Existence, multiplicity and concentration of positive solutions for a modified Schrödinger equation with critical exponent *

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ABSTRACT. In this paper, we concern the modified Schrödinger equations

$$-\varepsilon^2\Delta u + V(x)u - \varepsilon^2 u\Delta u^2 = |u|^{22^*-2}u + g(u),\ x\ \in \mathbb{R}^N.$$

First, a existence result of ground state positive solutions is given. Next, we research multiplicity and concentration of positive solutions. Where $N \geq 2$, ε is positive parameters and $2^* = \frac{2N}{N-2}$ is the critical exponent, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, $g \in C(\mathbb{R}, \mathbb{R})$. Our results improve corresponding results in [10] (X. He, A. Qian, W. Zou, Existence and concentration of positive solutions for quasilinear Schrödinger equations with critical growth, Nonlinearity, 26(2013), 3137-3168).

Key Words: modified Schrödinger equations; gound state solutions; critical expo-

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nent.

1 Introduction and Preliminaries

This paper deals with the existence, multiplicity and concentration of positive solutions for modified Schrödinger equations with critical growth

$$-\varepsilon^2\Delta u + V(x)u - \varepsilon^2\Delta(u^2)u = |u|^{22^*-2}u + g(u),\ x\ \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $2^* = \frac{2N}{N-2}$. Moreover, $V \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$ satisfy the following assumptions:

(V) $0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x) < \lim_{|x|\to\infty} V(x) := V_\infty \leq \infty$;

(G) (g_1) $g(s) = o(|s|)$ as $|s| \to 0$;

(g_2) there exist $q \in (4, 22^*)$ and $\sigma > P_N$ such that

$$\lim_{s\to+\infty} \frac{g(s)}{s^{q-1}} = 0, \quad g(s) \geq C_0 s^{\sigma-1}, \quad \forall\ s > 0,$$

where $P_N = 4$ if $N \geq 6$ and $P_N = \frac{2(N+2)}{N-2}$ if $N = 2, 3, 4, 5$.

(g_3) $\frac{g(s)}{s^3}$ is increasing in $(0, +\infty)$, $g(s) = 0$ for $s \leq 0$.

Solutions of (1.1) are related to the standing wave solutions of the form

$$\psi(x,t) = e^{-iEt/\varepsilon}u(x)$$

for the Schrödinger equations

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\varepsilon^2\triangle\psi + V(x)\psi - \varepsilon^2\triangle(\psi^2)\psi - f(\psi), \quad x \in \mathbb{R}^N. \tag{1.2}$$

The equation (1.2) appears naturally in mathematical physics and had been derived as models of several physical phenomena. We refer the reader to [2, 4, 6] and references therein for more physical motivations and development of physical aspects. The equation (1.1) has been extensively studied in recent years, for example, see [1, 3, 5, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Particularly, in [10], the multiplicity and concentration of positive solutions of the equation (1.1) were studied, where except the conditions (V) and (G), authors add the following two hash conditions on the function g:

(g_4) $g \in C^1(\mathbb{R}, \mathbb{R})$;

(g_5) there exists $\theta \in (4, 22^*)$ such that

$$0 < \theta G(s) := \theta \int_0^s g(t)dt \leq sg(s), \quad \forall\ s \in (0, +\infty).$$

Motivated by the above reason, in the present paper, our aim is to research the existence, multiplicity and concentration of positive solutions for problem (1.1) without (g_4) and (g_5). Our results show that the two conditions (g_4) and (g_5) are no need.

Our main results as follows:

Theorem 1 Suppose that (V) and (G) are satisfied. Then there exist $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, the problem (1.1) possesses a ground state positive solution.

Theorem 2 Suppose that (V) and (G) are satisfied. Then for any $\delta > 0$ there exist $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, the problem (1.1) has at least $cat_{\Lambda_\delta}(\Lambda)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $\zeta_\varepsilon \in \mathbb{R}^N$ is its global maximum, then $\lim\limits_{\varepsilon \to 0} V(\zeta_\varepsilon) = V_0$ and $\lim\limits_{|x| \to \infty} u_\varepsilon(x) = 0$. Where $\Lambda := \{x \in \mathbb{R}^N : V(x) = V_0\}$ and $\Lambda_\delta := \{x \in \mathbb{R}^N : d(x, \Lambda) \leq \delta\}$.

We need the following preliminaries.

The Sobolev space $H^1(\mathbb{R}^N)$ is defined by

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$$

with the natural norm

$$\|u\|_{H^1(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx)^{\frac{1}{2}}.$$

Moreover, we define the homogeneous Sobolev space

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$$

with the norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

which can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\cdot\|_{D^{1,2}(\mathbb{R}^N)}$. We have the following results.

Lemma 1.1 *([26]) For $N \geq 2$, there exists a constant $C = C(N) > 0$ such that $\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \leq C\|u\|_{D^{1,2}(\mathbb{R}^N)}$ for every $u \in D^{1,2}(\mathbb{R}^N)$. Moreover the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for any $s \in [2, 2^*]$, and is locally compact whenever $s \in [2, 2^*)$.*

Lemma 1.2 *([26]) Assume that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 dx = 0,$$

where $R > 0$. Then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for every $2 \leq s < 2^*$.

Remark 1.1 *Similarly, at the case that the sequence $\{|u_n|^{2^*}\}$ is vanishing, we can prove that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for every $2 < s \leq 2^*$.*

The equation (1.1) is equivalent to the equation

$$-\Delta u + V(\varepsilon x)u - u\Delta u^2 = |u|^{22^*-2}u + g(u), \ x \ \in \mathbb{R}^N. \tag{1.3}$$

Set

$$X = \{u \in E_\varepsilon : u^2 \in H^1(\mathbb{R}^N)\},$$

where E_ε is defined as

$$E_\varepsilon = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx < +\infty\}$$

with the norm

$$\|u\|_\varepsilon = \Big(\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(\varepsilon x)u^2\big)dx\Big)^{\frac{1}{2}}.$$

Define the functional

$$\widetilde{J}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |u|^{22^*} dx - \int_{\mathbb{R}^N} G(u)dx,$$

where $G(u) := \int_0^u g(t)dt$. Then $\widetilde{J}_\varepsilon$ is well defined on X. We say that $u \in X$ is a weak solution of (1.3) if

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N} \big(\nabla u \nabla \varphi + V(\varepsilon x) u\varphi\big) dx + 2\int_{\mathbb{R}^N} (u^2 \nabla u \nabla \varphi + |\nabla u|^2 u\varphi) dx - \int_{\mathbb{R}^N} |u|^{22^*-2} u\varphi dx \\
&\quad - \int_{\mathbb{R}^N} g(u)\varphi dx \\
&= \langle \widetilde{J}'_\varepsilon(u), \varphi\rangle
\end{aligned}
$$

for all $\varphi \in X$. Once we get a solution u_ε of (1.3), then the function $v_\varepsilon(x) := u_\varepsilon(\frac{x}{\varepsilon})$ is a solution of (1.1). Set

$$
J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u^2 dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |u^+|^{22^*} dx - \int_{\mathbb{R}^N} G(u) dx,
$$

where $u^+(x) := \max\{u(x), 0\}$. Then

$$
\begin{aligned}
\langle J'_\varepsilon(u), \varphi\rangle = &\int_{\mathbb{R}^N} \big(\nabla u \nabla \varphi + V(\varepsilon x) u\varphi\big) dx + 2\int_{\mathbb{R}^N} (u^2 \nabla u \nabla \varphi + |\nabla u|^2 u\varphi) dx \\
&- \int_{\mathbb{R}^N} |u^+|^{22^*-1} \varphi dx - \int_{\mathbb{R}^N} g(u)\varphi dx
\end{aligned}
$$

for all $u \in X$ and $\varphi \in X$.

Since X is not a linear space, the critical point theory can not be direct applied for the functional J_ε on X. Moreover, as $V_\infty = +\infty$, the continuous embedding $E_\varepsilon \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \leq s < 2^*$ (see also Lemma 3.4 in [28]), so that the study of the problem is more easy, and hence, we consider only the case $V_\infty < +\infty$.

If u is a positive weak solution of (1.3) and $J_\varepsilon(u) = \inf\{J_\varepsilon(v) : v > 0 \textit{ and } J'_\varepsilon(v) = 0\}$, then u is called a ground state positive solution of (1.3). In order to overcome the difficulty cause by the nonlinearity of space X, we will adopt the dual method proposed by Liu-Wang-Wang [15] and Colin-Jeanjean in [5].

Set $\widetilde{f}(s) := \int_0^s \sqrt{1+2t^2}dt$. Then $\widetilde{f}$ is positive, strictly increasing, convex and C^∞ in $[0, +\infty)$. Hence, we set $f = \widetilde{f}^{-1}$ for $s \geq 0$. For $s \leq 0$, we put $f(s) := -f(-s)$, and hence

$$
f'(s) = \frac{1}{\sqrt{1+2f^2(s)}}, \quad \forall\, s \in (-\infty, +\infty).
$$

Set

$$\Phi_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) f^2(v(x)) dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(v^+(x))|^{22^*} dx$$
$$- \int_{\mathbb{R}^N} G(f(v(x))) dx.$$

Then Φ is well defined in E_ε and $0 \le v \in E_\varepsilon$ is a critical point of Φ_ε if and only if $u = f(v)$ is a nonnegative critical point of J_ε.

For completeness we collect here some properties of $f(t)$.

Lemma 1.3 *(see, [5, 10, 15, 18]) The function $f(t)$ enjoys the following properties:*

(1) f is uniquely defined C^∞ function and invertible.

(2) $|f'(t)| \le 1$ for all $t \in \mathbb{R}$.

(3) $|f(t)| \le |t|$ for all $t \in \mathbb{R}$.

(4) $\lim_{t\to 0} \frac{f(t)}{t} = 1$.

(5) $\lim_{t\to +\infty} \frac{f(t)}{\sqrt{t}} = 2^{\frac{1}{4}}$.

(6) $\frac{1}{2}f(t) \le tf'(t) \le f(t)$ for all $t \ge 0$, $\frac{1}{2}f(t) \ge tf'(t) \ge f(t)$ for all $t \le 0$;

(7) $|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$.

(8) $f^2(t)$ is strictly convex.

(9) There exists a positive constant C such that

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{\frac{1}{2}}, & |t| \ge 1. \end{cases}$$

(10) There exists a positive constants $C_0 > 1$ such that

$$|t| \le C_0[|f(t)| + |f(t)|^2], \quad \forall\ t \in \mathbb{R}.$$

(11) $|f(t)f'(t)| \le \frac{1}{\sqrt{2}}$ for all $t \in \mathbb{R}$.

(12) For each $\lambda > 0$, there is a constant $C(\lambda) > 0$ such that $f^2(\lambda t) \leq C(\lambda) f^2(t)$ for all $t \in \mathbb{R}$.

(13) $\frac{f(t)f'(t)}{t}$ is strictly decreasing for $t > 0$.

(14) $\frac{f^q(t)f'(t)}{t}$ is strictly increasing for $q \geq 3$ and $t > 0$.

(15) $\lim\limits_{t\to+\infty}[t - \frac{1}{4\sqrt{2}}\ln t - \frac{1}{\sqrt{2}}f^2(t)] = c_0 > 0$.

(16) $f^{22^*}(t) = 2^{\frac{N}{N-2}}t^{2^*} - ct^{2^*-1}\ln t + O(t^{2^*-1})$ as $t \to +\infty$.

In the following, without loss of generality, we assume $0 \in \Lambda$. By the condition (V) we know that Λ is compact. Throughout the paper, we denote distinct constants by C and C_i.

2 Preliminary Results

Set $E_\varepsilon^+ = \{v \in E_\varepsilon : v^+(x) \neq 0\}$ and $S_\varepsilon^+ := S_\varepsilon \bigcap E_\varepsilon^+$, where S_ε is the unit sphere of E_ε.

Lemma 2.1 *There exist constants $C, \rho > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(v) dx \geq C\|v\|_\varepsilon^2$$

whenever $\|v\|_\varepsilon \leq \rho$.

Proof: If this is false, then there is a sequence $\{v_n\} \subset E_\varepsilon$ such that $v_n \to 0$ in E_ε and

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(v_n) dx < \frac{1}{n}\|v_n\|_\varepsilon^2.$$

Set $w_n := \frac{v_n}{\|v_n\|_\varepsilon}$. Then

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) w_n^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x)[\frac{f^2(v_n)}{v_n^2} - 1] w_n^2 dx < \frac{1}{n}.$$

Notice that, up to a subsequence, one has $v_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$. Hence, for each $\delta > 0$, the measure $|\{x \in \mathbb{R}^N : |v_n(x)| > \delta\}| \to 0$ as $n \to \infty$, and hence,

$$\int_{|v_n(x)|>\delta} w_n^2 dx \leq |\{x \in \mathbb{R}^N : |v_n(x)| > \delta\}|^{\frac{2^*-2}{2^*}} (\int_{\mathbb{R}^N} |w_n|^{2^*} dx)^{\frac{2}{2^*}} \to 0$$

as $n \to \infty$. Consequently, by Lemma 1.3 (4) we know that

$$\int_{\mathbb{R}^N} V(\varepsilon x)[\frac{f^2(v_n)}{v_n^2} - 1]w_n^2 dx \to 0$$

as $n \to \infty$, and hence we get $1 = \|w_n\|_\varepsilon \to 0$, a contradiction. This completes the proof. ▮

Lemma 2.2 *The functional Φ_ε satisfies the mountain pass geometry, that is*

(i) There exist $\beta, \rho > 0$ such that

$$\Phi_\varepsilon(v) \geq \beta, \quad as \quad \|v\|_\varepsilon = \rho;$$

(ii) There exists $e \in E_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $\Phi_\varepsilon(e) < 0$.

Proof. (i) For any $v \in E_\varepsilon \backslash \{0\}$, we have

$$\begin{aligned} \Phi_\varepsilon(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v|^2 + V(\varepsilon x)f^2(v)\big)dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(v^+)|^{22^*}dx - \int_{\mathbb{R}^N} G(f(v))dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} V(\varepsilon x)f^2(v)dx - C_2\|v\|_\varepsilon^{2^*}. \end{aligned} \tag{2.1}$$

By Lemma 2.1 we get first conclusion of Lemma 2.2.

(ii) For each $t > 0$ and each $v \in E_\varepsilon$ with $\|v\|_\varepsilon = 1$ and $v > 0$, one has

$$\begin{aligned} \Phi_\varepsilon(tv) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)f^2(tv)dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(tv)|^{22^*}dx - \int_{\mathbb{R}^N} G(f(tv))dx \\ &\leq \frac{t^2}{2}[\int_{\mathbb{R}^N} |\nabla v|^2 dxdy + \int_{\mathbb{R}^N} V(\varepsilon x)v^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} t^{2^*-2}|v|^{2^*} \cdot \frac{f^{22^*}(tv)}{t^{2^*}|v|^{2^*}}dx]. \end{aligned}$$

By Fatou Lemma and Lemma 1.3 (5), we know

$$\liminf_{t\to+\infty} \int_{\mathbb{R}^N} t^{2^*-2}v^{2^*} \cdot \frac{f^{22^*}(tv)}{t^{2^*}v^{2^*}}dx = +\infty.$$

Hence $\lim\limits_{t\to+\infty} \Phi_\varepsilon(tv) = -\infty$, and hence the conclusion (ii) holds. □

Lemma 2.3 *The following properties hold:*

(1) For each $v \in E_\varepsilon^+$ and $t > 0$, set $h_v(t) := \Phi_\varepsilon(tv)$. Then there exists an unique $t_v > 0$ such that $h_v(t_v) = \max_{t\geq 0} h_v(t)$, $h_v'(t_v) = 0$, $h_v'(t) > 0$ in $(0, t_v)$, $h_v'(t) < 0$ in $(t_v, +\infty)$ and $tv \in \mathcal{N}_\varepsilon$ if and only if $t = t_v$, where $\mathcal{N}_\varepsilon := \{v \in E_\varepsilon^+ : \langle \Phi_\varepsilon'(v), v\rangle = 0\}$.

(2) There exists a $\tau > 0$ independent of v such that $t_v > \tau$ for all $v \in S_\varepsilon^+$. Moreover, for each compact set $D \subset S_\varepsilon^+$ there exists $C_D > 0$ such that $t_v \leq C_D$ for all $v \in D$.

(3) The map $\widehat{m}_\varepsilon : E_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\widehat{m}(v) = t_v v$ is continuous and $m_\varepsilon := \widehat{m}_\varepsilon|_{S_\varepsilon^+}$ is a homeomorphism between S_ε^+ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(v) = \frac{v}{\|v\|_\varepsilon}$.

Proof: (1) From the proof of Lemma 2.2 we know that $h_v(0) = 0$, $h_v(t) > 0$ for small $t > 0$ and $\lim_{t\to+\infty} h_v(t) = -\infty$. Hence, there exists a $t_v > 0$ such that $h_v(t_v) = \max_{t\geq 0} h_v(t)$ and $h_v'(t_v) = 0$. Notice that

$$h_v'(t) = 0 \Leftrightarrow tv \in \mathcal{N}_\varepsilon \Leftrightarrow$$

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^2 dx = -\int_{\mathbb{R}^N} V(\varepsilon x)\frac{f(tv)f'(tv)}{tv} v^2(x,0)dx + \int_{\mathbb{R}^N} \frac{f^{22^*-1}(tv^+)f'(tv^+)}{tv^+}|v^+|^2 dx \\ + \int_{\mathbb{R}^N} \frac{g(f(tv^+))}{f^3(tv^+)} \cdot \frac{f^3(tv^+)f'(tv^+)}{tv^+}|v^+|^2 dx.\end{aligned}$$

By Lemma 1.3 (13)-(14) and the condition (g_3) we know that the right side is strictly increasing in $t > 0$. Hence t_v is unique. This completes the proof of (1).

(2) By Lemma 2.1 there exist constants $C, \rho > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(v) dx \geq C\|v\|_\varepsilon^2$$

whenever $\|v\|_\varepsilon \leq \rho$. If the first conclusion of (2) is false, then there exists a sequence $\{v_n\} \subset S_\varepsilon^+$ such that $t_n := t_{v_n} \to 0^+$. Hence, for large n, one has $0 < t_n < \rho$. Notice that

$$\begin{aligned}& t_n^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f(t_n v_n) f'(t_n v_n) t_n v_n dx \\ &= \int_{\mathbb{R}^N} f^{22^*-1}(t_n v_n^+) f'(t_n v_n^+) t_n v_n^+ dx + \int_{\mathbb{R}^N} g(f(t_n v_n^+)) f'(t_n v_n^+) t_n v_n^+ dx.\end{aligned}$$

By (g_1), (g_2) and Lemma 1.3 (6)-(7) we know that

$$\begin{aligned}
&\frac{1}{4}Ct_n^2\\
&\leq \frac{1}{4}[\int_{\mathbb{R}^N} |\nabla(t_n v_n)|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(t_n v_n) dx]\\
&\leq \int_{\mathbb{R}^N} |\nabla(t_n v_n)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) f(t_n v_n)) f'(t_n v_n) t_n v_n dx\\
&\leq C_3 \int_{\mathbb{R}^N} f^{22^*-1}(t_n v_n^+) f'(t_n v_n^+) t_n v_n^+ dx\\
&\leq C_4 \int_{\mathbb{R}^N} |t_n v_n|^{2^*} dx\\
&\leq C_5 t_n^{2^*}.
\end{aligned}$$

This contradicts that $t_n \to 0^+$. Hence the first conclusion of (2) holds.

Now, we prove the second conclusion of (2). If this is false, then there exists a sequence $\{v_n\} \subset D$ such that $t_n := t_{v_n} \to +\infty$. Since D is compact, we can assume that $v_n \to v \in D$. From the proof of Lemma 2.2 (ii) we know that $\lim\limits_{n\to\infty} \Phi_\varepsilon(t_n v_n) = -\infty$. By Remark 1.1 in [25] and (g_3) we know that

$$tg(t) - 4G(t) \geq 0, \quad \forall t \in \mathbb{R}.$$

Hence, by Lemma 1.3 (6), one has

$$\begin{aligned}
\Phi_\varepsilon(t_n v_n) &= \Phi_\varepsilon(t_n v_n) - \frac{1}{2}\langle \Phi'_\varepsilon(t_n v_n), t_n v_n\rangle\\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)[f^2(t_n v_n) - f(t_n v_n) f'(t_n v_n) t_n v_n] dx\\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{2}|f(t_n v_n^+)|^{22^*-1} f'(t_n v_n^+) t_n v_n^+ - \frac{1}{22^*}|f(t_n v_n^+)|^{22^*}] dx\\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{2} g(f(t_n v_n)) f'(t_n v_n) t_n v_n(x,0) - G(f(t_n v_n))] dx\\
&\geq \int_{\mathbb{R}^N} [\frac{1}{4} - \frac{1}{22^*}] |f(t_n v_n^+)|^{22^*} dx\\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{4} g(f(t_n v_n)) f(t_n v_n) - G(f(t_n v_n))] dx\\
&\geq 0.
\end{aligned}$$

This contradicts that $\lim\limits_{n\to\infty} \Phi_\varepsilon(t_n v_n) = -\infty$. Hence, the second conclusion of (2) holds.

(3) Obviously, $\widehat{m}_\varepsilon, m_\varepsilon$, m_ε^{-1} are well defined, and m_ε^{-1} is continuous. Since

$$m_\varepsilon^{-1}(m_\varepsilon(v)) = v, \quad \forall\ v \in S_\varepsilon^+,$$

$m_\varepsilon : S_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is a bijection. Now, we prove that $\widehat{m}_\varepsilon : E_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous. Indeed, let $\{v_n\} \subset E_\varepsilon^+$ be such that $v_n \to v$ in E_ε^+. By the conclusion (2) we know that, up to a subsequence, $t_n := t_{v_n} \to t_0 > 0$ and

$$\begin{aligned} & t_n^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f(t_n v_n) f'(t_n v_n) t_n v_n dx \\ & = \int_{\mathbb{R}^N} f^{22^*-1}(t_n v_n^+)) f'(t_n v_n^+) t_n v_n^+ dx + \int_{\mathbb{R}^N} g(f(t_n v_n)) f'(t_n v_n) t_n v_n dx. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above equality, we get

$$\begin{aligned} & t_0^2 \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f(t_0 v) f'(t_0 v) t_0 v dx \\ & = \int_{\mathbb{R}^N} f^{22^*-1}(t_0 v^+) f'(t_0 v^+) t_0 v^+ dx + \int_{\mathbb{R}^N} g(f(t_0 v)) f'(t_0 v) t_0 w dx. \end{aligned}$$

This means that $t_0 v \in \mathcal{N}_\varepsilon$, and hence $t_0 = t_v$. Consequently, $\widehat{m}_\varepsilon(v_n) \to \widehat{m}_\varepsilon(v)$ in E_ε^+. This shows that $\widehat{m}_\varepsilon : E_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous. This completes the proof of (3). ∎

Now, we define the functional $\widehat{\Psi}_\varepsilon : E_\varepsilon^+ \to \mathbb{R}$ by $\widehat{\Psi}_\varepsilon(v) = \Phi_\varepsilon(\widehat{m}_\varepsilon(v))$ and $\Psi_\varepsilon := \widehat{\Psi}_\varepsilon|_{S_\varepsilon^+}$. By Lemma 2.3, similar to Lemma 2.3 in [23] we can prove the following Lemma.

Lemma 2.4 *(1)* $\widehat{\Psi}_\varepsilon \in C^1(E_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}'_\varepsilon(w)v = \frac{\|\widehat{m}_\varepsilon(w)\|_\varepsilon}{\|w\|_\varepsilon} \Phi'_\varepsilon(\widehat{m}_\varepsilon(w))v, \quad \forall\ w \in E_\varepsilon^+ \ \text{ and } \ \forall\ v \in E_\varepsilon.$$

(2) $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*

$$\Psi'_\varepsilon(w)v = \|m_\varepsilon(w)\|_\varepsilon \Phi'_\varepsilon(m_\varepsilon(w))v, \quad \forall\ w \in S_\varepsilon^+ \ \text{ and } \ \forall\ v \in T_w S_\varepsilon^+,$$

where

$$T_w S_\varepsilon^+ := \{v \in E_\varepsilon : \langle w, v \rangle_\varepsilon = 0\}.$$

(3) If $\{w_n\}$ *is a* $(C)_d$ *sequence of* Ψ_ε*, that is* $\Psi_\varepsilon(w_n) \to d$ *and* $(1 + \|w_n\|_\varepsilon)\Psi'_\varepsilon(w_n) \to 0$*, then* $\{m_\varepsilon(w_n)\}$ *is a* $(C)_d$ *sequence of* Φ_ε*. If* $\{w_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(C)_d$ *sequence of* Φ_ε*, then* $\{m_\varepsilon^{-1}(w_n)\}$ *is a* $(C)_d$ *sequence of* Ψ_ε*.*

(4) w *is a critical point of* Ψ_ε *if and only if* $m_\varepsilon(w)$ *is a critical point of* Φ_ε*. Moreover, corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Phi_\varepsilon.$$

Using standard methods, we can prove the following Lemma 2.5.

Lemma 2.5

$$c_\varepsilon := \inf_{\mathcal{N}_\varepsilon} \Phi_\varepsilon = \inf_{v\in E_\varepsilon^+} \max_{t>0} \Phi_\varepsilon(tv) = \inf_{v\in S_\varepsilon^+} \max_{t>0} \Phi_\varepsilon(tv) > 0.$$

Lemma 2.6 *Let $\{w_n\} \subset E_\varepsilon$ and $A_n^2 := \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n)dx$. If $\{A_n\}$ is bounded, then there exists constant $C > 0$ such that*

$$A_n^2 \geq C\|w_n\|_\varepsilon^2. \tag{2.2}$$

Proof: We may assume that $w_n \neq 0$ (Otherwise, the conclusion is trivial). If this conclusion is not true, passing to a subsequence, we have $\frac{A_n^2}{\|w_n\|_\varepsilon^2} \to 0$. Set $v_n = \frac{w_n}{\|w_n\|_\varepsilon}$ and $g_n(x) = \frac{f^2(w_n(x))}{\|w_n\|_\varepsilon^2}$. Then

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) g_n(x) dx \to 0.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to 0, \qquad \int_{\mathbb{R}^N} V(\varepsilon x) g_n(x) dx \to 0, \qquad \int_{\mathbb{R}^N} V(\varepsilon x) v_n^2 dx \to 1.$$

We assert that for each $\delta > 0$, there exists a constant $C_1 > 0$ independent of n such that $meas(\Omega_n) < \delta$, where $\Omega_n := \{x \in \mathbb{R}^N : |w_n(x)| \geq C_1\}$. Otherwise, there is a $\delta_0 > 0$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that for any positive integer k,

$$meas(\{x \in \mathbb{R}^N : |w_{n_k}(x)| \geq k\}) \geq \delta_0 > 0.$$

Set $\Omega_{n_k} := \{x \in \mathbb{R}^N : |w_{n_k}(x)| \geq k\}$. By Lemma 1.3 (9) and (V)

$$A_{n_k}^2 \geq \int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_{n_k}) dx \geq \int_{\Omega_{n_k}} V(\varepsilon x) f^2(w_{n_k}) dx \geq Ck\delta_0 \to +\infty$$

as $k \to \infty$, a contradiction. Hence the assertion is true.

Obviously, we can assume $C_1 > 1$. Hence, as $|w_n(x)| \leq C_1$, by Lemma 1.3 (9) and (12), one has

$$\frac{w_n^2(x)}{C_1^2} \leq C_0 f^2(\frac{1}{C_1} w_n(x)) \leq C_2 f^2(w_n(x)).$$

Hence there exists a constant $C_3 > 0$ such that

$$\int_{\mathbb{R}^N \setminus \Omega_n} V(\varepsilon x) v_n^2 dx \leq C_3 \int_{\mathbb{R}^N} V(\varepsilon x) g_n(x) dx \to 0.$$

By the integral absolutely continuity, there exists $\delta > 0$ such that whenever $\Omega \subset \mathbb{R}^N$ and $meas(\Omega) < \delta$, $\int_\Omega V(\varepsilon x) v_n^2 dx < \frac{1}{2}$. For this δ, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} V(\varepsilon x) v_n^2 dx &= \int_{\Omega_n} V(\varepsilon x) v_n^2 dx + \int_{\mathbb{R}^N \setminus \Omega_n} V(\varepsilon x) v_n^2 dx \\
&\leq \frac{1}{2} + \int_{\mathbb{R}^N \setminus \Omega_n} V(\varepsilon x) v_n^2 dx,
\end{aligned}$$

which implies $1 \leq \frac{1}{2}$, a contradiction. Hence there exists a constant $C > 0$ such that $A_n^2 \geq C \|w_n\|_\varepsilon^2$. This completes the proof of Lemma 2.6. ▮

Lemma 2.7 *Let $\{w_n\} \subset E_\varepsilon$ is a $(C)_d$ sequence of Φ_ε. Then $\{w_n\}$ is bounded and $\{w_n^-\} = o_n(1)$.*

Proof: Since

$$\begin{aligned}
d + o_n(1) &\geq \Phi_\varepsilon(w_n) - \frac{1}{2}\langle \Phi_\varepsilon'(w_n), w_n \rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)[f^2(w_n) - f(w_n) f'(w_n) w_n] dx \\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{2}|f(w_n^+)|^{22^*-1} f'(w_n^+) w_n^+ - \frac{1}{22^*}|f(w_n^+)|^{22^*}] dx \\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{2} g(f(w_n)) f'(w_n) w_n - G(f(w_n))] dx \\
&\geq \int_{\mathbb{R}^N} [\frac{1}{4} - \frac{1}{22^*}] |f(w_n^+)|^{22^*} dx,
\end{aligned}$$

the sequence $\{\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*} dx\}$ is bounded. Since again

$$\begin{aligned}
& d + o_n(1) \\
&= \Phi_\varepsilon(w_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n) dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*} dx - \int_{\mathbb{R}^N} G(f(w_n)) dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n) dx - C\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*} dx,
\end{aligned}$$

the sequence $\{A_n\}$ is bounded, where $A_n^2 := \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n) dx$. By Lemma 2.6 we know that the sequence $\{w_n\}$ is bounded in E_ε.

Next, we prove that $\{w_n^-\} = o_n(1)$. Indeed, since $\{w_n\}$ is bounded in E_ε, so does $\{w_n^-\}$. Hence, similar to (2.2) we have

$$\int_{\mathbb{R}^N} |\nabla w_n^-|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n^-) dx \geq C\|w_n^-\|_\varepsilon^2. \tag{2.3}$$

Consequently,

$$\begin{aligned} o_n(1) &= \langle \Phi_\varepsilon'(w_n), w_n^- \rangle \\ &= [\int_{\mathbb{R}^N} |\nabla w_n^-|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f(w_n^-) f'(w_n^-) w_n^- dx] \\ &\geq \frac{1}{2}[\int_{\mathbb{R}^N} |\nabla w_n^-|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(w_n^-) dx] \\ &\geq C\|w_n^-\|_\varepsilon^2. \end{aligned}$$

This completes the proof of Lemma 2.7. ▮

Lemma 2.8 *There exists a constant $r > 0$ such that $\|w\|_\varepsilon \geq r$ for all $\varepsilon \geq 0$ and $w \in \mathcal{N}_\varepsilon$.*

Proof: If this false, then there is a sequence $\{w_n\} \subset \mathcal{N}_{\varepsilon_n}$ such that $\|w_n\|_{\varepsilon_n} \to 0$. Notice that for each $\delta > 0$ there is $C_\delta > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_0 f^2(w_n) dx \\ &\leq \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x) f^2(w_n) dx \\ &\leq \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) f(w_n) f'(w_n) w_n dx \\ &= \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*-1} f'(w_n^+) w_n dx + \int_{\mathbb{R}^N} g(f(w_n)) f'(w_n) w_n dx \\ &\leq C_\delta \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*} dx + \delta \int_{\mathbb{R}^N} f^2(w_n) dx. \end{aligned}$$

Hence, for small $\delta > 0$, there is $C_\delta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} V_0 f^2(w_n) dx \leq C_\delta \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*} dx.$$

By Lemma 2.6 we know that

$$\begin{aligned}\frac{1}{4}C\int_{\mathbb{R}^N}(|\nabla w_n|^2+w_n^2)dx &\le \frac{1}{4}[\int_{\mathbb{R}^N}|\nabla w_n|^2dx+\int_{\mathbb{R}^N}V_0f^2(w_n)dx]\\ &\le \int_{\mathbb{R}^N}|\nabla w_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^N}V_0f^2(w_n)dx\\ &\le C_1\int_{\mathbb{R}^N}|f(w_n^+)|^{22^*}dx\\ &\le C_2\int_{\mathbb{R}^N}|w_n|^{2^*}dx\\ &\le C_3(\int_{\mathbb{R}^N}(|\nabla w_n|^2+w_n^2)dx)^{\frac{2^*}{2}}.\end{aligned}$$

It contradicts that $\|w_n\|_{\varepsilon_n}\to 0$. This completes the proof of Lemma 2.8. ∎

When $V\equiv 1$, set $E:=E_\varepsilon$, $E^+:=E_\varepsilon^+$. For $\mu>0$ and $v\in E$, set

$$\Upsilon_\mu(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{\mu}{2}\int_{\mathbb{R}^N}f^2(v)dx-\frac{1}{22^*}\int_{\mathbb{R}^N}|f(v^+)|^{22^*}dx-\int_{\mathbb{R}^N}G(f(v))dx,$$

$$\mathcal{M}_\mu:=\{w\in E^+:\langle\Upsilon'_\mu(w),w\rangle=0\},\quad \widetilde{c}_\mu:=\inf_{\mathcal{M}_\mu}\Upsilon_\mu.$$

The following the proofs of Lemmas 2.9 and 2.10 are similar to Lemmas 2.10 and 2.12 in [10], respectively. In order to completeness, we give its proof, too.

Lemma 2.9 *For any $\mu>0$, there exists $w\in E^+$ such that*

$$\max_{t\ge 0}\Upsilon_\mu(tw)<\frac{1}{2N}S^{\frac{N}{2}},$$

where $S:=\inf_{u\in D^{1,2}(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{(\int_{\mathbb{R}^N}|u|^{2^*}dx)^{\frac{2}{2^*}}}$.

Proof: We consider the functional

$$J_\mu(u)=\Upsilon_\mu(f^{-1}(u))=\frac{1}{2}\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2dx+\frac{\mu}{2}\int_{\mathbb{R}^N}u^2dx-\frac{1}{22^*}|u^+|^{22^*}dx-\int_{\mathbb{R}^N}G(u)dx.$$

Then, it is sufficient to prove there exists $0\ne w\in X$ such that

$$\sup_{t\ge 0}J_\mu(tw)<\frac{1}{2N}S^{\frac{N}{2}}.$$

Indeed, since $\Upsilon_\mu(f^{-1}(tw)) = J_\mu(tw) \to -\infty$ as $t \to \infty$, there is $t^* > 0$ such that $\Upsilon_\mu(f^{-1}(t^*w)) < 0$. Then for $\gamma^*(t) := f^{-1}(tt^*w)$, there holds

$$\widetilde{c}_\mu \leq \sup_{t\in[0,1]} \Upsilon_\mu(\gamma^*(t)) \leq \sup_{t\geq 0} \Upsilon(f^{-1}(tw)) = \sup_{t\geq 0} J_\mu(tw) < \frac{1}{2N} S^{\frac{N}{2}}.$$

For $\delta > 0$, set

$$U_\delta(x) := \frac{[N(N-2)\delta]^{\frac{N-2}{4}}}{[\delta + |x|^2]^{\frac{N-2}{2}}}.$$

By [10] we know that S can achieved by U_δ. Moreover, U_δ satisfies

$$\int_{\mathbb{R}^N} |\nabla U_\delta|^2 dx = \int_{\mathbb{R}^N} |U_\delta|^{2^*} dx = S^{\frac{N}{2}}.$$

Let $\phi_0 \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a non-increasing cut-off function such that

$$\phi_0(x) = 1 \ \ if \ \ |x| < 1, \ \ \phi_0(x) = 0 \ \ if \ \ |x| \geq 2.$$

Define the function

$$w_\delta(x) = \phi_0(x) U_\delta^{\frac{1}{2}}(x) = \frac{\phi_0(x)[N(N-2)\delta]^{\frac{N-2}{8}}}{[\delta + |x|^2]^{\frac{N-2}{4}}}.$$

Set $\eta_\delta = \frac{w_\delta}{\|w_\delta\|_{2^*}^{\frac{1}{2}}}$. From [10] we know that

$$\int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx = S + O(\delta^{\frac{N-2}{2}}),$$

$$\int_{\mathbb{R}^N} |\eta_\delta|^p dx = \begin{cases} O(\delta^{\frac{p(N-2)}{8}}), & if \ 1 < p < 2^*, \\ O(\delta^{\frac{N}{4}} |ln\delta|), & if \ p = 2^*, \\ O(\delta^{\frac{N}{2} - \frac{p(N-2)}{8}}), & if \ 2^* < p < 22^*. \end{cases}$$

Therefore, $\eta_\delta \in X$ and by (g_2), we get

$$J_\mu(t\eta_\delta) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla \eta_\delta|^2 + \mu \eta_\delta^2) dx + \frac{t^4}{4} \int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx - \frac{C_0}{\sigma} t^\sigma \int_{\mathbb{R}^N} |\eta_\delta|^\sigma dx - \frac{t^{22^*}}{22^*} := h(t).$$

It is easy to verify that $\lim_{t\to+\infty} h(t) = -\infty$ and $h(t) > 0$ when $t > 0$ small enough. Consequently, there exists $t_\delta > 0$ such that $\max_{t\geq 0} h(t) = h(t_\delta)$ and $h'(t_\delta) = 0$, from which we deduce that

$$\int_{\mathbb{R}^N} (|\nabla \eta_\delta|^2 + \mu \eta_\delta^2) dx + t_\delta^2 \int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx = C_0 t_\delta^{\sigma-2} \int_{\mathbb{R}^N} |\eta_\delta|^\sigma dx + t^{22^*-2}.$$

Then the equality implies that there exists $T_1 > 0$ such that $t_\delta \geq T_1$ and $\{t_\delta\}$ is bounded.

Set $l(t) = \frac{t^4}{4}\int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx - \frac{t^{22^*}}{22^*}$. Then function attains its unique global maximum at $t_0 := (\int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx)^{\frac{1}{22^*-4}}$. Thus, by the properties of g, for $\delta > 0$ small enough, we deduce

$$\begin{aligned}
\max_{t\geq 0} J_\mu(t\eta_\delta) &\leq h(t_\delta) \\
&\leq l(t_0) + \frac{t_\delta^2}{2}\int_{\mathbb{R}^N} (|\nabla \eta_\delta|^2 + \eta_\delta^2)dx - \frac{C_0}{\sigma} t_\delta^\sigma \int_{\mathbb{R}^N} |\eta_\delta|^\sigma dx \\
&\leq (\frac{1}{4} - \frac{1}{22^*})(\int_{\mathbb{R}^N} |\nabla(\eta_\delta^2)|^2 dx)^{\frac{22^*}{22^*-4}} + C\int_{\mathbb{R}^N} (|\nabla \eta_\delta|^2 + \eta_\delta^2)dx - C\int_{\mathbb{R}^N} |\eta_\delta|^\sigma dx \\
&= \frac{1}{2N}(S + O(\delta^{\frac{N-2}{2}}))^{\frac{N}{2}} + C\int_{\mathbb{R}^N} (|\nabla \eta_\delta|^2 + \eta_\delta^2)dx - C\int_{\mathbb{R}^N} |\eta_\delta|^\sigma dx \\
&= \frac{1}{2N} S^{\frac{N}{2}} + O(\delta^{\frac{N-2}{2}}) + O(\delta^{\frac{N-2}{4}}) - O(\delta^{\frac{N}{2} - \frac{\sigma(N-2)}{8}}) \\
&< \frac{1}{2N} S^{\frac{N}{2}}.
\end{aligned}$$

Thus, the proof is completed. ▮

Lemma 2.10 *Let $\{w_n\} \subset E_\varepsilon$ be a $(PS)_d$ sequence of Φ_ε with $d < \frac{1}{2N} S^{\frac{N}{2}}$ and $w_n \rightharpoonup 0$ in E_ε. Then one of the following conclusions holds:*

(a) $w_n \to 0$ in E_ε;

(b) There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and positive constants r, β such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |w_n(x)|^2 dx \geq \beta.$$

Proof: If (b) does not occur, then for each $r > 0$, up to a subsequence,

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_r(z)} |w_n|^2 dx = 0.$$

Hence, by Lemma 1.2, one has

$$w_n \to 0 \;\; in \;\; L^t(\mathbb{R}^N), \;\; \forall\, t \in [2, 2^*).$$

By (g_1), (g_2) and Lemma 1.3 (3) (7) we know that

$$\int_{\mathbb{R}^N} G(f(w_n))dx = o_n(1), \;\; \int_{\mathbb{R}^N} g(f(w_n))f(w_n)dx = o_n(1). \tag{2.4}$$

Set $\phi = \frac{f(w_n)}{f'(w_n)}$. Then $|\phi| = |f(w_n)\sqrt{1+2f^2(w_n)}| \le 3|w_n|$ and

$$|\nabla\phi| = |[1+\frac{2f^2(w_n)}{1+2f^2(w_n)}]\nabla w_n| \le 2|\nabla w_n|.$$

Hence $\|\phi\|_\varepsilon \le C\|w_n\|_\varepsilon$. Consequently, $\langle \Phi'_\varepsilon(w_n), \phi\rangle = o_n(1)$, and hence

$$\begin{aligned}
&\int_{\mathbb{R}^N}[1+\frac{2f^2(w_n)}{1+2f^2(w_n)}]|\nabla w_n|^2dx + \int_{\mathbb{R}^N} V(\varepsilon x)f^2(w_n)dx\\
&= \int_{\mathbb{R}^N} g(f(w_n))f(w_n)dx + \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*}dx\\
&= \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*}dx + o_n(1).
\end{aligned}$$

Since $\{w_n\}$ is bounded in E_ε, up to a subsequence, there is a number $l \ge 0$ such that

$$\int_{\mathbb{R}^N}[1+\frac{2f^2(w_n)}{1+2f^2(w_n)}]|\nabla w_n|^2dx + \int_{\mathbb{R}^N} V(\varepsilon x)f^2(w_n)dx \to l$$

and

$$\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*}dx \to l.$$

If $l > 0$, then

$$\begin{aligned}
S &\le \frac{\int_{\mathbb{R}^N} |\nabla f^2(w_n^+)|^2dx}{(\int_{\mathbb{R}^N} |f^2(w_n^+)|^{2^*}dx)^{\frac{2}{2^*}}}\\
&\le \frac{\int_{\mathbb{R}^N}[1+\frac{2f^2(w_n)}{1+2f^2(w_n)}]|\nabla w_n|^2dx + \int_{\mathbb{R}^N} V(\varepsilon x)f^2(w_n)dx}{(\int_{\mathbb{R}^N} |f^2(w_n^+)|^{2^*}dx)^{\frac{2}{2^*}}}\\
&\to l^{\frac{2}{N}}
\end{aligned}$$

as $n \to \infty$. Hence $l \ge S^{\frac{N}{2}}$. Consequently, by (2.4), one has

$$\begin{aligned}
d &= \lim_{n\to\infty} \Phi_\varepsilon(w_n)\\
&= \lim_{n\to\infty}[\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla w_n|^2 + V(\varepsilon x)f^2(w_n)\big)dx - \int_{\mathbb{R}^N} G(f(w_n))dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*}dx]\\
&\ge \lim_{n\to\infty}\{\frac{1}{4}[\int_{\mathbb{R}^N}(1+\frac{2f^2(w_n)}{1+2f^2(w_n)})|\nabla w_n|^2dx + \int_{\mathbb{R}^N} V(\varepsilon x)f^2(w_n)dx]\\
&\qquad - \frac{1}{22^*}\int_{\mathbb{R}^N} |f(w_n^+)|^{22^*}dx\}\\
&= \frac{l}{2N} \ge \frac{1}{2N}S^{\frac{N}{2}},
\end{aligned}$$

a contradiction. Hence $l = 0$. Consequently, by the boundedness of $\{w_n\}$ in E_ε and Lemma 2.6, one has $w_n \to 0$ in E_ε, ie (a) holds. This completes the proof. ▮

Lemma 2.11 *Let $\{w_n\} \subset E_\varepsilon$ be a $(PS)_d$ sequence of Φ_ε with $d < \frac{1}{2N} S^{\frac{N}{2}}$ and $w_n \rightharpoonup 0$ in E_ε. Then $w_n \to 0$ in E_ε.*

Proof: By Lemma 2.7 we can assume $w_n \geq 0$. Consider any subsequence of $\{w_n\}$, still denoted by $\{w_n\}$. Since $w_n \rightharpoonup 0$ in E_ε, up to a subsequence, we can assume $w_n \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for $s \in [2, 2^*)$ and $w_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$. If $w_n \not\to 0$ in E_ε, then, we may assume that $w_n \in E^+_\varepsilon$ for each n, and by Lemma 2.10, there exists a sequence $\{x_n\} \subset \mathbb{R}^N$ and positive constants r, τ such that

$$\liminf_{n\to\infty} \int_{B_r(x_n)} |w_n(x)|^2 dx \geq \tau. \tag{2.5}$$

Hence the sequence $\{x_n\}$ is unbounded, and hence, we can assume

$$|x_n| := k_n \to \infty.$$

Notice that for each $j \in \mathbb{N}$, one has

$$\lim_{n\to\infty} \int_{B_{2k_j}(0)} |w_n(x)|^2 dx = 0.$$

Hence there is a $m_j \in \mathbb{N}$ such that

$$\int_{B_{2k_j}(0)} |w_n(x)|^2 dx < \frac{\tau}{2}$$

for all $n = m_j + i, i = 1, 2, \cdots$. Without the loss of generality, we can assume $m_{j+1} > m_j$. Set $n_j := m_j + j$. Then

$$\int_{B_{2k_j}(0)} |w_{n_j}(x)|^2 dx < \frac{\tau}{2}.$$

Hence, up to a subsequence, we have

$$\limsup_{n\to\infty} \int_{B_{2k_n}(0)} |w_n(x)|^2 dx \leq \frac{\tau}{2}. \tag{2.6}$$

Notice that $k_n \to +\infty$. Hence, for large n, one has $B_r(x_n) \subset B_{2k_n}(0)$, and hence (2.6) contradicts (2.5). This shows that $w_n \to 0$ in E_ε. This completes the proof. ∎

Lemma 2.12 *Let $\{w_n\} \subset E_\varepsilon$ be a (PS) sequence of Φ_ε with $w_n \rightharpoonup w$ in E_ε. Set $\widetilde{w}_n = w_n - w$. Then*

(i) $\Phi_\varepsilon(\widetilde{w}_n) = \Phi_\varepsilon(w_n) - \Phi_\varepsilon(w) + o_n(1)$,

(ii) $\|\Phi'_\varepsilon(\widetilde{w}_n)\| = o_n(1)$.

Proof: The proof is similar to Lemma 2.14 in [10], we omit it. ▮

Lemma 2.13 Φ_ε *satisfies the $(PS)_d$ condition at any level $d \le \widetilde{c}_{V_\infty}$.*

Proof: Let $\{w_n\} \subset E_\varepsilon$ be a $(PS)_d$ sequence of Φ_ε. Then, by Lemma 2.7, $\{w_n\}$ is bounded in E_ε and we can assume $w_n \ge 0$. Hence, up to a subsequence, there is $w \in E_\varepsilon$ such that $w_n \rightharpoonup w$ in E_ε, $w_n \to w$ in $L^s_{loc}(\mathbb{R}^N)$ for each $s \in [2, 2^*)$, $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$ and $\Phi'_\varepsilon(w) = 0$. Set $\widetilde{w}_n = w_n - w$. Then, by Lemma 2.12,

$$\Phi_\varepsilon(\widetilde{w}_n) = \Phi_\varepsilon(w_n) - \Phi_\varepsilon(w) + o_n(1) = d - \Phi_\varepsilon(w) + o_n(1) := a + o_n(1)$$

and $\|\Phi'_\varepsilon(\widetilde{w}_n)\| = o_n(1)$. By (g_3) and Lemma 1.3 (6), we have

$$\begin{aligned}
&\Phi_\varepsilon(w) \\
&= \Phi_\varepsilon(w) - \frac{1}{2}\langle \Phi'_\varepsilon(w), w\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)[f^2(w) - f(w)f'(w)w]dx + \int_{\mathbb{R}^N} [\frac{1}{2}g(f(w))f'(w)w - G(f(w))]dx \\
&\quad + \int_{\mathbb{R}^N} [\frac{1}{2}|f(w^+)|^{22^*-1}f'(w^+)w^+ - \frac{1}{22^*}|f(w^+)|^{22^*}]dx \\
&\ge 0.
\end{aligned}$$

Hence, by Lemma 2.9, $a := d - \Phi_\varepsilon(w) \le d \le \widetilde{c}_{V_\infty} < \frac{1}{2N}S^{\frac{N}{2}}$. By Lemma 2.11 we know $w_n \to w$ in E_ε. This completes the proof. ▮

From the proof of Lemma 2.13 we have

Lemma 2.14 Φ_ε *satisfies the $(PS)_d$ condition at any level $d < \frac{1}{2N}S^{\frac{N}{2}}$.*

3 The proof of Theorem 1

Proof: By Lemma 2.2 we know that the functional Φ_ε satisfies the mountain pass geometry, then using a version of the mountain pass theorem (e. g. see Theorem 6.3.4 in [27]), there exists a sequence $\{w_n\} \subset E_\varepsilon$ such that $\lim_{n\to\infty} \Phi_\varepsilon(w_n) = c_\varepsilon$ and $(1 + \|w_n\|_\varepsilon)\|\Phi'_\varepsilon(w_n)\| = o_n(1)$. By condition (V) we can assume that $V_0 = V(0) = \inf_{x\in\mathbb{R}^N} V(x)$. For any $\mu \in \mathbb{R}$ with $V_0 < \mu < V_\infty$, we have $\widetilde{c}_{V_0} < \widetilde{c}_\mu < \widetilde{c}_{V_\infty}$. By Lemma 2.9, $\widetilde{c}_\mu < \frac{1}{2N} S^{\frac{N}{2}}$. By virtue of Lemmas 2.2, 2.7 and Theorem 6.3.4 in [27] we know that $\widetilde{c}_\mu$ is a critical value of Υ_μ with corresponding positive critical point $w \in E$. For any $r > 0$, take $\eta_r \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta_r = 1$ if $|x| < r$ and $\eta_r = 0$ if $|x| \geq 2r$. Set $v_r := \eta_r w$, it is easy to verify that $v_r \in E$ for each $r > 0$. By Lemma 2.3 there exists $t_r > 0$ such that $\widetilde{v}_r := t_r v_r \in \mathcal{M}_\mu$. Hence there is $r_0 > 0$ such that $\widetilde{v} = \widetilde{v}_{r_0}$ satisfies $\Upsilon_\mu(\widetilde{v}) < \widetilde{c}_{V_\infty}$. In fact, if this is false, then $\Upsilon_\mu(\widetilde{v}_r) = \Upsilon_\mu(t_r v_r) \geq \widetilde{c}_{V_\infty}$ for all $r > 0$. Notice that $v_r \to w$ in E as $r \to +\infty$ and $w \in \mathcal{M}_\mu$. We can deduce that $t_r \to 1$ as $r \to +\infty$. Hence,

$$\widetilde{c}_{V_\infty} \leq \liminf_{r\to+\infty} \Upsilon_\mu(t_r v_r) = \Upsilon_\mu(w) = \widetilde{c}_\mu < \widetilde{c}_{V_\infty},$$

a contradiction. This shows $\Upsilon_\mu(\widetilde{v}) < \widetilde{c}_{V_\infty}$. Notice that $V_0 = V(0) < \mu$ and $supp(\widetilde{v})$ is compact. By the continuity of V, there is an $\varepsilon^* > 0$ such that

$$V(\varepsilon x) < \mu, \ \ \forall \ \ \varepsilon \in (0, \varepsilon^*) \ \ \textit{and} \ \ x \in supp(\widetilde{v}).$$

Hence,

$$\Phi_\varepsilon(t\widetilde{v}) \leq \Upsilon_\mu(t\widetilde{v}), \ \ \forall \ \ \varepsilon \in (0, \varepsilon^*) \ \ \textit{and} \ \ t \geq 0,$$

and

$$\max_{t\geq 0} \Phi_\varepsilon(t\widetilde{v}) \leq \max_{t\geq 0} \Upsilon_\mu(t\widetilde{v}) = \Upsilon_\mu(\widetilde{v}) < \widetilde{c}_{V_\infty}, \ \ \forall \ \ \varepsilon \in (0, \varepsilon^*).$$

Consequently,

$$c_\varepsilon < \widetilde{c}_{V_\infty}, \ \ \forall \ \ \varepsilon \in (0, \varepsilon^*).$$

By virtue of Lemma 2.13, up to a subsequence, one has $w_n \to v$ in E_ε. Hence $\Phi'_\varepsilon(v) = 0$ and $\Phi_\varepsilon(v) = c_\varepsilon$. Moreover, Harnack inequality (see [24]) implies $v(x) > 0$ in $\mathbb{R}^N$. Hence

$u := f(v)$ is a ground positive solution of (1.3). This completes the proof of Theorem 1. $\blacksquare$

Remark 3.1 *From the above proof we know that all non zero critical points of Φ_ε are positive.*

4 The proof of Theorem 2

Since $V_0 > 0$, by Lemma 2.9, $\widetilde{c}_{V_0} < \frac{1}{2N} S^{\frac{N}{2}}$. From the proof of Theorem 1 we know that $\widetilde{c}_{V_0}$ is a critical value of Υ_{V_0} with corresponding positive critical point $w \in E$.

Let η be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that $\eta(s) = 1$ if $0 \leq s \leq \frac{1}{2}$ and $\eta(s) = 0$ if $s \geq 1$. For each $z \in \Lambda$, let

$$\varrho_{\varepsilon,z}(x) = \eta(|\varepsilon x - z|) w(\frac{\varepsilon x - z}{\varepsilon}), \quad \forall\, x \in \mathbb{R}^N.$$

Then $\varrho_{\varepsilon,z} \in E_\varepsilon \backslash \{0\}$ for all $z \in \Lambda$. In fact, using the change of variable $\widetilde{z} = x - \frac{z}{\varepsilon}$, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} V(\varepsilon x) \varrho_{\varepsilon,z}^2(x) dx =& \int_{\mathbb{R}^N} V(\varepsilon x) \eta^2(|\varepsilon x - z|) w^2(\frac{\varepsilon x - z}{\varepsilon}) dx \\
=& \int_{|\varepsilon \widetilde{z}| \leq 1} V(\varepsilon \widetilde{z} + z) \eta^2(|\varepsilon \widetilde{z}|) w^2(\widetilde{z}) d\widetilde{z} \\
\leq& C \int_{\mathbb{R}^N} w^2(x) dx < +\infty.
\end{aligned}$$

Let $r = \frac{1}{\varepsilon}$. Then $\varrho_{\varepsilon,z}(x) = \eta(\frac{|x - \frac{z}{\varepsilon}|}{r}) w(x - \frac{z}{\varepsilon})$ and

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \varrho_{\varepsilon,z}(x)|^2 dx =& \int_{\mathbb{R}^N} |\frac{1}{r} w(x) \nabla \eta(\frac{|x|}{r}) + \eta(\frac{|x|}{r}) \nabla w(x)|^2 dx \\
\leq& C(\int_{\frac{r}{2} < |x| < r} w^2 dx + \int_{\mathbb{R}^N} |\nabla w|^2 dx) \\
\leq& C \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) dx < +\infty.
\end{aligned}$$

Hence, for each $\varepsilon > 0$ there exists unique $t_\varepsilon > 0$ such that

$$\max_{t \geq 0} \Phi_\varepsilon(t \varrho_{\varepsilon,z}) = \Phi_\varepsilon(t_\varepsilon \varrho_{\varepsilon,z})$$

by Lemma 2.3.

We introduce the map $\lambda_\varepsilon : \Lambda \to \mathcal{N}_\varepsilon$ by setting $\lambda_\varepsilon(z) = t_\varepsilon \varrho_{\varepsilon,z}$. Then $\lambda_\varepsilon(z)$ has a compact support for any $z \in \Lambda$. Moreover, we have the following fact for λ_ε.

Lemma 4.1 $\lim\limits_{\varepsilon\to 0} \Phi_\varepsilon(\lambda_\varepsilon(z)) = \widetilde{c}_{V_0}$, *uniformly in* $z \in \Lambda$.

Proof: Suppose that the result is false, then there exists $\zeta_0 > 0, \{z_n\} \subset \Lambda$ and $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$|\Phi_{\varepsilon_n}(\lambda_{\varepsilon_n}(z_n)) - \widetilde{c}_{V_0}| \geq \zeta_0 > 0. \tag{4.1}$$

By Lemmas 2.6 and 2.8 we know that there is a $r_0 > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} [g(f(\lambda_{\varepsilon_n}(z_n)))f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n) + |f(\lambda_{\varepsilon_n}(z_n))|^{22^*-1}f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n)]dx \\
&= \int_{\mathbb{R}^N} |\nabla(\lambda_{\varepsilon_n}(z_n))|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) f(\lambda_{\varepsilon_n}(z_n))f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n)dx \\
&\geq \frac{1}{2}[\int_{\mathbb{R}^N} |\nabla(\lambda_{\varepsilon_n}(z_n))|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) f^2(\lambda_{\varepsilon_n}(z_n))dx] \\
&\geq r_0.
\end{aligned} \tag{4.2}$$

Hence $t_{\varepsilon_n} \nrightarrow 0$, and hence there exists a $t_0 > 0$ such that $t_{\varepsilon_n} \geq t_0$. If $t_{\varepsilon_n} \nrightarrow \infty$, then, for large n, one has

$$\begin{aligned}
C\|w\|_E^2 &\geq \int_{\mathbb{R}^N} |\nabla \varrho_{\varepsilon_n,z_n}|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) \frac{f(t_{\varepsilon_n}\varrho_{\varepsilon_n,z_n})f'(t_{\varepsilon_n}\varrho_{\varepsilon_n,z_n})t_{\varepsilon_n}\varrho_{\varepsilon_n,z_n}}{t_{\varepsilon_n}^2} dx \\
&= \int_{\mathbb{R}^N} \frac{g(f(\lambda_{\varepsilon_n}(z_n)))f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n)}{t_{\varepsilon_n}^2} + \int_{\mathbb{R}^N} \frac{|f(\lambda_{\varepsilon_n}(z_n))|^{22^*-1}f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n)}{t_{\varepsilon_n}^2} dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} \frac{|f(t_{\varepsilon_n}\eta(|\varepsilon_n x - z_n|)w(\frac{\varepsilon_n x - z_n}{\varepsilon_n}))|^{22^*}}{t_{\varepsilon_n}^2} dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \frac{|f(t_{\varepsilon_n}\eta(\varepsilon_n|x|)w(x))|^{22^*}}{t_{\varepsilon_n}^2} dx \\
&\geq \frac{1}{2}\int_{|x|\leq \frac{1}{2\varepsilon_n}} \frac{|f(t_{\varepsilon_n}w(x))|^{22^*}}{t_{\varepsilon_n}^2} dx \\
&\geq C t_{\varepsilon_n}^{2^*-2}\int_{\frac{1}{2}<|x|<1} w^{2^*} dx \to +\infty,
\end{aligned}$$

a contradiction. Hence, $t_{\varepsilon_n} \leq C$. Consequently, we can assume that $t_{\varepsilon_n} \to T > 0$. By Lebesgue's theorem, one has

$$\begin{aligned}
&\lim_{n\to\infty}[\int_{\mathbb{R}^N} |\nabla\lambda_{\varepsilon_n}(z_n)|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) f(\lambda_{\varepsilon_n}(z_n)) f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n) dx] \\
&= T^2 \int_{\mathbb{R}^N} |\nabla w|^2 dxdy + \int_{\mathbb{R}^N} V_0 f(Tw(x)) f'(Tw(x)) Tw(x) dx,
\end{aligned}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |f(\lambda_{\varepsilon_n}(z_n))|^{22^*-1} f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n) dx = \int_{\mathbb{R}^N} |f(Tw(x))|^{22^*-1} f'(Tw(x)) Tw(x) dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(f(\lambda_{\varepsilon_n}(z_n))) f'(\lambda_{\varepsilon_n}(z_n))\lambda_{\varepsilon_n}(z_n) dx = \int_{\mathbb{R}^N} g(f(Tw(x))) f'(Tw(x)) Tw(x) dx.$$

Consequently, from (4.2), one has

$$\begin{aligned}
&T^2 \int_{\mathbb{R}^N} |\nabla w|^2 dx + V_0 \int_{\mathbb{R}^N} f(Tw(x)) f'(Tw(x)) Tw(x) dx \\
&= \int_{\mathbb{R}^N} g(f(Tw(x))) f'(Tw(x)) Tw(x) dx + \int_{\mathbb{R}^N} |f(Tw(x))|^{22^*-1} f'(Tw(x)) Tw(x) dx.
\end{aligned}$$

This shows $Tw \in \mathcal{M}_{V_0}$. Notice that $w \in \mathcal{M}_{V_0}$. Lemma 2.3 implies that $T = 1$. Moreover, similar to the above arguments, we can prove that

$$\lim_{n\to\infty} \Phi_{\varepsilon_n}(\lambda_{\varepsilon_n}(z_n)) = \Upsilon_{V_0}(w) = \widetilde{c}_{V_0}.$$

This contradicts to 4.1. This completes the proof of Lemma 4.1. ∎

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $\Lambda_\delta \subset B_\rho(0)$. Define $\chi : \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$\chi(x) = \begin{cases} x, & |x| \leq \rho, \\ \frac{\rho}{|x|}x, & |x| \geq \rho. \end{cases}$$

Moreover, we also define the map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\beta_\varepsilon(w) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|w(x)|^2 dx}{\int_{\mathbb{R}^N} |w(x)|^2 dx}.$$

We have the following fact for β_ε, its proof similar to Lemma 5.2 in [22].

Lemma 4.2 $\lim_{\varepsilon\to 0}\beta_\varepsilon(\lambda_\varepsilon(z)) = z$ *uniformly in* $z\in\Lambda$.

Lemma 4.3 *For each* $\mu>0$, *let* $\{w_n\}\subset\mathcal{M}_\mu$ *with* $\Upsilon_\mu(w_n)\to\widetilde{c}_\mu$. *Then* $\{w_n\}$ *has a subsequence strongly convergent in* E, *In particular, there exists a minimizer for* $\widetilde{c}_\mu$.

Proof: From the proof of Lemma 2.7 we know that $\{w_n\}$ is bounded in E. By Lemma 2.9, $\widetilde{c}_\mu < \frac{1}{2N}S^{\frac{N}{2}}$. Notice that

$$S_\mu^+ := \{w\in E^+ : \int_{\mathbb{R}^N}|\nabla w|^2dx + \mu\int_{\mathbb{R}^N}|w(x)|^2dx - 1 := B(w) = 0\}$$

and

$$\langle B'(w), w\rangle = 2[\int_{\mathbb{R}^N}|\nabla w|^2dx + \mu\int_{\mathbb{R}^N}|w(x)|^2dx] = 2$$

for all $w\in S_\mu^+$. Hence $B'(w)\neq 0$ for all $w\in S_\mu^+$. By Proposition 9 in [23],

$$\widehat{\Upsilon} := \Upsilon_\mu\circ\widehat{m}_\mu : E\backslash\{0\}\to\mathbb{R}$$

is a C^1 functional. Set $\{v_n\} := \{m_\mu^{-1}(w_n)\}\subset S_\mu^+$. Since $\Psi_\mu(v_n)\to\widetilde{c}_\mu$, for $\frac{1}{k^2}$, up to a subsequence, one has

$$\widetilde{c}_\mu \le \Psi_\mu(v_k) \le \widetilde{c}_\mu + \frac{1}{k^2}.$$

By Theorem 1.1 in [8], there exists a sequence $\{\widetilde{v}_k\}\subset S_\mu^+$ such that

$$\Psi_\mu(\widetilde{v}_k)\le\Psi_\mu(v_k),\quad \|v_k-\widetilde{v}_k\|_E\le\frac{1}{k}$$

and for each $v\in E\backslash\{0\}$ with $v\neq\widetilde{v}_k$, one has

$$\widehat{\Upsilon}(v) > \Psi_\mu(\widetilde{v}_k) - \frac{1}{k}\|v-\widetilde{v}_k\|_E.$$

Hence, for any $v\in S_\mu^+$, we have

$$\Psi_\mu(v) > \Psi_\mu(\widetilde{v}_k) - \frac{1}{k}\|v-\widetilde{v}_k\|_E.$$

Consequently, similar to the proof of Theorem 3.1 in [8], we can prove that there is a $\delta_k\in\mathbb{R}$ such that

$$\|\Psi_\mu'(\widetilde{v}_k)\| = \|\widehat{\Upsilon}'(\widetilde{v}_k) - \delta_k B'(\widetilde{v}_k)\| \le \frac{1}{k}.$$

Therefore,

$$\widetilde{v}_k = v_k + o_k(1), \quad \Psi_\mu(\widetilde{v}_k) \to \widetilde{c}_\mu, \quad \Psi'_\mu(\widetilde{v}_k) = o_k(1).$$

Hence we may assume that $\{v_n\}$ is a $(PS)_{\widetilde{c}_\mu}$ sequence of Ψ_μ. By Lemma 2.4, $\{w_n\}$ is a bounded $(PS)_{\widetilde{c}_\mu}$ sequence of Υ_μ. Hence, by Lemma 2.7, we may assume that $w_n \geq 0$. By Lemma 2.14, going to a subsequence if necessary, we may assume that $w_n \to w$ in E. This completes the proof. ∎

Lemma 4.4 *Let $\varepsilon_n \to 0$ and $w_n \in \mathcal{N}_{\varepsilon_n}$ be such that $\lim_{n\to\infty} \Phi_{\varepsilon_n}(w_n) = \widetilde{c}_{V_0}$. Then, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $w_n(\cdot + z_n)$ has a convergent subsequence in E and $\tilde{z}_n = \varepsilon_n z_n \to z \in \Lambda$.*

Proof: By Lemma 2.8, we know that $\|w_n\|_{\varepsilon_n} \nrightarrow 0$. Moreover, by $w_n \in \mathcal{N}_{\varepsilon_n}$ and $\lim_{n\to\infty} \Phi_{\varepsilon_n}(w_n) = \widetilde{c}_{V_0}$, from the proof of Lemma 2.7 we know that $\{w_n\}$ is bounded in E. We claim that there exist $\{z_n\} \subset \mathbb{R}^N$ and $r > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(z_n)} |w_n|^2 dx = \tau > 0. \tag{4.3}$$

Indeed, if this is false, then for any $r > 0$, one has

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_r(z)} |w_n|^2 dx = 0.$$

By Lemma 1.2 we know that $w_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in [2, 2^*)$. Notice that $w_n \in \mathcal{N}_{\varepsilon_n}$. We have

$$\int_{\mathbb{R}^N} \big(|\nabla w_n|^2 + V(\varepsilon_n x) f(w_n) f'(w_n) w_n\big) dx = \int_{\mathbb{R}^N} |f(w_n^+)|^{22^*-1} f'(w_n^+) w_n^+ dx + o_n(1).$$

Since $\{w_n\}$ is bounded in E, up to a subsequence, we can assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} [|\nabla w_n|^2 + V(\varepsilon_n x) f(w_n) f'(w_n) w_n] dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |f(u_n^+)|^{22^*-1} f'(u_n^+) u_n^+ dx = L.$$

As the proof of corresponding part of Lemma 2.10, we can prove $L = 0$, and hence

$$\lim_{n\to\infty} \|w_n\|_{\varepsilon_n} = 0,$$

a contradiction. Hence (4.3) holds.

Let $\widetilde{w}_n = w_n(\cdot + z_n)$. By the boundedness of $\{w_n\}$ in E and (4.3), up to a subsequence, we have $\widetilde{w}_n \rightharpoonup \widetilde{w} \neq 0$ in E and $\widetilde{w}_n(x) \to \widetilde{w}(x)$ a.e. in $\mathbb{R}^N$. Let $\{t_n\} \subset (0, \infty)$ be such that $v_n := t_n \widetilde{w}_n \in \mathcal{M}_{V_0}$ and set $\tilde{z}_n = \varepsilon_n z_n$. By (V_1) and $w_n \in \mathcal{N}_{\varepsilon_n}$, we get

$$
\begin{aligned}
\widetilde{c}_{V_0} \leq & \Upsilon_{V_0}(t_n \widetilde{w}_n) \\
= & \frac{1}{2}[t_n^2 \int_{\mathbb{R}^N} |\nabla \widetilde{w}_n|^2 dx + \int_{\mathbb{R}^N} V_0 f^2(t_n \widetilde{w}_n) dx] \\
& - [\int_{\mathbb{R}^N} G(f(t_n \widetilde{w}_n)) dx + \frac{1}{22^*} \int_{\mathbb{R}^N} |f(t_n \widetilde{w}_n^+)|^{22^*} dx] \\
\leq & \frac{1}{2}[t_n^2 \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x) f^2(t_n w_n) dx] \\
& - [\int_{\mathbb{R}^N} G(f(t_n w_n)) dx + \frac{1}{22^*} \int_{\mathbb{R}^N} |f(t_n w_n^+)|^{22^*} dx] \\
= & \Phi_{\varepsilon_n}(t_n w_n) \leq \Phi_{\varepsilon_n}(w_n) = \widetilde{c}_{V_0} + o_n(1).
\end{aligned}
$$

Hence

$$
\lim_{n\to\infty} \Upsilon_{V_0}(v_n) = \lim_{n\to\infty} \Upsilon_{V_0}(t_n \widetilde{w}_n) = \widetilde{c}_{V_0} > 0.
$$

By Lemma 4.3, up to a subsequence, one has $v_n \to v$ in E. Moreover, by (4.3), one has

$$
\begin{aligned}
\tau = & \liminf_{n\to\infty} \int_{B_r(z_n)} |w_n|^2 dx = \liminf_{n\to\infty} \int_{B_r(0)} |\widetilde{w}_n(z)|^2 dz \\
\leq & \liminf_{n\to\infty} \|\widetilde{w}_n\|_E^2.
\end{aligned}
$$

Consequently, for large n, one has $0 < \frac{\tau}{2} < \|\widetilde{w}_n\|_E^2$, and hence

$$
0 < \frac{\tau}{2} t_n^2 < \|t_n \widetilde{w}_n\|_E^2 = \|v_n\|_E^2 \leq C.
$$

Hence $\{t_n\}$ is bounded. Therefore, without loss of generality, we may assume that $t_n \to t^*$. By $\lim_{n\to\infty} \Upsilon_{V_0}(v_n) = \widetilde{c}_{V_0} > 0$ we get $t^* > 0$, hence, up to a subsequence, we have $v_n \to v = t^* \widetilde{w} \neq 0$ in E. $\widetilde{w}_n \to \frac{1}{t^*} v = \widetilde{w}$ in E. The first conclusion is proved.

Now, we show that $\{\tilde{z}_n\}$ is bounded in $\mathbb{R}^N$. Otherwise, up to a subsequence, one has $|\tilde{z}_n| \to \infty$. Notice that, up to a subsequence, we have $v_n \to v \neq 0$ in E.

By Fatou's lemma we have

$$
\begin{aligned}
\widetilde{c}_{V_0} =& \Upsilon_{V_0}(v) < \Upsilon_{V_\infty}(v) = \Upsilon_{V_\infty}(v) - \frac{1}{2}\langle \Upsilon'_{V_0}(v), v\rangle \\
=& \frac{1}{2}\int_{\mathbb{R}^N}(V_\infty f^2(v) - V_0 f(v)f'(v)v)dx + \int_{\mathbb{R}^N}(\frac{1}{2}g(f(v))f'(v)v - G(f(v)))dx \\
&+ \int_{\mathbb{R}^N}(\frac{1}{2}|f(v^+)|^{22^*-1}f'(v^+)v^+ - \frac{1}{22^*}|f(v^+)|^{22^*})dx \\
\le& \liminf_{n\to\infty}[\Phi_{\varepsilon_n}(v_n) - \frac{1}{2}\langle \Upsilon'_{V_0}(v_n), v_n\rangle] \\
=& \liminf_{n\to\infty}\Phi_{\varepsilon_n}(v_n) \le \lim_{n\to\infty}\Phi_{\varepsilon_n}(w_n) = \widetilde{c}_{V_0}.
\end{aligned}
$$

This is a contradiction. This shows that $\{\tilde{z}_n\}$ is bounded in $\mathbb{R}^N$. Hence, up to a subsequence, $\tilde{z}_n \to z \in \mathbb{R}^N$. If $z \in \mathbb{R}^N\backslash\Lambda$, then $V_0 < V(z)$. As the above, we can get a contradiction. Hence, $z \in \Lambda$ and the lemma is proved. ∎

Let

$$h(\varepsilon) := \max_{z\in\Lambda}|\Phi_\varepsilon(\lambda_\varepsilon(z)) - \widetilde{c}_{V_0}|.$$

Then, by Lemma 4.1, $\lim\limits_{\varepsilon\to 0^+} h(\varepsilon) = 0$. Define the set

$$\widetilde{\mathcal{N}}_\varepsilon = \{w \in \mathcal{N}_\varepsilon : \Phi_\varepsilon(w) \le \widetilde{c}_{V_0} + h(\varepsilon)\}.$$

Then, for each $z \in \Lambda$, we have $\lambda_\varepsilon(z) \in \widetilde{\mathcal{N}}_\varepsilon$ for each $\varepsilon > 0$.

By Lemma 4.4, similar to Lemma 4.5 in [10], we can prove the following Lemma.

Lemma 4.5 *For any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0}\sup_{w\in\widetilde{\mathcal{N}}_\varepsilon} d(\beta_\varepsilon(w), \Lambda_\delta) = 0.$$

Proof of theorem 2. For each $\delta > 0$, Lemma 4.2 implies that there is an $\varepsilon_\delta > 0$ such that

$$|\beta_\varepsilon(\lambda_\varepsilon(z)) - z| < \delta, \quad \forall\ \varepsilon \in (0, \varepsilon_\delta), \quad \forall\ z \in \Lambda.$$

Notice that for each $z \in \Lambda$, one has $\lambda_\varepsilon(z) \in \widetilde{\mathcal{N}}_\varepsilon$ for each $\varepsilon > 0$. The diagram

$$\beta_\varepsilon \circ \lambda_\varepsilon : \Lambda \to \widetilde{\mathcal{N}}_\varepsilon \to \Lambda_\delta$$

is well defined for all $\varepsilon \in (0, \varepsilon_\delta)$. Similar to corresponding part in [11] we can prove

$$cat_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq cat_{\Lambda_\delta}(\Lambda).$$

Notice that as E_ε^+ is a nonempty open set of a Banach space, Theorem 5.19 in [26] holds, too. For each $\varepsilon \in (0, \varepsilon_\delta)$, take $V = S_\varepsilon^+$, $X = E_\varepsilon^+$ and $\varphi = \widehat{\Psi}_\varepsilon$ in this theorem. Then, by Lemma 2.4, $\varphi \in C^1(X, \mathbb{R})$, $\varphi|_V = \Psi_\varepsilon \in C^1(V, \mathbb{R})$ and

$$\inf_V \varphi = \inf_{S_\varepsilon^+} \Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Phi_\varepsilon = \widetilde{c}_{V_0}.$$

Now, we prove that, for small $\varepsilon > 0$, Ψ_ε satisfies the $(PS)_d$ condition for all $d \in [\widetilde{c}_{V_0}, \widetilde{c}_{V_0} + h(\varepsilon)]$. In fact, let $\{w_n\} \subset S_\varepsilon^+$ be a $(PS)_d$ sequence of Ψ_ε with $d \in [\widetilde{c}_{V_0}, \widetilde{c}_{V_0} + h(\varepsilon)]$. Then, by Lemma 2.4, $\{v_n\} := \{m_\varepsilon(w_n)\} \subset \mathcal{N}_\varepsilon$ is a $(PS)_d$ sequence of Φ_ε. Notice that for small $\varepsilon > 0$, one has $\widetilde{c}_{V_0} + h(\varepsilon) < \widetilde{c}_{V_\infty}$. We can assume that for each $\varepsilon \in (0, \varepsilon_\delta)$, $\widetilde{c}_{V_0} + h(\varepsilon) < \widetilde{c}_{V_\infty}$. Then, by Lemma 2.13, we may assume $v_n \to v$ in E_ε. Hence $w_n \to w := m_\varepsilon^{-1}(v)$. This shows that Ψ_ε satisfies the $(PS)_d$ condition. Consequently, by Theorem 5.20 in [26], Ψ_ε has at least $cat_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon)$ critical points in $\Psi_\varepsilon^{\widetilde{c}_{V_0} + h(\varepsilon)} := \{w \in S_\varepsilon^+ : \Psi_\varepsilon(w) \leq \widetilde{c}_{V_0} + h(\varepsilon)\}$, and hence Φ_ε has at least $cat_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq cat_{\Lambda_\delta}(\Lambda)$ positive critical points in $\widetilde{\mathcal{N}}_\varepsilon$ by Remark 3.1. If $w_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ denotes such a critical point of Φ_ε, then $u_\varepsilon := f(w_\varepsilon)$ is a positive solution of (1.3). Consequently, the function $v_\varepsilon(x) := u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of (1.1). Therefore, the problem (1.1) has at least $cat_{\Lambda_\delta}(\Lambda)$ positive solutions.

Moreover, the maximum point ζ_ε of v_ε is related to the maximum point ξ_ε of u_ε with $\zeta_\varepsilon = \varepsilon \xi_\varepsilon$. Hence, to prove the concentration property of solution for (1.1) we just need to show $\lim\limits_{\varepsilon \to 0^+} V(\varepsilon \xi_\varepsilon) = V_0$.

Let $\{\varepsilon_n\} \subset \mathbb{R}_+$ be such that $\varepsilon_n \to 0^+$. Set $w_n := w_{\varepsilon_n}$. Then

$$\widetilde{c}_{V_0} \leq c_{\varepsilon_n} \leq \Phi_{\varepsilon_n}(w_n) \leq \widetilde{c}_{V_0} + h(\varepsilon_n) \to \widetilde{c}_{V_0}.$$

Hence $\Phi_{\varepsilon_n}(w_n) \to \widetilde{c}_{V_0}$. By Lemma 4.4 there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $w_n(\cdot + z_n)$ has a convergent subsequence in E and $\tilde{z}_n = \varepsilon_n z_n \to z \in \Lambda$. Set $\psi_n(x) := w_n(x + z_n)$ for all $x \in \mathbb{R}^N$. Then we can assume that $\psi_n \to \psi \neq 0$ in E. We claim that $\psi_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that

$$\|\psi_n\| \leq C, \quad \forall\, n \in \mathbb{N}.$$

In fact, for any $R > 0$, $0 < r \leq \frac{R}{2}$, take $\eta \in C^\infty(\mathbb{R}^N, [0,1])$ such that

$$\eta(x) = \begin{cases} 0, & if\ |x| \leq R - r, \\ 1, & if\ |x| \geq R, \end{cases}$$

and $|\nabla \eta| \leq \frac{2}{r}$. For $l > 0$, set

$$\psi_{l,n}(x) = \begin{cases} \psi_n(x), & if\ \psi_n(x) \leq l, \\ l, & if\ \psi_n(x) \geq l, \end{cases}$$

and

$$\phi_{l,n} = \eta^2 \psi_{l,n}^{2(\theta-1)} \psi_n, \quad \omega_{l,n} = \eta \psi_{l,n}^{\theta-1} \psi_n$$

with $\theta > 1$ to be determined later. For convenience' sake, we shall omit dx and dy in the following integrals.

Since w_n is a critical point of Φ_{ε_n}, take $\phi_{l,n}$ as the test function, one has

$$\begin{aligned} 0 = &\int_{\mathbb{R}^N} \nabla \psi_n \nabla \phi_{l,n} + \int_{\mathbb{R}^N} V(\varepsilon_n(x + z_n)) f(\psi_n) f'(\psi_n) \phi_{l,n} \\ &- \int_{\mathbb{R}^N} |f(\psi_n)|^{22^*-1} f'(\psi_n) \phi_{l,n} - \int_{\mathbb{R}^N} g(f(\psi_n)) f'(\psi_n) \phi_{l,n}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^N} \eta^2 \psi_{l,n}^{2(\theta-1)} |\nabla \psi_n|^2 \\ = &\int_{\mathbb{R}^N} |f(\psi_n)|^{22^*-1} f'(\psi_n) \phi_{l,n} + \int_{\mathbb{R}^N} g(f(\psi_n)) f'(\psi_n) \phi_{l,n} - 2 \int_{\mathbb{R}^N} \eta \psi_{l,n}^{2(\theta-1)} \psi_n \nabla \psi_n \nabla \eta \\ &- \int_{\mathbb{R}^N} V(\varepsilon_n(x + z_n)) f(\psi_n) f'(\psi_n) \phi_{l,n} - 2(\theta - 1) \int_{\mathbb{R}^N} \eta^2 \psi_{l,n}^{2\theta-3} \psi_n \nabla \psi_n \nabla \psi_{l,n}. \end{aligned}$$

Using Lemma 1.3, similar to proof of Lemma 4.6 in [10], we can prove that there is $K > 0$ such that

$$\|\psi_n\|_{L^\infty} \leq K, \quad \forall\ n \in \mathbb{N}$$

and $\lim_{|x| \to \infty} \psi_n(x) = 0$, uniformly in $n \in \mathbb{N}$.

Let P_n be the global maximum of ψ_n. Then $\xi_{\varepsilon_n} := P_n + z_n$ is the global maximum of w_n. Since $\lim_{|x| \to \infty} \psi_n(x) = 0$ uniformly in $n \in \mathbb{N}$, the sequence $\{P_n\} \subset \mathbb{R}^N$ is bounded. Otherwise, we may assume $|P_n| \to \infty$, and hence $\lim_{n\to\infty} \psi_n(P_n) = 0$. Hence, for any $0 < \tau < \frac{1}{4}V_0$, one has

$$2^{\frac{2^*}{2}} \psi_n^{2^*-2}(P_n) < \frac{\tau}{2}$$

for large n. Moreover, by (g_1), there exists $\rho_0 > 0$ such that $|\frac{g(s)}{s}| < \frac{\tau}{2}$ whenever $0 < |s| < \rho_0$. Notice that for large n, one has $0 < f(\psi_n(x)) \le f(\psi_n(P_n)) \le \psi_n(P_n) < \rho_0$ for all $x \in \mathbb{R}^N$. Hence, for large n,

$$\begin{aligned}
&\frac{1}{2}[\int_{\mathbb{R}^N} |\nabla\psi_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n(x+z_n))f^2(\psi_n)] \\
\le& \int_{\mathbb{R}^N} |\nabla\psi_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n(x+z_n))f(\psi_n)f'(\psi_n)\psi_n \\
=& \int_{\mathbb{R}^N} |f(\psi_n)|^{22^*-1} f'(\psi_n)\psi_n + \int_{\mathbb{R}^N} g(f(\psi_n))f'(\psi_n)\psi_n \\
\le& 2^{\frac{2^*}{2}} \psi_n^{2^*-2}(P_n) \int_{\mathbb{R}^N} f^2(\psi_n) + \int_{\mathbb{R}^N} \frac{g(f(\psi_n))}{f(\psi_n)} f(\psi_n)f'(\psi_n)\psi_n \\
\le& \tau \int_{\mathbb{R}^N} f^2(\psi_n),
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n x)f^2(w_n) = \int_{\mathbb{R}^N} |\nabla\psi_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n(x+z_n))f^2(\psi_n) = 0$$

for large n. Therefore,

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 + \int_{\mathbb{R}^N} V_0 f^2(w_n) = 0$$

for large n. By Lemma 2.6 there is a constant $C > 0$ such that

$$0 = \int_{\mathbb{R}^N} |\nabla w_n|^2 + \int_{\mathbb{R}^N} V_0 f^2(w_n) \ge C\|w_n\|_E.$$

Hence, $w_n = 0$ for large n, a contradiction. This shows that the sequence $\{P_n\} \subset \mathbb{R}^N$ is bounded. Consequently, $\varepsilon_n \xi_{\varepsilon_n} \to z \in \Lambda$. Therefore, $\lim_{n\to\infty} V(\varepsilon_n \xi_{\varepsilon_n}) = V(z) = V_0$. This completes the proof of Theorem 2. □

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