

$$\lim_{n\to\infty}I(\Phi(r_n,v_n))=c\ \ and\ \ (I\circ\Phi)'(r_n,v_n)\to 0.$$

Notice that

$$\langle(I\circ\Phi)'(r_n,v_n),(r,v)\rangle=\langle I'(\Phi(r_n,v_n)),\Phi(r_n,v)\rangle+rP(\Phi(r_n,v_n)),\quad\forall\ (r,v)\in\mathbb{R}\times E.$$

We have

$$I'(\Phi(r_n,v_n))\to 0,\quad P(\Phi(r_n,v_n))\to 0.$$

Set $w_n:=\Phi(r_n,v_n)$. Then

$$\begin{aligned}
&c+o_n(1)=I(w_n)-\frac{1}{2N-\mu}P(w_n)\\
=&\frac{e^{(N-2)r_n}}{2}\cdot\frac{N+2-\mu}{2N-\mu}\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\frac{e^{Nr_n}}{2}\cdot\frac{N-\mu}{2N-\mu}\int_{\mathbb{R}^N}V(e^{r_n}x)f^2(v_n)dx\\
&-\frac{e^{(N+2)r_n}}{2(2N-\mu)}\int_{\mathbb{R}^N}(x\cdot\nabla V(e^{r_n}x))f^2(v_n)dx.
\end{aligned}$$

For any $x\in\mathbb{R}^N$, set $h(r):=V(rx)$. Then $0\geq h'(r)=x\cdot\nabla V(rx)$ for all $r\in\mathbb{R}$. Hence

$$\begin{aligned}
&c+o_n(1)=I(w_n)-\frac{1}{2N-\mu}P(w_n)\\
\geq&\frac{e^{(N-2)r_n}}{2}\cdot\frac{N+2-\mu}{2N-\mu}\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\frac{e^{Nr_n}}{2}\cdot\frac{N-\mu}{2N-\mu}\int_{\mathbb{R}^N}V(e^{r_n}x)f^2(v_n)dx\\
=&\frac{N+2-\mu}{2(2N-\mu)}\int_{\mathbb{R}^N}|\nabla w_n|^2dx+\frac{N-\mu}{2(2N-\mu)}\int_{\mathbb{R}^N}V(x)f^2(w_n)dx\\
\geq&\frac{N-\mu}{2(2N-\mu)}A^2(w_n)\geq C\|w_n\|_E^2.
\end{aligned}$$

This shows $\{w_n\}\subset E$ is bounded.

Next, we prove that there exist $\sigma_0 > 0$, $\delta_0 > 0$ and $\{x_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_{\sigma_0}(x_n)} |w_n|^2 \geq \delta_0. \tag{3.17}$$

Indeed, if the conclusion is not true, then by Lemma 1.21 in [28] we know that

$$w_n \to 0 \ \ in \ \ L^s(\mathbb{R}^N), \ \ \forall\ s \in [2, 2^*).$$

Similar to the case (1^0) of Theorem 3.1 we can prove

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} [(|x|^{-\mu} * G(f(w_n)))g(f(w_n))f(w_n)dx = 0.$$

Hence

$$\begin{aligned}
o_n(1) =& \langle I'(w_n), \frac{f(w_n)}{f'(w_n)} \rangle \\
=& \int_{\mathbb{R}^N} [1 + (2\alpha - 1) \cdot \frac{2\alpha |f(w_n)|^{2(2\alpha-1)}}{1 + 2\alpha |f(w_n)|^{2(2\alpha-1)}}] |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x) f^2(w_n) dx + o_n(1) \\
\geq& C\|w_n\|_E^2 + o_n(1),
\end{aligned}$$

and hence $\|w_n\|_E \to 0$. Consequently, $0 < c = \lim\limits_{n\to\infty} I(w_n) = 0$, a contradiction. Therefore, (3.17) holds.

By the boundedness of $\{w_n\}$ and (3.17) we know that, up to a subsequence, there exists a $v_0 \in E\backslash\{0\}$ such that $w_n \rightharpoonup v_0$ in E, $w_n \to v_0$ in $L^t_{loc}(\mathbb{R}^N)$ for each $t \in [1, 2^*)$ and $w_n(x) \to v_0(x)$ a.e. $x \in \mathbb{R}^N$. Form the proof of the case (2^0) of Theorem 3.1 we know $w_n \to v_0$ in E. Consequently, $I(v_0) = c$, $I'(v_0) = 0$ and $P(v_0) = 0$. This shows that v_0 is a nontrivial solution of (1.5). Consequently, $u_0 := f(v_0)$ is a nontrivial solution of (1.1).

If $V \equiv 1$, then set $\mathcal{M} := \{v \in E\backslash\{0\} : I'(v) = 0\}$ and $c^* := \inf\limits_{v\in\mathcal{M}} I(v)$. Since $v_0 \in \mathcal{M}$, $c \geq c^*$. On the other hand, for any $w_0 \in M$, one has Pohozaev identity $P(w_0) = 0$. Define $\gamma : [0, +\infty) \to E$ by

$$\gamma(t)(x) = \begin{cases} w_0(\frac{x}{t}), & t > 0, \\ 0, & t = 0. \end{cases}$$

We claim that $\gamma : [0, +\infty) \to E$ is continuous. Indeed, if $t_0 \in (0, +\infty)$, $0 < t_n \to t_0$, then

$$\|\gamma(t_n)\|_E^2 = t_n^{N-2} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + t_n^N \int_{\mathbb{R}^N} |w_0|^2 dx \to \|\gamma(t_0)\|_E^2.$$

Hence there exists a $v \in E$ such that $\gamma(t_n) \rightharpoonup v$ in E, $\gamma(t_n) \to v$ in $L^p_{loc}(\mathbb{R}^N)$ for each $p \in [1, 2^*)$, $\gamma(t_n)(x) \to v(x)$ *a.e.* $x \in \mathbb{R}^N$. Notice that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} \gamma(t_n)(x)\varphi(x)dx = \int_{\mathbb{R}^N} t_n^N w_0(x)\varphi(t_n x))dx \to \int_{\mathbb{R}^N} \gamma(t_0)(x)\varphi(x)dx.$$

By the density of $C_0^\infty(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ we know that $\gamma(t_n) \rightharpoonup \gamma(t_0)$ in $L^2(\mathbb{R}^N)$. Consequently, $\gamma(t_0) = v$, and hence $\gamma(t_n) \to \gamma(t_0)$ in E. This shows that $\gamma : (0, +\infty) \to E$ is continuous. Moreover, as $0 < t_n \to 0$, we have $\|\gamma(t_n)\|_E \to 0$, obviously. Therefore, $\gamma : [0, +\infty) \to E$ is continuous.

Notice that as $t > 0$,

$$\begin{aligned} I(\gamma(t)) =& \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + \frac{t^N}{2} \int_{\mathbb{R}^N} f^2(w_0)dx \\ &- \frac{t^{2N-\mu}}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(w_0)))G(f(w_0))dx \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} I(\gamma(t)) =& \frac{(N-2)t^{N-3}}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + \frac{Nt^{N-1}}{2} \int_{\mathbb{R}^N} f^2(w_0)dx \\ &- \frac{(2N-\mu)t^{2N-\mu-1}}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(w_0)))G(f(w_0))dx. \end{aligned}$$

Since $0 < \mu < 2 < N$,

$$\begin{aligned} \frac{d}{dt} I(\gamma(t))|_{t=1} =& \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} f^2(w_0)dx \\ &- \frac{2N-\mu}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(w_0)))G(f(w_0))dx \\ =& P(w_0) = 0, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} I(\gamma(t))|_{t<1} =& t^{2N-\mu-1} [\frac{N-2}{2t^{N+2-\mu}} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + \frac{N}{t^{N-\mu}2} \int_{\mathbb{R}^N} f^2(w_0)dx \\ &- \frac{(2N-\mu)}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(w_0)))G(f(w_0))dx] \\ >& t^{2N-\mu-1} P(w_0) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} I(\gamma(t))|_{t>1} =& t^{2N-\mu-1} [\frac{N-2}{2t^{N+2-\mu}} \int_{\mathbb{R}^N} |\nabla w_0|^2 dx + \frac{N}{2t^{N-\mu}} \int_{\mathbb{R}^N} f^2(w_0)dx \\ &- \frac{2N-\mu}{2} \int_{\mathbb{R}^N} (|x|^{-\mu} * G(f(w_0)))G(f(w_0))dx] \\ <& t^{2N-\mu-1} P(w_0) = 0. \end{aligned}$$

Hence $I(\gamma(1)) = \max_{t\geq 0} I(\gamma(t))$. Moreover, obviously, there exists a $t^* > 0$ such that $I(\gamma(t^*)) < 0$. Define $\gamma_1 : [0,1] \to E$ by $\gamma_1(t) := \gamma(tt^*)$. Then $\gamma_1 \in \Gamma$. Hence

$$c \leq \sup_{t\in[0,1]} I(\gamma_1(t)) = \sup_{t\in[0,1]} I(\gamma(tt^*)) \leq I(\gamma(1)) = I(w_0),$$

and hence $c \leq c^*$. Therefore, $c = c^*$. This shows v_0 is a ground state solution (1.5), and hence $u_0 = f(v_0)$ is a ground state solution (1.1).

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