m-Parameter Mittag-Leffler function, its various properties and relation with fractional calculus operators

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Abstract

Mittag-Leffler functions has many applications in various areas of Physical, biological ,applied, earth Sciences and Engineering. It is used in solving problems of fractional order differential, integral and difference equations. In this paper, we aim to define the m-parameter Mittag-Leffler function, which can be reduced to various already known extensions of Mittag-Leffler function. We then, discuss its various properties like recurrence relations, differentiation formula and integral representations. We also represent the new m-parameter Mittag-Leffler function in terms of some known special functions such as Generalized hypergeometric function, Mellin Barnes integral, Wright hypergeometric function and Fox H-function. We also discuss its various integral transforms like Euler-Beta, Whittaker, Laplace and Mellin transforms. Further, fractional differential and integral operators are considered to discuss few properties of m-parameter Mittag-Leffler function. Also, we use the m-parameter Mittag-Leffler function to define a generalization of Prabhakar integral and discuss its properties. Further, relation of m-parameter Mittag Leffler function with various other functions such as exponential, trigonometric, hypergeometric and algebraic functions is obtained and represented graphically using MATHEMATICA 12.

Keywords
m-parameter Mittag-Leffler function, Integral transforms, fractional calculus, Generalized Prabhakar integral.

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1 Introduction

Gosta Mittag-Leffler introduced the function (see [21]),

$$E_{\alpha_1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha_1 n + 1)}, \tag{1.1}$$

where, t is a complex variable, $\alpha_1 \geq 0$ and Γ is gamma function given by

$$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} dt, \qquad \Re(m) > 0.$$

Wiman studied the generalization of $E_{\alpha_1}(t)$, known as Wiman function (see [36, p.191]). For $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\Re(\alpha_1), \Re(\alpha_2) > 0$,

$$E_{\alpha_1,\alpha_2}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha_1 n + \alpha_2)}. \tag{1.2}$$

A three parameter Mittag-Leffler function was introduced by Prabhakar, now known as Prabhakar function (see [24, eq. (1.3), p.7]), For $\Re(\alpha_1), \Re(\alpha_2) > 0$ and $\mu_1 > 0$.

$$E^{\mu_1}_{\alpha_1,\alpha_2}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n t^n}{\Gamma(\alpha_1 n + \alpha_2) n!}, \tag{1.3}$$

where, $(\mu_1)_n$ is the Pochhammer symbol defined by (see [26, eq.(1), p.22; eq.(3), p.23]), for $\mu_1 \neq 0, -1, -2, \ldots$

$$(\mu_1)_n = \begin{cases} \prod_{r=1}^{n} (\mu_1 + r - 1), n \in \mathbb{N} \\ 1, n = 0 \end{cases} \qquad \text{and} \qquad (\mu_1)_n = \frac{\Gamma(\mu_1 + n)}{\Gamma(\mu_1)}.$$

Shukla and Prajapati (see [29, eq (1.4), p.798]) gave a generalization of Prabhakar function,

$$E^{\mu_1,\mu_2}_{\alpha_1,\alpha_2}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} t^n}{\Gamma(\alpha_1 n + \alpha_2) n!}, \tag{1.4}$$

where, $min\,(\Re(\alpha_1), \Re(\alpha_2), \Re(\mu_1)) > 0, \alpha_1, \alpha_2, \mu_1, \mu_2 \in \mathbb{C}$, $(\mu_1)_{\mu_2 n} = \frac{\Gamma(\mu_1+\mu_2 n)}{\mu_1}$ is the Generalized Pochhammer symbol, which in particular reduces to, $\mu_2^{\mu_2 n} \prod_{i=1}^{\mu_2} \left(\frac{\mu_1+i-1}{\mu_2}\right)_n$, if $\mu_2 \in \mathbb{N}$.

Khan and Ahmed (see [12, eq (1.7),p.2]) defined an another generalization of the Mittag-Leffler function,

$$E^{\mu_1,\mu_2}_{\alpha_1,\alpha_2,\alpha_3}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} t^n}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_n}, \tag{1.5}$$

where, $min\,(\Re(\alpha_1), \Re(\alpha_2), \Re(\alpha_3), \Re(\mu_1)) > 0$ and $\mu_2 \in (0,1) \cup \mathbb{N}$. In the same paper, (see [12, eq (1.9), p.2]), they gave a further generalization of the above equation,

$$E^{\mu_1,\mu_2,\mu_3,\mu_4}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} t^n}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} (\alpha_5)_{\alpha_6 n}}, \tag{1.6}$$

where, $\mu_1, \mu_2, \mu_3, \alpha_1, \ldots, \alpha_5 \in \mathbb{C}$, $\mu_4, \alpha_6 > 0$, $\Re(\alpha_1) + \alpha_6 \geq \mu_4$, $min\,(\Re(\mu_1), \Re(\mu_2), \Re(\mu_3), \Re(\alpha_1), \ldots, \Re(\alpha_5)) > 0$.

In this paper, we will use some of the well known results:

- *Euler Beta Transform* is defined in [31] as,

$$B(f(t); \psi_1, \psi_2) = \int_0^1 t^{\psi_1 - 1}(1-t)^{\psi_2 - 1} f(t)dt, \qquad \Re(\psi_1), \Re(\psi_2) > 0. \tag{1.7}$$

- *Laplace Transform* of a piecewise continuous function f is defined in [31] as,

$$L\left[f(t); \psi\right] = \int_0^\infty e^{-\psi t} f(t)dt, \qquad \Re(\psi) > 0. \tag{1.8}$$

- *Whittaker Transform* is defined in [35] as,

$$\int_0^\infty e^{\frac{-z}{2}} z^{p-1} W_{\kappa,\omega}(z)dz = \frac{\Gamma(\frac{1}{2}+\omega+p)\Gamma(\frac{1}{2}-\omega+p)}{\Gamma(1-\kappa+p)}, \tag{1.9}$$

  where, $\Re(\omega+p), \Re(\omega-p) > 0$ and $W_{\kappa,\omega}$ is Whittaker confluent hypergeometric function.

- *Mellin Transform* is defined in [31] as,

$$M\left[f(t); \psi\right] = f^*(\psi) = \int_0^\infty t^{\psi-1} f(t)dt, \tag{1.10}$$

  and, the inverse Mellin transform is given by,

$$f(t) = M^{-1}\left[f^*(\psi); t\right] = \frac{1}{2\pi i}\int_C f^*(\psi) t^{-\psi} d\psi, \tag{1.11}$$

  where, C is a contour of integration that begins at $-i\infty$ and ends at $i\infty$.

- The *Generalized hypergeometric function* is defined in [26, eq.2, pg-73] as,

$${}_rF_s\left[\begin{matrix}\delta_1 & \delta_2 & \ldots\delta_r\\ \psi_1 & \psi_2 & \ldots\psi_s\end{matrix}; t\right] = \sum_{n=0}^\infty \left(\frac{\prod_{i=1}^r (\delta_i)_n}{\prod_{j=1}^s (\psi_j)_n}\frac{t^n}{n!}\right), \tag{1.12}$$

  which converges for all $t \in \mathbb{C}$ when $r \leq s$.

- The *Wright Generalized hypergeometric function* is defined in [33] as,

$${}_r\Psi_s\left[\begin{matrix}(\alpha_1, A_1) & (\alpha_2, A_2) & \ldots(\alpha_r, A_r)\\ (\beta_1, B_1) & (\beta_2, B_2) & \ldots(\beta_s, B_s)\end{matrix}; t\right] = \sum_{n=0}^\infty \left(\frac{\prod_{i=1}^r \Gamma(\alpha_i + A_i n)}{\prod_{j=1}^s \Gamma(\beta_j + B_j n)}\frac{t^n}{n!}\right). \tag{1.13}$$

- The *Fox's H-function* is defined in [13] as,

$$H^{x,y}_{r,s}\left[t\left|\begin{matrix}(\alpha_1,A_1) & (\alpha_2,A_2) & \dots(\alpha_r,A_r)\\ (\beta_1,B_1) & (\beta_2,B_2) & \dots(\beta_s,B_s)\end{matrix}\right.\right]$$
$$=\frac{1}{2\pi i}\int\limits_C\left(\frac{\prod_{j=1}^{x}\Gamma(\beta_j+B_j\delta)\prod_{j=1}^{y}\Gamma(1-\alpha_j-A_j\delta)}{\prod_{j=x+1}^{s}\Gamma(1-\beta_j-B_j\delta)\prod_{j=y+1}^{r}\Gamma(\alpha_j+A_j\delta)}\right)t^{-\delta}d\delta. \tag{1.14}$$

Fractional calculus is used in the study of non-integer orders of differentiation, in the fields of science and engineering to solve integral equations, ordinary differential equations and partial differential equations.
The left and right sided fractional integral operators of Riemann- Liouville type (see [15, eqs. (2.1.1), (2.1.2), p.69]) are

$$\left(I^{\xi}_{c+}g\right)(y)=\frac{1}{\Gamma(\xi)}\int_c^y\frac{g(\eta)}{(y-\eta)^{1-\xi}}d\eta \qquad : y>c, \tag{1.15}$$

and,

$$\left(I^{\xi}_{d-}g\right)(y)=\frac{1}{\Gamma(\xi)}\int_y^d\frac{g(\eta)}{(y-\eta)^{1-\xi}}d\eta \qquad : y<d, \tag{1.16}$$

where, $g\in L[c,d]$, where $L[c,d]$ is the space of Lebesgue measurable functions :
$L[c,d]=g:\|g\|_1=\int_c^d g(y)dy$, $\xi\in\mathbb{C}$ and $\Re(\xi)>0$.

For $g\in L[c,d],\xi\in\mathbb{C},\Re(\xi)>0$ and $m=\Re(\xi)+1$, left and right sided Riemann Liouville fractional derivatives (see [15, eqs. (2.1.5),(2.1.6), p.69] are given by:

$$\left(D^{\xi}_{c+}g\right)(y)=\left(\frac{d}{dy}\right)^m\left(I^{m-\xi}_{c+}g\right)(y), \tag{1.17}$$

and,

$$\left(D^{\xi}_{d-}g\right)(y)=\left(-\frac{d}{dy}\right)^m\left(I^{m-\xi}_{d-}g\right)(y). \tag{1.18}$$

Generalized form of fractional differential operator D^{ξ}_{c+} is defined as, (see[15])

$$\left(D^{\xi,\upsilon}_{c+}g\right)(y)=\left(I^{\upsilon(1-\xi)}_{c+}\frac{d}{dy}\left(I^{(1-\upsilon)(1-\xi)}_{c+}g\right)\right)(y), \tag{1.19}$$

where, $0<\xi,\upsilon<1$.

In this section of fractional calculus, we suppose $\varpi=\eta-c$ and $\upsilon=y-c$. We will require the following two results for fractional integral and derivative of power function:

1. For $\xi,\mu\in\mathbb{C},\Re(\xi)>0,\Re(\mu)>0$, (see [18])

$$\left(I^{\xi}_{c+}(\varpi)^{\mu-1}\right)(y)=\frac{\Gamma(\mu)}{\Gamma(\xi+\mu)}(\upsilon)^{\xi+\mu-1}. \tag{1.20}$$

2. For $y > c, 0 < \xi < 1, 0 \leq \upsilon \leq 1, \Re(\mu) > 0, \Re(\xi) > 0$, (see [34])

$$\left(D_{c+}^{\xi,\upsilon}(\varpi)^{\mu-1}\right)(y) = \frac{\Gamma(\mu)}{\Gamma(\xi-\mu)}(\upsilon)^{\mu-\xi-1}. \tag{1.21}$$

The Prabhakar integral is defined by (see [14, eq. 1.6, p-32]) -

$$\left(\mathfrak{E}_{\alpha_1,\alpha_2,c+}^{\zeta,\mu_1}g\right)(y) = \int_c^y (y-\eta)^{\alpha_2-1} E_{\alpha_1,\alpha_2}^{\mu_1}\left[\zeta(y-\eta)^{\alpha_1}\right] g(\eta)d\eta, \tag{1.22}$$

where, $\Re(\alpha_1), \Re(\alpha_2) > 0$ and $E_{\alpha_1,\alpha_2}^{\mu_1}$ is given by (1.3). The further generalization of Prabhakar fractional integral is given by (see [34, eq. (2.12), p. 202]

$$\left(\mathfrak{E}_{\alpha_1,\alpha_2,c+}^{\zeta,\mu_1,\mu_2}g\right)(y) = \int_c^y (y-\eta)^{\alpha_2-1} E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}\left[\zeta(y-\eta)^{\alpha_1}\right] g(\eta)d\eta, \tag{1.23}$$

where, $\Re(\alpha_1), \Re(\alpha_2) > 0$ and $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}$ is given by (1.4).
We will require the following Dirichlet's formula in our study (see [20, eq. (4.1) p. 56]).
If $H(u,v)$ is continuous on $[c,d] \times [c,d]$,

$$\int_c^d du \int_c^u H(u,v)dv = \int_c^d dv \int_v^d H(u,v)du. \tag{1.24}$$

For various interesting extensions and properties of Mittag-Leffler functions, readers may refer [2], [25], [32], [37] and many other research papers.

2 m-parameter Mittag-Leffler function

In this section, we introduce a new Mittag-Leffler function with m-parameters contained in the following definition:

Definition 2.1. *The generalized Mittag-Leffler function with m parameters is defined as,*

$$E_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}^{\mu_1,\ldots,\mu_{m-2}}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\cdots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\cdots(\alpha_{m-1})_{\alpha_m n}} t^n, \tag{2.1}$$

where, t is a complex variable, $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\left(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})\right) > 0$ and $(\mu_1)_{\mu_2 n} = \frac{\Gamma(\mu_1+\mu_2 n)}{\mu_1}$ is the Generalized Pochhammer symbol, which in particular reduces to, $\mu_2^{\mu_2 n}\prod_{i=1}^{\mu_2}\left(\frac{\mu_1+i-1}{\mu_2}\right)_n$, if $\mu_2 \in \mathbb{N}$.

The generalized Mittag- Leffler function with m parameters reduces to the following *special cases* on giving specific values to the various parameters:

(a) For $\mu_k = \alpha_{k+2}$ for $1 \leq k \leq m-2$ and $\alpha_2 = 1$, equation (2.1) reduces to the Gosta Mittag Leffler function ([21]) given by (1.1).

(b) For $\mu_k = \alpha_{k+2}$ for $1 \leq k \leq m-2$, equation (2.1) reduces to the Wiman function ([36, p.191])given by (1.2).

(c) For $\mu_k = \alpha_{k+2}$ for $2 < k \leq m-2$ and $\mu_2, \alpha_3, \alpha_4 = 1$, equation (2.1) reduces to the Prabhakar function ([24, eq. (1.3), p.7]) given by (1.3).

(d) For $\mu_k = \alpha_{k+2}$ for $2 < k \leq m-2$ and $\alpha_3, \alpha_4 = 1$, equation (2.1) reduces to the generalization of Prabhakar function given by Shukla and Prajapati ([29, eq (1.4), p.798]) and defined as, (1.4).

(e) For $\mu_k = \alpha_{k+2}$ for $2 < k \leq m-2$ and $\alpha_4 = 1$, equation (2.1) reduces to the generalization given by Khan and Ahmed ([12, eq (1.7),p.2]), defined by (1.5).

(f) For $\mu_k = \alpha_{k+2}$ for $4 < k \leq m-2$, equation (2.1) reduces to the further generalization given by Khan and Ahmed ([12, eq (1.9), p.2]), defined by (1.6).

3 Properties of Generalized Mittag-Leffler function

Here, we discuss various properties of m-parameter Mittag-Leffler function such as recurrence relations, differentiation formulae and integral representations.

Theorem 3.1 (Recurrence Relation). *If $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\Re(\alpha_1), \ldots, \Re(\alpha_{m-1}) > 0$, $\Re(\mu_1), \ldots, \Re(\mu_{m-3}) > 0$, $\alpha_m, \mu_{m-2} > 0$ and $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, then,*

$$\text{(a)}\quad \alpha_2 E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+1,\alpha_3\ldots,\alpha_m}(t) + \alpha_1 t \frac{d}{dt} E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+1,\alpha_3\ldots,\alpha_m}(t) = E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t). \tag{3.1}$$

$$\begin{aligned}\text{(b)}\quad E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) = \frac{1}{\Gamma(\alpha_2)} + &\frac{(\mu_1)_{\mu_2}(\mu_3)_{\mu_4}\ldots(\mu_{m-3})_{\mu_{m-2}}}{(\alpha_3)_{\alpha_4}\ldots(\alpha_{m-1})_{\alpha_m}} t \\ &\times E^{\mu_1+\mu_2,\mu_2,\mu_3+\mu_4,\mu_4,\ldots,\mu_{m-3}+\mu_{m-2},\mu_{m-2}}_{\alpha_1,\alpha_1+\alpha_2,\alpha_3+\alpha_4,\alpha_4,\ldots,\alpha_{m-1}+\alpha_m,\alpha_m}(t).\end{aligned} \tag{3.2}$$

$$\begin{aligned}\text{(c)}\quad E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2-\alpha_1,\alpha_3,\ldots,\alpha_m}(t) - E^{\mu_1,\ldots,\mu_{m-4},\mu_{m-3}-1,\mu_{m-2}}_{\alpha_1,\alpha_2-\alpha_1,\alpha_3,\ldots,\alpha_m}(t) = \mu_{m-2} t \frac{(\mu_1)_{\mu_2}(\mu_3)_{\mu_4}\ldots(\mu_{m-5})_{\mu_{m-4}}}{(\alpha_3)_{\alpha_4}\ldots(\alpha_{m-1})_{\alpha_m}} \\ \times \sum_{n=0}^{\infty} \left(\frac{(n+1)(\mu_1+\mu_2)_{\mu_2 n}\ldots(\mu_{m-5}+\mu_{m-4})_{\mu_{m-4}n}(\mu_{m-3})_{\mu_{m-2}n+\mu_{m-2}-1} t^n}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3+\alpha_4)_{\alpha_4 n}\ldots(\alpha_{m-1}+\alpha_m)_{\alpha_m n}} \right).\end{aligned} \tag{3.3}$$

Proof.

$$\begin{aligned}\text{(a)}\quad LHS =& \alpha_2 E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+1,\alpha_3\ldots,\alpha_m}(t) + \alpha_1 t \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+1)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\ =& \alpha_2 E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+1,\alpha_3\ldots,\alpha_m}(t) + \alpha_1 \sum_{n=0}^{\infty} \frac{n(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+1)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\ =& (\alpha_1 n+\alpha_2) \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+1)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\ =& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n = RHS.\end{aligned}$$

This gives the desired result (3.1).

$$
\begin{aligned}
(b)\quad LHS =& E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\
=& \frac{1}{\Gamma(\alpha_2)} + \sum_{n=1}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\
=& \frac{1}{\Gamma(\alpha_2)} + \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n+\mu_2}(\mu_3)_{\mu_4 n+\mu_4}\ldots(\mu_{m-3})_{\mu_{m-2}n+\mu m-2}}{\Gamma(\alpha_1 n+\alpha_1+\alpha_2)(\alpha_3)_{\alpha_4 n+\alpha_4}\ldots(\alpha_{m-1})_{\alpha_m n+\alpha_m}} t^{n+1} \\
=& \frac{1}{\Gamma(\alpha_2)} + \frac{(\mu_1)_{\mu_2}(\mu_3)_{\mu_4}\ldots(\mu_{m-3})_{\mu_{m-2}}}{(\alpha_3)_{\alpha_4}\ldots(\alpha_{m-1})_{\alpha_m}} t \\
&\times \sum_{n=1}^{\infty} \frac{(\mu_1+\mu_2)_{\mu_2 n}(\mu_3+\mu_4)_{\mu_4 n}\ldots(\mu_{m-3}+\mu_{m-2})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_1+\alpha_2)(\alpha_3+\alpha_4)_{\alpha_4 n}\ldots(\alpha_{m-1}+\alpha_m)_{\alpha_m n}} t^n = RHS.
\end{aligned}
$$

This proves our result (3.2).

$$
\begin{aligned}
(c)\quad LHS =& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1(n-1)+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\
&- \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3}-1)_{\mu_{m-2}n}}{\Gamma(\alpha_1(n-1)+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \\
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-5})_{\mu_{m-4}n}}{\Gamma(\alpha_1(n-1)+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} t^n \left((\mu_{m-3})_{\mu_{m-2}n} - (\mu_{m-3}-1)_{\mu_{m-2}n}\right)
\end{aligned}
$$

On Simplifying, we get,

$$
\begin{aligned}
LHS =& \mu_{m-2} t \sum_{n=0}^{\infty} \frac{(n+1)(\mu_1)_{\mu_2 n+\mu_2}(\mu_3)_{\mu_4 n+\mu_4}\ldots(\mu_{m-3})_{\mu_{m-2}n+\mu m-2-1}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n+\alpha_4}\ldots(\alpha_{m-1})_{\alpha_m n+\alpha_m}} t^n \\
=& \left(\sum_{n=0}^{\infty} \frac{(n+1)(\mu_1)_{\mu_2}(\mu_1+\mu_2)_{\mu_2 n}\ldots(\mu_{m-5})_{\mu_{m-4}}(\mu_{m-5}+\mu_{m-4})_{\mu_{m-4}n}(\mu_{m-3})_{\mu_{m-2}n+\mu m-2-1}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4}(\alpha_3+\alpha_4)_{\alpha_4 n}\ldots(\alpha_{m-1}+\alpha_m)_{\alpha_m n}(\alpha_{m-1})_{\alpha_m}} t^n\right) \\
&\times \mu_{m-2} t \\
=& RHS.
\end{aligned}
$$

This proves our result (3.3).

□

Theorem 3.2 (Differentiation formula)**.** *If* $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\Re(\alpha_1), \ldots, \Re(\alpha_{m-1}) > 0$, $\Re(\mu_1), \ldots, \Re(\mu_{m-3}) > 0$, $\alpha_m, \mu_{m-2} > 0$ *and* $\Re(\alpha_1)+\alpha_m \geq \mu_{m-2}$, *then for* $r \in \mathbb{N}$,

$$(a) \quad \left(\frac{d}{dt}\right)^r E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) = \frac{(1)_r(\mu_1)_{\mu_2 r}(\mu_3)_{\mu_4 r}\ldots(\mu_{m-3})_{\mu_{m-2}r}}{(\alpha_3)_{\alpha_4 r}\ldots(\alpha_{m-1})_{\alpha_m r}} \times \sum_{n=0}^{\infty}\frac{(\mu_1+\mu_2 r)_{\mu_2 n}(\mu_3+\mu_4 r)_{\mu_4 n}\ldots(\mu_{m-3}+\mu_{m-2}r)_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_1 r+\alpha_2)(\alpha_3+\alpha_4 r)_{\alpha_4 n}\ldots(\alpha_{m-1}+\alpha_m r)_{\alpha_m n}} \times \frac{(1+r)_n}{n!}t^n. \tag{3.4}$$

$$(b) \quad \left(\frac{d}{dt}\right)^r \left[t^{\alpha_2-1}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(\xi t^{\alpha_1})\right] = t^{\alpha_2-r-1}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2-r\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(\xi t^{\alpha_1}). \tag{3.5}$$

Proof.

$$\begin{aligned}
(a) \quad LHS &= \left(\frac{d}{dt}\right)^r E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) = \left(\frac{d}{dt}\right)^r \left[\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}t^n\right]\\
&=\sum_{n=r}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}n!}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}(n-r)!}t^{n-r}\\
&=\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n+\mu_2 r}(\mu_3)_{\mu_4 n+\mu_4 r}\ldots(\mu_{m-3})_{\mu_{m-2}n+\mu_{m-2}r}(n+r)!}{\Gamma(\alpha_1 n+\alpha_1 r+\alpha_2)(\alpha_3)_{\alpha_4 n+\alpha_4 r}\ldots(\alpha_{m-1})_{\alpha_m n+\alpha_m r}n!}t^n\\
&=\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 r}(\mu_1+\mu_2 r)_{\mu_2 n}\ldots(\mu_{m-3})_{\mu_{m-2}r}(\mu_{m-3}+\mu_{m-2}r)_{\mu_{m-2}n}(1)_r(1+r)_n}{\Gamma(\alpha_1 n+\alpha_1 r+\alpha_2)(\alpha_3)_{\alpha_4 r}(\alpha_3+\alpha_4 r)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m r}(\alpha_{m-1}\alpha_m r)_{\alpha_m n}n!}t^n\\
&=RHS.
\end{aligned}$$

This proves the result (3.4).

$$\begin{aligned}
(b) \quad LHS &= \left(\frac{d}{dt}\right)^r \left[t^{\alpha_2-1}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}\xi^n t^{\alpha_1 n}\right]\\
&=\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}(\alpha_1 n+\alpha_2-1)!}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}(\alpha_1 n+\alpha_2-1-r)!}\xi^n t^{\alpha_1 n+\alpha_2-r-1}\\
&=\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2-r)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}\xi^n t^{\alpha_1 n+\alpha_2-r-1}\\
&=RHS.
\end{aligned}$$

This proves the result (3.5).

□

Theorem 3.3. *(Differentiation formula) Let $\alpha_1 = \frac{s}{r}$ where s and r are coprime and $s, r \in \mathbb{N}$, $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\Re(\alpha_1), \ldots, \Re(\alpha_{m-1}) > 0$, $\Re(\mu_1), \ldots, \Re(\mu_{m-3}) > 0$, $\alpha_m, \mu_{m-2} > 0$ and $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, then,*

$$\frac{d^s}{dt^s}\left[E^{\mu_1,\ldots,\mu_{m-2}}_{\frac{s}{r},\alpha_2,\ldots,\alpha_{m-1},\alpha_m}(t^{\frac{s}{r}})\right]=\frac{\Gamma(\alpha_3)\ldots\Gamma(\alpha_{m-3})}{\Gamma(\mu_1)\ldots\Gamma(\mu_{m-5})}\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\ldots\Gamma(\mu_{m-5}+\mu_{m-4}n)}{\Gamma(\frac{sn}{r}+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\ldots\Gamma(\alpha_{m-3}+\alpha_{m-2}n)}$$
$$\times\frac{(\mu_{m-3})_{\mu_{m-2}n}t^{s\left(\frac{n}{r}-1\right)}\Gamma(\frac{sn}{r}+1)}{(\alpha_{m-1})_{\alpha_m n}\Gamma(\frac{sn}{r}-s+1)}. \tag{3.6}$$

Proof.

$$\begin{aligned}LHS=&\frac{d^s}{dt^s}\left[E^{\mu_1,\ldots,\mu_{m-2}}_{\frac{s}{r},\alpha_2,\ldots,\alpha_{m-1},\alpha_m}(t^{\frac{s}{r}})\right]=\frac{d^s}{dt^s}\left[\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\frac{sn}{r}+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}t^{\frac{sn}{r}}\right]\\
=&\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}\left(\frac{sn}{r}\right)!}{\Gamma(\frac{sn}{r}+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}\left(\frac{sn}{r}-s\right)!}t^{\frac{sn}{r}-s}\\
=&\frac{\Gamma(\alpha_3)\ldots\Gamma(\alpha_{m-3})}{\Gamma(\mu_1)\ldots\Gamma(\mu_{m-5})}\\
&\times\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\ldots\Gamma(\mu_{m-5}+\mu_{m-4}n)(\mu_{m-3})_{\mu_{m-2}n}t^{s\left(\frac{n}{r}-1\right)}\Gamma(\frac{sn}{r}+1)}{\Gamma(\frac{sn}{r}+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\ldots\Gamma(\alpha_{m-3}+\alpha_{m-2}n)(\alpha_{m-1})_{\alpha_m n}\Gamma(\frac{sn}{r}-s+1)}\\
=&RHS.\end{aligned}$$

This proves our result (3.6).

□

Theorem 3.4 (Integral Representation)**.** *If $\mu_1,\mu_2,\ldots,\mu_{m-3},\alpha_1,\ldots,\alpha_{m-1}\in\mathbb{C}$, $\Re(\alpha_1)$, $\ldots,\Re(\alpha_{m-1})>0$, $\Re(\mu_1),\ldots,\Re(\mu_{m-3})>0$, $\alpha_m,\mu_{m-2}>0$ and $\Re(\alpha_1)+\alpha_m\geq\mu_{m-2}$, then*

(a) $$\frac{1}{\Gamma(\psi)}\int_0^1\varsigma^{\alpha_2-1}(1-\varsigma)^{\psi-1}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t\varsigma^{\alpha_1})d\varsigma=E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+\psi,\alpha_3,\ldots,\alpha_{m-1},\alpha_m}(t). \tag{3.7}$$

(b) $$\begin{aligned}\frac{1}{\Gamma(\psi)}\int_t^{\kappa}(\kappa-s)^{\psi-1}(s-t)^{\alpha_2-1}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-1},\alpha_m}(\varrho(s-t)^{\alpha_1})ds=&(\kappa-t)^{\psi+\alpha_2-1}\\
&\times E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+\psi,\alpha_3,\ldots,\alpha_m}(\varrho(\kappa-t)^{\alpha_1}).\end{aligned} \tag{3.8}$$

(c) $$\int_0^t\zeta^{\alpha_2-1}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(\omega\zeta^{\alpha_1})d\zeta=t^{\alpha_2}E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+1,\alpha_3,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(\omega t^{\alpha_1}). \tag{3.9}$$

Proof.

(a) $$\begin{aligned}LHS=&\frac{1}{\Gamma(\psi)}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}t^n\int_0^1\varsigma^{\alpha_1 n+\alpha_2-1}(1-\varsigma)^{\psi-1}d\varsigma\\
=&\frac{1}{\Gamma(\psi)}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}t^n B(\alpha_1 n+\alpha_2,\psi)\\
=&\frac{1}{\Gamma(\psi)}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}t^n\frac{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\psi)}{\Gamma(\alpha_1 n+\alpha_2+\psi)}\\
=&RHS.\end{aligned}$$

This proves the result (3.7).

(b) Let us change the variable from s to $\varsigma = \frac{s-t}{\kappa-t}$. Then substituting $(s-t) = \varsigma(\kappa-t)$ and $(\kappa-s) = (\kappa-t)(1-\varsigma)$ in (3.8), we get,

$$\begin{aligned}
LHS =& \frac{1}{\Gamma(\psi)} \int_0^1 (\kappa-t)^{\psi+\alpha_2-1}(1-\varsigma)^{\psi-1}\varsigma^{\alpha_2-1} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \\
& \times \varrho^n(\kappa-t)^{\alpha_1 n}\varsigma^{\alpha_1 n} d\varsigma \\
=& \frac{(\kappa-t)^{\psi+\alpha_2-1}}{\Gamma(\psi)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \left[\varrho(\kappa-t)^{\alpha_1}\right]^n B(\alpha_2+\alpha_1 n, \psi) \\
=& \frac{(\kappa-t)^{\psi+\alpha_2-1}}{\Gamma(\psi)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \left[\varrho(\kappa-t)^{\alpha_1}\right]^n \frac{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\psi)}{\Gamma(\alpha_1 n+\alpha_2+\psi)} \\
=& RHS.
\end{aligned}$$

This proves the result (3.8).

$$\begin{aligned}
(c) \quad LHS =& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \omega^n \int_0^t \zeta^{\alpha_1 n+\alpha_2-1} d\zeta \\
=& t^{\alpha_2} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+1)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \omega^n t^{\alpha_1 n} = RHS.
\end{aligned}$$

This proves the result (3.9).

□

4 Representation in terms of some known Special functions

Here, we determine the relation of m-parameter Mittag-Leffler function with some known special functions.

Theorem 4.1. *The m-parameter Mittag-Leffler function can be expressed in terms of Generalized hypergeometric function as:*

$$\begin{aligned}
& E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) \\
=& \frac{1}{\Gamma(\alpha_2)} {}_{\mu_2+\mu_4+\mu_6+\dots+\mu_{m-2}+1}F_{\alpha_1+\alpha_4+\alpha_6+\dots+\alpha_m} \\
& \left[\begin{matrix} \Delta(\mu_2;\mu_1) & \Delta(\mu_4;\mu_3) & \dots\Delta(\mu_{m-2};\mu_{m-3}) & (1,1) \\ \Delta(\alpha_1;\alpha_2) & \Delta(\alpha_4;\alpha_3) & \dots\Delta(\alpha_m,\alpha_{m-1}) & \end{matrix}; \frac{\mu_2^{\mu_2}\mu_4^{\mu_4}\dots\mu_{m-2}^{\mu_{m-2}}}{\alpha_1^{\alpha_1}\alpha_4^{\alpha_4}\alpha_6^{\alpha_6}\dots\alpha_m^{\alpha_m}}t\right],
\end{aligned} \tag{4.1}$$

where, $\Delta(\mu_2;\mu_1)$ *is a* μ_2 *tuple -* $\frac{\mu_1}{\mu_2}, \frac{\mu_1+1}{\mu_2}, \dots, \frac{\mu_1+\mu_2-1}{\mu_2}$; $\Delta(\mu_4;\mu_3)$ *is a* μ_4 *tuple -* $\frac{\mu_3}{\mu_4}, \frac{\mu_3+1}{\mu_4}, \dots, \frac{\mu_3+\mu_4-1}{\mu_4}$ *and so on.*

Proof.

$$LHS = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n} \dots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \dots (\alpha_{m-1})_{\alpha_m n}} t^n = \frac{1}{\Gamma(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n} \dots (\mu_{m-3})_{\mu_{m-2} n}}{(\alpha_2)_{\alpha_1 n}(\alpha_3)_{\alpha_4 n} \dots (\alpha_{m-1})_{\alpha_m n}} t^n$$

$$= \frac{1}{\Gamma(\alpha_2)} \sum_{n=0}^{\infty} \left[\frac{\prod_{i_1=1}^{\mu_2} \left(\frac{\mu_1+i_1-1}{\mu_2}\right)_n \prod_{i_2=1}^{\mu_4} \left(\frac{\mu_3+i_2-1}{\mu_4}\right)_n \dots \prod_{i_{\frac{m-2}{2}}=1}^{\mu_{m-2}} \left(\frac{\mu_{m-3}+i_{\frac{m-2}{2}}-1}{\mu_{m-2}}\right)_n}{\prod_{j_1=1}^{\alpha_1} \left(\frac{\alpha_2+j_1-1}{\alpha_1}\right)_n \prod_{j_2=1}^{\alpha_4} \left(\frac{\alpha_3+j_2-1}{\alpha_4}\right)_n \dots \prod_{j_{\frac{m}{2}}=1}^{\alpha_m} \left(\frac{\alpha_{m-1}+j_{\frac{m}{2}}-1}{\alpha_m}\right)_n} \right]$$

$$\times \left(\frac{\mu_2^{\mu_2} \mu_4^{\mu_4} \dots \mu_{m-2}^{\mu_{m-2}}}{\alpha_1^{\alpha_1} \alpha_4^{\alpha_4} \alpha_6^{\alpha_6} \dots \alpha_m^{\alpha_m}} \right)^n \frac{t^n (1)_n}{n!} = RHS.$$

Convergence conditions:

1. If $\mu_2 + \mu_4 + \mu_6 + \dots + \mu_{m-2} + 1 \leq \alpha_1 + \alpha_4 + \alpha_6 + \dots + \alpha_m$, then, ${}_{\mu_2+\mu_4+\mu_6+\dots+\mu_{m-2}+1}F_{\alpha_1+\alpha_4+\alpha_6+\dots+\alpha_m}$ converges for all finite t.
2. If $\mu_2 + \mu_4 + \mu_6 + \dots + \mu_{m-2} + 1 = \alpha_1 + \alpha_4 + \alpha_6 + \dots + \alpha_m + 1$, then, ${}_{\mu_2+\mu_4+\mu_6+\dots+\mu_{m-2}+1}F_{\alpha_1+\alpha_4+\alpha_6+\dots+\alpha_m}$ converges for $|t| < 1$ and diverges for $|t| > 1$.
3. If $\mu_2 + \mu_4 + \mu_6 + \dots + \mu_{m-2} + 1 > \alpha_1 + \alpha_4 + \alpha_6 + \dots + \alpha_m + 1$, then, ${}_{\mu_2+\mu_4+\mu_6+\dots+\mu_{m-2}+1}F_{\alpha_1+\alpha_4+\alpha_6+\dots+\alpha_m}$ is divergent for $|t| \neq 0$.
4. If $\mu_2 + \mu_4 + \mu_6 + \dots + \mu_{m-2} + 1 = \alpha_1 + \alpha_4 + \alpha_6 + \dots + \alpha_m + 1$, then, ${}_{\mu_2+\mu_4+\mu_6+\dots+\mu_{m-2}+1}F_{\alpha_1+\alpha_4+\alpha_6+\dots+\alpha_m}$ is absolutely convergent on the circle $|t| = 1$ if,
$$\Re \left(\sum_{j_1=1}^{\alpha_1} \dots \sum_{j_{\frac{m}{2}}=1}^{\alpha_m} \left(\frac{\alpha_2+j_1-1}{\alpha_1}\right) \dots \left(\frac{\alpha_{m-1}+j_{\frac{m}{2}}-1}{\alpha_m}\right) - \sum_{i_1=1}^{\mu_2} \dots \sum_{i_{\frac{m-2}{2}}=1}^{\mu_{m-2}} \left(\frac{\mu_1+i_1-1}{\mu_2}\right) \dots \left(\frac{\mu_{m-3}+i_{\frac{m-2}{2}}-1}{\mu_{m-2}}\right) \right)$$
> 0.

□

Theorem 4.2. *Let $\mu_1, \mu_2, \dots, \mu_{m-3}, \alpha_1, \dots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\,(\Re(\mu_1), \dots, \Re(\mu_{m-3}), \Re(\alpha_1), \dots, \Re(\alpha_{m-1})) > 0$. Then, the Mellin Barnes Integral representation of the m-parameter Mittag-Leffler function $E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t)$ is given by,*

$$E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t) = \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} \frac{1}{2\pi i} \times \int_C \frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)} (-t)^{-p} dp, \tag{4.2}$$

where, $|\arg z| < \pi$, C is the contour of integration which begins at $-i\infty$ and ends at $+i\infty$ and indented to distinguish integrand poles at $p = -n$, for all $n \in \mathbb{N}_0$ from those at $p = \frac{\mu_{m-3}+n}{\mu_{m-2}}$.

Proof. Consider,

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}(-t)^{-p}dp$$

Evaluating as the sum of residues at the poles $p=0,-1,-2,\dots$

$$\begin{aligned}
&=\sum_{n=0}^{\infty} Res_{p=-n}\left[\frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}(-t)^{-p}\right]\\
&=\sum_{n=0}^{\infty}\lim_{p\to -n}\frac{\pi(p+n)}{sin\pi p}\left[\frac{\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}(-t)^{-p}\right]\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)}(-t)^n\\
&=\frac{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}}t^n\\
&=\frac{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t).
\end{aligned}$$

This proves our result (4.2).

□

Theorem 4.3. *The m-parameter Mittag-Leffler function can be expressed in terms of Wright hypergeometric function as:*

$$E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t)=\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}{}_{\frac{m}{2}}\Psi_{\frac{m}{2}}\left[\begin{matrix}(\mu_1,\mu_2) & (\mu_3,\mu_4) & \dots(\mu_{m-3},\mu_{m-2}) & (1,1)\\ (\alpha_2,\alpha_1) & (\alpha_3,\alpha_4) & \dots(\alpha_{m-1},\alpha_m) & \end{matrix};t\right]. \tag{4.3}$$

Proof.

$$\begin{aligned}
LHS=&E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t)=\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}}t^n\\
&=\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)}t^n\\
&=\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)}\frac{\Gamma(n+1)}{n!}t^n\\
&=RHS.
\end{aligned}$$

□

Theorem 4.4. *In terms of Fox H-function, the m-parameter Mittag-Leffler function can be expressed as:*

$$\begin{aligned}
&E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(t)\\
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}H^{1,\frac{m-2}{2}}_{\frac{m-2}{2},\frac{m+2}{2}}\left[-t\left|\begin{matrix}(0,1) & (1-\mu_1,\mu_2) & (1-\mu_3,\mu_4) & \dots(1-\mu_{m-3},\mu_{m-2})\\ (0,1) & (1-\alpha_2,\alpha_1) & (1-\alpha_3,\alpha_4) & \dots(1-\alpha_{m-1},\alpha_m)\end{matrix}\right.\right].
\end{aligned} \tag{4.4}$$

Proof. The equations (4.2) and (1.14) when combined, gives the desired result. $\square$

5 Integral Transforms

In this section, we discuss various integral transforms of the m-parameter Mittag Leffler function-

Theorem 5.1. *(Euler-Beta Transform-) Let* $\Re(a), \Re(b) > 0$, $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\,(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})) > 0$, *then Euler-Beta transform of* $E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(xt^\gamma)$ *is given by-*

$$\begin{aligned}
&B(E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(xt^\gamma), a, b)\\
=&\int_0^1 t^{a-1}(1-t)^{b-1} E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(xt^\gamma)dt\\
=&\frac{\Gamma(b)\Gamma(\alpha_3)\Gamma(\alpha_5)\ldots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\ldots\Gamma(\mu_{m-3})}\,{}_{\frac{m+2}{2}}\Psi_{\frac{m+2}{2}}\\
&\left[\begin{matrix}(\mu_1,\mu_2) & (\mu_3,\mu_4) & \ldots(\mu_{m-3},\mu_{m-2}) & (a,\gamma) & (1,1)\\ (\alpha_2,\alpha_1) & (\alpha_3,\alpha_4) & \ldots(\alpha_{m-1},\alpha_m) & (a+b,\gamma) & \end{matrix};x\right].
\end{aligned} \tag{5.1}$$

Proof.

$$\begin{aligned}
LHS =&\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}x^n\int_0^1 t^{a+\gamma n-1}(1-t)^{b-1}dt\\
=&\sum_{n=0}^{\infty}\frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}}x^n B(a+\gamma n, b)\\
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\ldots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\ldots\Gamma(\mu_{m-3})}\\
&\times\sum_{n=0}^{\infty}\left(\frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\ldots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\ldots\Gamma(\alpha_{m-1}+\alpha_m n)}\frac{\Gamma(a+\gamma n)\Gamma(b)}{\Gamma(a+b+\gamma n)}\right)x^n\\
=&RHS.
\end{aligned}$$

This proves our result (5.1).

$\square$

Theorem 5.2. *(Whittaker Transform-)Let* $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\,(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})) > 0$, *then Whittaker transform of m-parameter Mittag-Leffler function is given by-*

$$\begin{aligned}
&\int_0^\infty e^{\frac{-\chi t}{2}} t^{\xi-1} W_{\kappa,\omega}(\chi t) E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(pt^\gamma)dt\\
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\ldots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\ldots\Gamma(\mu_{m-3})}\chi^{-\xi}\,{}_{\frac{m+4}{2}}\Psi_{\frac{m+2}{2}}\\
&\left[\begin{matrix}(\mu_1,\mu_2) & (\mu_3,\mu_4) & \ldots(\mu_{m-3},\mu_{m-2}) & (\frac{1}{2}-\omega+\xi,\gamma) & (\frac{1}{2}+\omega+\xi,\gamma) & (1,1)\\ (\alpha_2,\alpha_1) & (\alpha_3,\alpha_4) & \ldots(\alpha_{m-1},\alpha_m) & (1-\kappa+\xi,\gamma) & & \end{matrix};\frac{p}{\chi^\gamma}\right].
\end{aligned} \tag{5.2}$$

Proof. Let $\chi t = \eta$, then

$$\begin{aligned}
LHS =& \int_0^\infty e^{\frac{-\eta}{2}} \left(\frac{\eta}{\chi}\right)^{\xi-1} W_{\kappa,\omega}(\eta) \sum_{n=0}^\infty \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \frac{p^n \eta^{\gamma n}}{\chi^{\gamma n}} \frac{d\eta}{\chi} \\
=& \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} \chi^{-\xi} \sum_{n=0}^\infty \frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)} \\
&\times \left(\frac{p}{\chi^\gamma}\right)^n \int_0^\infty e^{\frac{-\eta}{2}} \eta^{\xi+\gamma n-1} W_{\kappa,\omega}(\eta) d\eta
\end{aligned}$$

Using equation (1.9),

$$\begin{aligned}
LHS =& \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} \chi^{-\xi} \sum_{n=0}^\infty \frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)} \\
&\times \left(\frac{p}{\chi^\gamma}\right)^n \frac{\Gamma(\frac{1}{2}-\omega+\xi+\gamma n)\Gamma(\frac{1}{2}+\omega+\xi+\gamma n)}{\Gamma(1-\kappa+\xi+\gamma n)} \\
=& \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} \chi^{-\xi} \sum_{n=0}^\infty \frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)} \\
&\times \frac{\Gamma(\frac{1}{2}-\omega+\xi+\gamma n)\Gamma(\frac{1}{2}+\omega+\xi+\gamma n)}{\Gamma(1-\kappa+\xi+\gamma n)} \frac{\Gamma(n+1)}{n!} \left(\frac{p}{\chi^\gamma}\right)^n = RHS.
\end{aligned}$$

This proves our result (5.2).

□

Theorem 5.3. *(Laplace Transform-)Let* $\Re(p) > 0$, $\mu_1, \mu_2, \dots, \mu_{m-3}, \alpha_1, \dots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\left(\Re(\mu_1), \dots, \Re(\mu_{m-3}), \Re(\alpha_1), \dots, \Re(\alpha_{m-1})\right) > 0$, *then Laplace transform of m-parameter Mittag-Leffler function is given by-*

$$\begin{aligned}
& L\left[t^{a-1} E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(xt^\eta); p\right] \\
=& \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} p^{-a} {}_{\frac{m+2}{2}}\Psi_{\frac{m}{2}} \\
& \left[\begin{matrix} (\mu_1,\mu_2) & (\mu_3,\mu_4) & \dots(\mu_{m-3},\mu_{m-2}) & (a,\eta) & (1,1) \\ (\alpha_2,\alpha_1) & (\alpha_3,\alpha_4) & \dots(\alpha_{m-1},\alpha_m) & & \end{matrix}; \frac{x}{p^\eta}\right].
\end{aligned} \tag{5.3}$$

Proof.

$$\begin{aligned}
LHS =& \int_0^\infty t^{a-1} e^{-pt} E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(xt^\eta) dt \\
=& \sum_{n=0}^\infty \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} x^n \int_0^\infty t^{a+\eta n-1} e^{-pt} dt \\
=& \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})} \sum_{n=0}^\infty \frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)} \frac{\Gamma(a+\eta n)}{p^{a+\eta n}} x^n
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}p^{-a}\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\dots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\dots\Gamma(\alpha_{m-1}+\alpha_m n)}\\
&\times\frac{\Gamma(a+\eta n)\Gamma(n+1)}{n!}\left(\frac{x}{p^\eta}\right)^n\\
=&RHS.
\end{aligned}
$$

This proves our result (5.3).

□

Theorem 5.4. *(Mellin Transform-) Let $\mu_1,\mu_2,\dots,\mu_{m-3},\alpha_1,\dots,\alpha_{m-1}\in\mathbb{C}$, $\mu_{m-2},\alpha_m>0$, $\Re(\alpha_1)+\alpha_m\geq\mu_{m-2}$, $min\left(\Re(\mu_1),\dots,\Re(\mu_{m-3}),\Re(\alpha_1),\dots,\Re(\alpha_{m-1})\right)>0$. Then, the Mellin transform of m-parameter Mittag-Leffler function is given by-*

$$
\begin{aligned}
&M\left[E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(-\kappa t);p\right]\\
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}\frac{1}{2\pi i}\\
&\times\int_C\frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}(-\kappa t)^{-p}dp.
\end{aligned}
\tag{5.4}
$$

Proof. From (4.2), we have

$$
\begin{aligned}
E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(-\kappa t)=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}\frac{1}{2\pi i}\\
&\times\int_C\frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}\\
&\times(-\kappa t)^{-p}dp\\
=&\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\dots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\dots\Gamma(\mu_{m-3})}\frac{1}{2\pi i}\int_C f^*(p)(t)^{-p}dp,
\end{aligned}
$$

where,

$$
f^*(p)=\frac{\Gamma(p)\Gamma(1-p)\Gamma(\mu_1-\mu_2 p)\Gamma(\mu_3-\mu_4 p)\dots\Gamma(\mu_{m-3}-\mu_{m-2}p)}{\Gamma(\alpha_2-\alpha_1 p)\Gamma(\alpha_3-\alpha_4 p)\dots\Gamma(\alpha_{m-1}-\alpha_m p)}(-\kappa)^{-p}.
$$

Using (1.10) and (1.11), we get our desired result (5.4). □

6 Relation with fractional calculus operators

In this section, we derive results considering Riemann Liouville fractional integral and derivatives.

Theorem 6.1. *Suppose $y>c, c\in\mathbb{R}_+=[0,\infty]$, $\mu_1,\mu_2,\dots,\mu_{m-3},\alpha_1,\dots,\alpha_{m-1}\in\mathbb{C}$, $\mu_{m-2},\alpha_m>0$, $\Re(\alpha_1)+\alpha_m\geq\mu_{m-2}$, $min\left(\Re(\mu_1),\dots,\Re(\mu_{m-3}),\Re(\alpha_1),\dots,\Re(\alpha_{m-1})\right)>0$, then,*

$$
\left(I^{\xi}_{c+}(\varpi)^{\alpha_2-1}E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}(\zeta\varpi^{\alpha_1})\right)(y)=\upsilon^{\alpha_2+\xi-1}E^{\mu_1,\dots,\mu_{m-2}}_{\alpha_1,\alpha_2+\xi,\alpha_3,\dots,\alpha_{m-1},\alpha_m}(\zeta\upsilon^{\alpha_1}).\tag{6.1}
$$

Proof. Using equation (1.15), we have

$$
\begin{aligned}
LHS =& \frac{1}{\Gamma(\xi)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \int_c^y \varpi^{\alpha_1 n+\alpha_2-1}(y-\eta)^{\xi-1} d\eta \\
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \left(I_{c+}^{\xi} \varpi^{\alpha_1 n+\alpha_2-1}\right)(y)
\end{aligned}
$$

Using result (1.20), we get,

$$
\begin{aligned}
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+\xi)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \upsilon^{\alpha_1 n+\alpha_2+\xi-1} \\
=& \upsilon^{\alpha_2+\xi-1} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2+\xi)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \upsilon^{\alpha_1 n} \\
=& RHS.
\end{aligned}
$$

□

Theorem 6.2. *Suppose* $y > c, c \in \mathbb{R}_+ = [0, \infty]$, $\mu_1, \mu_2, \dots, \mu_{m-3}, \alpha_1, \dots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\,(\Re(\mu_1), \dots, \Re(\mu_{m-3}), \Re(\alpha_1), \dots, \Re(\alpha_{m-1})) > 0$, *then,*

$$
\left(D_{c+}^{\xi} (\varpi)^{\alpha_2-1} E_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\varpi^{\alpha_1})\right)(y) = \upsilon^{\alpha_2-\xi-1} E_{\alpha_1,\alpha_2-\xi,\alpha_3,\dots,\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\upsilon^{\alpha_1}). \quad (6.2)
$$

Proof. Using equation (1.17), we have,

$$
LHS = \left(\frac{d}{dy}\right)^n \left[I_{c+}^{n-\xi} \varpi^{\alpha_2-1} E_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\varpi^{\alpha_1})\right]
$$

Using (6.1),

$$
LHS = \left(\frac{d}{dy}\right)^n \upsilon^{\alpha_2+n-\xi-1} E_{\alpha_1,\alpha_2+n-\xi,\alpha_3,\dots,\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\upsilon^{\alpha_1})
$$

Using (3.5),we get our desired result (6.2). □

Theorem 6.3. *Suppose* $y > c, c \in \mathbb{R}_+ = [0, \infty]$, $\mu_1, \mu_2, \dots, \mu_{m-3}, \alpha_1, \dots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\,(\Re(\mu_1), \dots, \Re(\mu_{m-3}), \Re(\alpha_1), \dots, \Re(\alpha_{m-1})) > 0$, *then,*

$$
\left(D_{c+}^{\xi,\nu} (\varpi)^{\alpha_2-1} E_{\alpha_1,\dots,\alpha_{m-2},\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\varpi^{\alpha_1})\right)(y) = \upsilon^{\alpha_2-\xi-1} E_{\alpha_1,\alpha_2-\xi,\alpha_3,\dots,\alpha_{m-1},\alpha_m}^{\mu_1,\dots,\mu_{m-2}}(\zeta\upsilon^{\alpha_1}). \quad (6.3)
$$

Proof.

$$
\begin{aligned}
LHS =& D_{c+}^{\xi,\nu} \left[\sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \varpi^{\alpha_1 n+\alpha_2-1}\right](y) \\
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\dots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\dots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \left[D_{c+}^{\xi,\nu} \varpi^{\alpha_1 n+\alpha_2-1}\right](y)
\end{aligned}
$$

Using (1.21),we get our desired result (6.3). □

7 Generalized Prabhakar Integral

Here, we use the m-parameter Mittag-Leffler function to define the generalized Prabhakar integral and discuss it's various properties.

Definition 7.1. *If* $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\left(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})\right) > 0$, *then,*

$$\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} g\right)(y) = \int_c^y (y-\eta)^{\alpha_2-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(y-\eta)^{\alpha_1}\right] g(\eta)d\eta. \tag{7.1}$$

The generalized Prabhakar integral involving Mittag Leffler function with m-parameters reduces to the following *special cases* on giving specific values to the various parameters:

(a) When, $\mu_k = \alpha_{k+2}$ for $2 < k \leq m-2$ and $\mu_2, \alpha_3, \alpha_4 = 1$, equation (7.1) reduces to the Prabhakar integral ([14, eq. 1.6, p-32]) given by (1.22).

(b) When, $\mu_k = \alpha_{k+2}$ for $2 < k \leq m-2$ and $\alpha_3, \alpha_4 = 1$, equation (7.1) reduces to the generalization of Prabhakar function given by Srivastava and Tomovski ([34, eq. (2.12), p. 202]) defined as (1.23).

Remark 7.1. *Series representation of* $\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} g\right)(y)$ *is given as,*

$$\begin{aligned}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} g\right)(y) = &\frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\ldots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\ldots\Gamma(\mu_{m-3})} \\ &\sum_{n=0}^{\infty} \frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\ldots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\ldots\Gamma(\alpha_{m-1}+\alpha_m n)}\left[I^{\alpha_2+\alpha_1 n}_{c+} g\right](y).\end{aligned} \tag{7.2}$$

Proof.

$$\begin{aligned} LHS &= \int_c^y (y-\eta)^{\alpha_2-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(y-\eta)^{\alpha_1}\right] g(\eta)d\eta \\ &= \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\cdots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\cdots(\alpha_{m-1})_{\alpha_m n}}\zeta^n \int_c^y (y-\eta)^{\alpha_2+\alpha_1 n-1} g(\eta)d\eta \\ &= \frac{\Gamma(\alpha_3)\Gamma(\alpha_5)\ldots\Gamma(\alpha_{m-1})}{\Gamma(\mu_1)\Gamma(\mu_3)\ldots\Gamma(\mu_{m-3})}\sum_{n=0}^{\infty}\frac{\Gamma(\mu_1+\mu_2 n)\Gamma(\mu_3+\mu_4 n)\ldots\Gamma(\mu_{m-3}+\mu_{m-2}n)}{\Gamma(\alpha_1 n+\alpha_2)\Gamma(\alpha_3+\alpha_4 n)\ldots\Gamma(\alpha_{m-1}+\alpha_m n)} \\ &\quad \times \left[I^{\alpha_2+\alpha_1 n}_{c+} g\right](y) \\ &= RHS. \end{aligned}$$

□

7.1 Properties of Generalized Prabhakar Integral

Theorem 7.1. *If* $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1) + \alpha_m \geq \mu_{m-2}$, $min\left(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})\right) > 0$, *then,*

$$\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} (\varpi)^{\mu-1}\right)(y) = \upsilon^{\alpha_2+\mu-1}\Gamma(\mu) E^{\mu_1,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+\mu,\alpha_3,\ldots,\alpha_{m-1},\alpha_m}(\zeta\upsilon^{\alpha_1}). \tag{7.3}$$

Proof.

$$
\begin{aligned}
LHS =& \int_c^y (y-\eta)^{\alpha_2-1}(\varpi)^{\mu-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(y-\eta)^{\alpha_1}\right] d\eta. \\
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \int_c^y (y-\eta)^{\alpha_2+\alpha_1 n-1}(\varpi)^{\mu-1} \\
=& \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n}(\mu_3)_{\mu_4 n}\ldots(\mu_{m-3})_{\mu_{m-2}n}}{\Gamma(\alpha_1 n+\alpha_2)(\alpha_3)_{\alpha_4 n}\ldots(\alpha_{m-1})_{\alpha_m n}} \zeta^n \left(\Gamma(\alpha_1 n+\alpha_2) I^{\alpha_2+\alpha_1 n}_{c+}(\varpi)^{\mu-1}\right)(y) \\
=& RHS.
\end{aligned}
$$

Using (1.20) we get our desired result (7.3). □

Theorem 7.2. *If* $\mu_1,\mu_2,\ldots,\mu_{m-3},\alpha_1,\ldots,\alpha_{m-1}\in\mathbb{C}$, $\mu_{m-2},\alpha_m>0$, $\Re(\alpha_1)+\alpha_m\geq\mu_{m-2}$, $min\left(\Re(\mu_1),\ldots,\Re(\mu_{m-3}),\Re(\alpha_1),\ldots,\Re(\alpha_{m-1})\right)>0$, *then,*

$$
\left[I^{\xi}_{c+}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)\right](y) = \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}g\right)(y) = \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}\left[I^{\xi}_{c+}g\right]\right)(y). \tag{7.4}
$$

Proof. We first prove $\left[I^{\xi}_{c+}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)\right](y) = \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}g\right)(y)$.
Using (1.15) and (7.1), we get,

$$
\begin{aligned}
\left[I^{\xi}_{c+}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)\right](y) =& \frac{1}{\Gamma(\xi)}\int_c^y \frac{\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)(\tau)}{(y-\tau)^{1-\xi}} d\tau \\
=& \frac{1}{\Gamma(\xi)}\int_c^y (y-\tau)^{\xi-1} \\
&\times \left[\int_c^\tau (\tau-\eta)^{\alpha_2-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(\tau-\eta)^{\alpha_1}\right] g(\eta)d\eta\right] d\tau
\end{aligned}
$$

Interchanging the order of integration using (1.24),

$$
\begin{aligned}
&\left[I^{\xi}_{c+}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)\right](y) \\
=& \frac{1}{\Gamma(\xi)}\int_c^y \left[\int_\eta^y (y-\tau)^{\xi-1}(\tau-\eta)^{\alpha_2-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(\tau-\eta)^{\alpha_1}\right] d\tau\right] g(\eta)d\eta
\end{aligned}
$$

Let $(\tau-\eta)=\kappa$, then,

$$
\begin{aligned}
&\left[I^{\xi}_{c+}\left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}g\right)\right](y) \\
=& \frac{1}{\Gamma(\xi)}\int_c^y \left[\int_0^{y-\eta} (y-\eta-\kappa)^{\xi-1}(\kappa)^{\alpha_2-1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(\kappa)^{\alpha_1}\right] d\kappa\right] g(\eta)d\eta \\
=& \int_c^y \kappa^{\alpha_2-1}\left[\frac{1}{\Gamma(\xi)}\int_0^{y-\eta} \frac{E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\left[\zeta(\kappa)^{\alpha_1}\right]}{(y-\eta-\kappa)^{1-\xi}} d\kappa\right] g(\eta)d\eta
\end{aligned}
$$

Using (1.15) and (6.1),

$$\begin{aligned}\left[I_{c+}^{\xi}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)\right](y) &= \int_c^y \kappa^{\alpha_2-1} I_{c+}^{\xi}\left[E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(\kappa)^{\alpha_1}\right]\right] g(\eta)d\eta \\ &= \int_c^y \upsilon^{\alpha_2+\xi-1} E_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(\upsilon)^{\alpha_1}\right] g(\eta)d\eta \\ &= \int_c^y (y-\eta)^{\alpha_2+\xi-1} E_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(y-\eta)^{\alpha_1}\right] g(\eta)d\eta \\ &= \left(\mathfrak{E}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)(y).\end{aligned}$$

Hence we have proved,

$$\left[I_{c+}^{\xi}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)\right](y) = \left(\mathfrak{E}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)(y). \tag{7.5}$$

We now prove $\left(\mathfrak{E}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)(y) = \left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y)$.

$$\begin{aligned}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y) &= \int_c^y (y-\eta)^{\alpha_2-1} E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(y-\eta)^{\alpha_1}\right]\left(I_{c+}^{\xi}g\right)(\eta)d\eta. \\ &= \int_c^y (y-\eta)^{\alpha_2-1} E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(y-\eta)^{\alpha_1}\right] \\ &\quad \times \left(\frac{1}{\Gamma(\xi)}\int_0^y \frac{g(u)}{(\eta-u)^{1-\xi}du}\right)d\eta\end{aligned}$$

Interchanging the order of integration using (1.24),

$$\begin{aligned}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y) &= \int_c^y \frac{1}{\Gamma(\xi)}\left[\int_u^y (y-\eta)^{\alpha_2-1}(\eta-u)^{\xi-1} E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(y-\eta)^{\alpha_1}\right]d\eta\right] \\ &\quad \times g(u)du\end{aligned}$$

Let $(y-\eta)=\rho$, then,

$$\begin{aligned}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y) &= \int_c^y \frac{1}{\Gamma(\xi)}\left[\int_0^{y-u} (\rho)^{\alpha_2-1}(y-\rho-u)^{\xi-1} E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(\rho)^{\alpha_1}\right]d\rho\right] \\ &\quad \times g(u)du\end{aligned}$$

Using (1.15) and (6.1),

$$\begin{aligned}\left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y) &= \int_c^y \rho^{\alpha_2-1}\left[\int_0^{y-u}\frac{1}{\Gamma(\xi)}\frac{E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(\rho)^{\alpha_1}\right]}{(y-\rho-u)^{1-\xi}}d\rho\right] g(u)du \\ &= \int_c^y \rho^{\alpha_2-1} I_{c+}^{\xi}\left(E_{\alpha_1,\alpha_2,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(\rho)^{\alpha_1}\right]\right) g(u)du \\ &= \int_c^y (y-\eta)^{\alpha_2+\xi-1} I_{c+}^{\xi}\left(E_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m}^{\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[\zeta(y-\eta)^{\alpha_1}\right]\right) g(u)du \\ &= \left(\mathfrak{E}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)(y)\end{aligned}$$

Hence we have proved,

$$\left(\mathfrak{E}_{\alpha_1,\alpha_2+\xi,\alpha_3,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}g\right)(y) = \left(\mathfrak{E}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}\left[I_{c+}^{\xi}g\right]\right)(y). \tag{7.6}$$

From (7.5) and (7.6), we get our desired result (7.4). □

Theorem 7.3. *If* $\mu_1, \mu_2, \ldots, \mu_{m-3}, \alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, $\mu_{m-2}, \alpha_m > 0$, $\Re(\alpha_1)+\alpha_m \geq \mu_{m-2}$, *min* $(\Re(\mu_1), \ldots, \Re(\mu_{m-3}), \Re(\alpha_1), \ldots, \Re(\alpha_{m-1})) > 0$, *then,*

$$\| \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} \Psi \right) \|_1 \leq M \|\Psi\|_1, \tag{7.7}$$

where,

$$M = \left((d-c)^{\Re(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} \frac{(d-c)^{\alpha_1 n}}{\Re(\alpha_2)+\Re(\alpha_1)n} |\zeta^n| \right).$$

Proof. Using $\|g\|_1 = \int_c^d g(y)dy$, and then using (1.24), we get,

$$\begin{aligned} \| \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} \Psi \right) \|_1 &= \int_c^d \left| \int_c^y (y-\eta)^{\alpha_2 - 1} E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m} \left[\zeta (y-\eta)^{\alpha_1}\right] \Psi(\eta) d\eta \right| dy \\ &\leq \int_c^d \left[\int_\eta^d (y-\eta)^{\Re(\alpha_2)-1} \left| E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m} \left[\zeta (y-\eta)^{\alpha_1}\right] \right| dy \right] |\Psi(\eta)| d\eta. \end{aligned}$$

Let $(y-\eta) = u$, then,

$$\begin{aligned} &\| \left(\mathfrak{E}^{\zeta,\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m,c+} \Psi \right) \|_1 \\ &\leq \int_c^d \left[\int_0^{d-\eta} (u)^{\Re(\alpha_2)-1} \left| E^{\mu_1,\mu_2,\ldots,\mu_{m-2}}_{\alpha_1,\alpha_2,\ldots,\alpha_m} \left[\zeta (u)^{\alpha_1}\right] \right| du \right] |\Psi(\eta)| d\eta \\ &\leq \int_c^d \left[\sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} |\zeta^n| \int_0^{d-\eta} u^{\Re(\alpha_2)+\Re(\alpha_1)n-1} du \right] \\ &\quad \times \Psi(\eta)| d\eta \\ &\leq \int_c^d \left[\sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} |\zeta^n| \int_0^{d-c} u^{\Re(\alpha_2)+\Re(\alpha_1)n-1} du \right] \\ &\quad \times \Psi(\eta)| d\eta \\ &= \int_c^d \left[\sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} |\zeta^n| \left(\frac{u^{\Re(\alpha_2)+\Re(\alpha_1)n}}{\Re(\alpha_2)+\Re(\alpha_1)n} \right)_0^{d-c} \right] \\ &\quad \times \Psi(\eta)| d\eta \\ &= \left((d-c)^{\Re(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} \frac{(d-c)^{\alpha_1 n}}{\Re(\alpha_2)+\Re(\alpha_1)n} |\zeta^n| \right) \\ &\quad \times \int_0^{d-c} \Psi(\eta)| d\eta \\ &= \left((d-c)^{\Re(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} \frac{(d-c)^{\alpha_1 n}}{\Re(\alpha_2)+\Re(\alpha_1)n} |\zeta^n| \right) \\ &\quad \times \|\Psi\|_1 \\ &= M \|\Psi\|_1 \end{aligned}$$

where, $M = \left((d-c)^{\Re(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\mu_1)_{\mu_2 n} (\mu_3)_{\mu_4 n} \ldots (\mu_{m-3})_{\mu_{m-2} n}}{\Gamma(\alpha_1 n + \alpha_2)(\alpha_3)_{\alpha_4 n} \ldots (\alpha_{m-1})_{\alpha_m n}} \frac{(d-c)^{\alpha_1 n}}{\Re(\alpha_2)+\Re(\alpha_1)n} |\zeta^n| \right)$. Hence we get our desired result. □

Particular values							
α_i, μ_i	Fig-1(a)	Fig-1 (b)	Fig-2 (a)	Fig-2 (b)	Fig-3 (a)	Fig-3 (b)	Fig-4
α_1	1	1	2	4	2	2	2
α_2	1	2	2	2	2	2	2
α_3	1	1	1	1	1	1	2
α_4	1	1	1	1	1	1	1
α_5	1	1	1	1	1	1	1
α_6	1	1	1	1	1	1	1
α_7	1	1	1	1	1	1	1
α_8	1	1	1	1	1	1	1
α_9	1	1	1	1	1	1	1
μ_1	1	1	1	1	1	1	2
μ_2	1	1	1	1	1	1.5	2
μ_3	1	1	1	1	1	1.5	2
μ_4	1	1	1	1	2	2	2
μ_5	1	1	1	1	1	2	1
μ_6	1	1	1	1	0.5	0.5	1
μ_7	1	1	1	1	0.5	0.5	1
Function	e^t	$\frac{-1+e^t}{t}$	$\frac{\sinh(\sqrt{t})}{\sqrt{t}}$	$\frac{sin(t^{\frac{1}{4}})+\sinh(t^{\frac{1}{4}})}{2(t^{\frac{1}{4}})}$	${}_pF_q$	${}_rF_s$	$\frac{1-\sqrt{(1-4t)}}{2t\sqrt{1-4t}}$

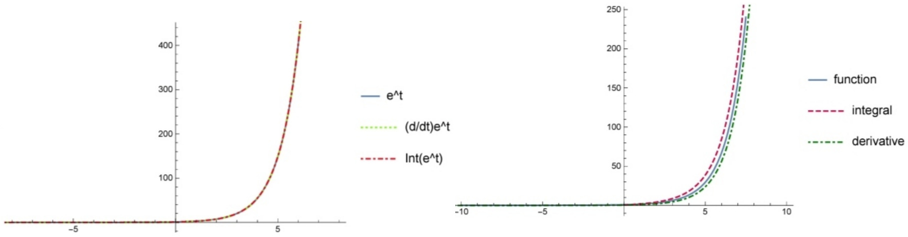
Table 1: Special cases of m-parameter Mittag Leffler function

8 Relation of m-parameter Mittag Leffler function with other functions and their graphical representation

In this section, Using MATHEMATICA 12, we reduce the m-parameter Mittag Leffler function in terms of other functions such as exponential, trigonometric, hypergeometric and algebraic functions by taking different values of the parameters. Also, we represent the functions, their integrals and derivatives graphically using MATHEMATICA 12.

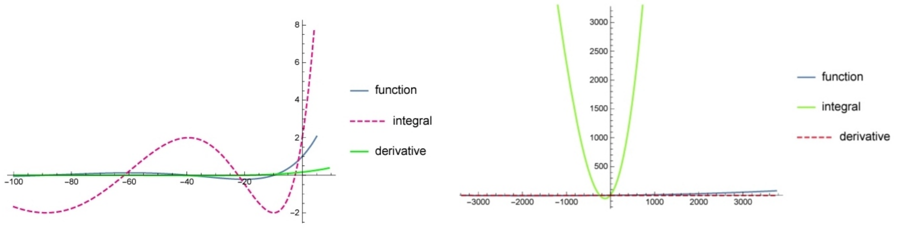
The following table gives us the list of functions after taking particular values of α_i's and μ_i's, once m is fixed.

Using MATHEMATICA 12, we now represent the functions mentioned in the above table, its integrals and derivatives graphically.



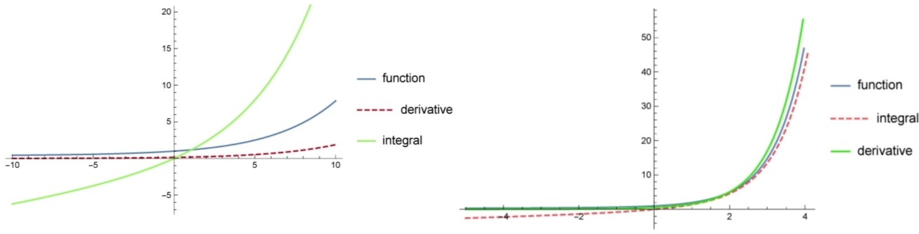
(a) e^t (b) $\frac{-1+e^t}{t}$

Figure 1: m-parameter Mittag Leffler function reduced to exponential function.



(a) $\frac{\sinh(\sqrt{t})}{\sqrt{t}}$ (b) $\frac{sin(t^{\frac{1}{4}})+\sinh(t^{\frac{1}{4}})}{2(t^{\frac{1}{4}})}$

Figure 2: m-parameter Mittag Leffler function reduced to trigonometric function.



(a) ${}_pF_q$ (b) ${}_rF_s$

Figure 3: m-parameter Mittag Leffler function reduced to hypergeometric function.

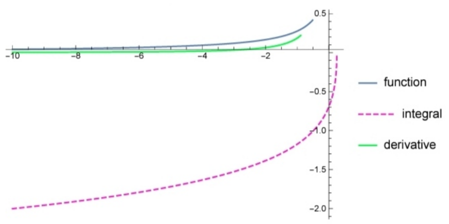


Figure 4: m-parameter Mittag Leffler function reduced to algebraic function $\frac{1-\sqrt{(1-4t)}}{2t\sqrt{1-4t}}$.

Similarly, we can take different values of the parameters and deduce the relation of m-parameter Mittag Leffler function with different functions, which cannot be done manually, using MATHEMATICA 12.

9 Applications

Mittag Leffler function has come into prominence due to its various applications in solving the problems of earth sciences, biological and physical engineering. It is used in various problems of fluid flow, rheology, diffusive transport, electric networks, probability and statistical distribution theory.
While moving from an integer order differential equation to fractional order differential equation, the solutions occur in terms of Mittag Leffler function and its generalizations. For instance, while investigating fractional generalization of the Kinetic equation, random walks, super-diffusive transport, study of complex systems and in Levy flights [7].
There is interpolation of Mittag Leffler function and its generalizations between a purely exponential law and power-law like behaviour of phenomena driven by ordinary Kinetic equations and their fractional counterparts (See, e.g. [1, 9, 10, 16, 17, 28]).
Certain boundary value problems that involve volterra type fractional integrodifferential equation has solutions involving Mittag Leffler function [27]. The solution of Abel-Volterra type equation has been expressed in terms of Mittag Leffler function by Hills and Tamarkin [11].
Mittag Leffler function is also used in Modelling. For instance, the one parameter Mittag Leffler function is used to describe the Philips curve, that is to describe the relations between unemployment rate and inflation rate [30]. Prabhakar function was used in dielectric relaxation models. Also used in mathematical models of filtration dynamics that are based on fractional equations constituting Hilfer-Prabhakar derivatives [4, 6, 22]. Recently, Prabhakar function was applied in the theory of non-linear heat conduction with memory and results were derived using generalized separating variable method [5, 23].

10 Conclusion

In our paper, we have defined the m-parameter Mittag-Leffler function, reduced it to known extensions of Mittag-Leffler function. We then, discussed its various properties, special function representations, integral representations and applied fractional calculus operators. We used this m-parameter Mittag-Leffler function to generalize the Prabhakar integral and discussed it various properties. We also obtained the relation of m-parameter Mittag Leffler function with various other functions such as exponential, trigonometric, hypergeometric and algebraic functions and represented graphically using MATHEMATICA 12. Mittag-Leffler function with one parameter has found its applications in the AB Model and in relaxation models involving exponential and power law behavious interpolation [19]. The data received from experimenting with models for cells and tissues is linked to the Wiman function [3]. Prabhakar function is associated with Havriliak-Negami relaxation [8]. Hence, we believe that the results of the m-parameter Mittag-Leffler function can also be used in simplification of certain important physical models in near future.

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