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RESEARCH ARTICLE

Study of convergence of reduced differential transform method for different classes of nonlinear differential equations†

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ABSTRACT

In this work, we study the sufficient condition for convergence of the reduced differential transform method for non-linear differential equations. The main power of this method is its ability and flexibility in solving non-linear problems properly and easily and obtain solutions both numerically and analytically. Simple approaches of reduced differential transform method and the convergence results for different classes of differential equations such as linear and non-linear ordinary, partial, fractional, and system of differential equations are briefly discussed. Eight examples are checked to confirm convergence results as well as the strength and efficiency of the method.

KEYWORDS:
Reduced differential transform method, Series solution, Convergence, Error estimate

1 | INTRODUCTION

The Reduced differential transform method (RDTM) is an analytical-numerical technique introduced for the first time by Keskin[17,18] to study the analytical solutions of linear and non-linear wave equations[19,20]. This suggested technique is highly efficient and powerful in obtaining the exact solutions as well as approximate solutions of mathematical modeling of many problems in technology, finance, engineering disciplines, natural sciences such as biology, physics, chemistry, and earth science, gives the solution in the form of rapidly convergent successive approximations and is capable of handling linear and non-linear equations in a similar manner. In recent years, the Reduced differential transform method has been widely adopted by many researchers such as in[22,26,31,6,2] and by the references therein. Also, it was shown by many authors[16,1,28,3,15] that the solution procedure of the RDTM is simpler and more straightforward than such as the homotopy perturbation method (HPM), differential transform method (DTM), variational iteration method (VIM), Adomian decomposition method (ADM), etc. On the other hand, the size of computational work has been reduced while still maintaining high precision from the numerical solution and rapid convergence has been guaranteed. The advantage of this method is its simplicity in using, it solves the equations straightforward and directly without using Adomian's polynomial, perturbation, linearization or any other transformation and restrictive conditions, gives the solution as convergent power series with simply determinable components. Therefore, the RDTM can overcome the foregoing limitations and restrictions of perturbation techniques, complicated computational, so that it provides us with a possibility to analyse accurately non-linear equations. The RDTM was successfully applied to ordinary differential equations[10], partial differential equations[30,32,33], fractional differential equations[4,21,24,27,29], Volterra integral equation[9,23,37] and integro-differential equations[8,34].

In the present work, in light of the above-mentioned method, we will study non-linear problems. In addition, our main objective is to study the sufficient condition for convergence of the method for non-linear equations. The main ideas explained in this

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paper are expected to be used for more non-linear models.
The rest of this study is presented in the following sections: In Section 2, we simply introduce the reduced differential transform method (RDTM). In Section 3, we prove the convergences of the considered method. In Section 4, RDTM approaches and convergence results are addressed. In Section 5, we apply this method to obtain the exact solutions for linear and non-linear ordinary, partial, fractional and system of differential equations. Finally, we offer some summaries and conclusions in section 6.

2 | SUMMARY OF THE METHOD

We present some important definitions and mathematical preliminaries operations of the reduced differential transform method in which can help to more understand of the stated method in this section. Now, consider the function of two variables $w(x,t)$ and assume that it can be expressed as the product of two different variable functions, i.e., $w(x,t) = \phi(x)\psi(t)$. The function $w(x,t)$ can be displayed due to the properties of differential transform as follows:

$$w(x,t) = \left(\sum_{i=0}^{\infty} \Phi(i)x^i\right)\left(\sum_{j=0}^{\infty} \Psi(j)t^j\right) = \sum_{k=0}^{\infty} W_k(x)t^k, \tag{1}$$

where $W_k(x)$ is the t-dimensional spectrum function of the original function $w(x,t)$.

Definition 1. The reduced differential transform function of $w(x,t)$ can be yield in the following form

$$W_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0}, \tag{2}$$

where $w(x,t)$ is analytic and differentiated continuously function with regard to space x and time t, in the domain of interest.

Definition 2. The reduced differential inverse transform of $W_k(x)$ is determined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^k. \tag{3}$$

Afterward, consolidating Eqs. (2) and (3) yields

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0} (t-t_0)^k. \tag{4}$$

By the help of the upper definitions, to illustrate the basic idea of the RDTM, consider the following operator form of non-linear partial differential equations written as

$$\mathcal{L}w(x,t) + \mathcal{R}w(x,t) + \mathcal{N}w(x,t) = h(x,t), \tag{5}$$

with considering the following initial condition

$$w(x,0) = g(x), \tag{6}$$

where $\mathcal{L} = \frac{\partial}{\partial t}$, $\mathcal{R}$ and $\mathcal{N}$ indicate linear and non-linear operators which has partial derivatives and h is a non-homogeneous term. After applying the RDTM definition on both sides of Eq. (5), we can write the following iteration formula

$$(k+1)W_{k+1}(x) = H_k(x) - \mathcal{R}W_k(x) - \mathcal{N}W_k(x), \tag{7}$$

where $\mathcal{L}W_k(x)$, $\mathcal{R}W_k(x)$, $\mathcal{N}W_k(x)$ and $H_k(x)$ are the reduced differential transform functions of $\mathcal{L}w(x,t)$, $\mathcal{R}w(x,t)$, $\mathcal{N}w(x,t)$ and $h(x,t)$ respectively.
Implementing the aforesaid method to the initial condition (6), we have

$$W_0(x) = g(x). \tag{8}$$

To discover the remaining iteration, we plugging Eq. (8) into Eq. (7) and by simple reiterative calculation, we tend to get the subsequent $W_k(x)$ values. Afterwards, the inverse transformation of the set of values $\{W_k(x)\}_{k=0}^{n}$ admits the n-terms approximation solution as bellows:

$$\tilde{w}_n(x,t) = \sum_{k=0}^{n} W_k(x)(t-t_0)^k. \tag{9}$$

Thus, the exact solution of the considered problem can be gained by

$$w(x,t) = \lim_{n\to\infty} \tilde{w}_n(x,t). \tag{10}$$

Table I contains the basic mathematical operations carried out by RDTM.

3 | CONVERGENCE OF METHOD

The principal main of this section is to survey the sufficient conditions for convergence of the reduced differential transform method, according to the approach described by this method for solving the non-linear Eq. (5) in the previous section. For this purpose, some theorems for convergence of the method, and the error computation is addressed.
The fundamental point views of RDTM for the solutions of non-linear models include ascertaining power series expansion with the initial time t_0,

$$w(x,t) = \sum_{k=0}^{\infty} a_k(x)(t-t_0)^k, \qquad t \in l, \tag{11}$$

where $l = (t_0, t_0 + r)$, $r > 0$. The important results are proposed in the below theorems.

Theorem 1. If $\varphi_k(x,t) = a_k(x)(t-t_0)^k$, then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, stated in Eq (11), $\forall k \in \mathbf{N} \cup \{0\}$

(I) It is convergent, if $\exists\, 0 < \lambda < 1$ such that $\|\varphi_{k+1}\| \le \lambda \|\varphi_k\|$,

(II) It is divergent, if $\exists\, \lambda > 1$ such that $\|\varphi_{k+1}\| \ge \lambda \|\varphi_k\|$.

Theorem 1 is a specific case of Banach's fixed point theorem. Using this theorem and a brief description of its proof, we investigate the truncation error of the series solution Eq. (11), as follows

Proof. Let $(C[l], \|.\|)$ be the Banach space of all continuous functions on l with the norm $\|\varphi_k(x,t)\| = \|a_k(x)(t-t_0)^k\|$. Also assume that $\|a_0(x)\| < N_0$, where N_0 is a positive number. Define the sequence of partial sums $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ as

$$\mathfrak{S}_n = \varphi_0 + \varphi_1 + \cdots + \varphi_n. \tag{12}$$

We want to show that $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To reach this goal, we take

$$\|\mathfrak{S}_{n+1} - \mathfrak{S}_n\| = \|\varphi_{n+1}\| \le \lambda \|\varphi_n\| \le \cdots \le \lambda^{n+1}\|\varphi_0\| \le \lambda^{n+1} N_0. \tag{13}$$

Therefore, for any $n, m \in \mathbb{N}$, $n \ge m$, making use of (13) and the triangle inequality successively, we have

$$\begin{aligned} \|\mathfrak{S}_n - \mathfrak{S}_m\| &= \|(\mathfrak{S}_n - \mathfrak{S}_{n-1}) + (\mathfrak{S}_{n-1} - \mathfrak{S}_{n-2}) + \cdots + (\mathfrak{S}_{m+1} - \mathfrak{S}_m)\| \\ &\le \|(\mathfrak{S}_n - \mathfrak{S}_{n-1})\| + \|(\mathfrak{S}_{n-1} - \mathfrak{S}_{n-2})\| + \cdots + \|(\mathfrak{S}_{m+1} - \mathfrak{S}_m)\| \\ &\le \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} \|\varphi_0\|, \end{aligned} \tag{14}$$

and because $0 < \lambda < 1$, we obtain

$$\lim_{n,m\to\infty} \|\mathfrak{S}_n - \mathfrak{S}_m\| = 0. \tag{15}$$

Hence, $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[l], \|.\|)$. Thus the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, defined in Eq. (11), is convergent and it completes the proof.

□

Remark 1. According to the assumptions in (II) and by using the ratio test, we have

$$\|\frac{\varphi_{k+1}}{\varphi_k}\| \ge \lambda > 1.$$

As a result, the series is divergent.

Remark 2. If the series $\sum_{k=0}^{\infty} a_k(x)(t-t_0)^k$ of the non-linear Eq. (5) converges then it is an exact solution.

Theorem 2. Suppose that the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, where $\varphi_k(x,t) = a_k(x)(t-t_0)^k$, converges to the solution $w(x,t)$. If the truncated series $\sum_{k=0}^{m} \varphi_k(x,t)$ is used as an approximation to the solution $u(x,t)$ and then the maximum absolute truncated error is estimated as

$$\left\| w(x,t) - \sum_{k=0}^{m} \varphi_k(x,t) \right\| \leq \frac{1}{1-\lambda} \lambda^{m+1} \|\varphi_0\|. \tag{16}$$

Proof. According to Theorem 1, we have the inequality Eq. (14) as follows:

$$\|\mathfrak{S}_n - \mathfrak{S}_m\| \leq \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} \|\varphi_0\|, \tag{17}$$

for $n \geq m$. Also, since $0 < \lambda < 1$, in the numerator, we have $1 - \lambda^{n-m} < 1$, therefore, the inequality Eq. (17) can be reduced to

$$\|\mathfrak{S}_n - \mathfrak{S}_m\| \leq \frac{1}{1-\lambda} \lambda^{m+1} \|\varphi_0\|. \tag{18}$$

It is clear when $n \to \infty$, $\mathfrak{S}_n \to w(x,t)$. Thus, inequality Eq. (16) is obtained and the Theorem is proved.

□

In summary, Theorems 1 and 2 state that the reduced differential transform solution of non-linear Eq. 5), obtained using the iteration formula (7) and (8), converges to an exact solution under the condition that $\exists 0 < \lambda < 1$ such that $\|\varphi_{k+1}\| \leq \lambda \|\varphi_k\|$, $\forall k \in \mathbb{N} \cup \{0\}$. In other words, if we define, for every $i \in \mathbb{N} \cup \{0\}$, the parameters,

$$\gamma_i = \begin{cases} \frac{||\varphi_{i+1}||}{||\varphi_i||}, & ||\varphi_i|| \neq 0, \\ 0, & ||\varphi_i|| = 0, \end{cases} \tag{19}$$

then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$ of the Eq. (5) converges to an exact solution $w(x,t)$, when $0 \leq \gamma_i < 1, \forall i \in \mathbb{N} \cup \{0\}$. In addition, the maximum absolute truncation error, as discussed in Theorem 2, is estimated to be

$$\left\| w(x,t) - \sum_{k=0}^{m} \varphi_k(x,t) \right\| \leq \frac{1}{\gamma - 1} \gamma^{j+1} \|\varphi_0\|,$$

where $\gamma = max\{\gamma_i,\ i = 0, 1, \cdots, j\}$.

Remark 3. The first finite terms have no effect the convergence of the series solution. In other words, if the first finite γ_i's, $i = 0, 1, \cdots, l$, are not less than one and $\gamma_i \leq 1$ for $i > l$, then, the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$ of the Eq. (5) converges to an exact solution. Because according to Theorem 1, we have

$$\|\mathfrak{S}_n - \mathfrak{S}_j\| \leq \frac{1-\lambda^{n-j}}{1-\lambda} \lambda^{j-l} \|\varphi_{l+1}\|, \tag{20}$$

and since $0 < \lambda < 1$, for $n \geq j$ and fixed l, we get $\lim_{n,j\to\infty} \|\mathfrak{S}_n - \mathfrak{S}_j\| = 0$. In this case, the convergence of the RDTM approach depends on γ_i, for $i > l$.

4 | RDTM APPROACHES AND CONVERGENCE RESULTS

The principal main of this section is to summarize RDTM inclusive convergence results of the method for solving different classes of differential equations.

4.1 | Ordinary differential equations (ODE)

Let us write the non-linear ordinary differential equation as the following

$$\frac{d^r}{dt^r} w(t) + \mathcal{R} w(t) + \mathcal{N} w(t) = h(t), \qquad t > 0, \tag{21}$$

where $r \in \mathbb{N}$, $\mathcal{R}$ is a linear operator, $\mathcal{N}$ is a non-linear operator and h is an non-homogeneous term, with the initial conditions $w^{(k)}(0) = c_k$, $k = 0, 1, \cdots, r-1$. According to the operations of differential transformation, the series solution $w(t) = \sum_{k=0}^{\infty} \varphi_k(t)$ is obtained using the following iteration formula

$$\frac{(k+r)!}{k!} W(k+r) + \mathcal{R} W(k) + \mathcal{N} W(k) = H(k), \tag{22}$$

converges to a solution of Eq. (21) if $\forall k \in \mathbb{N} \cup \{0\}$, $\exists 0 < \lambda < 1$ such that $\|\varphi_{k+1}\| \leq \lambda \|\varphi_k\|$.

4.2 | Partial differential equations (PDE)

Let us write the non-linear partial differential equation as the following

$$\frac{\partial^r}{\partial t^r} w(x,t) + \mathcal{R}w(x,t) + \mathcal{N}w(x,t) = h(x,t), \qquad t > 0, \tag{23}$$

where $r \in \mathbb{N}$, $\mathcal{R}$ is a linear operator, $\mathcal{N}$ is a non-linear operator and h is an non-homogeneous term, with the initial conditions $w^{(k)}(x,0) = g_k(x)$, $k = 0, 1, \cdots, r-1$. According to the operations of differential transformation, the series solution $w(x,t) = \sum_{k=0}^{\infty} \varphi_k(x,t)$ is obtained using the following iteration formula

$$\frac{(k+r)!}{k!} W_{k+r}(x) + \mathcal{R}W_k(x) + \mathcal{N}W_k(x) = H_k(x), \tag{24}$$

converges to a solution of Eq. (23) if $\forall k \in \mathbb{N} \cup \{0\}$, $\exists 0 < \lambda < 1$ such that $\|\varphi_{k+1}\| \leq \lambda \|\varphi_k\|$.

4.3 | Fractional partial differential equations (FPDE)

Let us write the non-linear fractional partial differential equation as the following

$$\frac{\partial^{\alpha r}}{\partial t^{\alpha r}} w(x,t) + \mathcal{R}w(x,t) + \mathcal{N}w(x,t) = h(x,t), \qquad t > 0, \quad m-1 < \alpha \leq m, \tag{25}$$

where $m \in \mathbb{N}$, $\frac{\partial^{\alpha r}}{\partial t^{\alpha r}}$ is the Caputo fractional derivative of order αr, $\mathcal{R}$ is a linear operator, $\mathcal{N}$ is a non-linear operator and h is an non-homogeneous term, with the initial conditions $w^{(k)}(x,0) = g_k(x)$, $k = 0, 1, \cdots, m-1$.

Definition 3. The Caputo fractional derivative operator is defined as

$$\mathcal{I}_a^{\alpha} w(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\xi)^{m-\alpha-1} w^{(m)}(\xi) \mathrm{d}\xi, \tag{26}$$

where $\alpha > 0$ and a are the order of the derivative and the initial value of function w respectively.

Properties of Caputo fractional derivative operator can be found in [11,12,13].
Also, to determine the result, we introduce the subsequent Riemann-Liouville fractional integral operator.

Definition 4. The Riemann-Liouville fractional integral operator is defined as

$$\mathcal{J}_a^{\alpha} w(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} w(\xi) \mathrm{d}\xi, \qquad t > 0, \quad \alpha > 0. \tag{27}$$

Properties of Riemann-Liouville fractional integral operator can be found in [5,25].

Definition 5. Let function $w(x,t)$ is analytic and differentiated continuously with respect to space x and time t in the domain of interest, the fractional reduced differential transform function (FRDTM) is

$$W_k^{\alpha}(x) = \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} w(x,t) \right]_{t=t_0}, \tag{28}$$

where $0 < \alpha \leq 1$, the t-dimensional spectrum function $W_k^{\alpha}(x)$ is the transformed function, and Γ is gamma function is defined as

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} \mathrm{d}x. \tag{29}$$

Definition 6. The fractional reduced differential inverse transform of $W_k^{\alpha}(x)$ is determined as follows:

$$w(x,t) = \sum_{k=0}^{\infty} W_k^{\alpha}(x)(t-t_0)^{\alpha k}. \tag{30}$$

Afterward, combining Eqs. (28) and (30) we write

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} w(x,t) \right]_{t=t_0} (t-t_0)^{\alpha k}, \tag{31}$$

which in practical application can be approximated by a finite series

$$\tilde{w}_n(x,t) = \sum_{k=0}^{n} W_k^{\alpha}(x)(t-t_0)^{\alpha k}. \tag{32}$$

Thus, the exact solution to the problem can be obtained by

$$w(x,t) = \lim_{n\to\infty} \tilde{w}_n(x,t). \tag{33}$$

In case of $\alpha = 1$, FRDTM is reduced to classical RDTM. The basic properties of mathematical operations performed by FRDTM can be found in[5]. According to RDTM and Caputo differential operator, then, following the same analysis presented in the previous section, the series solution $w(x,t) = \sum_{k=0}^{\infty} \varphi_k(x,t)$ is obtained using the following iteration formula

$$\frac{\Gamma(\alpha(k+r)+1)}{\Gamma(\alpha k+1)} W_{k+r}^{\alpha}(x) + \mathcal{R}W_k(x) + \mathcal{N}W_k(x) = H_k(x), \tag{34}$$

converges to a solution of Eq. (25) if $\forall k \in \mathbb{N}\cup\{0\}$, $\exists 0<\lambda<1$ such that $\|\varphi_{k+1}\| \le \lambda\|\varphi_k\|$.

4.4 | Systems of fractional partial differential equations

Let us write the following system of non-linear fractional partial differential equations,

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} w_1(x,t) + \mathcal{R}_1(w_1,w_2,\cdots,w_n) + \mathcal{N}_1(w_1,w_2,\cdots,w_n) = h_1(x,t), \\ \frac{\partial^\alpha}{\partial t^\alpha} w_2(x,t) + \mathcal{R}_2(w_1,w_2,\cdots,w_n) + \mathcal{N}_2(w_1,w_2,\cdots,w_n) = h_2(x,t), \\ \vdots \\ \frac{\partial^\alpha}{\partial t^\alpha} w_n(x,t) + \mathcal{R}_n(w_1,w_2,\cdots,w_n) + \mathcal{N}_n(w_1,w_2,\cdots,w_n) = h_n(x,t), \end{cases} \tag{35}$$

where $m-1<\alpha\le m$, $m,n\in\mathbb{N}$, $\mathcal{R}_1,\mathcal{R}_2,\cdots,\mathcal{R}_n$ are linear operators, $\mathcal{N}_1,\mathcal{N}_2,\cdots,\mathcal{N}_n$ are non-linear operators and $h_1,h_2,\cdots,h_n$ are the non-homogeneous terms, with the initial conditions

$$\begin{cases} w_1^{(k)}(x,0) = g_{1,k}(x), \\ w_2^{(k)}(x,0) = g_{2,k}(x), \\ \vdots \\ w_n^{(k)}(x,0) = g_{n,k}(x), \end{cases} \tag{36}$$

for $k=0,1,\cdots,m-1$. Then, for $i=1,2,\cdots,n$ the series solution $w_i(x,t) = \sum_{k=0}^{\infty} \varphi_{i,k}(x,t)$ obtained using the iteration formula,

$$\frac{\Gamma(\alpha(k+r)+1)}{\Gamma(\alpha k+1)} W_{i,k+1}^{\alpha}(x) + \mathcal{R}W_{i,k}(x) + \mathcal{N}W_{i,k}(x) = H_{i,k}(x), \tag{37}$$

converges to a solution of Eq. (35) if $\forall k \in \mathbb{N}\cup\{0\}$, $\exists 0<\lambda_i<1$ such that $\|\varphi_{i,k+1}\| \le \lambda_i\|\varphi_{i,k}\|$.

5 | APPLICATIONS

The principal main of this section is to apply the reduced differential transform method on the following examples to illustrate the accuracy of the presented method.

Example 1. We first consider the following linear ordinary differential equation

$$w''(t) + w(t) = 0, \qquad 0 < t \le 1, \tag{38}$$

subject to the initial conditions

$$w(0) = 0, \qquad w'(0) = 1. \tag{39}$$

According to the operations of differential transformation given in Table I for Eq. (38) we obtain the following recurrent relation

$$(k+1)(k+2)W(k+2) + W(k) = 0, \tag{40}$$

and from initial conditions (39), we write

$$W(0) = 0, \qquad W(1) = 1. \tag{41}$$

Substituting the above equations in Eq. (40), we drive the following results

$$W(2) = 0, \quad W(3) = -\frac{1}{3!}, \quad W(4) = 0, \quad W(5) = \frac{1}{5!}, \qquad \cdots .$$

Hence, the solution in series form is as follow

$$\tilde{w}_n(t) = \sum_{k=0}^{\infty} W(k)t^k = W(0) + W(1)t + W(2)t^2 + \cdots = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots ,$$

which converges efficiently to the exact solution $w(t) = \sin t$.

Example 2. As the second example, we consider the following non-linear ordinary differential equation

$$w'(t) = w^2(t) + 1, \qquad 0 < t \le 1, \tag{42}$$

subject to the initial condition

$$w(0) = 0. \tag{43}$$

According to the operations of differential transformation given in Table I for Eq. (42) we obtain the following recurrent relation

$$(k+1)W(k+1) = \sum_{r=0}^{k} W(r)W(k-r) + \delta(k), \tag{44}$$

and from initial condition (43), we write

$$W(0) = 0. \tag{45}$$

Substituting the above equation in Eq. (44), we drive the following results

$$W(1) = 1, \quad W(2) = 0, \quad W(3) = \frac{1}{3}, \quad W(4) = 0, \quad W(5) = \frac{2}{15}, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\tilde{w}_n(t) = \sum_{k=0}^{\infty} W(k)t^k = W(0) + W(1)t + W(2)t^2 + \cdots = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \cdots ,$$

which converges efficiently to the exact solution $w(t) = \tan t$.

Example 3. As the third example, we consider the following Burger's equation[36]

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2}, \qquad 0 \le t < \frac{1}{2}, \tag{46}$$

subject to the initial condition

$$w(x,0) = 2x. \tag{47}$$

According to the operations of differential transformation given in Table I for Eq. (46) we obtain the following recurrent relation

$$(k+1)W_{k+1}(x) + \sum_{r=0}^{k} W_r(x)\frac{\partial}{\partial x}W_{k-r}(x) = \frac{\partial^2}{\partial x^2}W_k(x), \tag{48}$$

and fromrom initial condition (47), we write

$$W_0(x) = 2x. \tag{49}$$

Substituting the above equation in Eq. (48), we drive the following results

$$W_1(x) = -4x, \quad W_2(x) = 8x, \quad W_3(x) = -16x, \quad W_4(x) = 32x, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\tilde{w}_n(x,t) = \sum_{k=0}^{\infty} W_k(x)t^k = W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots = 2x\Big(1 - 2t + (2t)^2 - \cdots\Big),$$

which converges efficiently to the exact solution $w(x,t) = \frac{2x}{1+2t}$.

Example 4. As the fourth example, we consider the following non-linear Klein-Gordon equation [14]

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + w^2 = -x\cos t + x^2\cos^2 t, \tag{50}$$

subject to the initial conditions

$$w(x,0) = x, \qquad \frac{\partial}{\partial t}w(x,0) = 0. \tag{51}$$

According to the operations of differential transformation given in Table I for Eq. (50) we obtain the following recurrent relation

$$\begin{aligned}(k+1)(k+2)W_{k+2}(x) - \frac{\partial^2}{\partial x^2}W_k(x) + \sum_{r=0}^{k} W_r(x)W_{k-r}(x) = -x\frac{\cos(\frac{k\pi}{2})}{k!}\\ + \frac{1}{2}x^2\Big(\delta(k) + \frac{2^k\cos(\frac{k\pi}{2})}{k!}\Big),\end{aligned} \tag{52}$$

and from initial conditions (51), we write

$$W_0(x) = x, \qquad W_1(x) = 0. \tag{53}$$

Substituting the above equation in Eq. (52), we drive the following results

$$W_2(x) = -\frac{x}{2!}, \quad W_3(x) = 0, \quad W_4(x) = \frac{x}{4!}, \quad W_5(x) = 0, \quad \cdots.$$

Hence, the solution in series form is as follow

$$\tilde{w}_n(x,t) = \sum_{k=0}^{\infty} W_k(x)t^k = W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots = x\Big(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\Big),$$

which converges efficiently to the exact solution $w(x,t) = x\cos t$.

Example 5. As the fifth example, we consider the following homogeneous non-linear time-fractional gas dynamics equation as [35,7]

$$\frac{\partial^\alpha w}{\partial t^\alpha} + w\frac{\partial w}{\partial x} - w + w^2 = 0, \tag{54}$$

subject to the initial condition

$$w(x,0) = e^{-x}. \tag{55}$$

According to the operations of differential transformation given in Table I for Eq. (54) we obtain the following recurrent relation

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}W^\alpha_{k+1}(x) + \sum_{r=0}^{k} W_r(x)\frac{\partial}{\partial x}W_{k-r}(x) - W_k + \sum_{r=0}^{k} W_r(x)W_{k-r}(x) = 0, \tag{56}$$

and from initial conditions (55), we write

$$W^\alpha_0(x) = e^{-x}. \tag{57}$$

Substituting the above equation in Eq. (56), we drive the following results

$$W^\alpha_1(x) = \frac{e^{-x}}{\Gamma(\alpha+1)}, \quad W^\alpha_2(x) = \frac{e^{-x}}{\Gamma(2\alpha+1)}, \quad W^\alpha_3(x) = \frac{e^{-x}}{\Gamma(3\alpha+1)}, \quad \cdots.$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{w}_n(x,t) = \sum_{k=0}^{\infty} W^\alpha_k(x)t^{\alpha k} &= W^\alpha_0(x) + W^\alpha_1(x)t^\alpha + W^\alpha_2(x)t^{2\alpha} + \cdots\\ &= e^{-x}\Big(1 + \frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} + \cdots\Big),\end{aligned}$$

which for $\alpha = 1$ converges efficiently to the exact solution $w(x,t) = e^{-x+t}$.

Example 6. As the sixth example, we consider the following system of inhomogeneous linear PDEs:

$$\begin{cases}\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} - w + v = -2,\\ \frac{\partial v}{\partial t} + \frac{\partial w}{\partial x} - w + v = -2,\end{cases} \tag{58}$$

subject to the initial condition

$$\begin{cases} w(x,0) = 1 + e^x, \\ v(x,0) = -1 + e^x. \end{cases} \tag{59}$$

According to the operations of differential transformation given in Table I for Eq. (58) we obtain the following recurrent relation

$$\begin{aligned} (k+1)W_{k+1}(x) - \frac{\partial}{\partial x}V_k(x) - W_k(x) + V_k(x) &= -2\delta(k), \\ (k+1)V_{k+1}(x) + \frac{\partial}{\partial x}W_k(x) - W_k(x) + V_k(x) &= -2\delta(k), \end{aligned} \tag{60}$$

and from initial conditions (59), we write

$$\begin{aligned} W_0(x) &= 1 + e^x, \\ V_0(x) &= -1 + e^x. \end{aligned} \tag{61}$$

Substituting the above equations in Eq. (60), we drive the following results

$$\begin{aligned} W_1(x) &= e^x, \quad W_2(x) = \frac{1}{2!}e^x, \quad W_3(x) = \frac{1}{3!}e^x, \quad W_4(x) = \frac{1}{4!}e^x, \quad \cdots, \\ V_1(x) &= -e^x, \quad V_2(x) = \frac{1}{2!}e^x, \quad V_3(x) = -\frac{1}{3!}e^x, \quad V_4(x) = \frac{1}{4!}e^x, \quad \cdots. \end{aligned}$$

Hence, the solutions in series form are as follow

$$\tilde{w}_n(x,t) = \sum_{k=0}^{\infty} W_k(x)t^k = W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots = 1 + e^x\Big(1 + t + \frac{t^2}{2!} + \cdots\Big),$$

and

$$\tilde{v}_n(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k = V_0(x) + V_1(x)t + V_2(x)t^2 + \cdots = -1 + e^x\Big(1 - t + \frac{t^2}{2!} - \cdots\Big),$$

which converges efficiently to the exact solutions $w(x,t) = 1 + e^{x+t}$, and $v(x,t) = -1 + e^{x-t}$.

Example 7. As the seventh example, we consider the following system of non-linear PDEs:

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}v - w\frac{\partial v}{\partial x} - w = 0, \\ \frac{\partial v}{\partial t} - w\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}v + v = 0, \end{cases} \tag{62}$$

subject to the initial conditions

$$\begin{cases} w(x,0) = e^x, \\ v(x,0) = e^x. \end{cases} \tag{63}$$

According to the operations of differential transformation given in Table I for Eq. (62) we obtain the following recurrent relation

$$\begin{aligned} (k+1)W_{k+1}(x) + \sum_{r=0}^{k} \frac{\partial}{\partial x}W_r(x)V_{k-r} - \sum_{r=0}^{k} W_r\frac{\partial}{\partial x}V_{k-r}(x) - W_k(x) &= 0, \\ (k+1)V_{k+1}(x) - \sum_{r=0}^{k} W_r\frac{\partial}{\partial x}V_{k-r}(x) + \sum_{r=0}^{k} \frac{\partial}{\partial x}W_r(x)V_{k-r} + V_k(x) &= 0, \end{aligned} \tag{64}$$

and from initial conditions (63), we write

$$\begin{aligned} W_0(x) &= e^x, \\ V_0(x) &= e^x. \end{aligned}$$

Substituting the above equations in Eq. (64), we drive the following results

$$\begin{aligned} W_1(x) &= e^x, \quad W_2(x) = \frac{1}{2!}e^x, \quad W_3(x) = \frac{1}{3!}e^x, \quad W_4(x) = \frac{1}{4!}e^x, \quad \cdots, \\ V_1(x) &= -e^x, \quad V_2(x) = \frac{1}{2!}e^x, \quad V_3(x) = -\frac{1}{3!}e^x, \quad V_4(x) = \frac{1}{4!}e^x, \quad \cdots. \end{aligned}$$

Hence, the solutions in series form are as follow

$$\tilde{w}_n(x,t) = \sum_{k=0}^{\infty} W_k(x)t^k = W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots = e^x\left(1 + t + \frac{t^2}{2!} + \cdots\right),$$

and

$$\tilde{v}_n(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k = V_0(x) + V_1(x)t + V_2(x)t^2 + \cdots = e^x\left(1 - t + \frac{t^2}{2!} - \cdots\right),$$

which converges efficiently to the exact solutions $w(x,t) = e^{x+t}$, and $v(x,t) = e^{x-t}$.

Example 8. Lastly, we consider the following system of homogeneous linear FPDEs:

$$\begin{cases} \frac{\partial^\alpha w}{\partial t^\alpha} - \frac{\partial v}{\partial x} + v + w = 0, \\ \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial w}{\partial x} + v + w = 0, \end{cases} \tag{65}$$

subject to the initial conditions

$$\begin{cases} u(x,0) &= \sinh x, \\ v(x,0) &= \cosh x. \end{cases} \tag{66}$$

According to the operations of differential transformation given in Table I for Eq. (65) we obtain the following recurrent relation

$$\begin{aligned} \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} W^\alpha_{k+1}(x) - \frac{\partial}{\partial x} V_k(x) + V_k(x) + W_k(x) = 0, \\ \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V^\alpha_{k+1}(x) - \frac{\partial}{\partial x} W_k(x) + V_k(x) + W_k(x) = 0, \end{aligned} \tag{67}$$

and from initial conditions (66), we write

$$\begin{aligned} W^\alpha_0(x) = \sinh x, \\ V^\alpha_0(x) = \cosh x. \end{aligned} \tag{68}$$

Substituting the above equations in Eq. (67), we drive the following results

$$\begin{aligned} W^\alpha_1(x) = -\frac{\cosh x}{\Gamma(\alpha+1)}, \quad W^\alpha_2(x) = \frac{\sinh x}{\Gamma(2\alpha+1)}, \quad W^\alpha_3(x) = -\frac{\cosh x}{\Gamma(3\alpha+1)}, \quad \cdots, \\ V^\alpha_1(x) = -\frac{\sinh x}{\Gamma(\alpha+1)}, \quad V^\alpha_2(x) = \frac{\cosh x}{\Gamma(2\alpha+1)}, \quad V^\alpha_3(x) = -\frac{\sinh x}{\Gamma(3\alpha+1)}, \quad \cdots. \end{aligned}$$

Hence, the solutions in series form are as follow

$$\begin{aligned} \tilde{w}_n(x,t) &= \sum_{k=0}^{\infty} W^\alpha_k(x)t^{\alpha k} = W^\alpha_0(x) + W^\alpha_1(x)t^\alpha + W^\alpha_2(x)t^{2\alpha} + W^\alpha_3(x)t^{3\alpha} + \cdots \\ &= \sinh x + \frac{(-1)t^\alpha}{\Gamma(\alpha+1)}\cosh x + \frac{(-1)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\sinh x + \frac{(-1)^3 t^{3\alpha}}{\Gamma(3\alpha+1)}\cosh x + \cdots, \end{aligned}$$

and

$$\begin{aligned} \tilde{v}_n(x,t) &= \sum_{k=0}^{\infty} V^\alpha_k(x)t^{\alpha k} = V^\alpha_0(x) + V^\alpha_1(x)t^\alpha + V^\alpha_2(x)t^{2\alpha} + V^\alpha_3(x)t^{3\alpha} + \cdots \\ &= \cosh x + \frac{(-1)t^\alpha}{\Gamma(\alpha+1)}\sinh x + \frac{(-1)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\cosh x + \frac{(-1)^3 t^{3\alpha}}{\Gamma(3\alpha+1)}\sinh x + \cdots. \end{aligned}$$

which for $\alpha = 1$ converges efficiently to the exact solutions $w(x,t) = \sinh(x-t)$, and $v(x,t) = \cosh(x-t)$.

Results for Examples 1–8 are reported in Figures 1 –8 and Tables 1 –8 , respectively. In these tables, the terms w_E, v_E, $w_{n,R}, v_{n,R}$ and $e(w), e(v)$ stand for exact solution, nth order approximate solution of RDTM and their absolute error respectively.

6 | CONCLUSIONS

In this work, the convergence of the reduced differential transform method (RDTM) for solving the linear and non-linear ordinary, partial, fractional differential equations and their's systems is discussed. The main strength of the RDTM is its fast convergence and under the assumption of Theorem 1, the method is convergence to the exact solution of the problem. The sufficient condition for convergence of the method and an error estimate has been addressed. For the efficiency of the RDTM, the form of the initial approximation is very important. We note that the RDTM solutions were computed via a simple algorithm and without involving the perturbation, linearization, or discretization provides a solution in both numerical and analytical manner. RDTM can be applied most of the biological, physical, engineering, etc. models as an alternative for obtaining reliable and fastest converge, useful approximations. Thus, it can be concluded the RDTM is a simple and powerful tool for solving functional equations.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Table I. The fundamental operations of RDTM

Original Form	Transformed Form
$w(x,t)$	$W_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} w(x,t)]_{t=0}$
$w(x,t) = \lambda u(x,t) \pm \gamma v(x,t)$	$W_k(x) = \lambda U_k(x) \pm \gamma V_k(x) \quad (\lambda, \gamma \text{ are constants})$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n), \qquad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = (k+1)\cdots(k+r) U_{k+r}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = e^{\lambda t}$	$W_k(x) = \frac{\lambda^k}{k!}$
$w(x,t) = \sin(\lambda t + \alpha x)$	$W_k(x) = \frac{\lambda^k}{k!} \sin(\frac{k\pi}{2} + \alpha x)$
$w(x,t) = \cos(\lambda t + \alpha x)$	$W_k(x) = \frac{\lambda^k}{k!} \cos(\frac{k\pi}{2} + \alpha x)$

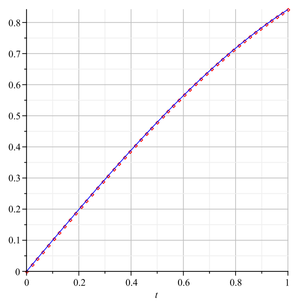


FIGURE 1 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 1

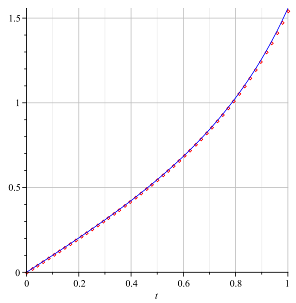


FIGURE 2 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 2

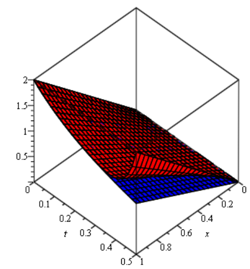


FIGURE 3 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 3

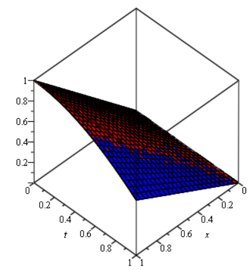


FIGURE 4 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 4

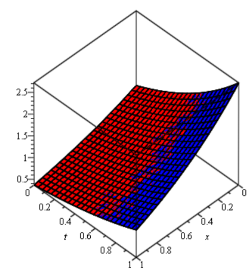
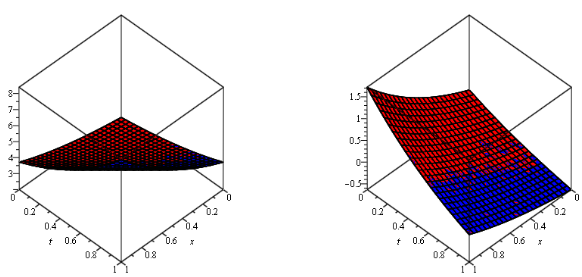


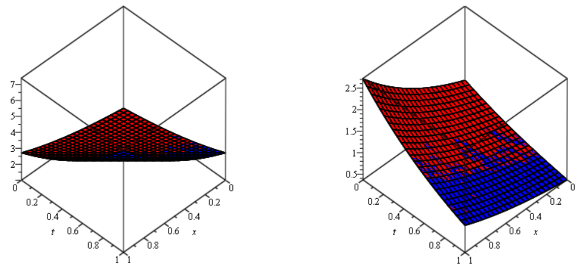
FIGURE 5 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 5 for $\alpha = 1$



(a) The solution of $w_n(x, t)$

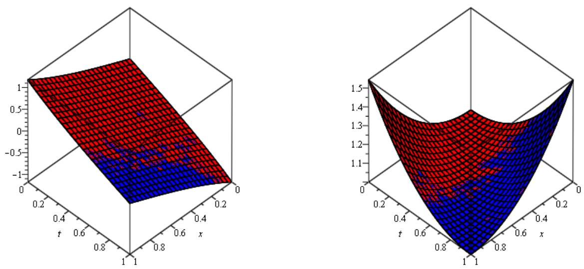
(b) The solution of $v_n(x, t)$

FIGURE 6 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 6



(a) The solution of $w_n(x,t)$ (b) The solution of $v_n(x,t)$

FIGURE 7 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 7



(a) The solution of $w_n(x,t)$ (b) The solution of $v_n(x,t)$

FIGURE 8 Comparison of the exact solution (blue) and the approximate solutions (red) of Example 8 for $\alpha = 1$

TABLE 1 Numerical result of Example 1

t	w_E	$w_{7,R}$	$w_{9,R}$	$n = 7,\ e(w)$	$n = 9,\ e(w)$
0	0	0	0	0	0
0.2	0.1986693308	0.1986693309	0.1986693309	7.02×10^{-12}	5.6×10^{-12}
0.4	0.3894183423	0.3894183415	0.3894183422	7.16×10^{-10}	6.36×10^{-12}
0.6	0.5646424734	0.5646424457	0.5646424735	2.77×10^{-8}	7.57×10^{-11}
0.8	0.7173560909	0.7173557232	0.7173560931	3.68×10^{-7}	2.13×10^{-9}
1	0.8414709848	0.8414682539	0.8414710096	2.73×10^{-6}	2.49×10^{-8}

TABLE 2 Numerical result of Example 2

t	w_E	$w_{7,R}$	$w_{9,R}$	$n = 7,\ e(w)$	$n = 9,\ e(w)$
0	0	0	0	0	0
0.2	0.2027100355	0.2027100242	0.2027100354	1.14×10^{-8}	1.77×10^{-10}
0.4	0.4227932187	0.4227870882	0.4227928212	6.13×10^{-6}	3.97×10^{-7}
0.6	0.6841368083	0.6838787657	0.6840991598	2.58×10^{-4}	3.76×10^{-5}
0.8	1.029638557	1.025675296	1.028610569	3.96×10^{-3}	1.03×10^{-3}
1	1.557407725	1.520634920	1.542504409	3.68×10^{-2}	1.49×10^{-2}

TABLE 3 Numerical result of Example 3

(x, t)	w_E	$w_{7,R}$	$w_{9,R}$	$n = 7,\ e(w)$	$n = 9,\ e(w)$
(0,0)	0	0	0	0	0
(0.2,0.1)	0.3333333334	0.33333248	0.3333332992	8.53×10^{-7}	3.41×10^{-8}
(0.4,0.2)	0.5714285714	0.57105408	0.5713686528	3.74×10^{-4}	5.9×10^{-5}
(0.6,0.3)	0.75	0.73740288	0.7454650368	1.26×10^{-2}	4.53×10^{-3}
(0.8,0.4)	0.8888888889	0.73975808	0.7934451712	1.49×10^{-1}	9.54×10^{-2}

TABLE 4 Numerical result of Example 4

(x, t)	w_E	$w_{6,R}$	$w_{10,R}$	$n = 6,\ e(w)$	$n = 10,\ e(w)$
(0.1,0)	0.1	0.1	0.1	0	0
(0.3,0.2)	0.2940199733	0.2940199733	0.2940199734	3.33×10^{-11}	5.24×10^{-11}
(0.5,0.4)	0.4605304970	0.4605304889	0.4605304970	8.11×10^{-9}	1.29×10^{-12}
(0.7,0.6)	0.5777349304	0.5777346400	0.5777349304	2.9×10^{-7}	3.36×10^{-11}
(0.9,0.8)	0.6270360384	0.6270323200	0.6270360383	3.72×10^{-6}	1.16×10^{-10}

TABLE 5 Numerical result of Example 5

(x, t)	w_E	$w_{7,R}$	$w_{10,R}$	$n = 7,\ e(w)$	$n = 10,\ e(w)$
(0.1,0)	0.9048374180	0.9048374180	0.9048374180	0	0
(0.3,0.2)	0.9048374180	0.9048374187	0.9048374187	2.34×10^{-11}	2.47×10^{-11}
(0.5,0.4)	0.9048374180	0.9048374079	0.9048374182	1.03×10^{-8}	3.13×10^{-11}
(0.7,0.6)	0.9048374180	0.9048371964	0.9048374184	2.21×10^{-7}	2.38×10^{-11}
(0.9,0.8)	0.9048374180	0.9048355630	0.9048374173	1.85×10^{-6}	9.49×10^{-10}

TABLE 6 Numerical result of Example 6

(x, t)	w_E	$w_{7,R}$	$w_{10,R}$	$n = 7,\ e(w)$	$n = 10,\ e(w)$
(0,0.1)	2.105170918	2.105170918	2.105170918	7.54×10^{-11}	7.57×10^{-11}
(0.2,0.3)	2.648721271	2.648721267	2.648721270	2.57×10^{-9}	5.14×10^{-10}
(0.4,0.5)	3.459603111	3.459602959	3.459603111	1.52×10^{-7}	6.81×10^{-10}
(0.6,0.7)	4.669296668	4.669293844	4.669296666	2.82×10^{-6}	2.09×10^{-9}
(0.8,0.9)	6.473947392	6.473921020	6.473947370	2.64×10^{-5}	2.06×10^{-8}
(x, t)	v_E	$v_{7,R}$	$v_{10,R}$	$n = 7,\ e(v)$	$n = 10,\ e(v)$
(0,0.1)	-0.951625820e-1	-0.951625819e-1	-0.951625819e-1	3.57×10^{-11}	3.59×10^{-11}
(0.2,0.3)	-0.951625820e-1	-0.951625840e-1	-0.951625822e-1	2.01×10^{-9}	8.46×10^{-11}
(0.4,0.5)	-0.951625820e-1	-0.951627186e-1	-0.951625816e-1	1.37×10^{-7}	2.19×10^{-10}
(0.6,0.7)	-0.951625820e-1	-0.951649979e-1	-0.951625810e-1	2.42×10^{-6}	6.6×10^{-10}
(0.8,0.9)	-0.951625820e-1	-0.951841642e-1	-0.951625658e-1	2.16×10^{-5}	1.62×10^{-8}

TABLE 7 Numerical result of Example 7

(x,t)	w_E	$w_{7,R}$	$w_{10,R}$	$n=7,\ e(w)$	$n=10,\ e(w)$
(0,0.3)	1.349858808	1.349858805	1.349858807	2.11×10^{-9}	4.23×10^{-10}
(0.2,0.4)	1.822118800	1.822118780	1.822118801	2.06×10^{-8}	1.55×10^{-10}
(0.4,0.5)	2.459603111	2.459602959	2.459603111	1.52×10^{-7}	6.81×10^{-10}
(0.6,0.6)	3.320116923	3.320116109	3.320116923	8.14×10^{-7}	1.14×10^{-9}
(0.8,0.7)	4.481689070	4.481685621	4.481689068	3.45×10^{-6}	1.37×10^{-9}
(x,t)	v_E	$v_{7,R}$	$v_{10,R}$	$n=7,\ e(v)$	$n=10,\ e(v)$
(0,0.3)	0.7408182207	0.7408182191	0.7408182206	1.59×10^{-9}	1.92×10^{-11}
(0.2,0.4)	0.8187307531	0.8187307340	0.8187307530	1.91×10^{-8}	1.32×10^{-10}
(0.4,0.5)	0.9048374180	0.9048372814	0.9048374184	1.37×10^{-7}	2.19×10^{-10}
(0.6,0.6)	1	0.9999992884	0.9999999994	7.11×10^{-7}	6.62×10^{-11}
(0.8,0.7)	1.105170918	1.105167967	1.105170919	2.95×10^{-6}	4.99×10^{-10}

TABLE 8 Numerical result of Example 8

(x,t)	w_E	$w_{7,R}$	$w_{9,R}$	$n=7,\ e(w)$	$n=9,\ e(w)$
(0.1,0.1)	0	$3.501211111\times10^{-11}$	$3.500934159\times10^{-11}$	3.5×10^{-11}	3.5×10^{-11}
(0.2,0.3)	-0.1001667500	-0.1001667505	-0.1001667502	4.17×10^{-10}	1.73×10^{-10}
(0.3,0.5)	-0.2013360025	-0.2013360265	-0.2013360026	2.4×10^{-8}	8.6×10^{-11}
(0.4,0.7)	-0.3045202934	-0.3045207633	-0.3045202961	4.7×10^{-7}	2.88×10^{-9}
(0.5,0.9)	-0.4107523258	-0.4107567262	-0.4107523679	4.4×10^{-6}	4.1×10^{-8}
(x,t)	v_E	$v_{7,R}$	$v_{9,R}$	$n=7,\ e(v)$	$n=9,\ e(v)$
(0.1,0.1)	1	1	1	4.76×10^{-11}	4.76×10^{-11}
(0.2,0.3)	1.005004168	1.005004166	1.005004168	1.18×10^{-9}	4.67×10^{-10}
(0.3,0.5)	1.020066756	1.020066655	1.020066756	1.01×10^{-7}	8.15×10^{-10}
(0.4,0.7)	1.045338514	1.045337006	1.045338506	1.51×10^{-6}	7.99×10^{-9}
(0.5,0.9)	1.081072372	1.081060784	1.081072265	1.16×10^{-5}	1.05×10^{-7}