Numerical solution of optimal control problems using Genocchi polynomials

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Abstract

In this paper, a novel numerical method for solving the optimal control problems (OCPs) are presented. This method uses Genocchi polynomials. Some properties of Genocchi polynomials are given and the operational matrix of derivative is constructed. This matrix helps us to convert the nonlinear constrained optimal control problem to the nonlinear programming one that can be solved by Maple programming software. The presented method is applied on some numerical examples in order to show its advantages.

Keywords: *Optimal control problems, Genocchi polynomials, Direct method*
AMS subject classification: 65K15, 74S25

1 INTRODUCTION

Optimal control problems, which often include an objective function and nonlinear constraints, need to be solved numerically using high-precision methods with low time and low cost, given the importance they have found in various engineering sciences. The numerical methods are divided into two parts, the indirect [5, 8, 25] and the direct methods [6, 7]. In the indirect methods, calculus of variations are used and the necessary conditions are achieved. Then we deal with a multi-point boundary value problem which must be solved. In this method, guessing the values of costate vectors are needed. In the direct methods both the control and the state vectors are discretized [16, 27]. After discretization of the vectors, a nonlinear optimization problem is obtained which can be solved by a suitable numerical optimization method [6]. Recently, most attention to solve such matters has been placed on the spectral methods. In these methods we expand the control and the state vectors as an unknown linear combination of a suitable base. Also the derivative of the control and the state vectors is obtained by the same base using the operational matrix of derivation. Vlassenbroeck has used Chebyshev polynomials to resolve control problems [29, 30]. Elnagar benefited from the legendre polynomials for approximation control and state vectors [14]. Edrisi et al. has used linear B-spline functions as polynomials for approximation [12]. In this paper we use Genocchi polynomials to approximate the unknown functions. The origin of the Genocchi numbers is provided by Anthony Genocchi (1817-1889). Useful properties of the Genocchi polynomials make us expect that the numerical solution of the optimal control problem have more accuracy. So that we can obtain an approximate

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solution for the objective function, system state equations, conditions and control vectors to transform the optimal control problem into a nonlinear programming problem. This method is based on the direct methods.

This paper's text order is as follows: Section 2, states the optimal control problem. In Section 3, the Genocchi numbers and polynomials and some properties of them are presented. The objective function and state equations are approximated in Section 4. In Section 5, some numerical examples are presented and solved by the presented method and finally, in Section 6, conclusion of the paper is stated.

2 Optimal control problem

Consider following OCP:
Minimizing the objective function

$$J = \int_0^1 h\left(x\left(t\right), u\left(t\right), t\right) \mathrm{d}t, \tag{1}$$

for finding the control vector, $u\left(t\right)$, and the corresponding state vector, $x\left(t\right)$, that apply to the following constraints

$$\dot{x}\left(t\right) = f\left(x\left(t\right), u\left(t\right), t\right), \tag{2}$$

$$g\left(x\left(t\right), u\left(t\right), t\right) \leq 0, \tag{3}$$

$$x(0) = x_0,\ x(1) = x_1, \tag{4}$$

where $x(t)$ and $u(t)$ are unknown vectors with dimensions $n \times 1$ and $m \times 1$ respectively which must be defined, h, f and g are known functions as $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, and x_0 and x_1 are $n \times 1$ known vectors.

3 Genocchi numbers and polynomials

The Genocchi numbers and polynomials are widely used in mathematics and physics. Genocchi numbers, G_n, and polynomials, $G_n(x)$, are defined respectively, by using exponential generating functions as [2–4, 18, 23]:

$$\begin{aligned} Q(t) &:= \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \\ Q(t,x) &:= \frac{2te^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} G_n\left(x\right) \frac{t^n}{n!}. \end{aligned} \tag{5}$$

Then, we can write the Genocchi polynomial of degree n as

$$G_n\left(x\right) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k} x^k, \tag{6}$$

where G_k in Eq.(6) is the Genocchi number [18–20] :

$$\begin{aligned}
&G_0 = 0, \\
&G_1 = 1, \\
&\left.\begin{aligned} &G_{2i} = 2iE_{2i-1}(0), \\ &G_{2i+1} = 0, \end{aligned}\right. \quad (i \geq 1)
\end{aligned}$$

and E_i is Euler's number, which is defined as [1,3,22]

$$E_{2i-1}(0) = 2^{1-2i} \sum_{j=1}^{2i-1} \left((-1)^{2i+j} . j^{2i-1} \left(\sum_{k=0}^{2i-1} \left(\begin{array}{c} 2i \\ k \end{array} \right) \right) \right) .$$

Also, we can write Genocchi numbers and polynomials in terms of Bernoulli numbers, B_n, and polynomials, $B_n(x)$ as

$$\begin{aligned}
G_n &= 2(1-2^n)B_n, \\
G_n(x) &= 2B_n(x) - 2^{n+1}B_n\left(\frac{x}{2}\right).
\end{aligned}$$

Some of the important properties of the Genocchi polynomials are as below:

$$\int_0^1 G_m(x)\, G_l(x)\, \mathrm{d}x = \frac{2(-1)^m m!l!}{(m+l)!} G_{m+l}, \qquad m,l \geq 1, \tag{7}$$

$$\frac{dG_i(x)}{dx} = iG_{i-1}(x), \qquad i \geq 1, \tag{8}$$

$$\begin{aligned}
&G_i(x+1) + G_i(x) = 2ix^{i-1}, \\
&G_i(1) + G_i(0) = 0, \qquad i > 1.
\end{aligned}$$

Here, we use Genocchi polynomials as basis polynomials to approximate the state and control variables. Let $G = Span\{G_1(t), G_2(t), \ldots, G_M(t)\}$ is generated by Genocchi polynomials. For every arbitrary element of $f(t) \in L^2[0,1]$, there is a unique best approximation in G named $f^*(t)$ such that

$$\forall g(t) \in G, \ \|f(t) - f^*(t)\| \leq \|f(t) - g(t)\|,$$

so, for every $g(t) \in G$

$$\langle f(t) - f^*(t), g(t) \rangle = 0, \tag{9}$$

where $\langle . \rangle$ is the inner product. In result of belonging $f^*(t)$ to G, unique coefficients $C_1, C_2, \ldots, C_M$ exist which we can approximate the arbitrary function $f(t)$ as [10,15]

$$f(t) \approx f^*(t) = \sum_{i=1}^{M} C_i G_i(t) = \mathbf{C}^T \mathbf{G}(t),$$

where $\mathbf{C} = [C_1, C_2, \ldots, C_M]^T$, $\mathbf{G}(t) = [G_1(t), G_2(t), \ldots, G_M(t)]^T$.

According to (9), we can write

$$\langle f(t) - \mathbf{C}^T \mathbf{G}(t), G_i(t) \rangle = 0, \ i = 1, 2, \ldots, M.$$

So, any arbitrary function $f(t) \in L^2[0,1]$ can be approximated by Genocchi basis polynomials as $f(t) = \mathbf{C}^T\mathbf{G}(t)$ where

$$\mathbf{C} = P^{-1}\langle f(t), \mathbf{G}(t)\rangle, \tag{10}$$

and

$$P = \langle \mathbf{G}(t), \mathbf{G}(t)\rangle = \int_0^1 \mathbf{G}(t)\mathbf{G}^T(t)\mathrm{d}t, \tag{11}$$

is a $M \times M$ matrix with entires obtained from (7) as

$$P = [p_{ij}]_{M\times M}, \quad p_{i,j} = \frac{2(-1)^i i!\,j!}{(i+j)!}G_{i+j}, \quad i,j = 1,2,\ldots,M.$$

For example for $M = 8$ we have :

$$P = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{17}{4} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{2}{5} & 0 & \frac{17}{14} & 0 & -\frac{62}{9} \\ -\frac{1}{2} & 0 & \frac{3}{10} & 0 & -\frac{17}{28} & 0 & \frac{31}{12} & 0 \\ 0 & -\frac{2}{5} & 0 & \frac{17}{35} & 0 & -\frac{31}{21} & 0 & \frac{1382}{165} \\ 1 & 0 & -\frac{17}{28} & 0 & \frac{155}{126} & 0 & -\frac{691}{132} & 0 \\ 0 & \frac{17}{14} & 0 & -\frac{31}{21} & 0 & \frac{691}{154} & 0 & -\frac{10922}{429} \\ -\frac{17}{4} & 0 & \frac{31}{12} & 0 & \frac{691}{132} & 0 & \frac{38227}{1716} & 0 \\ 0 & -\frac{62}{9} & 0 & \frac{1382}{165} & 0 & -\frac{10922}{429} & 0 & \frac{929569}{6435} \end{pmatrix}.$$

In order to obtain the derivative of Genocchi basis polynomials, $G(t)$, we use (8) and get

$$\mathbf{G}'(t) = \frac{d}{dt}(\mathbf{G}(t)) = D_G\mathbf{G}(t),$$

where D_G is a $M \times M$ operational matrix of derivative as

$$\mathbf{D_G} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & M-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & M & 0 \end{pmatrix}.$$

In order to approximate the derivative of the arbitrary function, we use operational matrix derivative of the Genocchi polynomials as below

$$f'(t) = \mathbf{C}^T\mathbf{G}'(t) = \mathbf{C}^T D_G\mathbf{G}(t).$$

4 Approximation of objective function and system constraints

Let

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T, \tag{12}$$

$$\dot{x}(t) = [\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]^T, \tag{13}$$

$$u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T, \tag{14}$$

and to generalize the discussion, suppose

$$\widehat{G}_n(t) = I_n \otimes \mathbf{G}(t), \tag{15}$$

$$\widehat{G}_{D,n}(t) = I_n \otimes D_G \mathbf{G}(t), \tag{16}$$

$$\widehat{G}_m(t) = I_m \otimes \mathbf{G}(t), \tag{17}$$

where I_n and I_m are identity matrix, $\mathbf{G}(t)$ is a vector at dimension $M \times 1$, $\otimes$ is the Kronecker product [24], $\widehat{G}_n(t)$, $\widehat{G}_{D,n}(t)$ are matrices with dimension $Mn \times n$ and $\widehat{G}_m(t)$ is a $Mm \times m$ matrix.
It is assumed that any $x_i, i = 1 \ldots n$ and $u_j, j = 1 \ldots m$, that in Eqs.(12)-(14), can be approximated as Genocchi basis polynomials

$$x_i(t) \approx \mathbf{G}^T(t)\mathbf{X}_i, \tag{18}$$

$$\dot{x}_i(t) \approx \mathbf{G}^T(t) D_G \mathbf{X}_i, \tag{19}$$

$$u_j(t) \approx \mathbf{G}^T(t)\mathbf{U}_j, \tag{20}$$

where $\mathbf{X}_i$ and $\mathbf{U}_j$ are $M \times 1$ vectors. So, from Eqs. (15)-(17) we have

$$x(t) \approx \widehat{G}_n^T(t) X, \tag{21}$$

$$\dot{x}(t) \approx \widehat{G}_{D,n}^T(t) X, \tag{22}$$

$$u(t) \approx \widehat{G}_m^T(t) U, \tag{23}$$

where X and U are matrices of dimension $n \times M$ and $m \times M$, respectively, and

$$X = [X_1, X_2, \ldots, X_n]^T,$$
$$U = [U_1, U_2, \ldots, U_m]^T.$$

Now, we want to approximate the objective function of OCP. To do this, we replace (21) and (23) in (1) and get

$$J \approx \int_0^1 h\left(\widehat{G}_n^T(t) X, \widehat{G}_m^T(t) U, t\right) \mathrm{d}t. \tag{24}$$

There are two cases:

(i) h in (24) is a quadratic function, then we have

$$J \approx \int_0^1 \left(x^T(t) Q x(t) + u^T(t) R u(t)\right) \mathrm{d}t \tag{25}$$

where Q and R are suitable matrices. By replacing (21) and (23) in (25) we get

$$J \approx X^T\left(\int_0^1 \widehat{G}_n(t) Q \widehat{G}_n^T(t) \mathrm{d}t\right) X + U^T\left(\int_0^1 \widehat{G}_m(t) R \widehat{G}_m^T(t) \mathrm{d}t\right) U. \tag{26}$$

We simplify (26) as

$$J \approx X^T \left(\int_0^1 Q \otimes G(t) G^T(t) \mathrm{d}t \right) X + U^T \left(\int_0^1 R \otimes G(t) G^T(t) \mathrm{d}t \right) U. \tag{27}$$

Finally, we change J with $J(X,U)$ and rewrite it as

$$J(X,U) \approx X^T (Q \otimes P) X + U^T (R \otimes P) U, \tag{28}$$

where P is the same as (11).

(ii) h in (24) is a time-varying quadratic function or an arbitrary one, then we use a suitable Newton-Cotes numerical integration method [28] and approximate the objective function as

$$J(X,U) = \sum_{i=0}^{r} \omega_i h\left(\widehat{G}_n^T(t_i) X, \widehat{G}_m^T(t_i) U, t_i \right), \quad t_i = \frac{i}{r}, i = 1, 2, \ldots, r, \tag{29}$$

where the weight ω_i is determined by

$$\omega_i = \int_0^1 l_i(t) \mathrm{d}t,$$

and each $l_i(t)$ is a Lagrange polynomial as

$$l_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{t - t_j}{t_i - t_j}.$$

For approximating the OCP constraints, we substitute (21) - (23) in (2) - (4) and we get

$$\widehat{G}_{D,n}^T(t) X \approx f\left(\widehat{G}_n^T(t) X, \widehat{G}_m^T(t) U, t \right), \tag{30}$$

$$g\left(\widehat{G}_n^T(t) X, \widehat{G}_m^T(t) U, t \right) \leq 0, \tag{31}$$

$$\widehat{G}_n^T(0) X \approx x_0, \ \widehat{G}_n^T(1) X \approx x_1. \tag{32}$$

By collocating Eqs. [28] and [29] at the chebyshev nodes

$$t_i = \frac{1}{2}\left[1 + \cos\left(\frac{2i-1}{2M} \pi \right) \right], \quad i = 1, \ldots, M, \tag{33}$$

We obtain

$$\widehat{G}_{D,n}^T(t_i) X = f\left(\widehat{G}_n^T(t_i) X, \widehat{G}_m^T(t) U, t_i \right), \tag{34}$$

$$g\left(\widehat{G}_n^T(t_i) X, \widehat{G}_m^T(t_i) U, t_i \right) \leq 0. \tag{35}$$

Therefore, solving an OCP turns to solving NLP such that we want to find X and U to minimize the objective function (28) or (29) and apply the constraints (32), (34) and (35).

5 Numerical Examples

Example 5.1. *Consider the following OCP [9,11,13]:*

$$\begin{aligned} \textit{minimize} \quad & J = \int_0^1 u^2(t)\,dt \\ \textit{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = u(t), \\ & x_1(0) = 1, \\ & x_2(0) = 1, \\ & x_1(1) = 0. \end{aligned}$$

The exact optimal value of this problem is $J^* = 12$ and its exact optimal solutions are

$$\begin{aligned} u^*(t) &= 6(t-1), \\ x^*{}_1(t) &= t^3 - 3t^2 + t + 1, \\ x^*{}_2(t) &= 3t^2 - 6t + 1. \end{aligned}$$

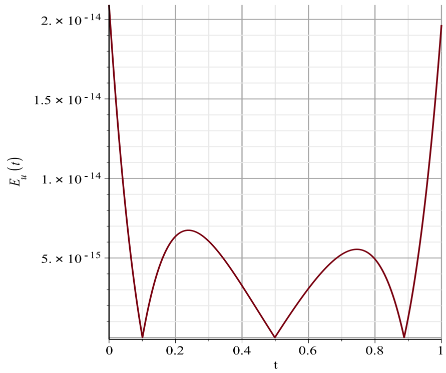
Assume $\hat{J}^*$, $\hat{x}_1{}^*(t)$, $\hat{x}_2{}^*(t)$, $\hat{u}^*(t)$ be the approximate optimal values, obtained by the presented method and J^*, $x_1{}^*(t)$, $x_2{}^*(t)$, $u^*(t)$ are the exact values of them.
Define error values in the form

$$\begin{aligned} E_J &= \left| J^* - \hat{J}^* \right|,, \\ E_{x_1} &= \|x_1{}^*(t) - \hat{x}_1{}^*(t)\|_{\infty,[0,1]}, \\ E_{x_2} &= \|x_2{}^*(t) - \hat{x}_2{}^*(t)\|_{\infty,[0,1]}, \\ E_u &= \|u^*(t) - \hat{u}^*(t)\|_{\infty,[0,1]}. \end{aligned}$$

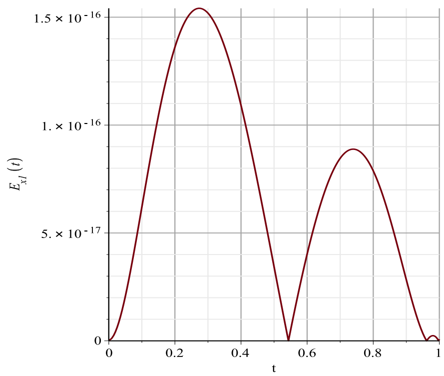
Table 1, shows the amount of these errors for different values of M. Table 2, shows the $'E_J'$ errors for different values of M for the methods presented in [13]. Comparing the value of objective function obtained by the presented method with two methods presented at [13] in Tab 2, we find that the presented method is better in accuracy and time saving in this example. The exact and approximate values of the optimal control and state vectors and related errors by using the Genocchi basis polynomials method are shown in Fig. 1

Table 1: Error values obtained by the presented method for Example 5.1

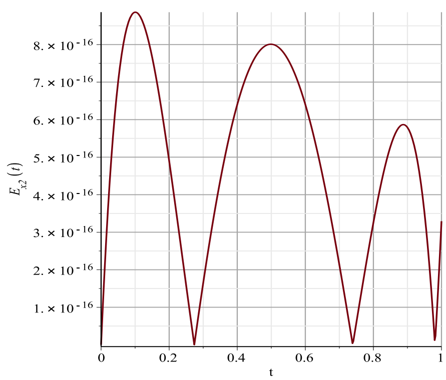
M	E_J	E_{x_1}	E_{x_2}	E_u	CPU Time
6	1.4×10^{-14}	2.04×10^{-12}	1.02×10^{-11}	2.11×10^{-10}	0.046
7	7.3×10^{-16}	6.83×10^{-15}	4.19×10^{-14}	1.12×10^{-12}	0.157
8	2.2×10^{-17}	1.51×10^{-16}	8.97×10^{-16}	2.09×10^{-14}	0.641
9	6.0×10^{-18}	2.61×10^{-15}	1.74×10^{-14}	1.07×10^{-12}	0.733



(a) Plot of $\left|\overset{*}{u}(t)-\hat{u}(t)\right|$



(b) Plot of $\left|\overset{*}{x}_1(t)-\hat{x}_1(t)\right|$



(c) Plot of $\left|\overset{*}{x}_2(t)-\hat{x}_2(t)\right|$

Figure 1: Absolute values of errors for $u(t)$, $x_1(t)$ and $x_2(t)$ using the presented method with $M=8$ for example 5.1

Table 2: Error values of the objective function derived from the methods in [13] for Example 5.1

M	Method 1 E_J	Method 1 CPU Time	Method 2 E_J	Method 2 CPU Time
6	3.67×10^{-2}	0.053	5.93×10^{-3}	0.163
7	1.86×10^{-2}	0.181	1.48×10^{-3}	0.401
8	9.34×10^{-3}	1.034	3.74×10^{-4}	1.377
9	4.68×10^{-3}	7.662	9.38×10^{-5}	7.400

Table 3: Error values obtained by the presented method for Example 5.2

M	E_J	E_{x_1}	E_{x_2}	E_u	CPU Time
6	1.12×10^{-3}	5.01×10^{-2}	7.02×10^{-2}	8.96×10^{-2}	0.402
7	1.06×10^{-3}	5.50×10^{-2}	7.64×10^{-2}	4.21×10^{-2}	0.577
8	1.16×10^{-3}	5.55×10^{-2}	7.44×10^{-2}	1.52×10^{-2}	0.609
9	1.09×10^{-3}	5.18×10^{-2}	7.45×10^{-2}	3.21×10^{-2}	0.639

Example 5.2. *Consider the following optimal control problem [12]*

$$\begin{aligned} &\textit{minimize} && J=\frac{1}{2}\int_0^1 \left({x_1}^2(t)+u^2(t)\right)\mathrm{d}t,\\ &\textit{subject to} && \dot{x}_1(t)=x_2(t),\\ &&& \dot{x}_2(t)=-x_2(t)+u(t),\\ &&& x_1(0)=0,\\ &&& x_2(0)=10,\\ &&& |u(t)|\leq 1. \end{aligned}$$

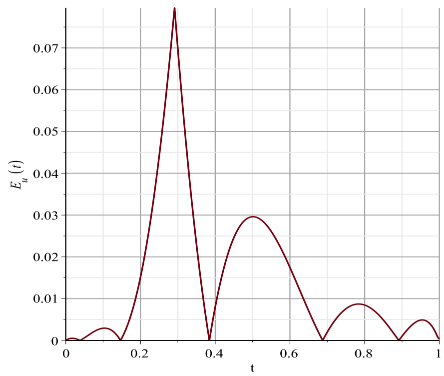
The optimal control is

$$u^*(t)=\begin{cases} -1 & {\lambda_2}^*(t)>1,\\ -{\lambda_2}^*(t) & -1<{\lambda_2}^*(t)<1,\\ +1 & {\lambda_2}^*(t)<-1. \end{cases}$$

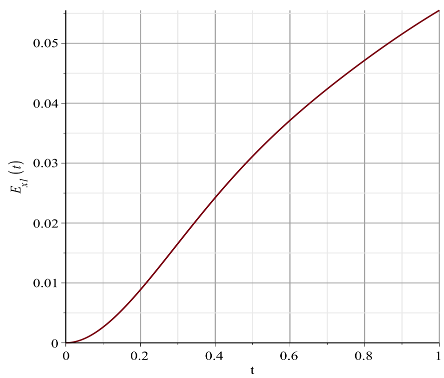
Tab. 3 shows the values of errors for $M=6,7,8,9$. Fig. 2 shows the error plots.

Example 5.3. *Consider the Breakwell problem from [11] as*

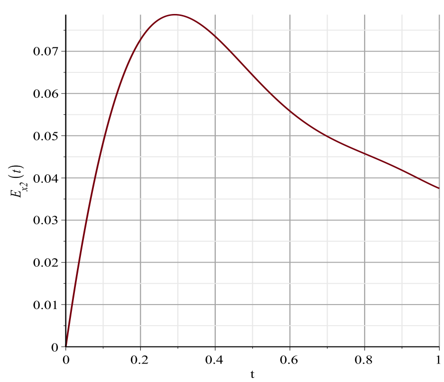
$$\begin{aligned} &\textit{minimize} && J=\frac{1}{2}\int_0^1 u^2(t)dt,\\ &\textit{subject to} && \dot{x}_1(t)=x_2(t),\\ &&& \dot{x}_2(t)=u(t),\\ &&& x_1(t)\leq 0.1,\\ &&& x_1(0)=x_1(1)=0,\\ &&& x_2(0)=-x_2(1)=1. \end{aligned}$$



(a) Plot of $\left|\overset{*}{u}(t)-\hat{u}(t)\right|$



(b) Plot of $\left|\overset{*}{x}_1(t)-\hat{x}_1(t)\right|$



(c) Plot of $\left|\overset{*}{x}_2(t)-\hat{x}_2(t)\right|$

Figure 2: Plot of absolute values of errors for $u(t)$, $x_1(t)$ and $x_2(t)$ using presenetd method with $M = 8$ for example 5.2

The exact solution of Breakwell problem is as follows

$$x_1^*(t) = \begin{cases} \frac{100}{27}t^3 - \frac{10}{3}t^2 + t, & 0 \le t \le 0.3, \\ \frac{1}{10}, & 0.3 \le t \le 0.7, \\ -\frac{100}{27}t^3 + \frac{70}{9}t^2 - \frac{49}{9}t + \frac{37}{27}, & 0.7 \le t \le 1, \end{cases}$$

$$x_2^*(t) = \begin{cases} \frac{100}{9}t^2 - \frac{20}{3}t + 1, & 0 \le t \le 0.3, \\ 0, & 0.3 \le t \le 0.7, \\ -\frac{100}{9}t^2 + \frac{140}{9}t - \frac{49}{9}, & 0.7 \le t \le 1, \end{cases}$$

$$u^*(t) = \begin{cases} \frac{200}{9}t - \frac{20}{3}, & 0 \le t \le 0.3, \\ 0, & 0.3 \le t \le 0.7, \\ -\frac{200}{9}t + \frac{140}{9}, & 0.7 \le t \le 1. \end{cases}$$

Also, this problem was numerically solved by using the pseudospectral method [17] and ChFD method [26], We solve it by the presented method and the results are reported in Table. 4. Error plots of the Breakwell problem for $M = 8$ has been shown in Fig. 3.

Table 4: Error values obtained by the presented method for Example 5.3

M	E_J	E_{x_1}	E_{x_2}	E_u	CPU Time
10	2.82×10^{-2}	3.08×10^{-3}	3.96×10^{-2}	5.89×10^{-1}	0.88
11	1.70×10^{-3}	4.53×10^{-4}	2.15×10^{-3}	2.19×10^{-1}	1.48

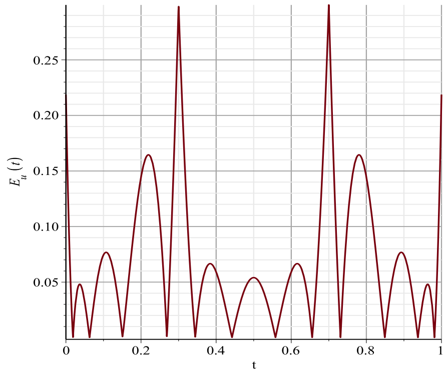
Table 5: Error values of the objective function derived from the method [11] for Example 5.3

Number of points	Method 1		Method 2	
	E_J	CPU Time	E_J	CPU Time
8	3.58×10^{-2}	0.702	8.98×10^{-2}	1.341
16	2.02×10^{-3}	3.701	2.19×10^{-2}	1.513

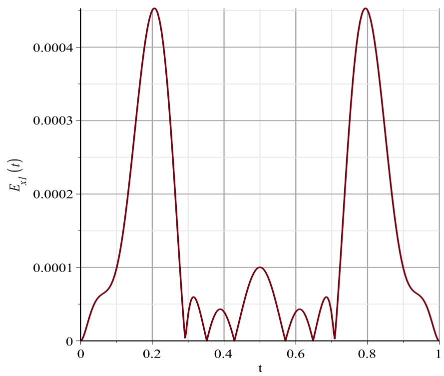
Example 5.4. *Consider the following optimal maneuvers of a rigid a symmetric spacecraft [26]. The Eulers equations for the angular velocities Y_1, Y_2, Y_3 of the spacecraft are given by*

$$\begin{aligned} \dot{Y}_1 &= -\frac{(I_3 - I_2)}{I_1} Y_2 Y_3 + \frac{u_1}{I_1}, \\ \dot{Y}_2 &= -\frac{(I_1 - I_3)}{I_2} Y_1 Y_3 + \frac{u_2}{I_2}, \\ \dot{Y}_3 &= -\frac{(I_2 - I_1)}{I_3} Y_1 Y_2 + \frac{u_3}{I_3}, \end{aligned}$$

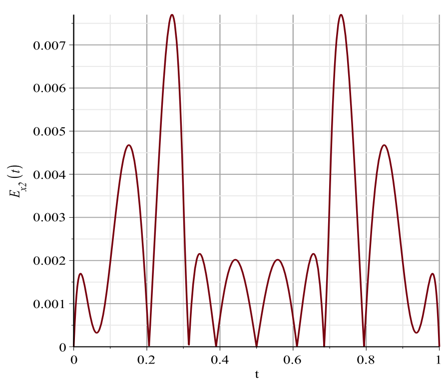
where u_1, u_2, u_3 are the control torques, and $I_1 = 86.24 kg\, m^2, I_2 = 85.07 kg\, m^2, I_3 = 113.59 kg\, m^2$ are the



(a) Plot of of $\left|\overset{*}{u}(t) - \hat{u}(t)\right|$



(b) Plot of $\left|\overset{*}{x}_1(t) - \hat{x}_1(t)\right|$



(c) Plot of $\left|\overset{*}{x}_2(t) - \hat{x}_2(t)\right|$

Figure 3: Plots of errors for $u(t)$, $x_1(t)$ and $x_2(t)$ obtained by the presented method with $M = 11$ for example 5.3

spacecraft principle inertia. The performance index to be minimized is given by

$$
\begin{aligned}
\textit{minimize} \quad & J = \frac{1}{2}\int_0^{100} \left(u_1{}^2(t) + u_2{}^2(t) + u_3{}^2(t)\right)\mathrm{d}t, \\
& Y_1(0) = 0.01, \\
& Y_2(0) = 0.005, \\
& Y_3(0) = 0.001, \\
& Y_1(100) = Y_2(100) = Y_3(100) = 0.
\end{aligned}
$$

In Table. 6, the values of the objective function J are presented and compared with two other methods presented in [21, 26].

Table 6: The values of the objective function for Example 5.3

Methods	J
Quasilinearization [21]	
N=6	0.0046878
Composite Chebyshev finite difference method [26]	
N=2, M=4	0.0046877953
Presented Method	
M=4	0.0046877856
M=6	0.0046877953
M=8	0.0046877953

Example 5.5. *Consider the following example from [17].*

$$
\begin{aligned}
\textit{minimize} \quad & J = \int_0^1 x_2(t)u(t)\mathrm{d}t, \\
\textit{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = -x_2(t) + u(t), \\
& x_2(t) \geq 0, \\
& 0 \leq u \leq 2, \\
& (x_1(0), x_2(0)) = (0, 1), \\
& (x_1(1), x_2(1)) = (1, 1).
\end{aligned}
$$

For this problem, we define the first and second constraints error in the following from

$$
\begin{aligned}
e_1 &= |\dot{x}_1(t) - x_2(t)| \\
e_2 &= |\dot{x}_2(t) + x_2(t) - u(t)|.
\end{aligned}
$$

The obtained results applying presented method for this problem are reported in Table. 7 and Figs. 4 and 5.

Table 7: Constraints error obtained by presented method for Example 5.5

M	$\max_t e_1$	$\max_t e_2$	CPU Time
7	2.08×10^{-18}	1.79×10^{-18}	0.88
8	2.18×10^{-18}	2.19×10^{-18}	1.48
9	1.34×10^{-17}	5.98×10^{-18}	2.02

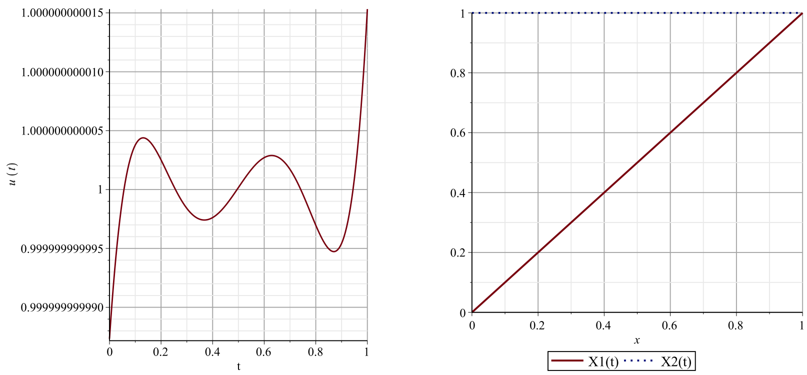


Figure 4: The exact and approximate values of the optimal control and state vectors for example 5.5

6 Conclusion

In the present study, the Genocchi polynomial basis was used to numerically solve the OCPs. We apply operational matrix of derivation of the Genocchi polynomials and change the solving an OCP to the solving NLP. The method can solve any arbitrary OCP with constant or free terminal time problems. Five examples are include to examine the performance and effectiveness of the new method. These examples show that the present method is superior in terms of accuracy and time saving.

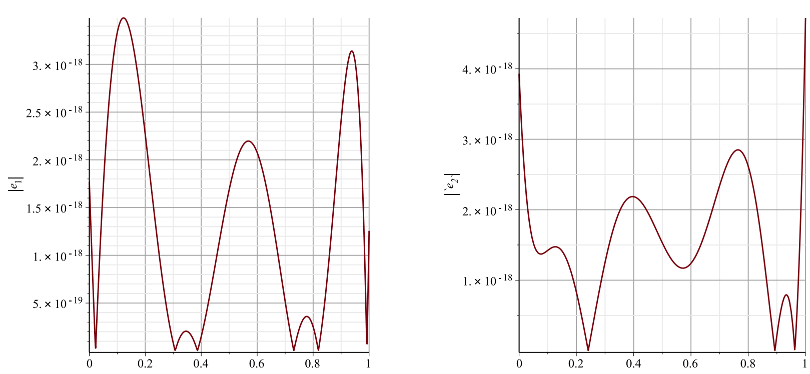


Figure 5: Results obtained for example 5.5

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