Simple inertial methods for solving split variational inclusions in Banach Spaces

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Abstract

In this paper, we introduce two simple inertial algorithms for solving the split variational inclusion problem in Banach spaces. Under mild and standard assumptions we establish the weak and strong convergence of the proposed methods, respectively. As theoretical realization we study existence of solutions of the split common fixed point problem in Banach spaces.

Several numerical examples in finite and infinite dimensional spaces compare and illustrate the performances of our schemes. Our work generalize and extend some recent relate results in the literature and also propose a simple and applicable method for solving split variational inclusions.

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1 Introduction

Censor et. al. [20] introduced the so-called *split inverse problem* (SIP) that consists of a model in which there are two spaces X, Y and a given linear and bounded mapping $A : X \to Y$.

Additionally, two inverse problems are involved, that is, one inverse problem (IP1) is formulated in the space X and another inverse problem (IP2) is formulated in the space Y. Given these data, the problem (SIP) is formulated as follows:

$$\begin{gathered}\text{Find a point } x^* \in X \text{ that solves the problem (IP1)}\\ \text{and such that}\\ \text{the point } y^* = Ax^* \in Y \text{ solves the problem (IP2).}\end{gathered}$$

The first instance of the problem (SIP), known as the *split convex feasibility problem* (SCFP), is introduced by Censor and Elfving [18] and in this case the inverse problems (IP1) and (IP2) are convex feasibility problems (CFP). Split feasibilities have been studied intensively both theoretically and practically, due to its applicability to real-world problems in image reconstruction, cancer treatment planning, computerized tomography and data compression, see, for example, Censor et al. [19, 20], Deepho et al. [23], Ceng [15], Xu [37], Shehu et al. [38], Combettes [12] and the references therein.

The split inverse problem reformulation is quite general and it enable to capture many different problems by choosing appropriate inverse problems (IP1) and (IP2). One recent example is Moudafi's [31] *split monotone variational inclusion problem* (SMVIP), that is formulated as follows. Let H_1 and H_2 be two real Hilbert spaces, $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ two operators and $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ two multi-valued maximal monotone mappings. In addition let $A : H_1 \to H_2$ be a (nonzero) linear and bounded operator. The SMVIP consists of finding a point x^* such that:

$$x^* \in H_1 \ \text{ such that } 0 \in f_1(x^*) + B_1(x^*) \tag{1.1}$$

and

$$y^* = Ax^* \in H_2 \ \text{ solve } 0 \in f_2(y^*) + B_2(y^*), \tag{1.2}$$

An interesting special case of (1.1)-(1.2) is when $f_1 = f_2 = 0$, this reduces to the well-known *split variational inclusion problem* (SVIP):

$$\text{Find a point } x^* \in H_1 \text{ such that } 0 \in B_1(x^*) \tag{1.3}$$

and the point

$$y^* = Ax^* \in H_2 \ \text{ solves } 0 \in B_2(y^*). \tag{1.4}$$

Other special cases of (1.1)-(1.2) are the variational inequality problem (VIP) [31], the convex feasibility problem (CFP) [18] and many constrained optimization problems as special cases, for more related problems see [10, 25, 32] as well as for applications in signal processing and image reconstruction, the reader can refer to [8, 15, 16] and the reference therein.

with respect to iterative algorithms for solving SIPs, we recall the equivalent fixed point reformulation of (1.3)-(1.4), that is.

$$x^* \text{ solves the problem } (1.3)-(1.4) \iff x^* = J_\lambda^{B_1}(x^* - \gamma A^*(I - J_\lambda^{B_2})Ax^*), \tag{1.5}$$

where for $\lambda > 0\,, \gamma > 0$, $J_\lambda^B = (I + \lambda B)^{-1}$ denotes the resolvent of a monotone operator B.

This reformulation yielded the *CQ algorithm* of Byrne for [10] for solving the two-sets split convex feasibility problem and more generally the *forward-backward algorithm*, which has the following update rule.

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \tag{1.6}$$

where A^* is the adjoint of A, the step size $\gamma \in (0, \frac{2}{L})$ with $L = \|A^*A\|$.

Many researchers have studied and proposed various algorithmic schemes close to (1.6), see, for example, Dong et al. [22], Sitthithakerngkiet et al. [40], Kazmi and Rizvi [29], Promluang and Kuman [39], Suantai et al. [41], Eslamian et al. [24], Thong et al. [42]).

A recent modification of (1.6), for solving SVIPs is the method proposed by Chuang in [14] and its iterative step is formulated as follows.

$$\begin{cases} y_n = J_{\beta_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Ax_n, \\ D(x_n, y_n) = x_n - y_n - \gamma_n[A^*(I - J_{\beta_n}^{B_2})Ax_n - A^*(I - J_{\beta_n}^{B_2})Ay_n], \\ x_{n+1} = J_\lambda^{B_1}(x_n - \alpha_n D(x,y_n)), \end{cases} \tag{1.7}$$

where $\alpha_n = \frac{\langle x_n - y_n, D(x_n,y_n)\rangle}{\|D(x_n,y_n)\|^2}$.

Alofi et al. [3] studied SVIPs in Banach spaces and incorporated the Halpern's iteration idea to propose the following iterative step.

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}^{B_1}(x_n - \lambda_n A^* J_E(I - J_\mu^{B_2})Ax_n), \tag{1.8}$$

where J_E is the duality mapping on a Banach space, $\{u_n\}$ is a sequence in a Hilbert space such that $u_n \to u$ and the step size λ_n satisfies $0 < \lambda_n L < 2$. Suantai et al. [41] proposed a viscosity modification in Banach spaces.

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}^{B_1}(x_n - \lambda_n A^* J_E(I - J_\mu^{B_2})Ax_n), \tag{1.9}$$

where $0 < \lambda_n L < 2$ and f is a contraction. Other related works include [26, 39, 44, 45] and the may references therein.

Motivated by second order time dynamical system, the heavy ball method (an implicit discretization), Alvarez [1] and Alvarez and Attouch [2] introduce an inertial term that encounter two previous iterates when updating the next iteration. This idea is studied intensively and is shown to have good convergence properties in the field of continues optimization. For some recent works applied to various fields see Ochs et al. [4, 5, 7, 35, 36].

As a relevant example of an inertial scheme for solving SVIPs in Banach spaces, Tang [46] introduced the following algorithm.

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^{B_1}(w_n - \lambda_n A^* J_E(I - J_\mu^{B_2})Aw_n), \end{cases}$$

where $\{\theta_n\}$ is in $(0, \bar{\theta_n})$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$,

$$\theta_n = \begin{cases} \min\{\theta, \epsilon_n \max\{\|x_n - x_{n-1}\|, \|x_n - x_{n-1}\|^2\}^{-1} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwsie.} \end{cases} \tag{1.10}$$

Based on the above, our goal in this paper is to establish two simple inertial methods for solving SIPs in Banach spaces. The outline of the paper is organized as follows. Some basic definitions and useful results are presented in Section 2. The two proposed methods are presented and analyzed in Section 3. The split common fixed point problem is presented in Section 4 as application and then in Section 5 some numerical experiments with comparisons to related methods demonstrate the algorithms' performances and suggested applicability. Final conclusions are reported in Section 6.

2 Preliminaries

Let E be a real Banach space and E^* be the dual space of E. A *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x^*\|^2 = \|x\|^2\}, \ \ \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*.

Let $S = \{x \in E : \|x\| = 1\}$. The norm $\| \cdot \|$ of E is said to be *Gateaux differentiable* if, for each $x, y \in S$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. In the case, E is called *smooth*. It is well known that E is smooth if and only if J is single-valued and, if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. We note that, in a Hilbert space, J is the identity operator.

A Banach space E is said to be *p-uniformly smooth* if, for any fixed real number $1 < p \leq 2$, there exists a constant $c > 0$ such that $\rho(t) = ct^p$ for all $t > 0$. From Chang et al. [17] and Chidume [13], we know that, if E is a 2-uniformly smooth Banach space, then, for all $x, y \in E$, there exists a constant $c > 0$ such that $\|Jx - Jy\| \leq c\|x - y\|$.

A multi-valued mapping $A : E \to 2^{E^*}$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \ \ \forall x, y \in D(A), \ x^* \in Ax, \ y^* \in Ay.$$

A monotone operator $A : E \to 2^{E^*}$ on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

The following theorem is due to Browder [9] (see also Takahashi [43]).

Theorem 2.1. (Browder [9]) *Let E be a uniformly convex and smooth Banach space and J be the normalized duality mapping of E into E^*. Let $A : E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if, for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with the Gateaux differentiable norm and $A : E \to 2^{E^*}$ be a maximal monotone operator. Now, we consider the *metric resolvent* of A given by

$$Q_\mu^A = (I + \mu J^{-1}A)^{-1}, \ \ \forall \mu > 0.$$

It is well known that the operator Q_μ^A is firmly nonexpansive and the fixed points of the operator Q_μ^A are the zero points of A (see, for example, Kohsaka and Takahashi [27, 28]). The resolvent plays an essential role in the approximation theory for zero points of maximal monotone operators in Banach spaces. According to the work of Aoyama et al. [6], we have the following properties:

$$\langle Q_\mu^A x - y, J(x - Q_\mu^A x)\rangle \geq 0, \ \ \forall y \in A^{-1}(0).$$

In particular, if E is a real Hilbert space, then

$$\langle J_\mu^A x - y, x - J_\mu^A x\rangle \geq 0, \ \ \forall y \in A^{-1}(0),$$

where $J_\mu^A = (I + \mu A)^{-1}$ is the general resolvent and $A^{-1}(0) = \{z \in E : 0 \in Az\}$. For more details on some properties of firmly nonexpansive mappings, one can see Aoyama et al. [6] and Bauschke et al. [11].

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For a sequence $\{x_n\}$ in H, we use the notations "$x_n \to x$" and "$x_n \rightharpoonup x$" to denote the strong and weak convergence to a point $x \in H$ of $\{x_n\}$, respectively. Moreover, we use the symbol $\omega_w(x_n)$ to denote the ω-weak limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \ \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The identity below is useful:

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ & - \alpha\beta\|x - y\|^2 - \beta\gamma\|y - z\|^2 - \gamma\alpha\|x - z\|^2 \end{aligned} \tag{2.1}$$

for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$.

Let C be a nonempty closed convex subset of H and P_C denote the *metric projection* from H onto C, that is,

$$P_C x = \arg\min\{\|x - y\| : y \in C\}, \ \ \forall x \in H.$$

The following are important characterizations of the projection P_C:

(1) For any $x \in H$ and $y \in C$,

$$P_C x = z \iff \langle x - z, y - z\rangle \leq 0, \ \forall y \in C. \tag{2.2}$$

(2) P_C is `firmly nonexpansive`, that is,

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H.$$

The following lemmas are useful to prove the main results in this paper.

Lemma 2.2. (Xu [47], Maingé [30]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \theta_n)a_n + \delta_n, \ \forall n \geq 0,$$

where $\{\theta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\theta_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. (Opial [34]) *Let H be a real Hilbert space and $\{x_n\}$ be a bounded sequence in H. Assume that there exists a nonempty subset $S \subset H$ satisfying the properties:*

(i) $\lim_{n\to\infty} \|x_n - z\|$ *exists for every* $z \in S$,

(ii) $\omega_w(x_n) \subset S$.

Then there exists $\bar{x} \in S$ such that $\{x_n\}$ converges weakly to $\bar{x}$.

Lemma 2.4. (Maingé [30]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at the infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, consider the sequence of integers $\{\sigma(n)\}_{n\geq n_0}$ defined by*

$$\sigma(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}.$$

Then $\{\sigma(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \sigma(n) = \infty$ and, for all $n \geq n_0$,

$$\max\{\Gamma_{\sigma(n)}, \Gamma_n\} \leq \Gamma_{\sigma(n)+1}.$$

Lemma 2.5. (Maingé [30]) *Let $\{l_n\}_{n=0}^{\infty} \subset [0, +\infty)$ and $\{\delta_n\}_{n=0}^{\infty}$ be the sequences satisfying the following conditions:*

(i) $l_{n+1} - l_n \leq \theta_n(l_n - l_{n-1}) + \delta_n$;

(ii) $\Sigma_{n=1}^{\infty}\delta_n < \infty$;

(iii) $\{\theta_n\} \subset [0, \theta]$, *where* $\theta \in (0, 1)$.

Then $\{l_n\}$ is a converging sequence and $\Sigma_{n=1}^{\infty}[l_{n+1} - l_n]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$ for any $t \in \mathbb{R}$.

3 Main results

Throughout the rest of this paper, H is a real Hilbert space and E is a 2-uniformly convex smooth Banach space. We rephrase the *split variational inclusion problem* (SVIP) as follows:

$$\text{Find a point } x^* \in H \ \text{ such that } 0 \in B_1(x^*) \tag{3.1}$$
$$\text{and such that the point}$$
$$y^* = Ax^* \in E \ \text{ solves } 0 \in B_2(y^*), \tag{3.2}$$

where $B_1 : H \to 2^H$ and $B_2 : E \to 2^{E^*}$ are two maximal monotone operators, respectively, and $A : H \to E$ is bounded linear operator with the adjoint operator A^* of A.

Denote by Ω the solution set of the problem (SVIP) (3.1)–(3.1), that is,

$$\Omega = \{x^* \in H : 0 \in B_1(x^*),\ 0 \in B_2(Ax^*)\}$$

and we always assume $\Omega \neq \emptyset$.

Next we present our two new methods.

Algorithm 1

Initialization: Choose the positive sequence $\{\epsilon_n\}$ satisfying $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Select arbitrary starting points $x_0, x_1 \in H$, two constants $\tau < \frac{1}{L}$ and $\theta \in [0, 1)$ and choose θ_n such that $0 < \theta_n < \bar{\theta_n}$, where $\bar{\theta_n}$ and L will be specified later on.
Iterative Step: After the n-iterate x_n is constructed, for any $r > 0$, compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_r^{B_1}(I - \tau A^* J_E(I - Q_\mu^{B_2})A)w_n, \\ d(w_n, y_n) = w_n - y_n - \tau[A^* J_E(I - Q_\mu^{B_2})Aw_n - A^* J_E(I - Q_\mu^{B_2})Ay_n] \end{cases} \tag{3.3}$$

and define the $(n + 1)$th iterate by

$$x_{n+1} = w_n - \alpha_n d(w_n, y_n), \tag{3.4}$$

where

$$\alpha_n = \begin{cases} \frac{\langle w_n - y_n, d(w_n, y_n)\rangle}{\|d(w_n, y_n)\|^2} & \text{if } d(w_n, y_n) \neq 0, \\ 0 & \text{if } d(w_n, y_n) = 0. \end{cases}$$

Algorithm 2

Initialization: Choose the positive sequence $\{\epsilon_n\}$ satisfying $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Select arbitrary starting points $x_0, x_1 \in H$, two constants $\tau < \frac{1}{L}$ and $\theta \in [0, 1)$ and choose θ_n such that $0 < \theta_n < \bar{\theta_n}$, where $\bar{\theta_n}$ and L will be specified later on.

Iterative Step: After the n-iterate x_n is constructed, for any $r > 0$, compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_r^{B_1}(I - \tau A^* J_E(I - Q_\mu^{B_2})A)w_n, \\ d(w_n, y_n) = w_n - y_n - \tau[A^* J_E(I - Q_\mu^{B_2})Aw_n - A^* J_E(I - Q_\mu^{B_2})Ay_n] \end{cases} \tag{3.5}$$

and define the $(n+1)$th iterate by

$$x_{n+1} = (1 - \beta_n - \gamma_n)w_n + \beta_n(w_n - \alpha_n d(w_n, y_n)), \tag{3.6}$$

where

$$\alpha_n = \begin{cases} \frac{\langle w_n - y_n, d(w_n, y_n)\rangle}{\|d(w_n, y_n)\|^2} & \text{if } d(w_n, y_n) \neq 0, \\ 0 & \text{if } d(w_n, y_n) = 0. \end{cases}$$

3.1 Convergence analysis

Now, we give some lemmas for the main results in this paper.

Lemma 3.1. (Tang [46]) *Let H be a real Hilbert space, E be a strictly convex reflexive and smooth Banach space and J be the normalized duality mapping on E. Let $B_1 : H \to 2^H$ and $B_2 : E \to 2^{E^*}$ be maximal operators such that $B_1^{-1}(0) \not\equiv \emptyset$ and $B_2^{-1}(0) \not\equiv \emptyset$, respectively. Let $A : H \to E$ be a bounded linear operator such that $A \not\equiv \emptyset$ and A^* be the adjoint operator of A. Suppose that $\Omega = B_1^{-1}(0) \bigcap A^{-1}(B_2^{-1}(0)) \not\equiv \emptyset$. Let λ, μ, $r > 0$ and $z \in H$. Then the following are equivalent:*

(1) $z \in B_1^{-1}(0) \bigcap A^{-1}(B_2^{-1}(0))$.
(2) $z = J_r^{B_1}(I - \lambda A^* J_E(I - Q_\mu^{B_2})A)z$, *where* $J_r^{B_1} = (I + rB_1)^{-1}$ *and* $Q_\mu^{B_2} = (I + \mu J^{-1}B_2)^{-1}$.

Lemma 3.2. (Tang [46]) *Let H be a real Hilbert space, E be a real 2-uniformly smooth Banach space and J_E be the normalized duality mapping on E. Let $B_1 : H \to 2^H$ and $B_2 : E \to 2^{E^*}$ be maximal operators such that $B_1^{-1}(0) \not\equiv \emptyset$ and $B_2^{-1}(0) \not\equiv \emptyset$. Let $A : H \to E$ be a bounded linear operator such that $A \not\equiv \emptyset$ and A^* be the adjoint operator of A. Assume that $A^{-1}(B_2^{-1}(0) \not\equiv \emptyset$. If $T = A^* J_E(I - Q_\mu^{B_2})A$, then T is Lipschitz continuous with constant $2c\|A\|^2$, where c is a constant such that*

$$\|Jx - Jy\| \leq c\|x - y\|.$$

Lemma 3.3. *Let $\{x_n\}$ be the sequence generated by* (3.4). *If $y_n = w_n$ or $d(w_n, y_n) = 0$, then $x_{n+1} \in \Omega$.*

Proof. Denote $T = A^* J_E(I - Q_\mu^{B_2})A$. It follows from (3.3) that

$$\begin{array}{rcl} \|d(w_n, y_n)\| & = & \|w_n - y_n - \tau(Tw_n - Ty_n\|) \\ & \geq & \|w_n - y_n\| - \tau\|Tw_n - Ty_n\| \\ & \geq & \|w_n - y_n\| - \tau \cdot 2c\|A\|^2\|w_n - y_n\| \\ & = & (1 - \tau L)\|w_n - y_n\|, \end{array}$$

where $L = 2c\|A\|^2$. In addition, we have

$$\begin{array}{rcl} \|d(w_n, y_n)\| & = & \|w_n - y_n - \tau(Tw_n - Ty_n\|) \\ & \leq & \|w_n - y_n\| + \tau\|Tw_n - Ty_n\| \\ & = & (1 - \tau L)\|w_n - y_n\|. \end{array}$$

So, it follows that $d(w_n, y_n) = 0$ if and only if $y_n = w_n$. When $y_n = w_n$ or $d(w_n, y_n) = 0$, from (3.3) and (3.4), we have

$$\begin{cases} w_n = J_r^{B_1}(I - \tau A^* J_E(I - Q_\mu^{B_2})A)w_n, \\ x_{n+1} = w_n, \end{cases}$$

which with Lemma 3.2 yields $x_{n+1} \in \Omega$. This completes the proof. $\square$

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by (3.4). Assume that $d(w_n, y_n) \neq 0$. If $z \in \Omega$, then we have the following:*

$$\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \|x_{n+1} - w_n\|^2 \tag{3.7}$$

and

$$\|w_n - y_n\|^2 \leq \frac{1 + \tau^2 L^2}{(1 - \tau L)^2}\|x_{n+1} - w_n\|^2. \tag{3.8}$$

Proof. First, denote $T = A^* J_E(I - Q_\mu^{B_2})A$, it follows out from (3.4) that

$$\begin{array}{rcl} \langle w_n - y_n, d(w_n, y_n)\rangle & = & \langle w_n - y_n, w_n - y_n - \tau(Tw_n - Ty_n)\rangle \\ & = & \|w_n - y_n\|^2 - \tau\langle w_n - y_n, Tw_n - Ty_n\rangle \\ & \geq & \|w_n - y_n\|^2 - \tau\|w_n - y_n\| \cdot \|Tw_n - Ty_n\| \\ & \geq & (1 - \tau L)\|w_n - y_n\|^2 \end{array} \tag{3.9}$$

and

$$\begin{array}{rcl} \|d(w_n, y_n)\|^2 & = & \|w_n - y_n - \tau(Tw_n - Ty_n)\|^2 \\ & = & \|w_n - y_n\|^2 + \tau\|Tw_n - Ty_n\|^2 - 2\tau\langle w_n - y_n, Tw_n - Ty_n\rangle \\ & \leq & (1 + \tau^2 L^2)\|w_n - y_n\|^2, \end{array} \tag{3.10}$$

where $L = 2c\|A\|^2$. Therefore, we have

$$\alpha_n = \frac{\langle w_n - y_n, d(w_n, y_n)\rangle}{\|d(w_n, y_n)\|^2} \geq \frac{1-\tau L}{1+\tau^2 L^2}.$$

Take $z \in \Omega$. Then $0 \in B_1 z$, $0 \in B_2(Az)$, $0 \in Tz$ and

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|w_n - z - \alpha_n d(w_n, y_n)\|^2 \\ &= \|w_n - z\|^2 - 2\alpha_n \langle w_n - z, d(w_n, y_n)\rangle + \alpha_n^2 \|d(w_n, y_n)\|^2.\end{aligned}$$

From (2.2), it follows that

$$\langle y_n - z, w_n - \tau T w_n - y_n\rangle \geq 0, \ \ \forall z \in A^{-1}(0). \tag{3.11}$$

In addition, it follows from $0 \in Tz$ and

$$\begin{aligned}\langle y_n - z, \tau(Ty_n - Tz)\rangle &= \tau\langle y_n - z, Ty_n\rangle \\ &= \tau\langle Ay_n - Az, J_E(I - Q_\mu^{B_2})Ay_n\rangle \\ &= \tau\langle Ay_n - Q_\mu^{B_2}Ay_n, J_E(I - Q_\mu^{B_2})Ay_n\rangle \\ &\quad + \langle Q_\mu^{B_2}Ay_n - Az, J_E(I - Q_\mu^{B_2})Ay_n\rangle \\ &\geq \tau\|Ay_n - Q_\mu^{B_2}Ay_n\|^2 \geq 0.\end{aligned} \tag{3.12}$$

Adding (3.11) and (3.12), one has

$$\langle y_n - z, w_n - y_n - \tau(Tw_n - Ty_n)\rangle = \langle y_n - z, d(w_n, y_n)\rangle \geq 0$$

and so

$$\begin{aligned}\langle w_n - z, d(w_n, y_n)\rangle &= \langle w_n - y_n, d(w_n, y_n)\rangle + \langle y_n - z, d(w_n, y_n)\rangle \\ &\geq \langle w_n - y_n, d(w_n, y_n)\rangle.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|w_n - \alpha_n d(w_n, y_n)\|^2 \\ &\leq \|w_n - z\|^2 - 2\alpha_n\langle w_n - z, d(w_n, y_n)\rangle + \alpha_n^2\|d(w_n, y_n)\|^2 \\ &\leq \|w_n - z\|^2 - 2\alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle + \alpha_n^2\|d(w_n, y_n)\|^2 \\ &= \|w_n - z\|^2 - 2\alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle + \alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle \\ &= \|w_n - z\|^2 - \alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle.\end{aligned} \tag{3.13}$$

In addition, we have

$$\alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle = \|\alpha_n d(w_n, y_n)\|^2 = \|x_{n+1} - w_n\|^2, \tag{3.14}$$

which reduces from (3.13) that $\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \|x_{n+1} - w_n\|^2$.

Second, it turns out from (3.14) that

$$\begin{aligned}\langle w_n - y_n, d(w_n, y_n)\rangle &= \frac{1}{\alpha_n}\|x_{n+1} - w_n\|^2 \\ &\le \frac{1+\tau^2 l^2}{1-\tau L}\|x_{n+1} - w_n\|^2.\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Assume that*

$$\bar{\theta_n} = \begin{cases} \min\{\theta, \epsilon_n(\max\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\})^{-1}\} & \text{if } x_n \neq x_{n-1} \\ \theta & \text{otherwise.} \end{cases}$$

Then the sequence $\{x_n\}$ *generated by* (3.4) *converges weakly to a solution of the split variational inclusion problem* (3.1)–(3.2).

Proof. Take $z \in \Omega$, it turns out from the recursion (3.3) that

$$\begin{aligned}\|w_n - z\|^2 &= \|(1+\theta_n)(x_n - z) - \theta_n(x_{n-1} - z)\|^2 \\ &= (1+\theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\quad +\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.15}$$

Hence it follows from (3.7) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le (1+\theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 \\ &\quad +\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \|x_{n+1} - w_n\|^2,\end{aligned}$$

that is,

$$\|x_{n+1} - z\|^2 - \|x_n - z\|^2 \le \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2.$$

According to the choice of $\{\theta_n\}$, we have

$$\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\|^2 < \infty.$$

Also, one can show from Lemma 2.5 that the limit of $\{\|x_n - z\|\}$ exists and

$$\lim_{n\to\infty}(\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = 0, \quad \lim_{n\to\infty}\|x_{n+1} - w_n\| = 0, \tag{3.16}$$

which in turn implies that $\{x_n\}$ is bounded. So, It turns out from (3.8) and (3.16) that

$$d(w_n, y_n) \to 0. \tag{3.17}$$

Next, we show that $\omega_{w_n}(x_n) \subset \Omega$. Let $\bar{x} \in \omega_{w_n}(x_n)$ be an arbitrary element. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $\bar{x}$. Note that

$$\theta_n\|x_n - x_{n-1}\| \leq \bar{\theta_n}\|x_n - x_{n-1}\| \leq \epsilon_n,$$

which implies that $\|w_n - x_n\| = \theta_n\|x_n - x_{n-1}\| \to 0$. Therefore, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ converges weakly to $\bar{x}$. Moreover, it follows from (3.17) that $\|w_n - y_n\| \to 0$ and

$$\begin{aligned} \|\bar{x} - J_r^{B_1}(I - \lambda A^* J_E(I - Q_\mu^{B_2})A)\bar{x}\| &= \liminf_{k\to\infty} \|w_{n_k} - y_{n_k}\| \\ &= \liminf_{k\to\infty} \|w_{n_k} - J_r^{B_1}(I - \lambda A^* J_E(I - Q_\mu^{B_2})A)w_{n_k}\| \\ &= 0, \end{aligned}$$

which implies that $\bar{x} \in \Omega$. Since the choice of $\bar{x}$ is arbitrary, we conclude that $\omega_{w_n}(x_n) \subset \Omega$. Hence it follows from Lemma 2.3 that the result holds. This completes the proof. □

Theorem 3.6. *Assume that the sequences $\{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following conditions:*

$$\beta_n + \gamma_n < 1, \ \lim_{n\to\infty} \gamma_n = 0, \ \Sigma_{n=1}^{\infty}\gamma_n = \infty, \ \epsilon_n = o(\gamma_n)$$

and

$$\bar{\theta_n} = \begin{cases} \min\{\theta, \epsilon_n(\max\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\})^{-1}\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases}$$

Then the sequence $\{x_n\}$ generated by (3.6) converges strongly to a point $z = P_\Omega(0)$.

Proof. Firs, we show that the sequence $\{x_n\}$ is bounded. Denote $u_n = w_n - \alpha_n d(w_n, y_n)$, then, for any $z \in \Omega$, it follows from (3.13) that

$$\begin{aligned} \|u_n - z\|^2 &\leq \|w_n - z\|^2 - \alpha_n\langle w_n - y_n, d(w_n, y_n)\rangle \\ &= \|w_n - z\|^2 - \|u_n - w_n\|^2. \end{aligned} \tag{3.18}$$

In addition, from (3.5), we have

$$\|w_n - z\| \leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\|.$$

Thus it follows from (3.6) that

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1 - \beta_n - \gamma_n)w_n + \beta_n(w_n - \alpha_n d(w_n, y_n)) - z\| \\ &\leq (1 - \beta_n - \gamma_n)\|w_n - z\| + \beta_n\|u_n - z\| + \gamma_n\|z\| \\ &\leq (1 - \beta_n - \gamma_n)\|w_n - z\| + \beta_n\|w_n - z\| + \gamma_n\|z\| \\ &\leq (1 - \gamma_n)(\|x_n - z\| + \theta_n\|x_n - x_{n-1}\|) + \gamma_n\|z\|. \end{aligned}$$

Denote $\sigma_n = \theta_n\|z_n - z_{n-1}\|$, then we have $\lim_{n\to\infty}\sigma_n = 0$ from the choice of $\{\theta_n\}$, which implies that $\{\sigma_n\}$ is bounded. Therefore, the sequences $\{\|x_n - z\|\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $x_n \to z$, where $z = P_\Omega(0)$. It follows from (2.1), (3.6) and (3.18) that

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
= \; & \|(1-\beta_n-\gamma_n)w_n + \beta_n u_n - z\|^2 \\
\leq \; & (1-\beta_n-\gamma_n)\|w_n - z\|^2 + \beta_n\|u_n - z\|^2 + \gamma_n\|z\|^2 - (1-\beta_n-\gamma_n)\|u_n - w_n\|^2 \\
\leq \; & (1-\beta_n-\gamma_n)\|w_n - z\|^2 + \beta_n[\|w_n - z\|^2 - \|u_n - w_n\|^2] \\
& +\gamma_n\|z\|^2 - (1-\beta_n-\gamma_n)\|u_n - w_n\|^2 \\
= \; & (1-\gamma_n)\|w_n - z\|^2 - \beta_n\|u_n - w_n\|^2 + \gamma_n\|z\|^2 - (1-\beta_n-\gamma_n)\|u_n - w_n\|^2 \\
= \; & (1-\gamma_n)\|w_n - z\|^2 + \gamma_n\|z\|^2 - (1-\gamma_n)\|u_n - w_n\|^2.
\end{aligned} \tag{3.19}$$

Using (3.15) in (3.19), we have

$$\begin{aligned}
& \|x_{n+1} - z\|^2 \\
\leq \; & (1-\gamma_n)[(1+\theta_n)\|x_n - z\|^2 - \theta_n\|x_{n-1} - z\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2] \\
& +\gamma_n\|z\|^2 - (1-\gamma_n)\|u_n - w_n\|^2 \\
\leq \; & \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 \\
& +\gamma_n\|z\|^2 - (1-\gamma_n)\|u_n - w_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\gamma_n)\|u_n - w_n\|^2 \leq \; & \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
& +2\theta_n\|x_n - x_{n-1}\|^2 + \gamma_n\|z\|^2
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\|x_{n+1} - z\|^2 - \|x_n - z\|^2 \leq \; & \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
& +2\theta_n\|x_n - x_{n-1}\|^2 + \gamma_n\|z\|^2 - (1-\gamma_n)\|u_n - w_n\|^2.
\end{aligned}$$

Next, we consider the following two cases:

Case I. The sequence $\{\|x_n - z\|\}$ is nonincreasing at the infinity, that is, there exists $n_0 \geq 0$ such that, for each $n \geq n_0$, $\|x_{n+1} - z\| \leq \|x_n - z\|$. This particularly implies that $\lim_{n\to\infty}\|x_n - z\|$ exists and thus

$$\lim_{n\to\infty}(\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = 0$$

and

$$\sum_{n=1}^{\infty}(\|x_{n+1} - z\|^2 - \|x_n - z\|^2) < \infty.$$

Now, due to the assumptions of the sequences $\{\theta_n\}$, $\{\gamma_n\}$, $\{\epsilon_n\}$ and the boundedness of $\{w_n\}$ and $\{u_n\}$, it follows from (3.20) that

$$\lim_{n\to\infty} \|u_n - w_n\| = 0, \tag{3.21}$$

which ensures that $\|x_{n+1} - w_n\| \le \beta_n \|u_n - w_n\| + \gamma_n \|w_n\| \to 0$, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - w_n + w_n - x_n\| \\ &\le \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &\le \|x_{n+1} - w_n\| + \theta_n \|x_n - x_{n-1}\| \to 0. \end{aligned}$$

By repeating the relevant part of the proof of Theorem 3.5, we get $\omega_w(x_n) \subset \Omega$.

Now, it is at the position to prove the strong convergence of $\{x_n\}$. Indeed, set $z_n = (1 - \beta_n)w_n + \beta_n u_n$, then $x_{n+1} = z_n - \gamma_n w_n = (1-\gamma_n)z_n - \gamma_n \beta_n (w_n - u_n)$. It follows from (3.18) that

$$\begin{aligned} \|z_n - z\|^2 &= \|(1-\beta_n)w_n + \beta_n u_n - z\|^2 \\ &\le (1-\beta_n)\|w_n - z\|^2 + \beta_n \|u_n - z\|^2 \\ &\le (1-\beta_n)\|w_n - z\|^2 + \beta_n(\|w_n - z\|^2 - \|u_n - w_n\|^2) \\ &\le \|w_n - z\|^2 - \beta_n \|u_n - w_n\|^2 \end{aligned}$$

Therefore, it follow from (3.15) that

$$\begin{aligned} & \|x_{n+1} - z\|^2 \\ = \ & \|(1-\gamma_n)z_n - \gamma_n z - \gamma_n \beta_n (w_n - u_n)\|^2 \\ \le \ & (1-\gamma_n)^2 \|z_n - z\|^2 - 2\langle \gamma_n \beta_n (w_n - u_n) + \gamma_n z, x_{n+1} - z\rangle \\ \le \ & (1-\gamma_n)^2 \|z_n - z\|^2 - 2\gamma_n \beta_n \langle w_n - u_n, x_{n+1} - z\rangle + 2\gamma_n \langle z, x_{n+1} - z\rangle \\ \le \ & (1-\gamma_n)^2 \|w_n - z\|^2 - (1-\gamma_n)^2 \beta_n \|u_n - w_n\|^2 \\ & -2\gamma_n \beta_n \langle w_n - u_n, x_{n+1} - z\rangle + 2\gamma_n \langle z, x_{n+1} - z\rangle \\ \le \ & (1-\gamma_n)^2 [(1+\theta_n)\|x_n - z\|^2 - \theta_n \|x_{n-1} - z\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2] \\ & -(1-\gamma_n)^2 \beta_n \|u_n - w_n\|^2 - 2\gamma_n \beta_n \langle w_n - u_n, x_{n+1} - z\rangle + 2\gamma_n \langle z, x_{n+1} - z\rangle \\ \le \ & (1-\gamma_n)\|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2 \\ & -2\gamma_n \beta_n \langle w_n - u_n, x_{n+1} - z\rangle + 2\gamma_n \langle z, x_{n+1} - z\rangle. \end{aligned} \tag{3.22}$$

Setting $a_n = \|x_n - z\|^2$ and

$$\begin{aligned} \delta_n &= \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2 \\ & \quad -2\gamma_n \beta_n \langle w_n - u_n, x_{n+1} - z\rangle + 2\gamma_n \langle z, x_{n+1} - z\rangle, \end{aligned}$$

we rewrite (3.22), equivalently, as

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n. \tag{3.23}$$

Since $\omega_w(x_n) \subset \Omega$ and $z = P_\Omega(0)$ which implies $\langle -z, q - z\rangle \le 0$ for all $q \in \Omega$, we deduce that

$$\limsup_{n\to\infty} \langle -z, x_{n+1} - z\rangle = \max_{q\in\omega_w(x_n)} \langle -z, q - z\rangle \le 0. \tag{3.24}$$

This enables us to apply Lemma 2.2 to (3.23) to obtain that $a_n \to 0$. That is, $x_n \to z$ in the norm and so the proof of Case I is complete.

Case II. The sequence $\{\|x_n - z\|\}$ is not nonincreasing at the infinity. In this case, we have, by Lemma 2.4, (taking $\Gamma_n := \|x_n - z\|$) a subsequence $\{\sigma(n)\}$ of positive integers such that $\sigma(n) \to \infty$ as $n \to \infty$ and with the properties:

$$\|x_{\sigma(n)} - z\| < \|x_{\sigma(n)+1} - z\|, \ \max\{\|x_{\sigma(n)} - z\|, \|x_n - z\|\} \le \|x_{\sigma(n)+1} - z\|.$$

Observe that, if $\|x_{n+1} - z\| > \|x_n - z\|$ for some $n \ge 0$, then it follows from (3.20) that

$$(1-\gamma_n)\|u_n - w_n\|^2 \le 2\theta_n\|x_n - x_{n-1}\|^2 + \gamma_n\|z\|^2.$$

Now this inequality holds for infinite many $n := \sigma(n)$. So replacing n with $\sigma(n)$ and taking the limit $n \to \infty$ yields (as $\gamma_{\sigma(n)} \to 0$)

$$\lim_{n\to\infty} \|u_{\sigma(n)} - w_{\sigma(n)}\| = 0. \tag{3.25}$$

Note that we still have $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0$. Note also that the relation (3.25) is sufficient to guarantee that $\omega_w(x_{\sigma(n)}) \subset \Omega$.

Next, we prove $x_{\sigma(n)} \to z$. As a matter of fact, observe that (3.22) holds for each $n \ge 0$. So, replacing n with $\sigma(n)$ in (3.22) and using the relation $\|x_{\sigma(n)} - z\|^2 < \|x_{\sigma(n)+1} - z\|^2$, we obtain

$$\begin{aligned}
& \|x_{\sigma(n)} - z\|^2 \\
\le\ & \frac{2\theta_{\sigma(n)}}{\gamma_{\sigma(n)}}\|x_{\sigma(n)} - x_{\sigma(n)-1}\|^2 + 2\beta_{\sigma(n)}\langle u_{\sigma(n)} - w_{\sigma(n)}, x_{\sigma(n)+1} - z\rangle + 2\langle z, x_{\sigma(n)+1} - z\rangle \\
\le\ & \frac{2\theta_{\sigma(n)}}{\gamma_{\sigma(n)}}\|x_{\sigma(n)} - x_{\sigma(n)-1}\|^2 + M\|u_{\sigma(n)} - w_{\sigma(n)}\| + 2\langle z, x_{\sigma(n)+1} - z\rangle,
\end{aligned} \tag{3.26}$$

where M is a constant such that $M \ge 2\|x_n - z\|$ for all $n \ge 0$. Now, since $\|u_{\sigma(n)} - w_{\sigma(n)}\| \to 0$ and $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle -z, x_{\sigma(n)+1} - z\rangle &= \limsup_{n\to\infty} \langle -z, x_{\sigma(n)} - z\rangle \\
&= \max_{q\in\omega_w(x_{\sigma(n)})} \langle -z, q - z\rangle \le 0
\end{aligned}$$

by virtue of the facts $z = P_\Omega(0)$ and $\omega(x_{\sigma(n)}) \subset \Omega$. Consequently, (3.26) assures that $x_{\sigma(n)} \to z$, which further implies that

$$\|x_n - z\| \le \|x_{\sigma(n)+1} - z\| \le \|x_{\sigma(n)+1} - x_{\sigma(n)}\| + \|x_{\sigma(n)} - z\| \to 0.$$

That is, $x_n \to z$ in the norm and the proof of Case II is complete. This completes the proof. $\square$

4 Application

The *split common fixed point problem* (SCFPP), which was introduced by Censor and Segal [21] in Euclidean spaces and extended by Moudafi [31] to Hilbert spaces, is formulated as finding a point x^* such that.

$$x^* \in Fix(U_1) \text{ and } Ax^* \in Fix(U_2), \tag{4.1}$$

where $U_1 : H \to 2^H$ and $U_2 : E \to 2^{E^*}$ are two mappings such that $Fix(U_1) \neq \emptyset$ and $Fix(U_2) \neq \emptyset$, respectively, and $A : H \to E$ is a (nonzero) bounded linear operator. Assume that the set of solutions of the problem (SCFPP) (4.1), denoted by Ω, is nonempty.

Recall that a mapping $T : E \to 2^{E^*}$ is said to be *κ-strictly pseudo-contractive* for some $\kappa \in [0, 1)$ if

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \kappa\|x - y - (Tx - Ty)\|^2, \ \forall x, y \in H.$$

Equivalently, $I - T$ is κ-inverse strongly monotone.

It is clear that the problem (SCFPP) (4.1) can be reduced to the problem SVIP (3.1)–(3.1) with $B_1 = I - U_1$ and $B_2 = I - U_2$. Consequently, we may apply Theorems 3.5 and 3.6 to get the following results.

Theorem 4.1. *Consider the problem* (SCFPP) (4.1) *where we assume that* $B_1 = I - U_1$ *and* $B_2 = I - U_2$ *with* $U_1 : H \to 2^H$ *and* $U_2 : E \to 2^{E^*}$ *being α- and β-strictly pseudo-contractive mappings, respectively. Assume that the sequences* $\{\epsilon_n\}$, $\{\theta_n\}$ *and* $\{\alpha_n\}$ *are same as in Theorem 3.5. Then the algorithm* $\{x_n\}$ *defined by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_r^{B_1}(I - \tau A^* J_E(I - Q_\mu^{B_2})A)w_n, \\ d(w_n, y_n) = w_n - y_n - \tau[A^* J_E(I - Q_\mu^{B_2})Aw_n - A^* J_E(I - Q_\mu^{B_2})Ay_n], \\ x_{n+1} = w_n - \alpha_n d(w_n, y_n) \end{cases}$$

converges weakly to a solution of the problem (SCFPP) (4.1).

Theorem 4.2. *Consider the problem* (SCFPP) (4.1) *where we assume that* $B_1 = I - U_1$ *and* $B_2 = I - U_2$ *with* $U_1 : H \to 2^H$ *and* $U_2 : E \to 2^{E^*}$ *being α- and β-strictly pseudo-contractive mappings, respectively. Assume that the sequences* $\{\epsilon_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are same as in Theorem 3.6. Then the algorithm* $\{x_n\}$ *defined by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_r^{B_1}(I - \tau A^* J_E(I - Q_\mu^{B_2})A)w_n, \\ d(w_n, y_n) = w_n - y_n - \tau[A^* J_E(I - Q_\mu^{B_2})Aw_n - A^* J_E(I - Q_\mu^{B_2})Ay_n], \\ x_{n+1} = (1 - \beta_n - \gamma_n)w_n + \beta_n(w_n - \alpha_n d(w_n, y_n)), \end{cases}$$

converges strongly to a solution of the problem (SCFPP) (4.1).

5 Numerical examples

In this section, we present numerical examples to illustrate and compare the applicability, efficiency and stability of our inertial algorithms. All the codes are written in MATLAB R2016b and are preformed on an LG dual core personal computer.

Example 5.1. *Suppose that $E = H = L^2[0,1]$ with norm $\|x\| := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$. Define the mappings A, B_1 and B_2 by $Ax(t) := x(t)$, $B_1x(t) := \frac{x(t)}{2}$ and $B_2x(t) := \frac{2x(t)}{3}$ for all $x \in L^2[0,1]$. In this example, we set the parameters of Algorithm 1 by $\epsilon_n = \frac{1}{(n+1)^2}$ for all $n \in N$.*

If $\theta < \epsilon_n(\max\{\|x_n - x_{n-1}\|, \|x_n - x_{n-1}\|^2)\}^{-1})$, then $\theta_n = \frac{\theta}{2}$. Otherwise, we take

$$\theta_n = \frac{1}{(n+2)^2} \max\{\|x_n - x_{n-1}\|, \|x_n - x_{n-1}\|^2)\}^{-1}.$$

At the same time, we set the parameters $\beta_n = \frac{n-1}{n+1}$, $\gamma_n = \frac{1}{n+1}$ in Algorithm 2.

We set the stopping criterion $\|x_{n+1} - x_n\| \leq 10^{-6}$ and test the performances of the algorithm for the following starting points.

Case I: $x_0(t) = \frac{sin(-3t)+cos(-10t)}{600}$, $x_1(t) = \frac{sin(-3t)+cos(-10t)}{1200}$;

Case II: $x_0(t) = \frac{t^2-e^{-t}}{100}$, $x_1(t) = \frac{sin(-3t)+cos(-10t)}{600}$;

Case III: $x_0(t) = \frac{t^2-e^{-t}}{100}$, $x_1(t) = \frac{t^2}{100}$;

The numerical results presented in Figures 1–3.

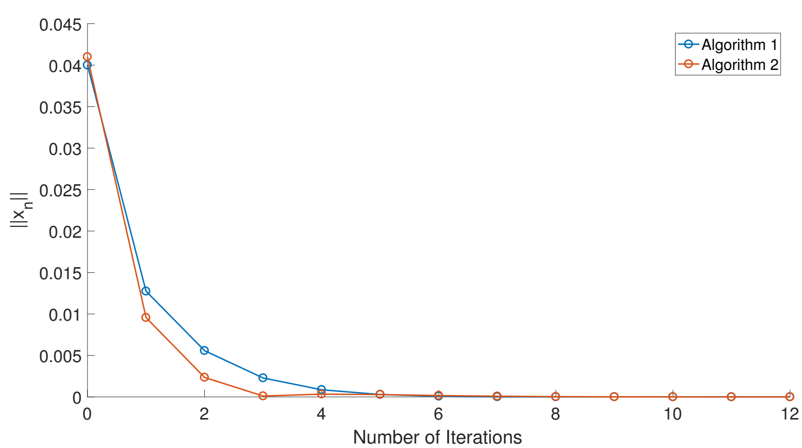

Figure 1: Our algorithms performances for Case I

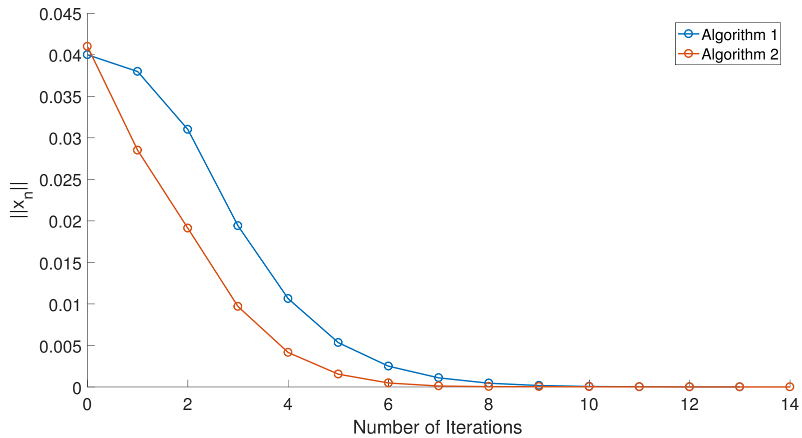

Figure 2: Our algorithms performances for Case II

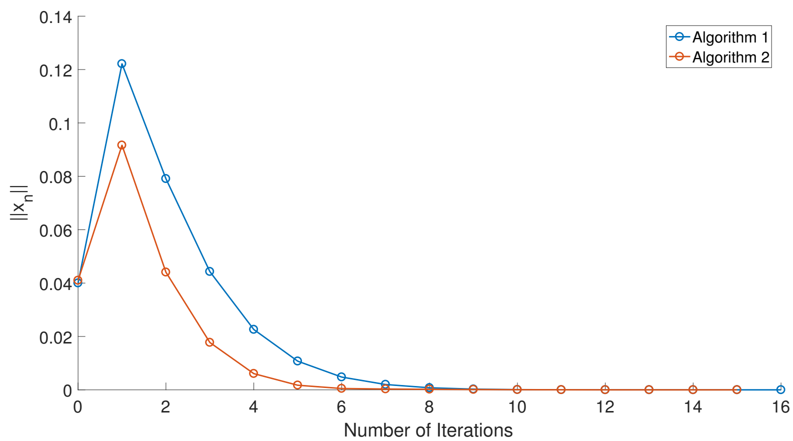

Figure 3: Our algorithms performances for Case III

Example 5.2. *Let $H_1 = H_2 = \mathbb{R}^3$. Define the operators A, B_1 and B_2 as follows:*

$$A = \begin{pmatrix} 6 & 3 & 1 \\ 8 & 7 & 5 \\ 3 & 6 & 2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

The parameters ϵ_n, θ_n are chosen as in the previous example.

First, we take the initial point $x_0 = (10, 0, -10), x_1 = (-10, 5, 10)$. The behavior of Algorithms 1 and 2 is reported in Figure 4. Next, we present several experiments to compare Algorithms 1 and 2 with the viscosity method of Suantai et al. [41] and the Halpern-type method of Alofi et al. [3]. Since $\|A\| = 14.87$, we choose the step size $\tau = 0.001$ for all algorithms. All results and comparisons are reported in Table 1 for the stopping rule $\|x_{n+1} - x_n\| \leq DOL$.

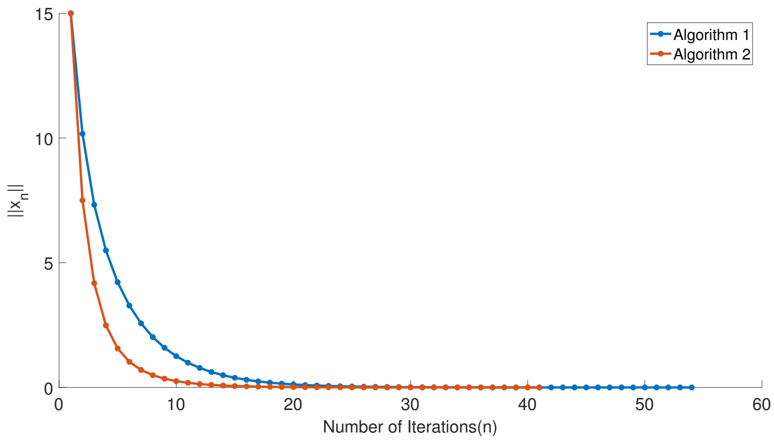

Figure 4: Our algorithms performances

Table 1: Comparison of Algorithms 1 and 2 with Suantai et al. [41] and Alofi et al. [3]

DOL	Method	Step size	Iter. (n)	CPU time (s)	$\frac{\|z-x_n\|}{\|x_0-x_{n+1}\|}$
10^{-6}	Algo. 1	0.001	54	5.49	$2.4258 * 10^{-7}$
	Algo. 2	0.001	41	4.658	$2.7943 * 10^{-7}$
	Suantai et al.	0.001	85	0.15577	$1.9452 * 10^{-7}$
	Alofi et al.	0.001	87	0.15677	$1.9582 * 10^{-7}$
10^{-8}	Algo. 1	0.001	66	5.53	$2.3993 * 10^{-9}$
	Algo. 2	0.001	52	5.2097	$2.7529 * 10^{-9}$
	Suantai et al.	0.001	114	0.916747	$1.9458 * 10^{-9}$
	Alofi et al.	0.001	116	0.168193	$2.01508 * 10^{-9}$

Example 5.3. *In this example we consider a problem from the field of compressed sensing, that is, recovery of a sparse and noisy signal from a limited number of sampling. Let $u_0 \in \mathbb{R}^n$ be K-sparse signal, $K \ll n$. The sampling matrix $A \in \mathbb{R}^{m\times n}$ $(m < n)$ is stimulated from the standard Gaussian distribution and vector $b = Ax + \epsilon$, where ϵ is additive noise. When $\epsilon = 0$, there is no noise in the observed data. Our task is to recover signal u_0 from data b. For further explanations, one can consult for example Nguyen and Shin [33]. In general, one can solve recover a sparse and noisy signal problem from LASSO problem, see Tibshirani [42].*

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2$$
$$\textit{subject to } \|x\|_1 < t,$$

where $t > 0$ is a given constant. Hence, with respect to the SVIP (3.1)-(3.2), we consider

$$B_1(x) = \begin{cases} \{u : \sup_{\|y\|_1 \le t} \langle y - x, u \rangle\} & \textit{if } x \in \mathbb{R}^n, \\ \emptyset & \textit{otherwise} \end{cases}$$

and

$$B_2(y) = \begin{cases} \mathbb{R}^m & \textit{if } y = b, \\ \emptyset & \textit{otherwise}. \end{cases}$$

Therefore, $B_1^{-1}(0) = \{x : \|x\| \le t\}$ and $B_2^{-1}(0) = b$. We test Algorithm 2 and compare it with the methods of Sitthithakerngkiet et al. [40] and Kazmi et al. [29]. For the experiment setting we choose the following parameters: $A \in \mathbb{R}^{m \times n}$ is generated randomly with $m = 2^7, n = 2^8$, $u_0 \in \mathbb{R}^n$ contains K-spikes with amplitude ± 1 distributed in the whole domain randomly. In addition, for simplicity, we take the viscosity function $h(x) = \frac{x}{2}$, $S = I, \alpha_i = \frac{1}{i+1}$ in Kazmi et al. [29], $S_i = I, \alpha_i = \frac{10^{-3}}{i=1}, \beta_i = 0.5 - \frac{1}{10i+2}$ in Sitthithakerngkiet et al. [40] and $\beta_i = \frac{2i-1}{2i+1}$, $\gamma_i = \frac{1}{2i+1}$ in our Algorithm 2. In addition, we take $t = K$ in all the algorithms and the stopping criterion $\|x_{n+1} - x_n\| \le DOL$ with $DOL = 10^{-4}$ and $DOL = 10^{-6}$, respectively. All the numerical results are presented in Figures 5-6 and Table 2.

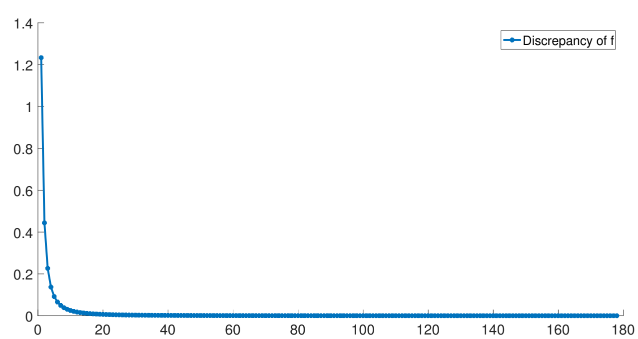

Figure 5: Discrepancy of the objective $f(x) = \frac{1}{2}\|Ax - b\|^2$

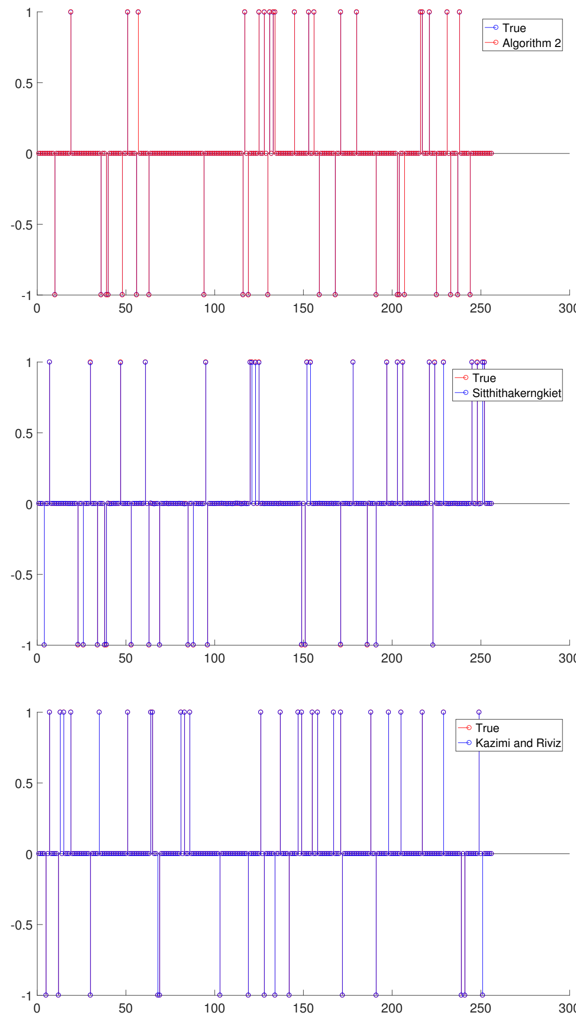

Figure 6: Comparison of Algorithms 2, Sitthithakerngkiet et al. [40], Kazmi et al. [29]

Table 2: Comparison of Algorithms 2, Sitthithakerngkiet et al. [40], Kazmi et al. [29]

K,m,n	DOL	Method	stepsize	Iter(n)
$K = 40, m = 2^7, n = 2^8$	10^{-4}	Algorithm 2	0.001	178
	10^{-4}	Sitthithakerngkiet	0.001	88
	10^{-4}	Kazmi et al.	0.001	9816
$K = 50, m = 2^7, n = 2^8$	10^{-6}	Algorithm 2	0.001	1881
	10^{-6}	Sitthithakerngkiet	0.001	15005
	10^{-6}	Kazmi et al.	0.001	335065

6 Conclusion

In this paper, we provide two simple inertial algorithms for solving split variational inclusions in Banach spaces. Weak and strong convergence theorems are established under standard assumptions. Our work extend and generalizes some related works in the literature as well as demonstrates good numerical behaviour.

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