Title	*Pricing To maximize SMR via optimal control theory*
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Abstract

In this paper, the porpuse is to determine an optimal premium in order to increase the insurer's SMR[1] using a control model. For this purpose, First, a stochastic dynamic model is introduced to describe the process of receiving premium and paying claims. Then, the premium variable is introduced as the problem control variable. Next, In order to increase SMR and control the premium, an appropriate objective function is defined for control variables and state variables. In the end, after deterministing and discreting the model, the optimal control problem by using particle swarm is solved.

Keywords: Optimal control, premium, Dynamical systems, solvency margin ratio

1. Introduction

The solvency ratio is one of the most important financial ratios for an insurer, which signals the overall health of the company. Accordingly, it is an important figure, which any stakeholder in the industry would like to watch closely. It is generally monitored either on a quarterly or an annual basis depending on the regulatory requirements of the specific country. Insurance companies which may be in a good financial position at a given Point of time may fall short of the solvency margin requirement in the next Period because of uncertainties and unforeseen factors. Although it is difficult to assess when such a situation for an insurance company could happens, it remains an important task to get best estimates possible with the available data and other factors[8]. In actuarial science, a premium principle equates the cost of a general insurance policy to the moments of the corresponding claim arrival and severity distributions[12]. In order to make a profit and cover their expenses, insurers add a loading to this cost price. Because many lines of insurance are highly competitive, the loading critically depends on the price of other insurers to ensure comparable costs of insurance policies. Insurance pricing is therefore an important factor in determining the type of insurance company customers select or change in the next insurance period. According to Taylor's model, that uses optimal control, the premium is set based

[1] **solvency margin ratio**

on the average insurance market price[1]. The Taylor model is based on, a discrete demand model for pricing in which the future average market price is exactly evaluated and new and existing premium holders are required to pay the same current premium rate[3]. It is difficult to determine premium price using a discrete deterministic model if the average market premium is a continuous stochastic process. Continuous time models can be sets up in several ways: one can either model the premium rate charged by the insurer for a unit of insurance cover, or charge a premium up front for a finite period of cover there after. In the former case, policyholders pay a premium P(t) continuously over the course of their policies[2]. It is important point to note is that pricing at a lower rate in the market can result in a negative premium[4]. This strategy of relatively low initial pricing in the market aims to generate sales; by controlling claims, a creditor's insurance can then increase after which he can increase his prices and profits. Emms and Haberman disucss different modeling constraints[5]. In mathematical finance, there is a similar supposition for the optimal wealth appropriation Problem[8]. It is assumed that the stock rate does not affect the allocation of the investor's stock. Never the less, most insurance lines are under the influence of several large insurers who control each other's prices and constantly update their prices. In such markets, competitive pricing models are not responsive and insurers should therefore pay attention to the response of insurers to insurance prices. The competitive pricing model defines the demand function that determines the relationship between premiums and the average Premium of the market[2]. If the insurer determines a premium Price lower than the average market price, policies will be sold. If the whole insurance market determines a high premium, this would lead to notable sales for an optimising insurer.

In reality however, customers do not pay more than the value of the insured, and they maybe at liberty to take or not to take out insurance. In some way, the demand for insurance policies depends on their premiums, and if the price of all policy sellers are far above the cost, our demand law dictates that very few policies will be sold.

In section 2, Initial stochastic optimal control model is presented . Section 3, the model presented in Section 2 is determined . in section 4, The model is solved and section 5 summarizes a conclusion is presented.

2. Initial stochastic optimal control model

In this section, based on the earlier work of Taylor and Emms [5,15], all prices and claims per unit of exposure varies according to the insured risk (where the exposure is the unit of risk for an insurer).

Suppose at time t the insurer's exposure is **q**, the insurer's premium (per unit exposure) is **P**, the market average premium (per unit exposure) is p, the wealth process is **w** and the mean claim size (per unit exposure) is u. Consider the continuous form of the Taylor model studied by Emms, Haberman & Savoulli :

$$du_t = u_t(\mu dt + \sigma dW_t) \tag{1}$$

$$dp_t = \gamma^{-1} du_t + \lambda(P_t - p_t)dt \tag{2}$$

$$dq_t = q_t(G - K)dt \tag{3}$$

$$dw_t = -\alpha w_t dt + q_t(GP_t - u_t)dt \tag{4}$$

policyholders pay a premium P(t) continuously over the course of their policies ,We assume P(t) t≥0 Per unit exposure, as a premium (P) at time t for a general insurance policy of fixed duration l. q(t) is the '***exposure***' that expresses a measure of the insurance company's potential liabilities '.It reflects the number of currently in force insurance policies and the potential size of the claims on these policies. The reserve, w(t), represents the amount of ***current capital held by the insurance company***, which increases with the sale of the insurance policy and decreases with payment of claims.

For simplicity we suppose the mean claim size rate u(t) is lognormally distributed with constant drift μ and volatility σ and $\{W_t\}$ is a standard Brownian motion. So, $x = (p(t), \mathrm{q}(t), \mathrm{w}(t), \mathrm{u}(t))^T$ is the current states vector.

Since, A **solvency ratio** measures the extent to which assets cover commitments for future payments, the liabilities.The **solvency ratio** of an insurance company is the size of its capital relative to all risks it has taken so insurance companies are always trying to maximize the SMR, Therefore, in this paper, the Premium is controlled to increase the SMR . For this purpose, we have the following model:

$$J = Max(A\int_{t_0}^{t_f} \frac{w_t}{w_f} dt \,, B\int_{t_0}^{t_f} \frac{1}{P_t} dt) \tag{5}$$

s.t

$$du_t = u_t(\mu dt + \sigma dW_t)$$
$$dp_t = \gamma^{-1} du_t + \lambda(P_t - p_t)dt$$
$$dq_t = q_t(G - K)dt$$
$$dw_t = -\alpha w_t dt + q_t(GP_t - u_t)dt$$

$\frac{w_t}{w_f}$ is the solvency margin ratio and A , B is constant. Suppose that G as the ***demand function***, which is associated with the premium. Since the above model is multi-objective stochastic model, so in the next section, we will deterministic it.

In the above model, G is a demand function. Since in the economics world, the supply and demand model is set for the competitive market where neither the buyers nor the sellers can have a drastic effect on the price, and the price is regarded as a data. The producer's produce and the consumer's demand are dependent on the market price of the product . The supply law states that if the other conditions are constant, the supply value is dependent on the price, and the higher the price; the more the supply and vice versa. The demand law also states that if other conditions are constant, at higher prices, demand is lower and at lower prices demand is higher. In the competitive market, the equilibrium price and the equilibrium amount are determined by the supply anddemand for that good in the market . At higher prices, there is a shortage of demand and a surplus of supply. This overhead puts pressure on prices and pushes the price back to equilibrium. At lower prices, the demand is more than the supply causing a surplus of demand [14]. This demand surplus will increase the price and therefore returns the price to its former value (equilibrium price). Once the price has reached a balance, this price will last forever. In this model the demand function is defined as follows:

$$G = \frac{1}{e^{P_t}} \tag{6}$$

In general, in the insurance market, the negative law of the demand also applies. Premiums assigned to the demand law are proportional to the amount of interest, i.e., people are more willing to purchase an insurance policy with lower premium[1]. One of the solutions to this problem is to consider G's function in terms of other variables such as premium, the market average premium, or claim size rate which

3. Deterministic the model

In this section, We transform the multi-objective model of Section 2 into a simpler model And then We will determine the model.

The model presented in Section 2 is considered as follows:

$$J=min(A\int_{t_0}^{t_f}\left|w_t - w_f\right| + B\int |P_t|) \tag{7}$$

s.t

$$du_t = u_t(\mu dt + \sigma dW_t)$$

$$dp_t = \gamma^{-1} du_t + \lambda(P_t - p_t)dt$$

$$dq_t = q_t(G - K)dt$$

$$dw_t = -\alpha w_t dt + q_t(GP_t - u_t)dt$$

the following stochastic differential equation[10]:

$$du = \mu u dt + \sigma u dW$$

Suppose W is a standard Wiener process so we put:

$$t_i = \frac{t.i}{n}\ ,\ \ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$$

If $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ And by the Wiener process, $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i} \sim N\left(0, \frac{t}{n}\right)$, In other words $var\left(\Delta W_{t_i}\right) = E\left(\Delta^2 W_{t_i}\right) = \frac{t}{n}$, therefore:

$$E(\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\Delta W_{t_i}\right)^2) = \lim_{n\to\infty}\sum_{i=0}^{n-1} E\left(\Delta W_{t_i}\right)^2 = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{t}{n} = t$$

On the other hand:

$$\int_0^t dW_s\,.\,dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = t = \int_0^t ds \Rightarrow \int_0^t (dW_s)^2 = \int_0^t ds$$

With a differential taking sides:

$$(dW_t)^2 = dt \Rightarrow dW_t = \sqrt{dt}$$

thus $du = \mu u dt + \sigma u\sqrt{dt}$ is approximated in discrete time by:

$$\delta u = \mu u \delta t + \sigma u \phi (\delta t)^{\frac{1}{2}}$$

where ϕ is drawn from a standardized normal distribution.This reasoning which seems to follow naturally from the definition of a Wiener process triggered some thoughts and questions I cannot solve. Think of the following general diffusion process:

$$du = a(t, u(t))dt + b(t, u(t))dW$$

If $\delta t \to 0$ then $\delta t \cong dt$:

$$\frac{u_t + u_{t+\delta t}}{\delta t}.\,dt \cong \mu u_t dt + \sigma u\sqrt{dt},$$

$$\frac{u_t + u_{t+\delta t}}{\delta t}.\sqrt{dt} \cong \mu u_t \sqrt{dt} + \sigma u,$$

$$u_t + u_{t+\delta t} \cong \mu u_t \sqrt{dt}\sqrt{dt} + \sigma u\sqrt{dt}.$$

Now transform the second term in a similar fashion as above and drop the stochastic component:

$$du_t = \mu u_t dt + \sigma u_t \sqrt{dt}$$

The last term is no Riemann-Stieltjes integral (that would e.g. be $d(t)^{\frac{1}{2}}$).

The problem will be as follows:

$$J = min(A\int_{t_0}^{t_f} |w(t) - w(f)|\, dt + B\int |P(t)|\, dt) \quad (8)$$

s.t

$$du_t = \mu u_t dt + \sigma u_t \sqrt{dt}$$

$$dp_t = \gamma^{-1} du_t + \lambda(P_t - p_t)dt$$

$$dq_t = q_t(G - K)dt$$

$$dw_t = -\alpha w_t dt + q_t(GP_t - u_t)dt$$

This model is Deterministic.

4. Solving the Deterministic optimal control problem

In this section, The model presented in Section 4, is discrete and the model is transformed into a nonlinear programming problem and then solved by using particle swarm optimization[6] in matlab.

The model is discrete in the interval $[0,1]$ and $\Delta t = 0.1$ is considerd.

The discrete model as follows:

$$\min \sum_{i=0}^{N} (A(\frac{|w_{i+1} - w_f| + |w_i - w_f|}{2}) + B\frac{|P_{i+1} + P_i|}{2})0.1 \quad (9)$$

s. t

$$\frac{u_{i+1} - u_i}{0.1} = \mu u_i 0.1 + \sigma u_i \sqrt{0.1}$$

$$\frac{p_{i+1} - p_i}{0.1} = \gamma^{-1}\frac{u_{i+1} - u_i}{0.1} + \lambda(P_i - p_i)0.1$$

$$\frac{q_{i+1} - q_i}{0.1} = q_i(G - k)0.1$$

$$\frac{w_{i+1} - w_i}{0.1} = -\alpha w_i 0.1 + q_i(GP_i - u_i)0.1$$

$$i = 1, \dots, N$$

Table 1. sample data set

Constant	value
Time horizon T	1 year
Depreciation of wealth α	0.05 p/a
Demand parameterisation a	1 p/a
Demand parameterisation b	1 p/a
Length of policy $l = k^{-1}$	1 year
Rate of market reaction λ	0.1 p/a
Loss ratio γ	0.9 p/a
A	0.8
B	0.2
w_f	2

We set the initial conditions and using table 1, designed state and demand function Then the problem is solved with MATLAB and the value of the objective function is obtained $J = 0.21445$. The state and control function are showed in Fig 1:

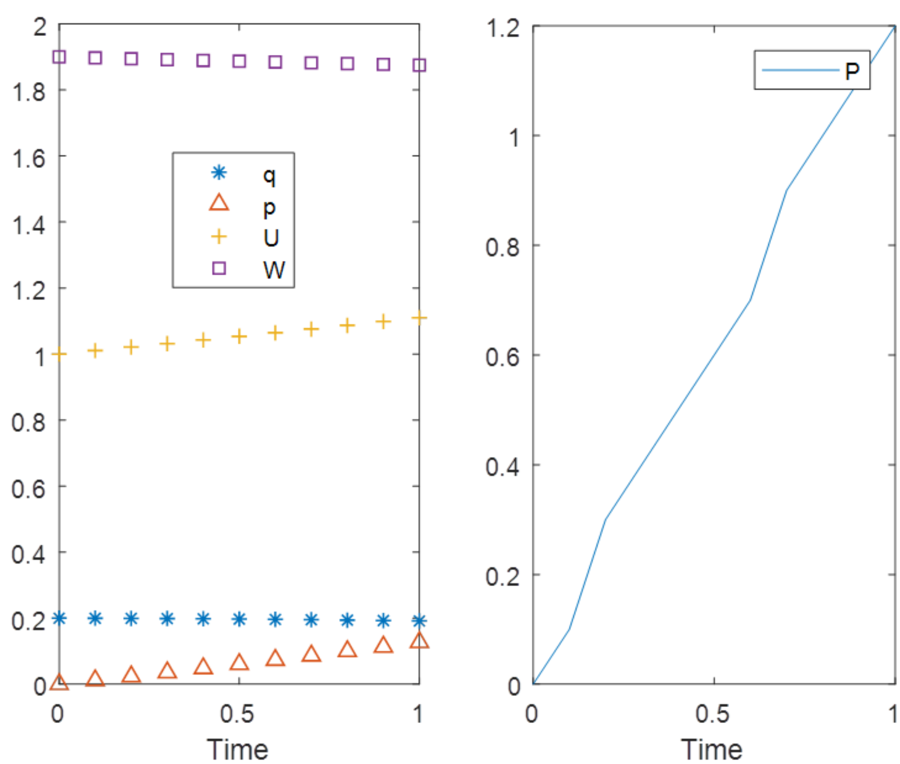

Figure.1

As shown in the above figure, The goals of the insurance company are fulfilled. w is close to w_f during the planning period, and that means that the SMR of the insurance company will not always be lower than its designated amount. In fact, with this model the insurance company will have a good SMR and it will have good financial conditions. and also the risk rate in the model presented always will be decrease. In fact, the insured are selected at an appropriate risk.

5. conclusion

In this paper, an optimal control problem was proposed based on an insurance model. We have calculated the premium strategy which maximises the objective of the insurersubject to a constraint on the control or constraints on the reserve that the insurermust hold. Since the model is very simple an analytical solutioncan be found if therelative premium is bounded. In this study, the wealth of the insurance company has been maximized by providing different stochastic control model To this end, the optimal control theory has been used along with the provision of appropriate premium in the planning period. In fact, a dynamic system has been used to describe the process of receiving and paying damages, and premium has been considered as a control variable. The stochastic model has been transformed from stochastic to deterministic in new way.In this method, the problem is converted to discrete problem by converting the stochastic differential equation to the deterministic equation and using the differential equations of the fractional order. In this case, finally, by defining demand function, the obtained model is solved using numerical methods.deterministic control theory has been used to identify an optimal premium strategy for Increase wealth. In fact, in this model we have tried to increase the wealth by using the maximize SMR of the insurance company. Finally, we showed that SMR was maximized and the risk curve went reduced.That is, we were able to control the risk in this model. On the other hand, reserve at the end of the period tends to be expected reserve(w_f). In addition, since we have found a smoothoptimal deterministic control under parametric restrictions, the deterministiccontrol is the optimal dynamic control using a Paul theorem. A similar analysis can also be carried out if we consider the objective of maximising the expected total utility of wealth with a utility functionwhich is in the wealth process. Further work is aimed at generalisingthe SMR to include both deterministic and stochastic models.

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