Approximate and Generalized Solutions of Conformable Type Coudrey-Dodd-Gibbon-Sawada-Kotera Equation

Mehmet Senol[1], **Lanre Akinyemi**[2], **Ayşe Ata**[3], **Olaniyi S. Iyiola**[4]

[1,3]**Department of Mathematics, Faculty of Science and Art, Nevsehir Haci Bektasi Veli University, Nevsehir, TURKEY**

[2]**Department of Mathematics, Ohio University, Athens, USA**

[4]**Department of Mathematics, Computer Science & Information System, California University of Pennsylvania, Pennsylvania, USA**

e-mail: msenol@nevsehir.edu.tr[1]

e-mail: la740411@ohio.edu[2]

e-mail: ayseata@nevsehir.edu.tr[3]

e-mail: iyiola@calu.edu and niyi4oau@gmail.com[4]

Abstract

In this study, we consider conformable type Coudrey-Dodd-Gibbon-Sawada-Kotera (CDGSK) equation. Three powerful analytical methods are employed to obtain generalized solutions of the nonlinear equation of interest. First, the sub-equation method is used as baseline where generalized closed form solutions are obtained and are exact for any fractional order α. Furthermore, Residual power series (RPSM) and q-homotopy (q-HAM) analysis techniques are then applied to obtain approximate solutions. These are possible using some properties of conformable derivative. These approximate methods are very powerful and efficient due to the absence of the need for linearization, discretization and perturbation. Numerical simulations are carried out showing error values, $\hbar$-curve for q-HAM and the effects of fractional order on the solution profiles.

Keywords: Residual Power Series Method, Coudrey-Dodd-Gibbon-Sawada-Kotera Equation, Conformable Type Derivative, q-Homotopy Analysis Method, Sub-equation Method

1 Introduction

Fractional analysis is the generalized state of classical integer order derivative and integral. Since L-Hospital's letter to Leibniz in 1695 after the initial work of L-Hospital and Leibniz, many mathematicians worked on the definition of fractional derivatives and using their representations, they have made definitions of fractional derivatives and integrals. Although, many definitions of derivative are used in fractional analysis, most studies have focused on the integral form of the fractional derivative. The most well known fractional derivatives are Caputo and Riemann-Liouville. But these two definitions have some disadvantages, one of them, the Riemann-Liouville definition, does not meet $\mathcal{D}^{\alpha}\rho = 0$, for ρ a constant when α is not a natural number and Caputo definition suppose that the function is differentiable. Both definitions does not meet the chain rule, index rule and the derivative of the product of two functions. Because of these shortcomings of the existing definitions, we wanted to use the conformable derivative definition. In [1,2], new results and properties of conformable derivative are presented.

Recently, it has been observed that nonlinear fractional differential equations (NFDEs) have been solved by some approximate methods and in many cases give results with very small error even exact in some cases. In the literature, the techniques used to find approximate solutions of nonlinear fractional partial differential equations include different methods like homotopy perturbation method [3], Laplace analysis method [4], homotopy analysis method [5–9], Adomian decomposition method [10], differential transformation method [11], perturbation-iteration algorithm [12], iterative Shehu transform method [13], residual power series method in [17–25] and also q-homotopy analysis transform method in [14–16].

In this study, exact and approximate solutions of CDGSK equation [26] given below, are obtained by using sub-equation, residual power series method and q-homotopy analysis method with conformable derivative definition.

$$\mathcal{D}_t^{\alpha} u + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \tag{1}$$

In this equation, the fractional order α, ($0 < \alpha < 1$), symbolize the conformable derivative and we obtained both analytic and approximate solutions for different α values and showed how close it is to the exact solution with tables and graphics. The CDGSK equation is a class of fifth order Korteweg-de Vries equation which has numerous application in nonlinear optics and quantum mechanics. In [27], physical understanding of CDGSK equation has been highlighted. Wazwaz has obtained one-soliton solution by the tanh-coth method in [28]. Hirota transformation of the CDGSK equation and its bilinear forms are discussed in [29–31]. Analytical solutions of CDGSK equation have been studied in many papers by different methods (See [32–34]).

The rest of the work is arranged as follows: Section 2, we give some basic definitions and notation useful for the work in sequel. In Section 3, we give a brief description and application of sub-equation method to CDGSK equation. Explanation of residual power series method is presented in Section 4. In Section 5, the fundamental idea of q-homotopy analysis method is discussed and implemented on CDGSK equation. We give a numerical comparison in Section 6 followed by conclusion in Section 7.

2 Preliminaries

Here, we present some basic definitions and notation used in this present work.

Definition 2.1. *[35] The Riemann–Liouville fractional derivative operator $D^{\alpha} f(x)$ for $\alpha > 0$ and $\eta - 1 < \alpha < \eta$ defined as*

$$\mathcal{D}^{\alpha} f(x) = \frac{d^{\eta}}{dx^{\eta}} \left[\frac{1}{\Gamma(\eta - \alpha)} \int_x^{\alpha} \frac{f(t)}{(x-t)^{\alpha+1-\eta}} dt \right]. \tag{2}$$

Definition 2.2. *[36] Caputo fractional derivative of order $\alpha > 0$ for $n \in N, n-1 < \alpha < n$ defined as*

$$\mathcal{D}_*^{\alpha} f(x) = J^{n-\alpha} \mathcal{D}^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^{\alpha} (x-t)^{n-\alpha-1} \left(\frac{d}{dt} \right)^n f(t)\ dt. \tag{3}$$

Definition 2.3. *The conformable fractional derivative of a function, $f : [0, \infty) \to R$ of order α is defined as [37]*

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - (f)(t)}{\varepsilon} \tag{4}$$

Theorem 2.4. *[38] Let $\alpha \in (0, 1]$ and f, g be α differentiable at a point $t > 0$. Then*

1. $T_{\alpha}(e_1 f + e_2 g) = e_1 T_{\alpha}(f) + e_2 T_{\alpha}(g)$, for $e_1, e_2 \in \mathbb{R}$.
2. $T_{\alpha}(t^p) = pt^{p-\alpha},\ \forall\ p \in \mathbb{R}$.
3. $T_{\alpha}(f.g) = fT_{\alpha}(g) + gT_{\alpha}(f)$.
4. $T_{\alpha}(\frac{f}{g}) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$.
5. $T_{\alpha}(\rho) = 0$, ρ *is constant.*
6. Furthermore, $T_{\alpha}(f)(t) = t^{1-\alpha}\ \frac{\partial f}{\partial t}$, if f is differentiable.

Definition 2.5. *[1] Let f be a function with n variables $x_1, ..., x_n$ and the conformable partial derivative of f of order $\alpha \in (0, 1]$ in x_i is defined as follows*

$$\frac{d^{\alpha}}{dx_i^{\alpha}} f(x_1, ..., x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, ..., x_n) - f(x_1, ..., x_n)}{\varepsilon}. \tag{5}$$

Definition 2.6. *The α-fractional integral of a function u starting from $\nu \geq 0$ is defined as*

$$\mathcal{J}_{\nu}^{\mu}(u)(t) = \int_{\nu}^{t} \frac{u(s)}{s^{1-\mu}}\, ds. \tag{6}$$

Here, $\alpha \in (0,1]$ and the integral is the usual Riemann improper integral.

3 Sub-equation Method for Fractional Order Derivative

In this section, we set out the most important steps of the fractional sub-equation method [39,40] for solving conformable time-fractional partial differential equations (CT-FPDEs). For a given CT-FPDE

$$\mathcal{F}\big(u, T_t^{\alpha}, T_x u, T_x^2 u, T_x^3 u, \cdots\big) = 0. \tag{7}$$

where T_t^{α} is a conformable derivative operator of arbitrary order.

Step 1: The fractional wave transformation [41] is in the form

$$u(x,t) = \mathrm{U}(\xi), \;\; \xi = kx + c\frac{t^{\alpha}}{\alpha}. \tag{8}$$

Here, c and k are constants to be determined subsequently.

Step 2: Through the use of chain rule, Eq. (7) can be rewrite to an integer order nonlinear fractional ODE

$$\widetilde{\mathcal{F}}\big(\mathrm{U}(\xi), \mathrm{U}'(\xi), \mathrm{U}''(\xi), \mathrm{U}'''(\xi), \cdots\big) = 0. \tag{9}$$

Step 3: The travelling wave solution of Eq. (9) is given as

$$\mathrm{U}(\xi) = a_0 + \sum_{r=1}^{N} a_r \vartheta^r(\xi), \;\; a_N \neq 0, \tag{10}$$

where $a_r(r = 0, 1, \cdots, N)$, k and c are are constants to be determined afterwards. The integer N can be found when we balance the nonlinear terms and highest-order derivative [42] in Eq. (9) and the function $\vartheta^r(\xi)$ with ϱ been a constant, fulfills the Riccati equation given by

$$\vartheta'(\xi) = \varrho + \vartheta^2(\xi). \tag{11}$$

Step 4: A solutions set which validate Eq. (11) is described below:

$$\vartheta(\xi) = \begin{cases} -\sqrt{-\varrho}\tanh(\sqrt{-\varrho}\,\xi), & \varrho < 0, \\ -\sqrt{-\varrho}\coth(\sqrt{-\varrho}\,\xi), & \varrho < 0, \\ \sqrt{\varrho}\tan(\sqrt{\varrho}\,\xi), & \varrho > 0, \\ -\sqrt{\varrho}\cot(\sqrt{\varrho}\,\xi), & \varrho > 0, \\ -\frac{1}{\xi + l}, \;\; \text{l is a constant}, & \varrho = 0. \end{cases} \tag{12}$$

Step 5: Substituting Eq. (10) into Eq. (9) and in addition use Eq. (11), then, setting the coefficients of $\vartheta^r(\xi)$ to zero, one gets the nonlinear algebraic system of equations in $a_r(r = 0, 1, \cdots, N)$, k, and c.

Step 6: Computing the solution of these nonlinear algebraic system of equations and substituting into Eq. (10), we get the exact travelling solutions for Eq. (1).

3.1 Implementation of Sub-equation Method to CDGSK Equation

Here, we first use Eq. (8) and chain rule to transform Eq. (1) to the below nonlinear fractional ODE

$$cU'(\xi)+k^5U^{(5)}(\xi)+30k^3U(\xi)U'''(\xi)+30k^3U'(\xi)U'''(\xi)+180kU^2(\xi)U'(\xi)=0. \tag{13}$$

After integrating the above ODE once, we have the reduced form

$$cU(\xi)+k^5U^{(4)}(\xi)+30k^3U(\xi)U''(\xi)+60kU^3(\xi)=0. \tag{14}$$

Balancing the nonlinear term $U^3(\xi)$ and the highest order derivative term $U^{(4)}(\xi)$ in Eq. (14), we obtain $N=2$. Thus, from Eq. (10), one get

$$U(\xi)=a_0+a_1\vartheta(\xi)+a_2\vartheta^2(\xi). \tag{15}$$

Substituting Eq. (15) together with Eq. (11) into Eq. (14), collecting the coefficients of $\vartheta^r(\xi)$ and set them to zero. A set of algebraic equations is obtained in a_0, a_1, a_2, k, and c as follows:

$$\begin{aligned}
\vartheta^0(\xi) &: \quad 60a_0^3k+60a_0a_2k^3\varrho^2+a_0c+16a_2k^5\varrho^3=0,\\
\vartheta^1(\xi) &: \quad 180a_0^2a_1k+60a_0a_1k^3\varrho+60a_1a_2k^3\varrho^2+a_1c+16a_1k^5\varrho^2=0,\\
\vartheta^2(\xi) &: \quad 180a_0^2a_2k+180a_0a_1^2k+240a_0a_2k^3\varrho+60a_1^2k^3\varrho+60a_2^2k^3\varrho^2+a_2c+136a_2k^5\varrho^2=0,\\
\vartheta^3(\xi) &: \quad 360a_0a_1a_2k+60a_0a_1k^3+60a_1^3k+300a_1a_2k^3\varrho+40a_1k^5\varrho=0,\\
\vartheta^4(\xi) &: \quad 180a_0a_2^2k+180a_0a_2k^3+180a_1^2a_2k+60a_1^2k^3+240a_2^2k^3\varrho+240a_2k^5\varrho=0,\\
\vartheta^5(\xi) &: \quad 180a_1a_2^2k+240a_1a_2k^3+24a_1k^5=0,\\
\vartheta^6(\xi) &: \quad 60a_2^3k+180a_2^2k^3+120a_2k^5=0.
\end{aligned} \tag{16}$$

Using Mathematica to solve the above algebraic equations, we achieved the following cases:

Case 1. *Let $a_0=-k^2\varrho, a_1=0, a_2=-k^2$, and $c=-16k^5\varrho^2$. Substitute these values into Eq. (15) and taking into account Eq. (8) and Eq. (12), we obtain the following solutions:*

$$\begin{aligned}
u_{11} &= -k^2\varrho+k^2\varrho\tanh^2\left(\sqrt{-\varrho}\Big(kx-\frac{16k^5\varrho^2}{\alpha}t^\alpha\Big)\right), \quad \varrho<0,\\
u_{12} &= -k^2\varrho+k^2\varrho\coth^2\left(\sqrt{-\varrho}\Big(kx-\frac{16k^5\varrho^2}{\alpha}t^\alpha\Big)\right), \quad \varrho<0,\\
u_{13} &= -k^2\varrho-k^2\varrho\tan^2\left(\sqrt{\varrho}\Big(kx-\frac{16k^5\varrho^2}{\alpha}t^\alpha\Big)\right), \qquad \varrho>0,\\
u_{14} &= -k^2\varrho-k^2\varrho\cot^2\left(\sqrt{\varrho}\Big(kx-\frac{16k^5\varrho^2}{\alpha}t^\alpha\Big)\right), \qquad \varrho>0,\\
u_{15} &= -\frac{k^2}{(l+kx)^2}, \qquad \varrho=0.
\end{aligned} \tag{17}$$

Case 2. *Let $a_0=-\frac{1}{30}\left(\sqrt{105}k^2\varrho+15k^2\varrho\right), a_1=0, a_2=-k^2$, and $c=2\left(\sqrt{105}k^5\varrho^2-11k^5\varrho^2\right)$. Substitute these values into Eq. (15) and taking into account Eq. (8) and Eq. (12), we obtain the following solutions:*

$$
\begin{aligned}
u_{21} &= -\frac{1}{30}\left(\sqrt{105}k^2\varrho+15k^2\varrho\right)+k^2\varrho\tanh^2\left(\sqrt{-\varrho}\left(kx+\frac{2\left(\sqrt{105}k^5\varrho^2-11k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho<0,\\
u_{22} &= -\frac{1}{30}\left(\sqrt{105}k^2\varrho+15k^2\varrho\right)+k^2\varrho\coth^2\left(\sqrt{-\varrho}\left(kx+\frac{2\left(\sqrt{105}k^5\varrho^2-11k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho<0,\\
u_{23} &= -\frac{1}{30}\left(\sqrt{105}k^2\varrho+15k^2\varrho\right)-k^2\varrho\tan^2\left(\sqrt{\varrho}\left(kx+\frac{2\left(\sqrt{105}k^5\varrho^2-11k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho>0,\\
u_{24} &= -\frac{1}{30}\left(\sqrt{105}k^2\varrho+15k^2\varrho\right)-k^2\varrho\cot^2\left(\sqrt{\varrho}\left(kx+\frac{2\left(\sqrt{105}k^5\varrho^2-11k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho>0.
\end{aligned}
\tag{18}
$$

Case 3. *Let* $a_0=\frac{1}{30}\left(\sqrt{105}k^2\varrho-15k^2\varrho\right), a_1=0, a_2=-k^2,$ *and* $c=-2\left(\sqrt{105}k^5\varrho^2+11k^5\varrho^2\right)$. *Substitute these values into Eq. (15) and taking into account Eq. (8) and Eq. (12), we obtain the following solutions:*

$$
\begin{aligned}
u_{31} &= \frac{1}{30}\left(\sqrt{105}k^2\varrho-15k^2\varrho\right)+k^2\varrho\tanh^2\left(\sqrt{-\varrho}\left(kx-\frac{2\left(11k^5\varrho^2+\sqrt{105}k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho<0,\\
u_{32} &= \frac{1}{30}\left(\sqrt{105}k^2\varrho-15k^2\varrho\right)+k^2\varrho\coth^2\left(\sqrt{-\varrho}\left(kx-\frac{2\left(11k^5\varrho^2+\sqrt{105}k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho<0,\\
u_{33} &= \frac{1}{30}\left(\sqrt{105}k^2\varrho-15k^2\varrho\right)-k^2\varrho\tan^2\left(\sqrt{\varrho}\left(kx-\frac{2\left(11k^5+\sqrt{105}k^5\varrho^2\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho>0,\\
u_{34} &= \frac{1}{30}\left(\sqrt{105}k^2\varrho-15k^2\varrho\right)-k^2\varrho\cot^2\left(\sqrt{\varrho}\left(kx-\frac{2\left(11k^5\varrho^2+\sqrt{105}k^5\varrho^2\right)}{\alpha}t^\alpha\right)\right), \quad \varrho>0.
\end{aligned}
\tag{19}
$$

Remark 3.1. *The result for u_{25} and u_{35} is omitted because they are the same as the result of u_{15} for the case when* $\varrho=0$.

4 *Description of Residual Power Series Method*

Here, we will present some major definitions and theorems about the method we use.

Definition 4.1. *[35] A power series expansion of the form*

$$\sum_{r=0}^{\infty} e_r(t-t_0)^{r\alpha}=e_0+e_1(t-t_0)^\alpha+e_2(t-t_0)^{2\alpha}+\cdots$$

where $t>t_0$ and $0<n-1<\alpha\leq n$, $n\in\mathbb{N}$ is known as the fractional power series about t_0. Here, e_n's are constant coefficients of the series.

Definition 4.2. *Assuming that f has a FPS representation at $t_0=0$ of the form*

$$f(t)=\sum_{r=0}^{\infty} e_r t^{r\alpha},\ 0<t<R^{\frac{1}{\alpha}},\ R>0, \tag{20}$$

Furthermore, suppose that f is infinitely conformable α differentiable function, for some $0<n-1<\alpha\leq n$, $n\in\mathbb{N}$ in a neighborhood of a point $t_0=0$. Then, the coefficients e_r in (20) is express as

$$e_r=\frac{f^{(r\alpha)}(0)}{\alpha^r r!}$$

where $f^{(r\alpha)}$ represent the application of the conformable fractional derivative r-times.

Definition 4.3. *[43] A power series of the form* $\sum_{r=0}^{\infty} f_r(x)t^{\alpha}$*, is defined as a multiple fractional power series about* $t_0 = 0$*, where* $f_r(x)$ *is a function of* x *called the coefficients of the series.*

Definition 4.4. *[43] Assume that* $u(x,t)$ *has a multiple FPS representation at* $t_0 = 0$*, then,*

$$u(x,t) = \sum_{r=0}^{\infty} f_r(x)t^{r\alpha},\ 0 < n-1 < \alpha \leq n,\ x \in I,\ 0 \leq t < R^{\frac{1}{\alpha}}. \tag{21}$$

If $u_t^{(r\alpha)}(x,t)$*,* $r = 0, 1, 2, \ldots$ *are continuous on* $I \times (0, \mathbb{R}^{\frac{1}{\alpha}})$ *and* $f_r(x) = \frac{u_t^{(r\alpha)}(x,0)}{\alpha^r r!}$.

To exemplify the central idea of RPSM, we consider a CT-FPDE in the structure given below

$$T_\alpha u(x,t) + \mathscr{L}u(x,t) + \mathcal{N}u(x,t) = g(x,t), \quad x \in \mathbb{R}, \quad n-1 < n\alpha \leq n, \quad t > 0, \tag{22}$$

having initial condition

$$u(x,0) = \theta_0(x) = \theta(x), \tag{23}$$

where $\mathscr{L}$ and $\mathcal{N}$ are respectively linear and nonlinear operators and $g(x,t)$ is a known continuous function. The RPSM make of declaratory the solution of the equation as the fractional power series expansion around $t = 0$ is given below as

$$\theta_{(r-1)}(x) = T_t^{(r-1)\alpha} u(x,0) = \theta(x). \tag{24}$$

The RPSM solution is

$$u(x,t) = \sum_{r=0}^{\infty} \theta_r(x) \frac{t^{r\alpha}}{\alpha^n r!}. \tag{25}$$

The k-th truncated series of $u(x,t)$ which is $u_k(x,t)$ can be expressed as

$$u_k(x,t) = \sum_{r=0}^{k} \theta_r(x) \frac{t^{r\alpha}}{\alpha^r r!}. \tag{26}$$

The 1^{st} *RPSM approximate solution* $u_1(x,t)$ *is written as:*

$$u_1(x,t) = \theta(x) + \theta_1(x) \frac{t^{\alpha}}{\alpha^r}. \tag{27}$$

Primarily, the residual function Res is defined as:

$$Res(x,t) = T_\alpha u(x,t) + \mathscr{L}u(x,t) + \mathcal{N}u(x,t) - g(x,t), \tag{28}$$

and the k-th residual function Res_k as:

$$Res_k(x,t) = T_\alpha u_k(x,t) + \mathscr{L}u_k(x,t) + \mathcal{N}u_k(x,t) - g(x,t),\ k = 1, 2, 3, \cdots$$

For $k = 1$ the expression $Res_1(x,t)$ is written. In this expression, $\theta_1(x)$ is obtained when $Res_1(x,0) = 0$ is regulated for $t = 0$. This expression results in the first solution of the RPSM approximation, $u_1(x,t)$. In each subsequent step, different $\theta_k(x)$ is obtained for each of $k = 1, 2, 3, \cdots$, as

$$\frac{\partial^{(r-1)\alpha}}{\partial t^{(r-1)\alpha}} Res_k(x,0) = 0,\ 0 < \alpha \leq 1,\ x \in I,\ 0 \leq t < R,\ r = 1, 2, 3, \cdots, \tag{29}$$

is an important processing step of the RPSM [23, 25, 44]. That is, in the second step, the first conformable derivative of each side according to α is taken and the expression is equal to zero for $t = 0$. In the third step, the $2nd$ conformable derivative of each side according to α is taken and thus, firstly the $\theta_k(x)$ values and then $u_k(x,t)$ approximate solutions are obtained respectively. In this method, it can be said that the exact result will be approached more in each step thanks to the fractional power series of the equation taken.

4.1 Approximate Solution of the CDGSK Equation Using RPSM

Time fractional CDGSK equation is expressed as [26]:

$$\mathcal{D}_t^{\alpha} u + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \tag{30}$$

having initial condition

$$u(x,0) = -k^2 \varrho - k^2 \varrho \tan^2 \left(\sqrt{\varrho} kx\right). \tag{31}$$

In order to find the values of $\theta_r(x)$ in Eq. (26), for $r = 1, 2, 3, \ldots, k$, $u(x,t)$ series expansion is performed. Residual function of time fractional CDGSK equation $Res(x,t)$ is

$$Res(x,t) = D_t^{\alpha} u + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x. \tag{32}$$

The k-th Residual function, Res_k is

$$Res_k(x,t) = D_t^{\alpha} u_k + u_{kxxxxx} + 30u_k u_{kxxx} + 30u_{kx} u_{kxx} + 180u_k^2 u_{kx}. \tag{33}$$

In the first step of the residual power series algorithm, $u_1(x,t)$ truncated series is placed into the equation $Res(x,t)$ which is solved. The expression $Res_1(x,t)$ is obtained as

$$Res_1(x,t) = D_t^{\alpha} u_1 + u_{1xxxxx} + 30u_1 u_{1xxx} + 30u_{1x} u_{1xx} + 180u_1^2 u_{1x}, \tag{34}$$

where

$$u_1(x,t) = \theta(x) + \theta_1(x)\frac{t^{\alpha}}{\alpha}. \tag{35}$$

Then,

$$\begin{aligned} Res_1(x,t) \quad = \quad & \theta_1 + 180\left(\theta + \frac{t^{\alpha}\theta_1}{\alpha}\right)^2 \left(\theta^{'} + \frac{t^{\alpha}\theta_1^{'}}{\alpha}\right) + 30\left(\theta^{'} + \frac{t^{\alpha}\theta_1^{'}}{\alpha}\right)\left(\theta^{''} + \frac{t^{\alpha}\theta_1^{''}}{\alpha}\right) \\ & + 30\left(\theta + \frac{t^{\alpha}\theta_1}{\alpha}\right)\left(\theta^{'''}(x) + \frac{t^{\alpha}\theta_1^{'''}}{\alpha}\right) + \theta^{(5)} + \frac{t^{\alpha}\theta_1^{(5)}}{\alpha}, \end{aligned} \tag{36}$$

where $\theta = \theta(x)$. Thus, for $Res_1(x,0) = 0$,

$$\theta_1(x) = -180\theta^2\theta' - 30\theta^{'}\theta^{''} - 30\theta^{'''}\theta^{'''}. \tag{37}$$

Hence, the $1^s t$ RPSM approximate solution of CDGSK equation as

$$u_1(x,t) = \theta(x) + \frac{t^{\alpha}\theta_1(x)}{\alpha}. \tag{38}$$

Similarly, $Res_k(x,t)$ for $k = 2$ is given as

$$Res_2(x,t) = D_t^{\alpha} u_2 + u_{2xxxxx} + 30u_2 u_{2xxx} + 30u_{2x} u_{2xx} + 180u_2^2 u_{2x}. \tag{39}$$

In the $Res_2(x,t)$, the following expression is written instead of $u_2(x,t)$ as

$$u_2(x,t) = \theta(x) + \theta_1(x)\frac{t^{\alpha}}{\alpha} + \theta_2(x)\frac{t^{2\alpha}}{2\alpha^2}. \tag{40}$$

Then,

$$
\begin{aligned}
Res_2(x,t) &= t^{1-\alpha}\left(t^{\alpha-1}\theta_1+\frac{t^{2\alpha-1}\theta_2}{\alpha}\right)+180\left(\theta(x)+\frac{t^\alpha\theta_1}{\alpha}+\frac{t^{2\alpha}\theta_2}{2\alpha^2}\right)^2 \\
&\times\left(\theta^{'}+\frac{t^\alpha\theta_1^{'}}{\alpha}+\frac{t^{2\alpha}\theta_2^{'}}{2\alpha^2}\right)+30\left(\theta^{'}+\frac{t^\alpha\theta_1^{'}}{\alpha}+\frac{t^{2\alpha}\theta_2^{'}}{2\alpha^2}\right) \\
&\times\left(\theta^{''}+\frac{t^\alpha\theta_1^{''}}{\alpha}+\frac{t^{2\alpha}\theta_2^{''}}{2\alpha^2}\right)+30\left(\theta+\frac{t^\alpha\theta_1}{\alpha}+\frac{t^{2\alpha}\theta_2}{2\alpha^2}\right) \\
&\times\left(\theta^{'''}+\frac{t^\alpha\theta_1^{'''}}{\alpha}+\frac{t^{2\alpha}\theta_2^{'''}}{2\alpha^2}\right)+\theta^{(5)}+\frac{t^\alpha\theta_1^{(5)}}{\alpha}+\frac{t^{2\alpha}\theta_2^{(5)}}{2\alpha^2}. \qquad (41)
\end{aligned}
$$

Now, applying T_α conformable derivative on both sides of (41) and equating to 0 for $t=0$ gives the $2nd$ approximate solution of RPSM as follows:

$$
\theta_2(x) = -360\theta\theta_1\theta^{'}-180\theta^2\theta_1^{'}-30\theta_1^{'}\theta^{''}-30\theta^{'}\theta_1^{''}-30\theta_1\theta^{'''}-30\theta\theta_1^{'''}-\theta_1^{(5)}, \qquad (42)
$$

and

$$
\begin{aligned}
u_2(x,t) &= \theta+\frac{t^\alpha\theta_1}{\alpha}+\frac{1}{2\alpha^2}t^{2\alpha}\left(-360\theta\theta_1\theta^{'}-180\theta^2\theta_1^{'}\right) \\
&+\frac{1}{2\alpha^2}t^{2\alpha}\left(-30\theta_1^{'}\theta^{''}-30\theta^{'}\theta_1^{''}\right) \qquad (43) \\
&+\frac{1}{2\alpha^2}t^{2\alpha}\left(-30\theta_1\theta^{'''}-30\theta\theta_1^{'''}-\theta_1^{(5)}\right).
\end{aligned}
$$

If the process steps are carried out in the same way, $\theta_3(x)$ and $u_3(x,t)$ are obtained respectively as

$$
\begin{aligned}
\theta_3(x) &= -360\theta_1^2\theta^{'}-360\theta\theta_2\theta^{'}-720\theta\theta_1\theta_1^{'}-180\theta^2\theta_2^{'}-30\theta_2^{'}\theta^{''} \\
&-60\theta_1^{'}\theta_1^{''}-30\theta^{'}\theta_2^{''}-30\theta_2\theta^{'''}-60\theta_1\theta_1^{'''}-30\theta\theta_2^{'''}-\theta_2^{(5)}, \qquad (44)
\end{aligned}
$$

and

$$
\begin{aligned}
u_3(x,t) &= \theta+\frac{t^\alpha\theta_1}{\alpha}+\frac{t^{2\alpha}\theta_2}{2\alpha^2}+\frac{t^{3\alpha}}{6\alpha^3}\left(-360\theta_1^2\theta^{'}-360\theta\theta_2\theta^{'}\right) \\
&+\frac{t^{3\alpha}}{6\alpha^3}\left(-720\theta\theta_1\theta_1^{'}-180\theta^2\theta_2^{'}-30\theta_2^{'}\theta^{''}\right) \\
&+\frac{t^{3\alpha}}{6\alpha^3}\left(-60\theta_1^{'}\theta_1^{''}-30\theta^{'}(x)\theta_2^{''}-30\theta_2\theta^{'''}\right) \qquad (45) \\
&+\frac{t^{3\alpha}}{6\alpha^3}\left(-60\theta_1\theta_1^{'''}-30\theta\theta_2^{'''}-\theta_2^{(5)}\right).
\end{aligned}
$$

5 Approximate Solution of the CDGSK Equation Using q-HAM

Here, application of q-homotopy analysis method [6–8] to time fractional CDGSK equation is presented. The time fractional CDGSK equation is expressed as

$$
\mathcal{D}_t^\alpha u+u_{xxxxx}+30uu_{xxx}+30u_xu_{xx}+180u^2u_x=0, \qquad (46)
$$

having initial condition

$$u(x,0) = -k^2\varrho - k^2\varrho\tan^2\left(\sqrt{\varrho}kx\right). \tag{47}$$

First, we choose the linear operator as

$$\mathscr{L}\big[\varphi(x,t;q)\big] = D_t^{\alpha}\varphi(x,t;q),$$

having property $\mathscr{L}[\mathcal{B}] = 0$, for a constant $\mathcal{B}$. The nonlinear operator $\mathcal{N}$, with $\varphi = \varphi(x,t;q)$ is define as

$$\mathcal{N}(\varphi) = \mathcal{D}_t^{\alpha}\varphi + \varphi_{xxxxx} + 30\varphi\varphi_{xxx} + 30\varphi_x\varphi_{xx} + 180\varphi^2\varphi_x. \tag{48}$$

Using Theorem 2.4, the above equation can be rewritten as follows:

$$\mathcal{N}(\varphi) = t^{1-\alpha}\frac{\partial\varphi}{\partial t} + \varphi_{xxxxx} + 30\varphi\varphi_{xxx} + 30\varphi_x\varphi_{xx} + 180\varphi^2\varphi_x \tag{49}$$

We set up the zero-order deformation equation as

$$(1-nq)\mathscr{L}\left(\varphi(x,t;q) - u_0(x,t)\right) = q\hbar\mathcal{H}(x,t)\mathcal{N}\big(\varphi(x,t;q)\big), \tag{50}$$

and for $\mathcal{H}(x,t) = 1$, the k-th order deformation equation is

$$\mathscr{L}\big[u_k - \mathcal{X}_k^* u_{k-1}\big] = \hbar\mathcal{R}_{1,k}\big(\vec{u}_{k-1}\big), \tag{51}$$

where

$$\begin{aligned} \mathcal{R}_k\big(\vec{u}_{k-1}\big) \quad &= \quad t^{1-\mu}\frac{\partial u_{k-1}}{\partial t} + u_{(k-1)xxxxx} + 30\sum_{i=0}^{k-1} u_i u_{(k-1-i)xxx} \\ &+ 30\sum_{i=0}^{k-1} u_{ix}u_{(k-1-i)xx} + \sum_{i=0}^{k-1}\sum_{j=0}^{i} u_j u_{i-j} u_{(k-1-i)x}. \end{aligned} \tag{52}$$

The solution to Eq. (46) for $k \geq 1$ result in

$$u_k(x,y,t) = \mathcal{X}_k^* u_{k-1} + \hbar\mathcal{J}_t^{\mu}\big[\mathcal{R}_k\big(\vec{u}_{k-1}\big)\big]. \tag{53}$$

Here,

$$\mathcal{X}_k^* = \begin{cases} 0 & k \leqslant 1, \\ \\ n & k > 1. \end{cases} \tag{54}$$

Upon solving Eq. (53) for $k = 1, 2, 3, \cdots$ and using Eq. (6), we derive the following

$$u_0(x,t) \quad = \quad -k^2\varrho - k^2\varrho\tan^2\left(\sqrt{\varrho}kx\right),$$

$$
\begin{aligned}
u_1(x,t) &= hJ_t^{\alpha}\big[t^{1-\alpha}\frac{\partial u_0}{\partial t} + u_{0xxxxx} + 30u_0u_{0xxx} \\
&\quad + 30u_{0x}u_{0xx} + 180u_0u_0u_{0x}\big] \\
&= -\frac{32\varrho^{\frac{7}{2}}\hbar k^7 t^{\alpha}}{\alpha}\tan\left(\sqrt{\varrho}kx\right)\sec^2\left(\sqrt{\varrho}kx\right),
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) &= nu_1 + hJ_t^{\alpha}\big[t^{1-\alpha}\frac{\partial u_1}{\partial t} + u_{1xxxxx} + 30u_0u_{1xxx} + 30u_1u_{0xxx} \\
&\quad + 30u_{0x}u_{1xx} + 30u_{1x}u_{0xx} + 180u_0u_0u_{1x} + 360u_0u_1u_{0x}\big] \\
&= (n+\hbar)u_1 + \frac{256\varrho^6\hbar^2k^{12}t^{2\alpha}}{\alpha^2}\Big(\cos\left(2\sqrt{\varrho}kx\right) - 2\Big)\sec^4\left(\sqrt{\varrho}kx\right),
\end{aligned}
$$

$$
\begin{aligned}
u_3(x,t) &= nu_2 + hJ_t^{\alpha}\big[t^{1-\alpha}\frac{\partial u_2}{\partial t} + u_{2xxxxx} + 30u_0u_{2xxx} + 30u_1u_{1xxx} + 30u_2u_{0xxx} \\
&\quad + 30u_{0x}u_{2xx} + 30u_{1x}u_{1xx} + 30u_{2x}u_{0xx} + 180u_0u_0u_{2x} + 180u_1u_0u_{1x} \\
&\quad + 180u_0u_1u_{1x} + 180u_1u_1u_{0x} + 180u_0u_2u_{0x} + 180u_2u_0u_{0x}\big] \\
&= (n+\hbar)u_2 + \frac{256\varrho^6\hbar^2(n+\hbar)k^{12}t^{2\alpha}}{\alpha^2}\Big(\cos\left(2\sqrt{\varrho}kx\right) - 2\Big)\sec^4\left(\sqrt{\varrho}kx\right) \\
&\quad + \frac{8192\varrho^{\frac{17}{2}}\hbar^3k^{17}t^{3\alpha}}{3\alpha^3}\Big(\cos\left(2\sqrt{\varrho}kx\right) - 5\Big)\tan\left(\sqrt{\varrho}kx\right)\sec^4\left(\sqrt{\varrho}kx\right).
\end{aligned}
$$

The rest of the solution can be achieved appropriately. Thus, the q-HAM series solution is

$$
\mathcal{U}^{[N]}(x,t;n;\hbar) = u_0(x,t) + \sum_{i=1}^{N} u_i(x,t)\left(\frac{1}{n}\right)^i
$$

6 Numerical Comparison

Here, we demonstrate the numerical simulation of the CDGSK equation. The exact solution taken from Eq. (17) is given as

$$
u(x,t) = -k^2\varrho - k^2\varrho\tan^2\left(\sqrt{\varrho}\Big(kx - \frac{16k^5\varrho^2}{\alpha}t^{\alpha}\Big)\right),\ \ 0<\alpha\le 1,\ \ t>0. \tag{55}
$$

In particularly, in Figs. 2 to 5, the graphical comparison of RPSM and q-HAM obtained solution with the exact solution for diverse fractional order α ($\alpha = 0.5, 0.75, 0.95$) is presented. We can see that for each fractional order, the cited graphs are almost indistinguishable. The error in the form of absolute error of the two proposed methods has been carried out in tabular form and displayed in Tabs. 1 to 3. From these table, we observed that the difference between the numerical values obtained by RPSM, q-HAM and the exact values are almost identical. The absolute error formula used for the numerical computation is

$$
Error = |\mathcal{U}^{[3]} - u(x,t)|. \tag{56}
$$

In order to ensure fast convergence of q-HAM series solution, the choice of the auxiliary parameter $\hbar$ is essential. In Fig. 1, the $\hbar$-curves which guarantee the optimal choice of $\hbar$ is depicted for different fractional order α and n. The horizontal line segment in the $\hbar$-curves presents the range for $\hbar$. These curves help to adjust and control the region of the convergence of q-HAM solution.

Table 1: Exact, approximate by RPSM and q-HAM ($n = 1, \hbar = -1$), $\mathcal{U}^{[3]}$-solution along with absolute errors for $t = 0.1$, $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.95$.

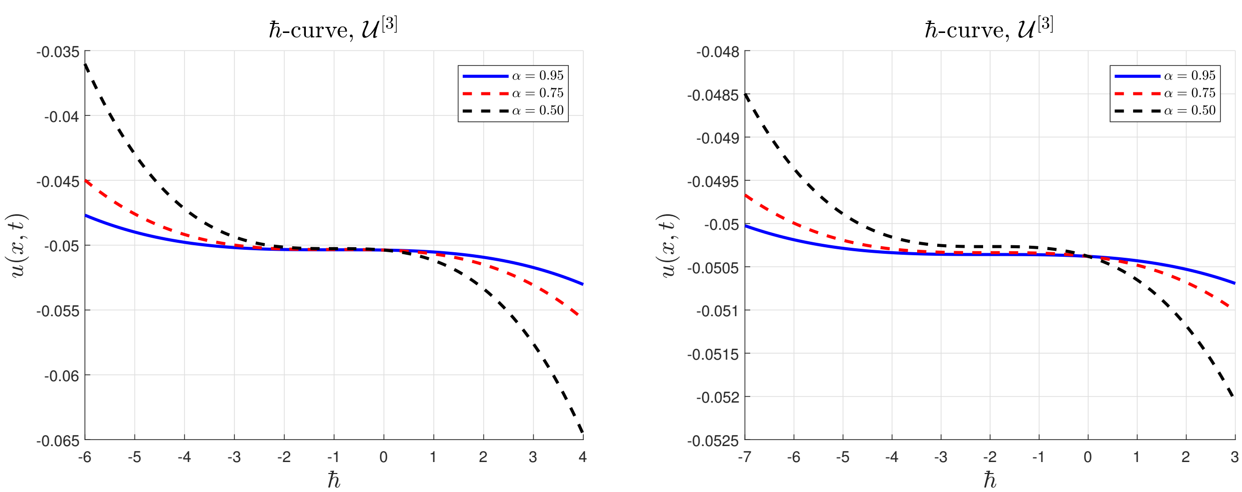
		Approximate solution		Absolute error	
x	Exact solution	RPSM	q-HAM	RPSM	q-HAM
0.1	-0.0480210836	-0.0480210836	-0.0480210836	2.660372×10^{-14}	2.660372×10^{-14}
0.2	-0.0480882993	-0.0480882993	-0.0480882993	2.692291×10^{-14}	2.692291×10^{-14}
0.3	-0.0482019495	-0.0482019495	-0.0482019495	2.747108×10^{-14}	2.747108×10^{-14}
0.4	-0.0483624729	-0.0483624729	-0.0483624729	2.824824×10^{-14}	2.824824×10^{-14}
0.5	-0.0485704918	-0.04857049178	-0.0485704918	2.926132×10^{-14}	2.926132×10^{-14}
0.6	-0.0488268172	-0.0488268172	-0.0488268172	3.053807×10^{-14}	3.053807×10^{-14}
0.7	-0.0491324555	-0.0491324555	-0.0491324555	3.209238×10^{-14}	3.209238×10^{-14}
0.8	-0.0494886173	-0.0494886173	-0.0494886173	3.393119×10^{-14}	3.393119×10^{-14}
0.9	-0.0498967272	-0.0498967272	-0.0498967272	3.611694×10^{-14}	3.611694×10^{-14}
1.0	-0.0503584364	-0.0503584364	-0.0503584364	3.862882×10^{-14}	3.862882×10^{-14}

Table 2: Exact, approximate by RPSM and q-HAM ($n = 1, \hbar = -1$), $\mathcal{U}^{[3]}$-solution along with absolute errors for $t = 0.1$, $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.75$.

		Approximate solution		Absolute error	
x	Exact solution	RPSM	q-HAM	RPSM	q-HAM
0.1	-0.0480191935	-0.0480191935	-0.0480191935	4.320225×10^{-13}	4.320225×10^{-13}
0.2	-0.0480843794	-0.0480843794	-0.0480843794	4.372613×10^{-13}	4.372613×10^{-13}
0.3	-0.0481959848	-0.0481959848	-0.0481959848	4.461015×10^{-13}	4.461015×10^{-13}
0.4	-0.0483544404	-0.0483544404	-0.0483544404	4.586886×10^{-13}	4.586886×10^{-13}
0.5	-0.0485603602	-0.0485603602	-0.0485603602	4.751893×10^{-13}	4.751893×10^{-13}
0.6	-0.0488145469	-0.0488145469	-0.0488145469	4.958603×10^{-13}	4.958603×10^{-13}
0.7	-0.0491179982	-0.0491179982	-0.0491179982	5.209722×10^{-13}	5.209722×10^{-13}
0.8	-0.0494719153	-0.0494719153	-0.0494719153	5.509065×10^{-13}	5.509065×10^{-13}
0.9	-0.0498777135	-0.0498777135	-0.0498777135	5.861353×10^{-13}	5.861353×10^{-13}
1.0	-0.0503370336	-0.0503370336	-0.0503370336	6.271650×10^{-13}	6.271650×10^{-13}

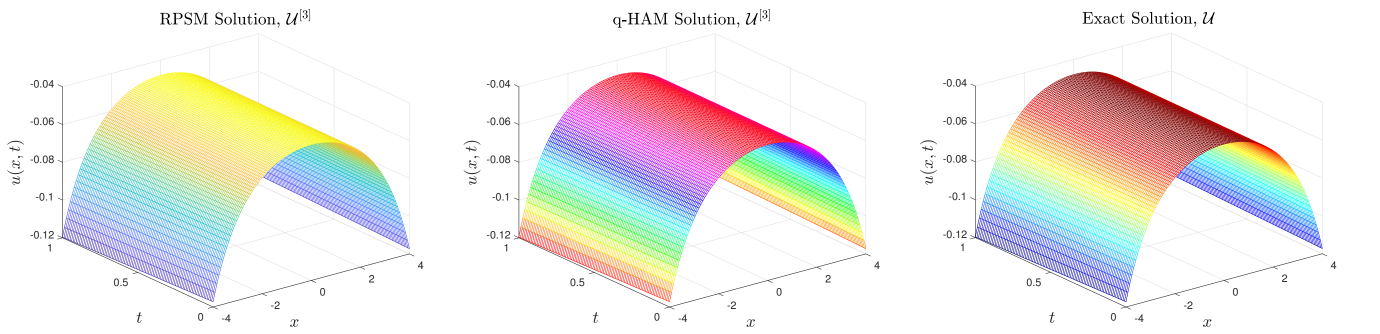
Table 3: Exact, approximate by RPSM and q-HAM ($n = 1, \hbar = -1$), $\mathcal{U}^{[3]}$-solution along with absolute errors for $t = 0.1$, $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.50$.

		Approximate solution		Absolute error	
x	Exact solution	RPSM	q-HAM	RPSM	q-HAM
0.1	-0.0480135515	-0.0480135515	-0.0480135515	2.186624×10^{-11}	2.186624×10^{-11}
0.2	-0.0480719974	-0.0480719973	-0.0480719973	2.212602×10^{-11}	2.212602×10^{-11}
0.3	-0.0481768150	-0.0481768149	-0.0481768149	2.256859×10^{-11}	2.256859×10^{-11}
0.4	-0.0483284086	-0.0483284086	-0.0483284086	2.320018×10^{-11}	2.320018×10^{-11}
0.5	-0.0485273655	-0.0485273654	-0.0485273654	2.402978×10^{-11}	2.402978×10^{-11}
0.6	-0.0487744602	-0.0487744602	-0.0487744602	2.506933×10^{-11}	2.506933×10^{-11}
0.7	-0.0490706617	-0.0490706617	-0.0490706617	2.633396×10^{-11}	2.633396×10^{-11}
0.8	-0.0494171410	-0.0494171410	-0.0494171410	2.784233×10^{-11}	2.784233×10^{-11}
0.9	-0.0498152813	-0.0498152813	-0.0498152813	2.961705×10^{-11}	2.961705×10^{-11}
1.0	-0.0502666898	-0.0502666897	-0.0502666897	3.168525×10^{-11}	3.168525×10^{-11}



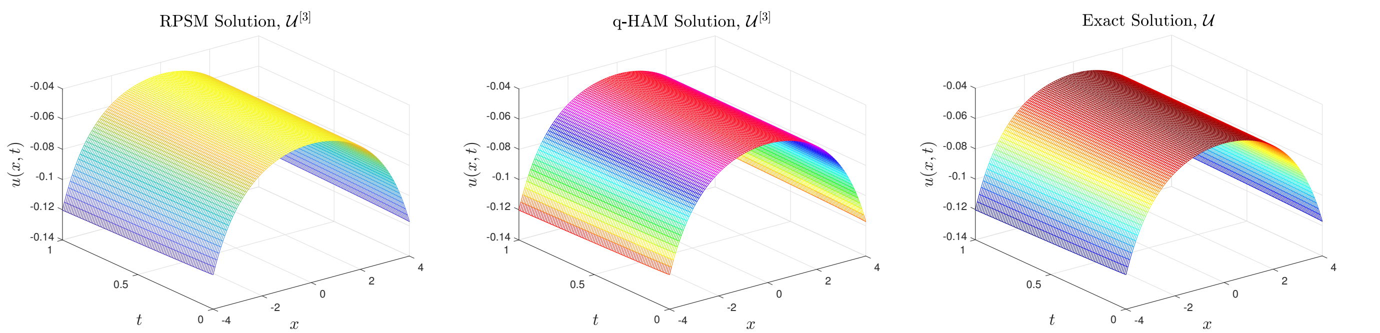
(a) (b)

Figure 1: $\hbar$-curves plot in (a) $n = 1$ and in (b) $n = 2$ when $x = 0.1,\ t = 0.01$ with different α



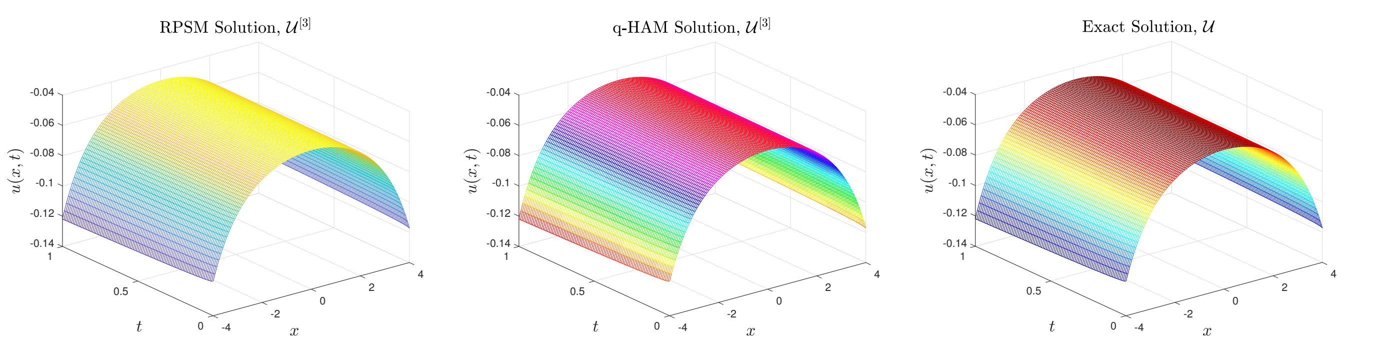
(a) (b) (c)

Figure 2: The graphical comparison of RPSM and q-HAM $(n = 1, \hbar = -1)$ solution with exact solution when $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.95$.



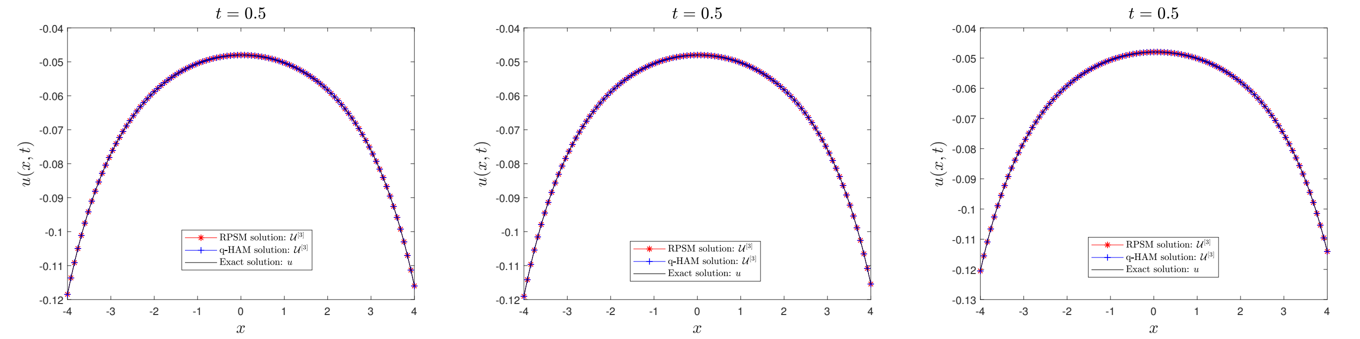
(a) (b) (c)

Figure 3: The graphical comparison of RPSM and q-HAM $(n = 1, \hbar = -1)$ solution with exact solution when $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.75$.



(a) (b) (c)

Figure 4: The graphical comparison of RPSM and q-HAM $(n = 1, \hbar = -1)$ solution with exact solution when $\varrho = 0.3$, $k = 0.4$, and $\alpha = 0.50$.



(a) (b) (c)

Figure 5: The 2D comparison of RPSM, q-HAM ($n = 1, \hbar = -1$), and exact solution in (a) $\alpha = 0.95$, (b) $\alpha = 0.75$, (c) $\alpha = 0.50$ when $\varrho = 0.3$, and $k = 0.4$.

7 Conclusion

In this study, exact and approximate solutions of Coudrey-Dodd-Gibbon-Sawada-Kotera (CDGSK) equation by sub-equation method, residual power series method (RPSM) and q-homotopy analysis method (q-HAM) is obtained. With these methods and the definition of the conformable derivative, it is shown that another complex method and definition are not required. From the cited graphs and tables, one can acknowledge that the obtained approximate solution shows a very good agreement with the exact solution for different α (precisely, $\alpha = 0.95, 0.75$ and 0.5). In addition, it is seen that conformable derivative are more open, simple and understandable than other derivative definitions. The exact and approximate solutions obtained can be used in understanding the physical phenomena of the proposed problem in mathematical physics. Finally, from the results obtained by the two proposed method, we can conclude that the proposed techniques are significantly efficient and can be employed to examine strong nonlinear fractional order mathematical models to understand the nature of complex phenomena.

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