Received ; | Revised ; | Accepted
DOI: xxx/xxxx

ARTICLE TYPE

Stationary distribution and extinction in a stochastic SIQR epidemic model with media coverage under Markovian switching †

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Summary

This paper is concerned with stationary distribution and extinction of a stochastic SIQR epidemic model with media coverage which is disturbed by both white and telegraph noises. By using the stochastic Lyapunov function method, we obtain sufficient conditions for the existence of a stationary distribution of the global positive solution to the model. Then we establish sufficient conditions for extinction of the disease. A stationary distribution means that all the individuals can be coexistent and persistent in the long term. Finally, numerical simulations are introduced to illustrate our theoretical results.

KEYWORDS:
SIQR epidemic model, Markovian switching, stationary distribution, extinction, media coverage

1 | INTRODUCTION

Recently, mathematical models have been widely used to analyze the mechanisms of infectious diseases, such as polio, diphtheria, tuberculosis, tetanus, pertussis, measles, hepatitis B, 2019 novel coronavirus (2019-nCoV) etc [1,2,3,4,5,6,7] and various epidemic models of population dynamics have been proposed [8,9,10,11,12,13,]. For example, Nistal et al. [11] studied the stability and equilibrium points of multistaged SI(n)R epidemic models. Zhang et al.[13] investigated the asymptotic behavior of global positive solution to a stochastic SIRS epidemic model incorporating media coverage and saturated incidence rate. Ma et al.[10] considered an SIQR epidemic model with standard incidence rate and their model can be expressed as follows

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \beta \frac{SI}{N} - \mu S, \\ \frac{dI}{dt} &= \beta \frac{SI}{N} - (\mu + \alpha + \delta + \gamma) I, \\ \frac{dQ}{dt} &= \delta I - (\mu + \alpha + \epsilon) Q, \\ \frac{dR}{dt} &= \gamma I + \epsilon Q - \mu R. \end{cases} \tag{1}$$

where S, I, R denote the number of susceptible, infective and removed, respectively, Q denotes the number of quarantined, $N = S + I + Q + R$ denotes the number of total population individuals. The parameter Λ denotes the recruitment rate of S corresponding to births and immigration, β is the disease transmission coefficient between

†The authors were supported by the Fundamental Research Funds for the Central Universities (XDJK2019C106), the Sponsored by Natural Science Foundation of Chongqing, China (cstc2019jcyj-bshX0122), the Project funded by China Postdoctoral Science Foundation (2019M653816XB), the National Natural Science Foundation of China (11901477, 11761019, 11361014), the Science Technology Foundation of Guizhou Education Department (20175736-001,2008038), the Project of High Level Creative Talents in Guizhou Province (20164035) and Major Research Projects on Innovative Groups in Guizhou Provincial Education Department ([2018]019).

compartments S and I, μ denotes the natural death rate, γ and ϵ are the recover rates from groups I, Q to R, δ represents the removal rate from I, α denotes the disease-caused death rate of I and Q. All parameters are assumed to be nonnegative and $\mu, \Lambda > 0$. Motivated by the system (1), liu et al.[14] developed a stochastic multigroup SIQR epidemic model with standard incidence rates and studied the existence of a stationary distribution of the positive solutions to the model, and established sufficient conditions for extinction of the disease.

When an infectious disease emerges and prevails in a region, one of the important prevention measures is educating people with the correct preventive knowledge of the disease through mass media and other platforms at the first opportunity[15]. Mass media including television, radio, newspaper, networks and so on potentially affect the behavior of the people, which can be used to deliver preventive healthcare messages for precaution and avoidance of negative behavior as a result of panic and to present updated information about the disease. Thus, media coverage is an urgent issue that needs attention [16,17,18]. Cui et al.[18] developed an SIS model to consider the impact of media and eduction on the spread of infectious disease. Liu and Li [19] proposed a drug model to discuss the impact of media coverage on the spread and control of drug addiction. In Ref.[20] , Liu and Zhang consider a SIS epidemic model on two patches incorporating media coverage. Recently, many mathematical models have been proposed to investigate the impact of media coverage on the transmission dynamics of infectious disease. Especially, Tchuenche et al.[21] incorporated a nonlinear function of the number infective individuals in their transmission term to investigate the effects of media coverage on the transmission dynamic where β_1 is the contact rate before media alert, the terms $\beta_2 I/(m+I)$ measure the effect of reduction of the contact rate when infectious individuals are reported in the media. The half-saturation constant $m > 0$ reflects the impact of media coverage on the contact transmission. Because the coverage report cannot prevent disease from spreading completely, we have $\beta_1 \geq \beta_2 > 0$. Hence, we consider the effects of media coverage on the transmission dynamic, model (1) can be modified as follows

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{SI}{N} - \mu S. \\ \frac{dI}{dt} &= \left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{SI}{N} - (\mu+\alpha+\delta+\gamma)I. \\ \frac{dQ}{dt} &= \delta I - (\mu+\alpha+\epsilon)Q. \\ \frac{dR}{dt} &= \gamma I + \epsilon Q - \mu R. \end{cases} \tag{2}$$

In addition, real life is full of randomness and stochasticity, epidemic models are always affected by the environmental noise in an ecosystem. Therefore, numerous scholars have used stochastic differential equations to study the dynamic behaviors of stochastic biological mathematical models[22,23,24,25,26,27,28,29]. For example, scholars obtained thresholds of the stochastic system which determine the extinction and persistence of the epidemic in [27,28]. Based on the discussion above, in this paper, we consider a stochastic non-autonomous SIQR model with periodic coefficients

$$\begin{cases} dS(t) &= \left[\Lambda(t) - \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t)+I(t)}\right)\frac{S(t)I(t)}{N} - \mu(t)S(t)\right]dt + \sigma_1(t)S(t)dB_1(t), \\ dI(t) &= \left[\left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t)+I(t)}\right)\frac{S(t)I(t)}{N} - (\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I(t)\right]dt + \sigma_2(t)I(t)dB_2(t), \\ dQ(t) &= \left[\delta(t)I(t) - (\mu(t)+\alpha(t)+\epsilon(t))Q(t)\right]dt + \sigma_3(t)Q(t)dB_3(t), \\ dR(t) &= \left[\gamma(t)I(t) + \epsilon(t)Q(t) - \mu(t)R(t)\right]dt + \sigma_4(t)R(t)dB_4(t). \end{cases} \tag{3}$$

Where $B_i(t)(i = 1,2,3,4)$ are independent Brownian motions and $\sigma_i(t)(i = 1,2,3,4)$ are the coefficients of the effects of environmental stochastic perturbations on $S(t), I(t), Q(t), R(t)$. The parameter functions $\Lambda(t), \beta_1(t), \beta_2(t), m(t), \mu(t), \alpha(t), \delta(t), \gamma(t), \epsilon(t)$ and $\sigma_i(t)(i = 1,2,3,4)$ are positive and continuous periodic functions with positive periodic $\mathbf{T}$.

In the real ecological systems, the population dynamics are usually influenced by a random switching in the external environments [30,31,32]. In paper [30], the switching between environmental regime is often memoryless and the waiting time for the next switching follows the exponential distribution. Usually, the random switching of environmental regimes is characterized the continuous-time Markov chain with value in a finite state space. Therefore, we propose the following stochastic SIQR model with regime switching

$$\begin{cases} dS(t) &= \Big[\Lambda(r(t)) - \Big(\beta_1(r(t)) - \frac{\beta_2(r(t))I(t)}{m(r(t))+I(t)}\Big)\frac{S(t)I(t)}{N} - \mu(r(t))S(t)\Big]dt + \sigma_1(r(t))S(t)dB_1(t), \\ dI(t) &= \Big[\Big(\beta_1(r(t)) - \frac{\beta_2(r(t))I(t)}{m(r(t))+I(t)}\Big)\frac{S(t)I(t)}{N} - (\mu(r(t)) + \alpha(r(t)) + \delta(r(t)) + \gamma(r(t)))I(t)\Big]dt \\ & \quad +\sigma_2(r(t))I(t)dB_2(t), \\ dQ(t) &= \Big[\delta(r(t))I(t) - (\mu(r(t)) + \alpha(r(t)) + \epsilon(r(t)))Q(t)\Big]dt + \sigma_3(r(t))Q(t)dB_3(t), \\ dR(t) &= \Big[\gamma(r(t))I(t) + \epsilon(r(t))Q(t) - \mu(r(t))R(t)\Big]dt + \sigma_4(r(t))R(t)dB_4(t). \end{cases} \tag{4}$$

where $r(t)$ is a right-continuous time Markov chain with values in finite state space $\mathcal{M} = \{1, 2, \cdots N\}$. For any $k \in \mathcal{M}$, the parameters $\Lambda(k), \beta_1(k), \beta_2(k), m(k), \mu(k), \alpha(k), \delta(k), \gamma(k), \epsilon(k)$ and $\sigma_i(k)(i = 1, 2, 3, 4)$ are all nonnegative constants.

The rest of the paper is organized as follows. In Section 2, we introduce some needed results throughout this paper. In Section 3, we show that there exists a unique global positive solution of system (4). In Section 4, we verify that there is an ergodic stationary distribution of system (4). In Section 5, we establish sufficient conditions for extinction of system (4). In Section 6, numerical simulations are given to illustrate our conclusion.

2 | PRELIMINARIES

In this section, we introduce the notations and lemmas which will be used in the whole paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

Let $r(t)$ is a right-continuous time Markov chain on the probability space taking values with in a finite state space $\mathcal{M} = \{1, 2, \cdots m\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$p\left(r(t+h) = j \mid r(t) = i\right) = \begin{cases} \gamma_{ij}h + o(h), & if \;\; i \neq j, \\ 1 + \gamma_{ij}h + o(h), & if \;\; i = j, \end{cases},$$

where $h > 0$. Hence $\gamma_{ij} > 0$ is the transition rate from i to j if $i \neq j$, while $\sum_{j=1}^{m} \gamma_{ij} = 0$. Assume further that the Markov chain $r(t)$ is irreducible and has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ which can be determined by equation

$$\pi\Gamma = 0, \tag{5}$$

subject to

$$\sum_{i=1}^{m} \pi_i = 1, \;\; \pi_i > 0, \;\; \forall i \in \mathcal{M}.$$

We assume that the Markov chain $r(t)$ is independent of the Brownian motion $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$. For any vector $g = (g(1), g(2), \cdots, g(m))$, define $\hat{g} = \min_{k\in\mathcal{M}} g(k)$,
$\check{g} = \max_{k\in\mathcal{M}} g(k)$

Let $(X(t), r(t))$ is the diffusion Markov process and satisfy the following equation

$$dX(t) = b(X(t), r(t))dt + \sigma(X(t), r(t))dB(t), \quad X(0) = x_0, \quad r(0) = r, \tag{6}$$

where $b(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^n, \sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^{n\times n}$ and $D(x, k) = \sigma(x, k)\sigma^{\mathrm{T}}(x, k) = (d_{ij}(x, k))$. For each $k \in \mathcal{M}$, let $V(\cdot, k)$ be any twice continuously differentiable function, the differential operator L of Eq.(6) is defined by

$$L(x,k) = \sum_{i=1}^{n} b_i(x,k)\frac{\partial V(x,k)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n} d_{ij}(x,k)\frac{\partial^2 V(x,k)}{\partial x_i \partial x_j} + \sum_{i=1}^{m} \gamma_{kl}V(x,l).$$

Lemma 1. [33] If the following conditions are satisfied

(a) $\gamma_{ij} > 0$, for any $i \neq j$;

(b) for each $k \in \mathcal{M}, D(x,k) = (d_{ij}(x,k))$ is symmetric and satisfies

$$\lambda \mid \zeta^2 \mid \leq (D(x,k)\zeta, \zeta) \leq \lambda^{-1} \mid \zeta^2 \mid, \;\; for \;\; all \;\; \zeta \in \mathbb{R}^n,$$

with some constant $\lambda \in (0, 1]$ for all $x \in \mathbb{R}^n$;

(c) there exists a nonempty open set $\mathcal{D}$ with compact closure, satisfying that, for each $k \in \mathcal{M}$, there is nonnegative function $V(\cdot, k) : \mathcal{D}^c \to \mathbb{R}$ such that $V(\cdot, k)$ is twice continuously differential and that for some $\varrho > 0$,

$$LV(x, k) \leq -\varrho, \quad (x, k) \in \mathcal{D}^c \times \mathcal{M}.$$

Then $(X(t), r(t))$ of system (4) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution $\pi(\cdot, \cdot)$ such that for any Borel measurable function $f(\cdot, \cdot) : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}$ satisfies

$$\sum_{k=1}^{m} \int_{\mathbb{R}^n} \mid f(x, k) \mid \pi(dx, k) < \infty,$$

we have

$$p\Big(\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(s), r(s))ds = \sum_{k=1}^{m} \int_{\mathbb{R}^n} \mid f(x, k) \mid \pi(dx, k)\Big) = 1.$$

Lemma 2. [34]. Let $M = \{M_t\}_t \geq 0$ be a real-valued continuous local martingale vanishing $t = 0$. Then

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

3 | EXISTENCE AND UNIQUENESS OF THE GLOBAL POSITIVE SOLUTION

In this section, we use the Lyapunov function method to prove that the solution of system (4) is global and positive.

Theorem 1. For any initial value $(S(0), I(0), Q(0), R(0), r(0)) \in \mathbb{R}_+^4 \times \mathcal{M}$, there is a unique positive solution $(S(t), I(t), Q(t), R(t), r(t))$ of system (4) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^4 \times \mathcal{M}$ with probability one, namely, $(S(t), I(t), Q(t), R(t), r(t)) \in \mathbb{R}_+^4 \times \mathcal{M}$ for all $t \geq 0$ almost surely.

Proof. Note that the coefficients of the model (4) are locally Lipschitz conditions, then for any given initial value $(S(0), I(0), Q(0), R(0), r(0)) \in \mathbb{R}_+^4 \times \mathcal{M}$, there ia a unique positive local solution $(S(t), I(t), Q(t), R(t), r(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [35]. To demonstrate that this solution is global, we only need to prove that $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for any initial value $S(0), I(0), Q(0)$ and $R(0)$ lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the following stopping time

$$\tau_k = \inf\Big\{t \in [0, \tau_e) : \min\{S(t), I(t), Q(t), R(t)\} \leq \frac{1}{k} \ or \ \max\{S(t), I(t), Q(t), R(t)\} \geq k\Big\}$$

where we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_e$ a.s. Next, we only need to verify $\tau_\infty = \infty$ a.s. If this statement is false, then there exist two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus there is an integer $k_1 \geq k_0$ such that $\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon$, for all $k \geq k_1$.

Define a C^2-function $V : \mathbb{R}_+^4 \times \mathcal{M} \to \mathbb{R}_+$ as follows

$$V(S, I.Q, R, r) = S - a - a\ln\frac{S}{a} + I - 1 - \ln I + Q - 1 - \ln Q + R - 1 - \ln R,$$

the nonnegativity of this function can be obtained from $x - 1 - \ln x \geq 0, x > 0$, and the parameter a will be determined later.

Applying Itô's formula yields

$$\begin{aligned} dV(S, I, Q, R, r) \;=\; & LVdt + (S-a)\sigma_1(t)dB_1(t) + (I-1)\sigma_2(t)dB_2(t) + (Q-1)\sigma_3(t)dB_3(t) \\ & +(R-1)\sigma_4(t)dB_4(t), \end{aligned}$$

where

$$\begin{aligned}
LV &= (1-\frac{a}{S})\Big[\Lambda(r(t))-\Big(\beta_1(r(t))-\frac{\beta_2(r(t))I(t)}{m(r(t))+I(t)}\Big)\frac{S(t)I(t)}{N}-\mu(r(t))S(t)\Big]+\frac{a\sigma_1^2(r(t))}{2}\\
&+(1-\frac{1}{I})\Big[\Big(\beta_1(r(t))-\frac{\beta_2(r(t))I(t)}{m(r(t))+I(t)}\Big)\frac{S(t)I(t)}{N}-(\mu(r(t))+\alpha(r(t))+\delta(r(t))\\
&+\gamma(r(t)))I(t)\Big]+\frac{\sigma_2^2(r(t))}{2}\\
&+(1-\frac{1}{Q})\Big[\delta(r(t))I(t)-(\mu(r(t))+\alpha(r(t))+\epsilon(r(t)))Q(t)\Big]+\frac{\sigma_3^2(r(t))}{2}\\
&+(1-\frac{1}{R})\Big[\gamma(r(t))I(t)+\epsilon(r(t))Q(t)-\mu(r(t))R(t)\Big]+\frac{\sigma_4^2(r(t))}{2},
\end{aligned}$$

which implies that

$$\begin{aligned}
LV &\le \Lambda(r(t))+\frac{aI(t)}{N}\beta_1(r(t))+a\mu(r(t))-(\mu(r(t))+\alpha(r(t)))I(t)+3\mu(r(t))+2\alpha(r(t))\\
&+\epsilon(r(t))+\gamma(r(t))+\delta(r(t))+\frac{a\sigma_1^2(r(t))}{2}+\frac{\sigma_2^2(r(t))}{2}+\frac{\sigma_3^2(r(t))}{2}+\frac{\sigma_4^2(r(t))}{2}\\
&\le \check{\Lambda}-(\hat{\mu}+\hat{\alpha}-\frac{a\check{\beta}_1}{N})I+a\check{\mu}+3\check{\mu}+2\check{\alpha}+\check{\epsilon}+\check{\gamma}+\check{\delta}+\frac{a\check{\sigma_1^2}+\check{\sigma_2^2}+\check{\sigma_3^2}+\check{\sigma_4^2}}{2}.
\end{aligned}$$

Choose $a=\frac{N(\hat{\mu}+\hat{\alpha})}{\check{\beta}_1}$ such that $\hat{\mu}+\hat{\alpha}-\frac{a\check{\beta}_1}{N}=0$, then

$$LV\le \check{\Lambda}+a\check{\mu}+3\check{\mu}+2\check{\alpha}+\check{\epsilon}+\check{\gamma}+\check{\delta}+\frac{a\check{\sigma_1^2}+\check{\sigma_2^2}+\check{\sigma_3^2}+\check{\sigma_4^2}}{2}:=K,$$

where K is a positive constant.

The remainder of the proof follows as that in [36]. The proof is completed. □

4 | EXISTENCE OF ERGODIC STATIONARY DISTRIBUTION OF MODEL (4)

In this section, we investigated the existence of an ergodic stationary distribution of model (4).

Define a parameter

$$\mathfrak{R}_1=\frac{\sum_{k=1}^m\pi_k\Lambda(k)(\beta_1(k)-\beta_2(k))}{N\sum_{k=1}^m\pi_k(\mu(k)+\frac{\sigma_1^2(k)}{2})\sum_{k=1}^m\pi_k(\mu(k)+\alpha(k)+\delta(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2})}.$$

Theorem 2. If $\mathfrak{R}_1>1$, then for any initial value $(S(0),I(0),Q(0),R(0),r(0))\in\mathbb{R}_+^4\times\mathcal{M}$, the solution $(S(t),I(t),Q(t),R(t),r(t))$ of model (4) admits a unique ergodic stationary distribution.

Proof. In order to proof Theorem 2, it suffices to verify conditions $(a),(b),(c)$ in lemma 1. Assumption $\gamma_{ij}>0$ for $i\neq j$ in section 2 implies that condition (a) in Lemma 1 is satisfied. On the other hand, we consider the following bounded open subset:

$$\mathcal{D}=(1/l,l)\times(1/l,l)\times(1/l,l)\times(1/l,l)\in\mathbb{R}_+^4,$$

where l is a sufficiently large number. Then $\bar{\mathcal{D}}\in\mathbb{R}_+^4$. We have $D(S,I,Q,R,k)=W(S,I,Q,R,k)$ $W^{\mathrm{T}}(S,I,Q,R,k)$ in which $W(S,I,Q,R,k)=diag(S\sigma_1(k),I\sigma_2(k),Q\sigma_3(k),R\sigma_4(k)),k\in\mathcal{M}$. Then $D(S,I,Q,R,k)$ is positive semi-definite and since $W(S,I,Q,R,k)$ is nonsingular matrix, we deduce that $D(S,I,Q,R,k)$ is positive definite. Hence

$$\lambda_{\max}(D(S,I,Q,R,k))\ge\lambda_{\min}(D(S,I,Q,R,k))>0,$$

and for all $\zeta\in\mathcal{D}$, we have

$$\lambda_{\min}(D(S,I,Q,R,k))\mid\zeta\mid^2\le\zeta^{\mathrm{T}}(D(S,I,Q,R,k))\zeta\le\lambda_{\max}(D(S,I,Q,R,k))\mid\zeta\mid^2.$$

It is easy to see that $\lambda_{\min}(D(S,I,Q,R,k))$ and $\lambda_{\max}(D(S,I,Q,R,k))$ are two continuous functions of S,I,Q,R. Therefore $\hat{\lambda}=\min_{(S,I,Q,R,k)\in\bar{\mathcal{D}}\times\mathcal{M}}\lambda_{\min}(D(S,I,Q,R,k))>0$ and $\check{\lambda}=\max_{(S,I,Q,R,k)\in\bar{\mathcal{D}}\times\mathcal{M}}\lambda_{\max}(D(S,I,Q,R,k))>$

0, which implies that

$$\lambda \mid \zeta^2 \mid \leq (D(x,k)\zeta,\zeta) \leq \lambda^{-1} \mid \zeta^2 \mid, \quad for\ all \quad \zeta \in \mathbb{R}^n,$$

where $\lambda = \min\{\hat{\lambda}, \check{\lambda}^{-1}\}$. Condition (b) is verified.

Now we verify condition (c). Define a C^2-function $V : [0,+\infty) \times \mathbb{R}^4_+ \to \mathbb{R}$:

$$V(S,I,Q,R,k) = M(V_1(S,I)+\omega(k)) + V_2(S,I,Q,R) + V_3(S) + V_4(Q) + V_5(R),$$

$$V_1(S,I) = -C_1 \ln S - C_2 \ln I, V_2(S,I,Q,R) = \frac{1}{\theta+1}(S+I+Q+R)^{\theta+1},$$

$$V_3(S) = -\ln S, V_4(Q) = -\ln Q, V_5(R) = -\ln R,$$

where

$$C_1 = \frac{\sum_{k=1}^m \pi_k \Lambda(k)}{\sum_{k=1}^m \pi_k(\mu(k) + \frac{\sigma_1^2(k)}{2})}, \quad C_2 = \frac{\sum_{k=1}^m \pi_k \Lambda(k)}{\sum_{k=1}^m \pi_k \Lambda(k)(\mu(k)+\alpha(k)+\delta(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2})},$$

and $\theta > 0$ is a sufficiently small constant such that

$$\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 + \check{\sigma}_2^2 + \check{\sigma}_3^2 + \check{\sigma}_4^2) > 0,$$

and $M > 0$ is a sufficiently large positive constant and satisfies the following condition

$$-M\lambda + C \leq -2,$$

where

$$\lambda = 2\sum_{k=1}^m \pi_k \Lambda(k)(\mathfrak{R}_1^{\frac{1}{2}} - 1),$$

and

$$\begin{aligned} C = & \sup_{(S,I,Q,R,k)\in R_+^4 \times \mathcal{M}} \Big\{ -\frac{1}{2}(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2))(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) + D \\ & +3\check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2 + \check{\sigma}_3^2 + \check{\sigma}_4^2}{2}\Big\}, \end{aligned}$$

where

$$D = \sup_{(S,I,Q,R,k)\in R_+^4 \times \mathcal{M}} \Big\{ \check{\Lambda}(S+I+Q+R)^{\theta} - \frac{1}{2}(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2))(S+I+Q+R)^{\theta+1}\Big\}.$$

Obviously, $V(S,I,Q,R,k)$ has a minimum value point (S_0,I_0,Q_0,R_0,k). Then we can define a nonnegative C^2-function $\overline{V}$:

$$\overline{V} = V(S,I,Q,R,k) - V(S_0,I_0,Q_0,R_0,k).$$

By the Itô's formula, we obtain

$$\begin{aligned}
LV_1 &= -\frac{C_1}{S}\Big[\Lambda(k)-\Big(\beta_1(k)-\frac{\beta_2(k)I}{m(k)+I}\Big)\frac{SI}{N}-\mu(k)S\Big]+\frac{C_1\sigma_1^2(k)}{2}\\
&\quad -\frac{C_2}{I}\Big[\Big(\beta_1(k)-\frac{\beta_2(k)I}{m(k)+I}\Big)\frac{SI}{N}-(\mu(k)+\alpha(k)+\delta(k)+\gamma(k))I\Big]+\frac{C_2\sigma_2^2(k)}{2}\\
&\le -\frac{C_1\Lambda(k)}{S}-\frac{C_2S}{N}(\beta_1(k)-\beta_2(k))+\frac{C_1\beta_1(k)I}{N}+C_1(\mu(k)+\frac{\sigma_1^2(k)}{2})\\
&\quad +C_2(\mu(k)+\alpha(k)+\delta(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2})\\
&\le -2\sqrt{\frac{C_1C_2\Lambda(k)}{N}(\beta_1(k)-\beta_2(k))}+\frac{C_1\beta_1(k)I}{N}+C_1(\mu(k)+\frac{\sigma_1^2(k)}{2})\\
&\quad +C_2(\mu(k)+\alpha(k)+\delta(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2})\\
&= B_0(k)+\frac{C_1\beta_1(k)I}{N},
\end{aligned}$$

where $B_0(k)=-2\sqrt{\frac{C_1C_2\Lambda(k)}{N}(\beta_1(k)-\beta_2(k))}+C_1(\mu(k)+\frac{\sigma_1^2(k)}{2})+C_2(\mu(k)+\alpha(k)+\delta(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2})$.

Let $(\overline{\omega}(1),\overline{\omega}(2),\cdots,\overline{\omega}(m))$ be the solution of the following Poisson system:

$$\Gamma\overline{\omega}=\sum_{l=1}^{m}\pi_l\overline{B}_0(l)-\overline{B}_0,$$

where $\overline{B}_0=(\overline{B}_0(1),\overline{B}_0(2),\cdots,\overline{B}_0(m))$. Therefore, combining the definitions of C_1,C_2, leads to

$$\begin{aligned}
L(V_1+\omega(t)) &\le -2\sum_{k=1}^{m}\pi_k\Lambda(k)(\Re_1^{\frac{1}{2}}-1)+\frac{C_1\beta_1(k)I}{N}\\
&\le -\lambda+\frac{C_1\check{\beta}_1 I}{N}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
LV_2 &= (S+I+Q+R)^{\theta}\Big[\Lambda(k)-\mu(k)S-(\mu(k)+\alpha(k))(I+Q)-\mu(k)R\Big]\\
&\quad +\frac{1}{2}\theta(S+I+Q+R)^{\theta-1}(\sigma_1^2(k)S^2+\sigma_2^2(k)I^2+\sigma_3^2(k)Q^2+\sigma_4^2(k)R^2)\\
&\le \Lambda(k)(S+I+Q+R)^{\theta}-\mu(k)(S+I+Q+R)^{\theta+1}+\frac{1}{2}\theta(S+I+Q+R)^{\theta+1}\\
&\quad \times(\sigma_1^2(k)\vee\sigma_2^2(k)\vee\sigma_3^2(k)\vee\sigma_4^2(k))\\
&= \Lambda(k)(S+I+Q+R)^{\theta}-\Big(\mu(k)-\frac{1}{2}\theta(\sigma_1^2(k)\vee\sigma_2^2(k)\vee\sigma_3^2(k)\vee\sigma_4^2(k))\Big)\\
&\quad \times(S+I+Q+R)^{\theta+1}\\
&\le D-\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1}),
\end{aligned}$$

$$\begin{aligned}
LV_3 &= -\frac{1}{S}\Big[\Lambda(k)-\Big(\beta_1(k)-\frac{\beta_2(k)I}{m(k)+I}\Big)\frac{SI}{N}-\mu(k)S\Big]+\frac{\sigma_1^2(k)}{2}\\
&\le -\frac{\Lambda(k)}{S}+\frac{\beta_1(k)I}{N}+\mu(k)+\frac{\sigma_1^2(k)}{2}\\
&\le -\frac{\hat{\Lambda}}{S}+\frac{\check{\beta}_1 I}{N}+\check{\mu}+\frac{\check{\sigma}_1^2}{2},
\end{aligned}$$

$$\begin{aligned}
LV_4 &= -\frac{1}{Q}\Big[\delta(k)I(t) - (\mu(k)+\alpha(k)+\epsilon(k))Q(t)\Big] + \frac{\sigma_3^2(k)}{2} \\
&= -\frac{\delta(k)I}{Q} + (\mu(k)+\alpha(k)+\epsilon(k)) + \frac{\sigma_3^2(k)}{2} \\
&\le -\frac{\hat{\delta}I}{Q} + \check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_3^2}{2},
\end{aligned}$$

and

$$\begin{aligned}
LV_5 &= -\frac{1}{R}\Big[\gamma(k)I + \epsilon(k)Q - \mu(k)R\Big] + \frac{\sigma_4^2(k)}{2} \\
&= -\frac{\gamma(k)I}{R} - \frac{\epsilon(k)Q}{R} + \mu(k) + \frac{\sigma_4^2(k)}{2} \\
&\le -\frac{\hat{\gamma}I}{R} + \check{\mu} + \frac{\check{\sigma}_4^2}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
L\overline{V} &= ML(V_1+\omega(k)) + LV_2 + LV_3 + LV_4 + LV_5 \\
&\le M(-\lambda + \frac{C_1\check{\beta}_1 I}{N}) - \frac{\hat{\Lambda}}{S} + \frac{\check{\beta}_1 I}{N} - \frac{\hat{\delta}I}{Q} - \frac{\hat{\gamma}I}{R} + 3\check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} \\
&\quad + D - \frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}), \\
&= -M\lambda + \frac{\check{\beta}_1 I}{N}(MC_1+1) - \frac{\hat{\Lambda}}{S} - \frac{\hat{\delta}I}{Q} - \frac{\hat{\gamma}I}{R} + 3\check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} \\
&\quad + D - \frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}).
\end{aligned}$$

Define the following bounded closed set

$$U = \Big\{(S,I,Q,R,k) \in R_+^4 \times \mathcal{M} : \varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I \le \frac{1}{\varepsilon}, \varepsilon^2 \le Q \le \frac{1}{\varepsilon^2}, \varepsilon^2 \le R \le \frac{1}{\varepsilon^2}\Big\},$$

where $\varepsilon > 0$ is a sufficiently small number. In the set $R_+^4 \setminus U$, we can choose ε sufficiently small such that

$$-\frac{\hat{\Lambda}}{\varepsilon} + E \le -1, \tag{7}$$

$$-M\lambda + \frac{\check{\beta}_1\varepsilon}{N}(MC_1+1) + C \le -1, \tag{8}$$

$$-\frac{\hat{\delta}}{\varepsilon} + E \le -1, \tag{9}$$

$$-\frac{\hat{\gamma}}{\varepsilon} + E \le -1, \tag{10}$$

$$-\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{\theta+1}} + F \le -1, \tag{11}$$

$$-\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{\theta+1}} + G \le -1, \tag{12}$$

$$-\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + H \le -1, \tag{13}$$

$$-\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + J \le -1, \tag{14}$$

where E, C, F, G, H, J are positive constants which can be found below. For the sake of convenience, we divide into eight domains

$$U_1=\Big\{(S,I,Q,R)\in R_+^4: 0<S<\varepsilon\Big\}, U_2=\Big\{(S,I,Q,R,)\in R_+^4: 0<I<\varepsilon\Big\},$$

$$U_3=\Big\{(S,I,Q,R)\in R_+^4: I>\varepsilon, 0<Q<\varepsilon^2\Big\}, U_4=\Big\{(S,I,Q,R)\in R_+^4: I>\varepsilon, 0<R<\varepsilon^2\Big\}.$$

$$U_5=\Big\{(S,I,Q,R)\in R_+^4: S>\frac{1}{\varepsilon}\Big\}, U_6=\Big\{(S,I,Q,R)\in R_+^4: I>\frac{1}{\varepsilon}\Big\},$$

$$U_7=\Big\{(S,I,Q,R)\in R_+^4: Q>\frac{1}{\varepsilon^2}\Big\}, U_8=\Big\{(S,I,Q,R)\in R_+^4: R>\frac{1}{\varepsilon^2}\Big\},$$

Next we will prove that $LV(S,I,Q,R,k)\le -1$ on $R_+^4\setminus U$, which is equivalent to proving it on the above eight domains.

Case 1. If $(S,I,Q,R,k)\in U_1$, one can see that

$$\begin{aligned} L\overline{V} \le\ & \frac{\check{\beta}_1 I}{N}(MC_1+1)-\frac{\hat{\Lambda}}{S}+3\check{\mu}+\check{\alpha}+\check{\epsilon}+\frac{\check{\sigma}_1^2}{2}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_4^2}{2}+D \\ & -\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1}) \\ \le\ & -\frac{\hat{\Lambda}}{\varepsilon}+E \end{aligned} \tag{15}$$

where

$$\begin{aligned} E =\ & \sup_{(S,I,Q,R)\in R_+^4}\Big\{\frac{\check{\beta}_1 I}{N}(MC_1+1)+3\check{\mu}+\check{\alpha}+\check{\epsilon}+\frac{\check{\sigma}_1^2}{2}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_4^2}{2}+D \\ & -\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\Big\}. \end{aligned}$$

By (7), we have $LV\le -1$ for all $(S,I,Q,R)\in U_1$.

Case 2. If $(S,I,Q,R)\in U_2$, one can obtain that

$$\begin{aligned} LV(S,I,Q,R) \le\ & -M\lambda+\frac{\check{\beta}_1 I}{N}(MC_1+1)+3\check{\mu}+\check{\alpha}+\check{\epsilon}+\frac{\check{\sigma}_1^2}{2}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_4^2}{2}+D \\ & -\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1}) \\ \le\ & -M\lambda+\frac{\check{\beta}_1 I}{N}(MC_1+1)+C \\ \le\ & -M\lambda+\frac{\check{\beta}_1 \varepsilon}{N}(MC_1+1)+C, \end{aligned} \tag{16}$$

where

$$\begin{aligned} C =\ & \sup_{(S,I,Q,R)\in R_+^4}\Big\{-\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1}) \\ & +3\check{\mu}+\check{\alpha}+\check{\epsilon}+\frac{\check{\sigma}_1^2}{2}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_4^2}{2}+D\Big\}. \end{aligned}$$

Accordingly to (8), we have $LV\le -1$ for all $(S,I,Q,R)\in U_2$.

Case 3. If $(S,I,Q,R)\in U_3$, we have

$$\begin{aligned} LV(S,I,Q,R) \le\ & \frac{\check{\beta}_1 I}{N}(MC_1+1)-\frac{\hat{\delta} I}{Q}+3\check{\mu}+\check{\alpha}+\check{\epsilon}+\frac{\check{\sigma}_1^2}{2}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_4^2}{2}+D \\ & -\frac{1}{2}\Big(\hat{\mu}-\frac{1}{2}\theta(\check{\sigma}_1^2\vee\check{\sigma}_2^2\vee\check{\sigma}_3^2\vee\check{\sigma}_4^2)\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1}) \\ \le\ & -\frac{\hat{\delta} I}{Q}+E. \end{aligned} \tag{17}$$

In view of (9), we have $LV \leq -1$ for all $(S,I,Q,R) \in U_3$.

Case 4. If $(S,I,Q,R) \in U_4$, one can derive that

$$\begin{aligned} LV(S,I,Q,R) &\leq -\frac{\gamma^l I}{R} + \frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} \\ &\quad + D - \frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) \\ &\leq -\frac{\gamma^l I}{R} + E \\ &\leq -\frac{\gamma^l}{\varepsilon} + E. \end{aligned} \tag{18}$$

By (10), we have $LV \leq -1$ for all $(S,I,Q,R) \in U_4$.

Case 5. If $(S,I,Q,R) \in U_5$, it follows that

$$\begin{aligned} LV(S,I,Q,R) &\leq -\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) \\ &\quad + \frac{\check{\beta}_1 I}{N}(MC_1+1) + 3\check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} + D \\ &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)S^{\theta+1} + F \\ &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{\theta+1}} + F, \end{aligned} \tag{19}$$

where

$$\begin{aligned} F = \sup_{(S,I,Q,R)\in R_+^4} \Big\{ &-\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(I^{\theta+1} + Q^{\theta+1}) + \frac{\check{\beta}_1 I}{N}(MC_1+1) + 3\check{\mu} \\ &+ \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} + D - \frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)S^{\theta+1}\Big\}. \end{aligned}$$

Together with (11), we have $LV \leq -1$ for all $(S,I,Q,R) \in U_5$.

Case 6. If $(S,I,Q,R) \in U_6$, we obtain

$$\begin{aligned} LV(S,I,Q,R) &\leq -\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) \\ &\quad + \frac{\check{\beta}_1 I}{N}(MC_1+1) + 3\check{\mu} + \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} + D \\ &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)I^{\theta+1} + G \\ &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{\theta+1}} + G, \end{aligned} \tag{20}$$

where

$$\begin{aligned} G = \sup_{(S,I,Q,R)\in R_+^4} \Big\{ &-\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + R^{\theta+1}) + \frac{\check{\beta}_1 I}{N}(MC_1+1) + 3\check{\mu} \\ &+ \check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} + D - \frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)I^{\theta+1}\Big\}. \end{aligned}$$

By virtue of (12), we have $LV \leq -1$ for all $(S,I,Q,R) \in U_6$.

Case 7. If $(S,I,Q,R) \in U_7$, we can deduce that

$$\begin{aligned} LV(S,I,Q,R) &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)Q^{\theta+1} + \frac{\check{\beta}_1 I}{N}(MC_1+1) + 3\check{\mu} \\ &\quad - \frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1}) + \check{\alpha} + \check{\epsilon} + D \\ &\quad - \frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)Q^{\theta+1} + \frac{\check{\sigma_1}^2}{2} + \frac{\check{\sigma_3}^2}{2} + \frac{\check{\sigma_4}^2}{2} \\ &\leq -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + H, \end{aligned} \tag{21}$$

where

$$H = \sup_{(S,I,Q,R)\in R_+^4} \Big\{ -\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1}) + \frac{\check{\beta}_1 I}{N}(MC_1 + 1) + 3\check{\mu}$$
$$+\check{\alpha} + \check{\epsilon} + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2} + D - \frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)Q^{\theta+1}\Big\}.$$

It follows from (13), we have $LV \leq -1$ for all $(S, I, Q, R) \in U_7$.

Case 8. If $(S, I, Q, R) \in U_8$, one can obtain that

$$\begin{aligned} LV(S,I,Q,R) \leq\ & -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)R^{\theta+1} + \frac{\check{\beta}_1 I}{N}(MC_1 + 1) + 3\check{\mu} \\ & -\frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1}) + \check{\alpha} + \check{\epsilon} + D \\ & -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)R^{\theta+1} + \frac{\check{\sigma_1}^2}{2} + \frac{\check{\sigma_3}^2}{2} + \frac{\check{\sigma_4}^2}{2} \\ \leq\ & -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)R^{\theta+1} + J \\ \leq\ & -\frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + J, \end{aligned} \tag{22}$$

where

$$J = \sup_{(S,I,Q,R)\in R_+^4} \Big\{\frac{\check{\beta}_1 I}{N}(MC_1 + 1) + 3\check{\mu} - \frac{1}{2}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)(S^{\theta+1} + I^{\theta+1})$$
$$+\check{\alpha} + \check{\epsilon} + D - \frac{1}{4}\Big(\hat{\mu} - \frac{1}{2}\theta(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2)\Big)R^{\theta+1} + \frac{\check{\sigma_1}^2}{2} + \frac{\check{\sigma_3}^2}{2} + \frac{\check{\sigma_4}^2}{2}\Big\}.$$

By (14), we can conclude that $LV \leq -1$ for all $(S, I, Q, R) \in U_8$.

Therefore, we have proof that for a sufficiently small $\varepsilon > 0$,

$$LV(S, I, Q, R) \leq -1, (S, I, Q, R) \in R_+^4 \setminus U.$$

Hence, (c) in Lemma 1 holds. This completes the proof of Theorem 2. □

5 | EXTINCTION OF MODEL (4)

In this section, we investigate the conditions for the extinction of model (4).

Define a parameter

$$\mathfrak{R}_2 = \frac{\check{\Lambda}\check{\beta}_1}{N\hat{\mu}\sum_{k=1}^m \pi_k(\mu(k) + \alpha(k) + \delta(k) + \gamma(k) + \frac{\sigma_2^2(k)}{2})}.$$

Theorem 3. Let$(S(t), I(t), Q(t), R(t), r(t))$ be a solution of model (4) with initial value $(S(0), I(0), Q(0), R(0), r(0)) \in R_+^4 \times \mathcal{M}$. If $\mathfrak{R}_2 < 1$, then the disease I goes to extinction exponentially with probability one, i.e.,

$$\lim_{t\to\infty} I(t) = 0 \quad a.s.$$

and also

$$\lim_{t\to\infty} \langle S\rangle_t \leq \frac{\check{\Lambda}}{\hat{\mu}}, \quad \lim_{t\to\infty} Q(t) = \lim_{t\to\infty} R(t) = 0. \quad a.s.$$

Proof. From model (4), we have

$$\frac{S(t) - S(0)}{t} = \langle\Lambda\rangle_t - \langle(\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{N}\rangle_t - \langle\mu S\rangle_t + \frac{\int_0^t \sigma_1(s)S(s)dB_1(s)}{t},$$

and

$$\frac{I(t) - I(0)}{t} = \langle(\beta_1 - \frac{\beta_2 I}{m + I})\frac{SI}{N}\rangle_t - \langle(\mu + \alpha + \delta + \gamma)I\rangle_t + \frac{\int_0^t \sigma_2(s)I(s)dB_2(s)}{t}.$$

Then

$$\begin{aligned}\frac{I(t)-I(0)}{t}+\frac{S(t)-S(0)}{t} &= \langle\Lambda\rangle_t-\langle\mu S\rangle_t-\langle(\mu+\alpha+\delta+\gamma)I\rangle_t+\frac{\int_0^t\sigma_1(s)S(s)dB_1(s)}{t}\\ &\quad+\frac{\int_0^t\sigma_2(s)I(s)dB_2(s)}{t}\\ &\le \check{\Lambda}-\hat{\mu}\langle S\rangle_t-(\hat{\mu}+\hat{\alpha}+\hat{\delta}+\hat{\gamma})\langle I\rangle_t+\frac{\check{\sigma_1}\int_0^t S(s)dB_1(s)}{t}\\ &\quad+\frac{\check{\sigma_2}\int_0^t I(s)dB_2(s)}{t}.\end{aligned}$$

It is easy to obtain

$$\langle S\rangle_t \le \frac{\check{\Lambda}}{\hat{\mu}}-\frac{\hat{\mu}+\hat{\alpha}+\hat{\delta}+\hat{\gamma}}{\hat{\mu}}\langle I\rangle_t+H(t), \tag{23}$$

where

$$H(t)=\frac{\frac{\check{\sigma_1}\int_0^t S(s)dB_1(s)}{t}}{\hat{\mu}}+\frac{\frac{\check{\sigma_2}\int_0^t I(s)dB_2(s)}{t}}{\hat{\mu}}-\frac{\frac{I(t)-I(0)}{t}+\frac{S(t)-S(0)}{t}}{\hat{\mu}}.$$

According to Lemma 2, we have

$$\lim_{t\to\infty}H(t)=0 \quad a.s. \tag{24}$$

By the Itô's formula, we obtain

$$\begin{aligned}d\ln I(t) &= \Big\{\frac{1}{I}\Big[(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I(t)\Big]\\ &\quad-\frac{\sigma_2^2(t)}{2}\Big\}dt+\sigma_2(t)dB_2(t)\\ &\le \Big(\frac{\beta_1(t)S}{N}-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))-\frac{\sigma_2^2(t)}{2}\Big)dt+\sigma_2(t)dB_2(t).\end{aligned} \tag{25}$$

Integrating (25) from 0 to t and dividing t on both sides, we get

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t} &\le \frac{\langle\beta_1(t)S\rangle_t}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}\\ &\le \frac{\check{\beta}_1\langle S\rangle_t}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}.\end{aligned}$$

Together with (23), we have

$$\begin{aligned}\frac{\ln I(t)}{t} &\le \frac{\check{\beta}_1}{N}\Big[\frac{\check{\Lambda}}{\hat{\mu}}-\frac{\hat{\mu}+\hat{\alpha}+\hat{\delta}+\hat{\gamma}}{\hat{\mu}}\langle I\rangle_t+H(t)\Big]-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t\\ &\quad+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}+\frac{\ln I(0)}{t}\\ &\le \frac{\check{\beta}_1\check{\Lambda}}{N\hat{\mu}}+\frac{\check{\beta}_1H(t)}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}+\frac{\ln I(0)}{t}.\end{aligned} \tag{26}$$

As ergodic properties of $\xi(t)$, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^1\mu(r(s))ds=\sum_{k=1}^N\pi_k\mu(k).$$

Taking the limit superior of both of (26) and using Lemma 2, which together with (24), we can obtain

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\ln I(t)}{t} &\leq \frac{\check{\beta}_1\check{\Lambda}}{N\hat{\mu}}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}\\ &=\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}\Big(\frac{\check{\beta}_1\check{\Lambda}}{N\hat{\mu}\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}-1\Big)\\ &=\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}(\mathfrak{R}_2-1)\\ &<0,\end{aligned}$$

which implies $\lim_{t\to\infty} I(t)=0$.

From (23), it is easy to get that

$$\lim_{t\to\infty}\langle S\rangle_t\leq\frac{\check{\Lambda}}{\hat{\mu}}.$$

From the third and fourth equations of model (4), it is easy to obtain that

$$\lim_{t\to\infty}Q(t)=0,\ \lim_{t\to\infty}R(t)=0.$$

This completes the proof. □

6 | NUMERICAL SIMULATIONS

In this section, we give two examples to support the theoretical prediction.

Example 6.1. In model (4), let $r(t)$ is a right-continuous Markov chain taking values $k=1,2$ and the generator Γ of the Markov chain is

$$\begin{pmatrix}-0.06 & 0.06\\ 0.04 & -0.04\end{pmatrix}.$$

By solving the linear equation (5), we obtain the unique stationary distribution

$$\pi=(\pi_1,\pi_2)=(0.4,0.6).$$

Choose the parameters values in model (4) as follows

$$\Lambda(1)=0.8,\beta_1(1)=0.6,\beta_2(1)=0.2,m(1)=1,N=1,\mu(1)=0.2,\alpha(1)=0.25,$$

$$\delta(1)=0.2,\gamma(1)=0.15,\varepsilon(1)=0.2,\sigma_1(1)=\sigma_2(1)=\sigma_3(1)=\sigma_4(1)=0.05.$$

and

$$\Lambda(2)=1,\beta_1(2)=0.7,\beta_2(2)=0.1,m(1)=1,N=1,\mu(2)=0.3,\alpha(2)=0.3,$$

$$\delta(2)=0.3,\gamma(2)=0.25,\varepsilon(2)=0.3,\sigma_1(2)=\sigma_2(2)=\sigma_3(2)=\sigma_4(2)=0.1.$$

Note that $\mathfrak{R}_1\approx 1.828>1$ holds, that is to say, the condition of Theorem 2 holds. Therefore, the stochastic model (4) has an ergodic stationary distribution. That means that the stochastic SIQR model (4) with regime switching has a unique stationary distribution and it has the ergodic property (see Figure 1).

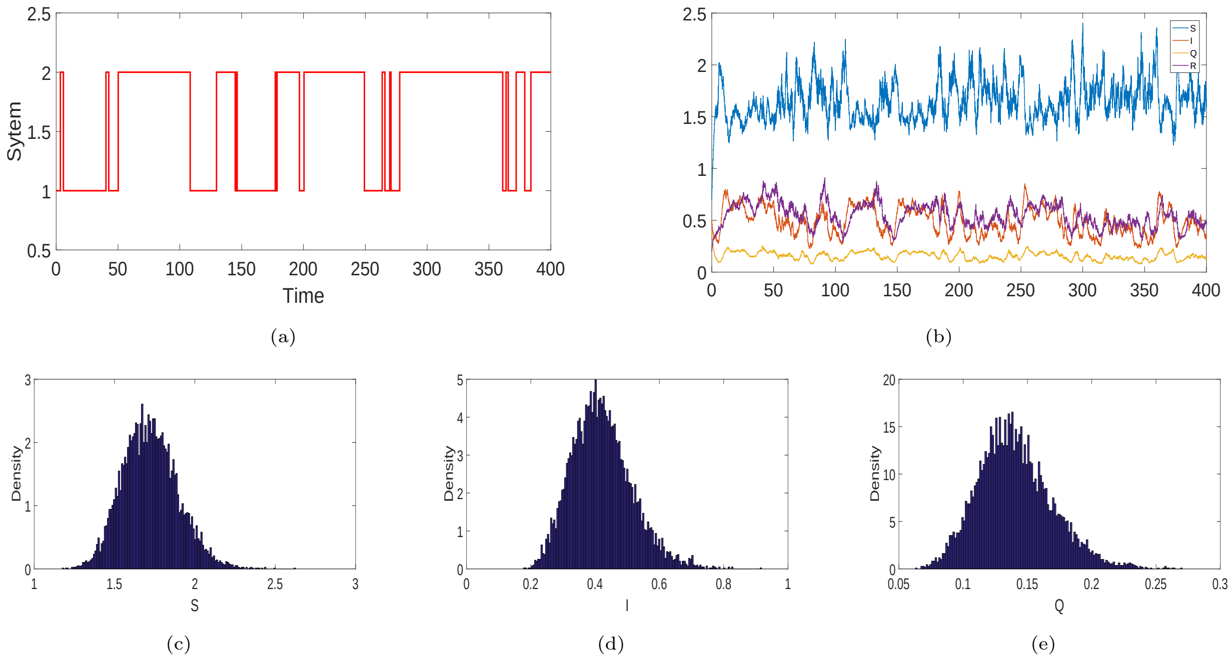

FIGURE 1 (a) is Markov chain, (b) is a stationary distribution of the stochastic model (4), (c) is the probability function of $S(t)$, (d) is the probability function of $I(t)$, (e) the probability function of $Q(t)$. The initial value is $(S(0), I(0), Q(0), R(0)) = (0.7, 0.5, 0.4, 0.2)$

Example 6.2. In model (4), let $r(t)$ is a right-continuous Markov chain taking values $k = 1, 2$ and the generator Γ of the Markov chain is

$$\begin{pmatrix} -0.06 & 0.06 \\ 0.04 & -0.04 \end{pmatrix}.$$

By solving the linear equation (5), we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2) = (0.4, 0.6).$$

Choose the parameters values in model (4) as follows

$$\Lambda(1) = 0.8, \beta_1(1) = 0.25, \beta_2(1) = 0.15, m(1) = 1, N = 1, \mu(1) = 0.25, \alpha(1) = 0.3,$$

$$\delta(1) = 0.3, \gamma(1) = 0.2, \varepsilon(1) = 0.2, \sigma_1(1) = \sigma_2(1) = \sigma_3(1) = \sigma_4(1) = 0.15.$$

and

$$\Lambda(2) = 1, \beta_1(2) = 0.3, \beta_2(2) = 0.2, m(1) = 1, N = 1, \mu(2) = 0.3, \alpha(2) = 0.35,$$

$$\delta(2) = 0.4, \gamma(2) = 0.4, \varepsilon(2) = 0.4, \sigma_1(2) = \sigma_2(2) = \sigma_3(2) = \sigma_4(2) = 0.2.$$

Note that $\mathfrak{R}_2 \approx 0.916 < 1$ holds, the condition of Theorem 3 is satisfied. Therefore, we obtain that the disease $I(t)$ will tend to zero exponentially with probability one (see Figure 2).

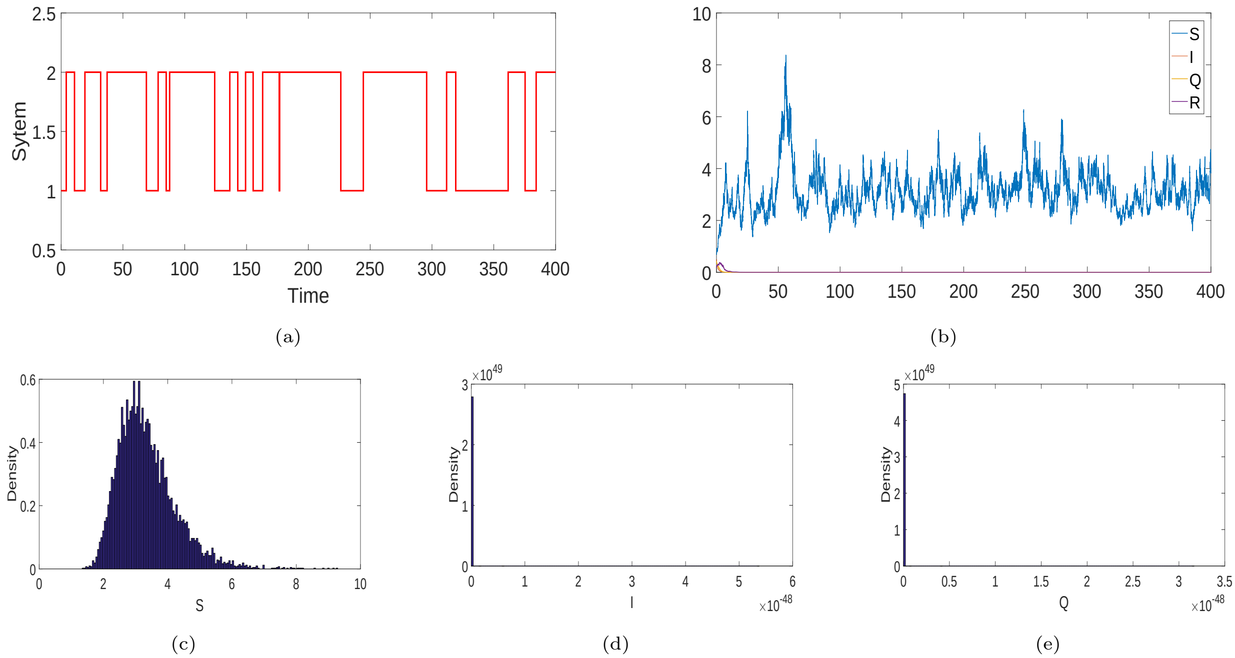

FIGURE 2 (a) is Markov chain, (b) is a stationary distribution of the stochastic model (4), (c) is the probability function of $S(t)$,(d) is the probability function of $I(t)$, (e) the probability function of $Q(t)$. The initial value is $(S(0), I(0), Q(0), R(0)) = (0.7, 0.5, 0.4, 0.2)$

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□