GLOBAL REGULARITY OF 3D TROPICAL CLIMATE MODEL WITH FRACTIONAL DIFFUSION AND NONLINEAR DAMPING

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Abstract. This paper investigates the three dimensional tropic climate equations with fractional diffusion and nonlinear damping. The global exixtence and unique strong solutions of the tropic climate equations have been build under the assumption that initial data $(u_0, v_0, \theta_0) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ with $\alpha > 1, \max(1, \frac{p+4}{2p}) < \beta, \gamma < 1 + \frac{3}{p+1}$ and $p, q > 3$.

Keywords: Global regularity; Tropic climate model; Fractional diffusion; damping

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1. Introduction

This paper aims to address the global regularity of the following three dimensional tropical climate model

$$
\begin{cases}
\partial_t u + u\cdot\nabla u + \mu\Lambda^{2\alpha}u + \nabla\pi + \nabla\cdot(v\otimes v) + \sigma_1|u|^{p-1}u = 0, & (x,t)\in\mathbf{R}^3\times(0,\infty),\\
\partial_t v + u\cdot\nabla v + \nu\Lambda^{2\beta}v + v\cdot\nabla u + \nabla\theta + \sigma_2|v|^{q-1}v = 0, & (x,t)\in\mathbf{R}^3\times(0,\infty),\\
\partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^{2\gamma}\theta + \nabla\cdot v = 0, & (x,t)\in\mathbf{R}^3\times(0,\infty),\\
\mathrm{div} u = 0, & (x,t)\in\mathbf{R}^3\times(0,\infty),\\
u|_{t=0} = u_0, v|_{t=0} = v_0, \theta|_{t=0} = \theta_0, & x\in\mathbf{R}^3,
\end{cases}
\tag{1.1}
$$

where $u = (u_1(t,x,y,z), u_2(t,x,y,z), u_3(t,x,y,z)), v = (v_1(t,x,y,z), v_2(t,x,y,z), v_3(t,x,y,z))$ denote the barotropic mode and the first baroclinic mode of the velocity field, $\pi(t,x,y,z)$ and $\theta(t,x,y,z)$ stand for the scalar pressure and scalar temperature. Here $v\otimes v$ is the standard tensor notation. The fractional power $\alpha, \beta, \gamma \geq 0$ are real constants. The damping exponents $p, q \geq 1$ are real parameters and σ_1, σ_2 are the coefficient of the damping. This model is widely used in different fields, for example the physical phenomena in hydrodynamics, molecular biology such as anomalous diffusion in semi-conductor growth, probability, finance and so on (see [2], [8], [9]).

In 2004, Frierson et al. [3] derived the original tropical climate model without any dissipation terms after a Galerkin truncation to the hydrostatic Boussinesq. Let us briefly review some previous results on the tropical climate model. Li and Titi [6] established a unique global strong solution to the two dimensional tropic climate equations with any initial data with $\mu = \nu = 1, \kappa = \sigma_1 = \sigma_2 = 0$. Later, Ye [12] got the global regularity of a 2D tropical climate model with weak dissipation and Laplace dissipations. There is a large amount of literature of the global regularity on the tropic climate model (see [10], [11]).

When $v = \theta = 0$, the system (1.1) becomes the following Navier Stokes equation with fractional dissipation and damping:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \Lambda^{2\alpha}u + \nabla\pi + |u|^{p-1}u = 0, \\ \mathrm{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.2}$$

There is a rich literature on the well-posedness and decay estimates of the system (1.2) with $\alpha = 1$ (see [4], [5], [14], [15] et al). Here we give some examples for the well-posedness. Cai and Jiu [1] showed the existence of a global weak solution for $p \geq 1$ and global strong solutions for $p \geq \frac{7}{2}$. However, the uniqueness was shown for any $\frac{7}{2} \leq p \leq 5$. Zhang [13] proved the Navier Stokes equation has a global strong solution under the assumption that $u_0 \in H^1(\mathbf{R}^3)\cap L^{p+1}(\mathbf{R}^3)$ with $p > 3$. But the uniqueness is required another strict assumption that $3 < p \leq 5$. Later, Zhou [16] showed that there exists a global strong solution when $p \geq 3$. In 2021, Liu, Li and Sun [7] proved the existence and unique strong solution of the incompressible Navier Stokes equations (1.2) under the assumption that $\frac{p+4}{p+1} < \alpha < \min\{\frac{5}{2}, \frac{3}{p+1} + 2\}$ and $p > 1$. Motivated by [7], [10] and [11], the purpose of this paper is to establish the global well-posedness of the three tropic climate model with fractional dissipation and nonlinear damping. Our result is stated as follows.

Theorem 1.1. *Let $\alpha > 1, \max(1, \frac{p+4}{2p}) < \beta, \gamma < 1 + \frac{3}{p+1}$ and $p, q > 3$. Suppose that $(u_0, v_0, \theta_0) \in H^1(R^3)$ satisfies $\nabla\cdot u_0 = 0$. Then the three-dimensional tropical climate model (1.1) admits a global unique strong solution (u, v, θ) satisfying for any $0 < T < \infty$,*

$$\begin{aligned} &(u, v, \theta) \in L^\infty(0, T; H^1(R^3)),\ \theta \in L^2(0, T; H^{\gamma+1}(R^3)), \\ &u \in L^{p+1}(0, T; L^{p+1}(R^3)) \cap L^2(0, T; H^{\alpha+1}(R^3)), \\ &v \in L^{q+1}(0, T; L^{q+1}(R^3)) \cap L^2(0, T; H^{\beta+1}(R^3)), \\ &\nabla|u|^{\frac{p+1}{2}} \in L^2(0, T; L^2(R^3)), \nabla u|u|^{\frac{p-1}{2}} \in L^2(0, T; L^2(R^3)), \\ &\nabla|v|^{\frac{q+1}{2}} \in L^2(0, T; L^2(R^3)), \nabla v|v|^{\frac{q-1}{2}} \in L^2(0, T; L^2(R^3)). \end{aligned} \tag{1.3}$$

Remark 1.2. *In fact, our theorem is also true for $\alpha = \beta = \gamma = 1$, similar result was established in [11], but the temperature equation of (1.1) lacks the damping term $|\theta|^{r-1}\theta$. Moreover, we extend the results in [11] to the three dimensional tropic climate model with fractional dissipation. When $\alpha = 1, v = \theta = 0$, the tropic climate model (1.1) becomes the classical incompressible Navier-Stokes equations, Theorem 1.1 improves the results in [1], [13] and [16]. We only need the initial data $u_0 \in H^1(\mathbf{R}^3)$ and enlarge the scope of the power p of the damping.*

Remark 1.3. *When $\alpha \neq 0, \beta = \gamma = 0$ and $v = \theta = 0$, Theorem 1.1 is the corresponding theorem in [7]. But the conditions imposed on α, p satisfy $\max\{1, \frac{3}{2(p+1)} + \frac{1}{2}\} < \alpha < \min\{\frac{3}{p+1} + 2, \frac{5}{2}\}$ and $p > 1$, which are different from the ones given in the Theorem 1.1 in [7]. The gap appeared in Page 3 of [7] is changed as follows.*

$$\begin{aligned} I_1(t) &= C\|u\cdot\nabla u\|^2_{L^{\frac{6}{2\alpha+1}}} \leq C\|u\|^2_{L^{p+1}}\|\nabla u\|^2_{L^{\frac{6(p+1)}{(2\alpha+1)(p+1)-6}}} \\ &\leq C\|u\|^2_{L^{p+1}}\|u\|_{L^2}^{\frac{2(2\alpha-1)(p+1)-6}{(\alpha+1)(p+1)}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{6-2(\alpha+2)(p+1)}{(\alpha+1)(p+1)}} \leq C\|u\|_{L^{p+1}}^{\frac{2(\alpha+1)(p+1)}{(2\alpha-1)(p+1)-3}}\|u\|^2_{L^2} + \frac{1}{4}\|\Lambda^{1+\alpha}u\|^2_{L^2} \end{aligned}$$

$$\le C(\|u\|_{L^{p+1}}^{p+1}+1)\|u\|_{L^2}^2+\frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2,$$

where we have used the fact $\frac{3}{2(p+1)}+\frac{1}{2}<\alpha<\frac{3}{p+1}+2$.

Remark 1.4. *It is interesting to consider the same questions of the system* (1.1) *with lower diffusion* $0<\alpha,\beta,\gamma<1$. *Unfortunately, our method doesn't work in this case.*

The rest of this paper is organized as follows. Some crucial lemmas can be found in Section 2. The proof of existence of Theorem 1.1 can be found in Section 3. The uniqueness part of Theorem 1.1 is represented in Section 4.

2. Some useful Lemmas

The following lemmas play a crucial role in estimating the nonlinear terms to three fractional dimensional tropical climate equation.

Lemma 2.1. *Let* $1<p,q,r<\infty, 1<\theta\le 1$, *and* $s,s_1,s_2\in\mathbf{R}$. *Then the following fractional Gagliardo-Nirenberg inequality*

$$\|\Lambda^s u\|_{L^p(\mathbf{R}^d)}\le \|\Lambda^{s_1}u\|_{L^q(\mathbf{R}^d)}^{1-\theta}\|\Lambda^{s_2}u\|_{L^r(\mathbf{R}^d)}^{\theta},$$

holds if and only if

$$\frac{1}{p}-\frac{s}{d}=(1-\theta)(\frac{1}{q}-\frac{s_1}{d})+\theta(\frac{1}{r}-\frac{s_2}{d}), s\le(1-\theta)s_1+\theta s_2.$$

In addition, we also need the following Lemma.

Lemma 2.2. *Let* $s\in[0,\frac{d}{2})$. *Then the space* $\dot{H}^s(\mathbf{R}^d)$ *is continuously embedded in* $L^{\frac{2d}{d-2s}}(\mathbf{R}^d)$.

3. The proof of the existence of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. By a priori estimates and taking limits of the Galerkin approximated solutions , we can obtain the existence of a global strong solution. Therefore , in the sequel, we only build the a priori estimates. Throughout this section, we always assume that $\mu=\nu\kappa=\sigma_1=\sigma_2=1$.

Step 1 L^2**-energy estimate** Taking the L^2-inner product of 1.1 with u,v and θ respectively, then summing them up to arrive at

(3.1)
$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2+\|\Lambda^\gamma\theta\|_{L^2}^2+\|u\|_{L^{p+1}}^{p+1}+\|v\|_{L^{q+1}}^{q+1}=0,$$

where we have used

$$(\nabla\cdot(v\otimes v),u)+(v\cdot\nabla u,v)=0,(\nabla\theta,v)+(\nabla\cdot v,\theta)=0.$$

Then, one has

$$\begin{aligned}(3.2)\quad &\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2+2\int_0^t\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2+\|\Lambda^\gamma\theta\|_{L^2}^2+\|u\|_{L^{p+1}}^{p+1}+\|v\|_{L^{q+1}}^{q+1}d\tau\\ &=\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2+\|\theta_0\|_{L^2}^2.\end{aligned}$$

Step 2 H^1**-energy estimate**

Taking the L^2 inner product of 1.1 with $-\Delta u, -\Delta v$ and $-\Delta\theta$ respectively, then summing them up to conclude

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+1}v\|_{L^2}^2+\|\Lambda^{\gamma+1}\theta\|_{L^2}^2\\
&\quad+\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\|\nabla u|u|^{\frac{p-1}{2}}\|_{L^2}^2+\frac{4(q-1)}{(q+1)^2}\|\nabla|v|^{\frac{q+1}{2}}\|_{L^2}^2+\|\nabla v|v|^{\frac{q-1}{2}}\|_{L^2}^2\\
&=(u\cdot\nabla u,\Delta u)+(\nabla\cdot(v\otimes v),\Delta u)+(u\cdot\nabla v,\Delta v)+(v\cdot\nabla u,\Delta v)+(u\cdot\nabla\theta,\Delta\theta)\\
&\triangleq I_1+I_2+I_3+I_4+I_5,
\end{aligned}\tag{3.3}$$

where we used

$$\begin{aligned}
(|u|^{p-1}u,-\Delta u)&=\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\|\nabla u|u|^{\frac{p-1}{2}}\|_{L^2}^2,\\
(|v|^{q-1}v,-\Delta v)&=\frac{4(q-1)}{(q+1)^2}\|\nabla|v|^{\frac{q+1}{2}}\|_{L^2}^2+\|\nabla v|v|^{\frac{q-1}{2}}\|_{L^2}^2.
\end{aligned}$$

Thanks to the Hölder inequality, the Young inequality and Lemma 2.1, it yields that

$$\begin{aligned}
I_1\le&\|u\cdot\nabla u\|_{L^2}\|\Delta u\|_{L^2}\le\|u\cdot\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
\le&\int_{\mathbf{R}^3}|u|^2|\nabla u|^{\frac{4}{p-1}}|\nabla u|^{2-\frac{4}{p-1}}dx+\|\Delta u\|_{L^2}^2\\
\le&\||u|^2|\nabla u|^{\frac{4}{p-1}}\|_{L^{\frac{p-1}{2}}}\||\nabla u|^{2-\frac{4}{p-1}}\|_{L^{\frac{p-1}{p-3}}}+\|\Delta u\|_{L^2}^2\\
=&\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^{\frac{4}{p-1}}\|\nabla u\|_{L^2}^{\frac{2(p-3)}{p-1}}+\|\Delta u\|_{L^2}^2\\
\le&\frac{1}{2}\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
\le&\frac{1}{2}\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^{\frac{2(\alpha-1)}{\alpha}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{2}{\alpha}}\\
\le&\frac{1}{2}\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2.
\end{aligned}\tag{3.4}$$

We infer from Leibntz Law, the Hölder inequality, the Young inequality and Lemma 2.1

$$\begin{aligned}
I_2\le&\||v||\nabla v|\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
=&\int_{\mathbf{R}^3}|v|^2|\nabla v|^{\frac{4}{q-1}}|\nabla v|^{2-\frac{4}{q-1}}dx+\|\Delta u\|_{L^2}^2\\
\le&\||v|^2|\nabla v|^{\frac{4}{q-1}}\|_{L^{\frac{q-1}{2}}}\||\nabla v|^{2-\frac{4}{q-1}}\|_{L^{\frac{q-1}{q-3}}}+\|\Delta u\|_{L^2}^2\\
\le&\frac{1}{2}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2+C\|\nabla v\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
\le&\frac{1}{2}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2+C\|\nabla v\|_{L^2}^2+\|\nabla u\|_{L^2}^{\frac{2(\alpha-1)}{\alpha}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{2}{\alpha}}\\
\le&\frac{1}{2}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2+C\|\nabla v\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2.
\end{aligned}\tag{3.5}$$

To handle the third term I_3. According to the Hölder inequality and the Young inequality, Lemma 2.1 and Lemma 2.2, one deduces

$$
\begin{aligned}
I_3 \leq & \|u\cdot\nabla v\|_{L^{\frac{6}{1+2\beta}}}\|\Delta v\|_{L^{\frac{6}{5-2\beta}}}\\
\leq & C\|u\cdot\nabla v\|_{L^{\frac{6}{1+2\beta}}}\|\Delta v\|_{\dot{H}^{\beta-1}}\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\cdot\nabla v\|_{L^{\frac{6}{1+2\beta}}}^2\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla v\|_{L^{\frac{6(p+1)}{(2\beta+1)(p+1)-6}}}^2\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla v\|_{L^2}^{\frac{2(p+1)(2\beta-1)-6}{\beta(p+1)}}\|\Lambda^{1+\beta}v\|_{L^2}^{\frac{2(p+1)(1-\beta)+6}{\beta(p+1)}}\\
\leq & \frac{1}{4}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^{\frac{2\beta(p+1)}{(p+1)(2\beta-1)-3}}\|\nabla v\|_{L^2}^2\\
\leq & \frac{1}{4}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C(1+\|u\|_{L^{p+1}}^{p+1})\|\nabla v\|_{L^2}^2.
\end{aligned}
\tag{3.6}
$$

By integration by parts, we write

$$
\begin{aligned}
I_4 =& \sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3} v_i\partial_i u_j\partial_k\partial_k v_j dx\\
=& -\sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3}\partial_k v_i\partial_i u_j\partial_k v_j dx - \sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3} v_i\partial_i\partial_k u_j\partial_k v_j dx\\
=& \sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3}(u_j\partial_k\partial_i v_i\partial_k v_j + u_j\partial_k v_i\partial_k\partial_i v_j)dx - \sum_{i,j,k=1}^{3}\int_{\mathbf{R}^3} v_i\partial_i\partial_k u_j\partial_k v_j dx\\
\leq & 2\int_{\mathbf{R}^3}|u||\nabla v||D^2 v|dx + \int_{\mathbf{R}^3}|v||\nabla v||D^2 u|dx\\
=: & J_1 + J_2.
\end{aligned}
$$

Similarly as I_3. Thanks to the Hölder inequality and the Young inequality, Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
J_1 \leq & 2\||u||\nabla v|\|_{L^{\frac{6}{1+2\beta}}}\|D^2 v\|_{L^{\frac{6}{5-2\beta}}}\\
\leq & C\||u||\nabla v|\|_{L^{\frac{6}{1+2\beta}}}\|\Delta v\|_{\dot{H}^{\beta-1}}\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\||u||\nabla v|\|_{L^{\frac{6}{1+2\beta}}}^2\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla v\|_{L^{\frac{6(p+1)}{(2\beta+1)(p+1)-6}}}^2\\
\leq & \frac{1}{8}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla v\|_{L^2}^{\frac{2(p+1)(2\beta-1)-6}{\beta(p+1)}}\|\Lambda^{1+\beta}v\|_{L^2}^{\frac{2(p+1)(1-\beta)+6}{\beta(p+1)}}\\
\leq & \frac{1}{4}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C\|u\|_{L^{p+1}}^{\frac{2\beta(p+1)}{(p+1)(2\beta-1)-3}}\|\nabla v\|_{L^2}^2\\
\leq & \frac{1}{4}\|\Lambda^{1+\beta}v\|_{L^2}^2 + C(1+\|u\|_{L^{p+1}}^{p+1})\|\nabla v\|_{L^2}^2.
\end{aligned}
\tag{3.7}
$$

Similarly as I_2. Thanks to the Hölder inequality and the Young inequality and Lemma2.1, we can verify

$$\begin{aligned}
J_2 \leq & \||v||\nabla v|\|_{L^2}\|D^2 u\|_{L^2} \leq \||v||\nabla v|\|_{L^2}\|\Delta u\|_{L^2} \\
\leq & \||v||\nabla v|\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \\
\leq & \||v|^2|\nabla v|^{\frac{4}{q-1}}\|_{L^{\frac{q-1}{2}}}\||\nabla v|^{2-\frac{4}{q-1}}\|_{L^{\frac{q-1}{q-3}}} + \|\nabla u\|_{L^2}^{\frac{2(\alpha-1)}{\alpha}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{2}{\alpha}} \\
\leq & \||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^{\frac{4}{q-1}}\|\nabla v\|_{L^2}^{\frac{2(q-3)}{q-1}} + C\|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\Lambda^{1+\alpha}u\|_{L^2}^2 \\
\leq & \frac{1}{4}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 \\
\leq & \frac{1}{4}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2.
\end{aligned}$$

which along with (3.7) gives rise to

$$\begin{aligned}
I_4 \leq & \frac{1}{4}\|\Lambda^{1+\beta}v\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{L^{p+1}}^{p+1}\|\nabla v\|_{L^2}^2 \\
& + \frac{1}{4}\||v|^{\frac{q-1}{2}}\nabla v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2
\end{aligned} \tag{3.8}$$

Similarly as I_5, one has

$$\begin{aligned}
I_5 \leq & \|u\cdot\nabla\theta\|_{L^{\frac{6}{1+2\gamma}}}\|\Delta\theta\|_{L^{\frac{6}{5-2\gamma}}} \\
\leq & C\|u\cdot\nabla\theta\|_{L^{\frac{6}{1+2\gamma}}}\|\Delta\theta\|_{\dot{H}^{\gamma-1}} \\
\leq & \frac{1}{4}\|\Lambda^{1+\gamma}\theta\|_{L^2}^2 + C\|u\cdot\nabla\theta\|_{L^{\frac{6}{1+2\gamma}}}^2 \\
\leq & \frac{1}{4}\|\Lambda^{1+\gamma}\theta\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla\theta\|_{L^{\frac{6(p+1)}{(2\gamma+1)(p+1)-6}}}^2 \\
\leq & \frac{1}{4}\|\Lambda^{1+\gamma}\theta\|_{L^2}^2 + C\|u\|_{L^{p+1}}^2\|\nabla\theta\|_{L^2}^{\frac{2(p+1)(2\gamma-1)-6}{\gamma(p+1)}}\|\Lambda^{1+\gamma}\theta\|_{L^2}^{\frac{2(p+1)(1-\gamma)+6}{\gamma(p+1)}} \\
\leq & \frac{1}{2}\|\Lambda^{1+\gamma}\theta\|_{L^2}^2 + C\|u\|_{L^{p+1}}^{\frac{2\gamma(p+1)}{(p+1)(2\gamma-1)-3}}\|\nabla\theta\|_{L^2}^2 \\
\leq & \frac{1}{2}\|\Lambda^{1+\gamma}\theta\|_{L^2}^2 + C(1+\|u\|_{L^{p+1}}^{p+1})\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{3.9}$$

Inserting (3.4)-(3.9) into (3.3), it holds that

$$\begin{aligned}
& \frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\Lambda^{\alpha+1}u\|_{L^2}^2 + \|\Lambda^{\beta+1}v\|_{L^2}^2 + \|\Lambda^{\gamma+1}\theta\|_{L^2}^2 \\
& \quad + \|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2 + \|\nabla u|u|^{\frac{p-1}{2}}\|_{L^2}^2 + \|\nabla|v|^{\frac{q+1}{2}}\|_{L^2}^2 + \|\nabla v|v|^{\frac{q-1}{2}}\|_{L^2}^2 \\
& \leq C(\|u\|_{L^{p+1}}^{p+1}+1)(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Applying the Gronwall inequality and (3.2) to get

$$\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2 + \|\Lambda^{\beta+1}v\|_{L^2}^2 + \|\Lambda^{\gamma+1}\theta\|_{L^2}^2 \\
& \quad + \|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2 + \|\nabla u|u|^{\frac{p-1}{2}}\|_{L^2}^2 + \|\nabla|v|^{\frac{q+1}{2}}\|_{L^2}^2 + \|\nabla v|v|^{\frac{q-1}{2}}\|_{L^2}^2)d\tau \\
& \leq C(t,u_0,v_0,\theta_0),
\end{aligned} \tag{3.10}$$

which completes the proof of the existence of the strong solution .

4. The proof of uniqueness of Theorem 1.1

This section is devoted to establish the uniqueness of the strong solution constructed in Theorem 1.1. Let $(u_1, v_1, \theta_1, \pi_1)$ and $(u_2, v_2, \theta_2, \pi_2)$ be two solutions of the system (1.1) with the same initial data (u_0, v_0, θ_0). Suppose that $(\bar{u}, \bar{v}, \bar{\theta}, \bar{\pi}) = (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2, \pi_1 - \pi_2)$, then $(\bar{u}, \bar{v}, \bar{\theta}, \bar{\pi})$ obeys

$$
\begin{cases}
\partial_t \bar{u} + u_2 \cdot \nabla \bar{u} + \Lambda^{2\alpha} \bar{u} + \nabla \bar{\pi} + (|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2) = -\bar{u} \cdot \nabla u_1 - \nabla \cdot (\bar{v} \otimes v_1 + \ v_2 \otimes \bar{v}), \\
\partial_t \bar{v} + u_2 \cdot \nabla \bar{v} + \Lambda^{2\beta} \bar{v} + \nabla \bar{\theta} + (|v_1|^{q-1} v_1 - |v_2|^{q-1} v_2) = -(v_2 \cdot \nabla \bar{u} + \bar{v} \cdot \nabla u_1) - \bar{u} \cdot \nabla v_1, \\
\partial_t \bar{\theta} + u_2 \cdot \nabla \bar{\theta} + \Lambda^{2\gamma} \bar{\theta} + \nabla \cdot \bar{v} = -\bar{u} \cdot \nabla \theta_1, \\
\mathrm{div} \bar{u} = 0, \\
\bar{u}|_{t=0} = 0, \bar{b}|_{t=0} = 0, \bar{\theta}|_{t=0} = 0,
\end{cases}
\tag{4.1}
$$

Taking L^2-inner product to the system (4.1) with $(\bar{u}, \bar{v}, \bar{\theta})$, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|\bar{u}\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \|\Lambda^\alpha \bar{u}\|_{L^2}^2 + \|\Lambda^\beta \bar{v}\|_{L^2}^2 + \|\Lambda^\gamma \bar{\theta}\|_{L^2}^2 \\
&\quad + (|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2, u_1 - u_2) + (|v_1|^{q-1} v_1 - |v_2|^{q-1} v_2, v_1 - v_2) \\
&= - (\bar{u} \cdot \nabla u_1, \bar{u}) - (\nabla \cdot (\bar{v} \otimes v_1), \bar{u}) - (\nabla \cdot (v_2 \otimes \bar{v}), \bar{u}) - (\bar{u} \cdot \nabla v_1, \bar{v}) \\
&\quad - (\bar{v} \cdot \nabla u_1, \bar{v}) - (v_2 \cdot \nabla \bar{u}, \bar{v}) - (\bar{u} \cdot \nabla \theta_1, \bar{\theta}) \\
&\triangleq K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7.
\end{aligned}
\tag{4.2}
$$

It is easy to prove that (which can be found in [7] and [11])

$$
\begin{aligned}
\int_{\mathbf{R}^3} (|u_1|^{p-1} u_1 &- |u_2|^{p-1} u_2)(u_1 - u_2) dx \geq (\|u_1\|_{L^{p+1}}^p - \|u_2\|_{L^{p+1}}^p)(\|u_1\|_{L^{p+1}} - \|u_2\|_{L^{p+1}}) \geq 0, \\
\int_{\mathbf{R}^3} (|v_1|^{q-1} v_1 &- |v_2|^{q-1} v_2)(v_1 - v_2) dx \geq (\|v_1\|_{L^{q+1}}^q - \|v_2\|_{L^{q+1}}^q)(\|v_1\|_{L^{q+1}} - \|v_2\|_{L^{q+1}}) \geq 0.
\end{aligned}
$$

Thanks to integration by parts, the Hölder inequality, the Young inequality, Lemma 2.1 and Lemma 2.2, it yields that

$$
\begin{aligned}
K_1 = (\nabla \cdot \bar{u} u, u_1) + (\bar{u} \cdot \nabla \bar{u}, u_1) &\leq \|\bar{u}\|_{L^3} \|\nabla \bar{u}\|_{L^2} \|u_1\|_{L^6} \\
&\leq C \|\nabla \bar{u}\|_{L^2}^2 + C \|\bar{u}\|_{L^3}^2 \|u_1\|_{L^6}^2 \\
&\leq C \|\nabla \bar{u}\|_{L^2}^2 + C_t \|\bar{u}\|_{L^3}^2 \\
&\leq C \|\bar{u}\|_{L^2}^{\frac{2\alpha-2}{\alpha}} \|\Lambda^\alpha \bar{u}\|_{L^2}^{\frac{2}{\alpha}} + C_t \|\bar{u}\|_{L^2}^{\frac{2\alpha-1}{\alpha}} \|\Lambda^\alpha \bar{u}\|_{L^2}^{\frac{1}{\alpha}} \\
&\leq C_t \|\bar{u}\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\alpha \bar{u}\|_{L^2}^2.
\end{aligned}
\tag{4.3}
$$

Applying integration by parts, the Hölder inequality, the Young inequality Lemma2.1 and Lemma 2.2 to deduce that

$$
\begin{aligned}
K_2 &\le \|v_1\|_{L^6}\|\bar{v}\|_{L^3}\|\nabla\bar{u}\|_{L^2} \\
&\le C\|\nabla\bar{u}\|_{L^2}^2 + \|v_1\|_{L^6}^2\|\bar{v}\|_{L^3}^2 \\
&\le C\|\nabla\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^3}^2 \\
&\le C\|\bar{u}\|_{L^2}^{\frac{2\alpha-2}{\alpha}}\|\Lambda^\alpha\bar{u}\|_{L^2}^{\frac{2}{\alpha}} + C_t\|\bar{v}\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{1}{\beta}} \\
&\le \|\bar{u}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta\bar{v}\|_{L^2}^2.
\end{aligned}
\tag{4.4}
$$

A similar argument to estimate K_2 gives

$$
\begin{aligned}
K_3 &\le \|\bar{v}\|_{L^3}\|v_2\|_{L^6}\|\nabla\bar{u}\|_{L^2} \\
&\le \|\nabla\bar{u}\|_{L^2}^2 + \|\bar{v}\|_{L^3}^2\|v_2\|_{L^6}^2 \\
&\le \|\nabla\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^3}^2 \\
&\le \|\bar{u}\|_{L^2}^{\frac{2\alpha-2}{\alpha}}\|\Lambda^\alpha\bar{u}\|_{L^2}^{\frac{2}{\alpha}} + C_t\|\bar{v}\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{1}{\beta}} \\
&\le \|\bar{u}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta\bar{v}\|_{L^2}^2.
\end{aligned}
\tag{4.5}
$$

Along the same line to bound K_3, we obtain

$$
\begin{aligned}
K_4 &\le \|\bar{u}\|_{L^3}\|\nabla\bar{v}\|_{L^2}\|v_1\|_{L^6} \\
&\le \|\nabla\bar{v}\|_{L^2}^2 + \|\bar{u}\|_{L^3}^2\|v_1\|_{L^6}^2 \\
&\le \|\nabla\bar{v}\|_{L^2}^2 + C_t\|\bar{u}\|_{L^3}^2 \\
&\le \|\bar{v}\|_{L^2}^{\frac{2\beta-2}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{2}{\beta}} + C_t\|\bar{u}\|_{L^2}^{\frac{2\alpha-1}{\alpha}}\|\Lambda^\alpha\bar{u}\|_{L^2}^{\frac{1}{\alpha}} \\
&\le C\|\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta\bar{v}\|_{L^2}^2 + C_t\|\bar{u}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha\bar{u}\|_{L^2}^2.
\end{aligned}
\tag{4.6}
$$

Thanks to Leibnitz law, the Hölder inequality, the Young inequality, Lemma 2.1 and Lemma 2.2 again, we get

$$
\begin{aligned}
K_5 &\le \|\nabla\cdot\bar{v}\|_{L^2}\|\bar{v}\|_{L^3}\|u_1\|_{L^6} + \|\bar{v}\|_{L^3}\|\nabla\bar{v}\|_{L^2}\|u_1\|_{L^6} \\
&\le 2\nabla\bar{v}\|_{L^2}\|\bar{v}\|_{L^3}\|\|u_1\|_{L^6} \\
&\le C\|\nabla\bar{v}\|_{L^2}^2 + C\|\bar{v}\|_{L^3}^2\|u_1\|_{L^6}^2 \\
&\le C\|\nabla\bar{v}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^3}^2 \\
&\le C\|\bar{v}\|_{L^2}^{\frac{2\beta-2}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{2}{\beta}} + C_t\|\bar{v}\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{1}{\beta}} \\
&\le C_t\|\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta\bar{v}\|_{L^2}^2.
\end{aligned}
\tag{4.7}
$$

According to integration by parts and the methods to bound K_5, we can verify

$$\begin{aligned} K_6 &= (\nabla\cdot v_2\cdot\bar{u},\bar{v}) + (v_2\cdot\bar{u},\nabla\cdot\bar{v}) \\ &\le \|v_2\|_{L^6}\|\nabla\bar{u}\|_{L^2}\|\bar{v}\|_{L^3} \\ &\le C\|\nabla\bar{u}\|_{L^2}^2 + \|v_2\|_{L^6}^2\|\bar{v}\|_{L^3}^2 \\ &\le C\|\nabla\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^3}^2 \\ &\le C\|\bar{u}\|_{L^2}^{\frac{2\alpha-2}{\alpha}}\|\Lambda^\alpha\bar{u}\|_{L^2}^{\frac{2}{\alpha}} + C_t\|\bar{v}\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^\beta\bar{v}\|_{L^2}^{\frac{1}{\beta}} \\ &\le C\|\bar{u}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha\bar{u}\|_{L^2}^2 + C_t\|\bar{v}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta\bar{v}\|_{L^2}^2. \end{aligned} \tag{4.8}$$

Similarly as (4.8), we get

$$\begin{aligned} K_7 &= (\bar{u}\cdot\nabla\bar{\theta},\theta_1) \le \|\bar{u}\|_{L^3}\|\nabla\bar{\theta}\|_{L^2}\|\theta_1\|_{L^6} \\ &\le \|\nabla\bar{\theta}\|_{L^2}^2 + \|\bar{u}\|_{L^3}^2\|\theta_1\|_{L^6}^2 \\ &\le C\|\nabla\bar{\theta}\|_{L^2}^2 + C_t\|\bar{u}\|_{L^3}^2 \\ &\le C\|\bar{\theta}\|_{L^2}^{\frac{2\gamma-2}{\gamma}}\|\Lambda^\gamma\bar{\theta}\|_{L^2}^{\frac{2}{\gamma}} + C_t\|\bar{u}\|_{L^2}^{\frac{2\alpha-1}{\alpha}}\|\Lambda^\alpha\bar{u}\|_{L^2}^{\frac{1}{\alpha}} \\ &\le C\|\bar{\theta}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\gamma\bar{\theta}\|_{L^2}^2 + C_t\|\bar{u}\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\alpha\bar{u}\|_{L^2}^2. \end{aligned} \tag{4.9}$$

where we used $(\nabla\cdot\bar{u}\bar{\theta},\theta_1)=0$. Inserting (4.3)-(4.9) into (4.2), it yields that

$$\begin{aligned} &\frac{d}{dt}(\|\bar{u}\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \|\Lambda^\alpha\bar{u}\|_{L^2}^2 + \|\Lambda^\beta\bar{v}\|_{L^2}^2 + \|\Lambda^\gamma\bar{\theta}\|_{L^2}^2 \\ &\le C_t(\|\bar{u}\|_{L^2}^2 + \|\bar{v}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2), \end{aligned}$$

which together with the Gronwall inequality ensures $\|(\bar{u},\bar{v},\bar{\theta})\|_{L^2}=0$. This completes the proof of Theorem1.1.

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