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Quasilinearized semiorthogonal B-spline wavelets method for solving multi-term nonlinear fractional order equations

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Abstract

In the present article, we implement a new numerical scheme, the quasilinearization semiorthogonal B-spline wavelets method, combining the semiorthogonal B-spline wavelets collocation method with the quasilinearization method, for a class of the multi-term nonlinear fractional order equations. The fractional order equations contain Riemann-Liouville fractional integral operator and Caputo fractional differential operator. The quasilinearization method is firstly utilized to convert the multi-term nonlinear fractional order equation into a multi-term linear fractional order equation, which is solved by means of semiorthogonal B-spline wavelets subsequently. Herein, we investigate the operational matrix and the convergence of the proposed scheme. Several numerical results are given to confirm the accuracy and efficiency of our scheme.

KEYWORDS:
multi-term nonlinear fractional order equations; fractional integral; Caputo derivatives; semiorthogonal B-spline wavelets; quasilinearization

1 | INTRODUCTION

Fractional calculus, generalization of the integer calculus, has been found more appropriate to describe some phenomena in the field of dynamics [1], physics [2,3], medicine [4], chemical [5] and other scientific areas [6]. There have been plenty of works, especially in physical systems of the real world, where Caputo type fractional derivatives and Riemann-Liouville fractional integral are more widely used for describing the materials transport. Since most fractional order equations cannot be resolved analytically, numerical methods have been taken into account to give their numerical solutions e.g. [7–10]. In recent years, multi-term fractional orders equations have received increasing attention because of its more extensive application. A great deal of papers are devoted to the numerical methods for approximating the multi-term fractional differential equations, such as Bernstein polynomials method [11], B-spline method [12], et al. In contrast to this, only a few papers concern multi-term fractional order equations with fractional integral and derivatives. In [13], Kojabad and Rezapour discussed the existence of solutions of multi-term fractional order equations by the Caputo differentiation, and used the Legendre and Chebyshev method to find the numerical solutions, respectively. Zheng et al. [14] studied the linear multi-term fractional order equations via discontinuous Galerkin finite element method.

In this paper, we consider the following problem I about multi-term fractional order equations

$$\sum_{i=1}^{q} a_i(x) D^{\alpha_i} u(x) + a_{q+1}(x) I^{\beta} u(x) = f\left(x, u(x), \cdots, u^{(p)}(x)\right) \tag{1}$$

with $p+1$ boundary conditions in $[0,b]$

$$g_k\left(0,u(0),\cdots,u^{(p)}(0)\right)=0,\quad k=1,\cdots,j-1 \tag{2}$$

and

$$g_k\left(b,u(b)\cdots,u^{(p)}(b)\right)=0,\quad k=j,\cdots,p+1. \tag{3}$$

where $0<\alpha_1<\alpha_2<\cdots<\alpha_q$, $\beta\geq 0$, $p<\alpha_q\leq p+1$ and $p,q\in\mathbb{N}$. The functions f, $g_k(k=1,\cdots,p+1)$ is nonlinear function of $u(x)$ and its derivatives $u^k(x)(k=1,\cdots,p)$. The operator D^{α_i} is Caputo type fractional derivatives operator of order α_i, and I^β is Riemann-Liouville fractional integral operator of order β.

Wavelet methods, which are the relatively novel approaches, have been applied for solving fractional calculus. Because of the structural characteristic that the base of wavelets can be achieved by dilation and translation of the mother wavelet functions, wavelet methods can also benefit the program operation. At present, some orthogonal wavelet methods have been developed to estimate the approximate solutions for fractional order equations [15,16]. Compared with orthogonal wavelet method, the semiorthogonal B-spline wavelet method (SOBWM) has the following advantages: compact support, explicit analytical form, and finite basis functions in any wavelet subspaces [17]. Due to the accuracy and efficiency, SOBWM is applied in several kinds of differential and integral equations. Maleknejad et al [18] adopted this method to solve nonlinear Fredholm-Hammerstein integral equations of the second kind; Aram et al [19] used the method to deal with integral integro-difference equations, and Liu et al [20] approximated multi-term linear fractional differential equations by the method.

Quasilinearization method [21,22] and homotopy method [23,24] are two common approaches for linearizing nonlinear functions in existing studies. For the multi-term fractional order equations, our simulation results show that the solutions obtained by homotopy method are easy to diverge. By contrast, the quasilinearization method is more suitable for nonlinear multi-term fractional order equations. In this paper, we utilize it to linearize the nonlinear fractional equations in problem I. Then, we solve the linearized fractional order equations with semiorthogonal B-spline wavelet collocation method (SOBWCM).

The remainder of this article is as follows. In Section 2, we introduce some basic notations, definitions, and lemmas about fractional calculus. The definitions of the semiorthogonal B-spline wavelets (SOBW) and related theorems, properties are shown in Section 3. Section 4 presents the implementation process of the quasilinearized semiorthogonal B-spline wavelets method (QSOBWM). The convergence of QSOBWM is analyzed in Section 5. In Section 6, the validity of the presented scheme are examined by illustrative examples. In Section 7, we draw a concise conclusion.

2 | DEFINITONS AND PROPERTIES OF FRACTIONAL CALCULUS

In this section, the definitions of Riemann-Liouville fractional integral and Caputo type derivative are presented first, and then some related properties are provided [25,26].

Definition 1. The Riemann-Liouville fractional integral operator of order $\beta>0$ is defined as

$$I^\beta u(x)=\frac{1}{\Gamma(\beta)}\int_0^x\frac{u(t)dt}{(x-t)^{1-\beta}}. \tag{4}$$

Definition 2. The Caputo fractional derivative of order $\alpha>0$ is defined as

$$D^\alpha u(x)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{u^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, & n-1<\alpha<n,\\ u^{(n)}(t), & \alpha=n\end{cases} \tag{5}$$

where $n\in\mathbb{N}_+$.

Definition 3. The function $\hat{u}(s)$ of the variable s defined by

$$\hat{u}(s)=\mathcal{L}\left[u(x)\right]=\int_0^\infty e^{-sx}u(x)dx,\quad s\in\mathbb{R} \tag{6}$$

is called the Laplace transform of the function $u(x)$.

Definition 4. The inverse Laplace transform of $\hat{u}(s)$ is

$$u(x) = \mathcal{L}^{-1}\left[\hat{u}(s)\right] = \int_{c-i\infty}^{c+i\infty} e^{sx}\hat{u}(s)ds, \quad c = Re(s) > c_0 \tag{7}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral Equation (6).

Lemma 1. The Laplace transform of the Riemann-Liouville fractional integral:

$$\mathcal{L}\left[I^\beta u(x)\right] = s^{-\beta}\hat{u}(s). \tag{8}$$

Lemma 2. The Laplace transform formula for the Caputo fractional derivative:

$$\mathcal{L}\left[D^\alpha u(x)\right] = s^\alpha\hat{u}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}u^{(k)}(0). \tag{9}$$

3 | SEMIORTHOGONAL B-SPLINE WAVELETS IN $[0, B]$

In this section, we describe the construction of SOBW[20,27] in $[0, b]$. The integral and derivative for SOBW in $[0, b]$ are also presented.

3.1 | Construction of semiorthogonal B-spline wavelets

Definition 5. For $j \in \mathbf{Z}$, let the knots sequence $\xi^{(j)} := \left\{\xi_k^{(j)}\right\}_{k=-m+1}^{2^j-m+1}$ in $[0, b]$, with

$$\xi_{-m+1}^{(j)} = \xi_{-m+2}^{(j)} = \cdots = \xi_0^{(j)} = 0, \tag{10}$$

$$\xi_k^{(j)} = \frac{kb}{2^j}, \quad k = 1, \cdots, 2^j - 1, \tag{11}$$

$$\xi_{2^j}^{(j)} = \xi_{2^j+1}^{(j)} = \cdots = \xi_{2^j+m-1}^{(j)} = b. \tag{12}$$

The mth order B-spline function is defined by

$$B_{i,m,j}(x) = (-1)^m(\xi_{i+m}^{(j)} - \xi_i^{(j)})\left[\xi_i^{(j)}, \xi_{i+1}^{(j)}, \cdots, \xi_{i+m}^{(j)}\right]_\xi (x-\xi)_+^{m-1} \tag{13}$$

here $[\cdot, \cdots, \cdot]_\xi$ is the mth divided difference of $(x-\xi)_+^{m-1}$ in regard to variable ξ, and $(x-\xi)_+^{m-1}$ is denoted as

$$(x-\xi)_+^{m-1} = \begin{cases} (x-\xi)^{m-1}, & x > \xi, \\ 0, & x \le \xi. \end{cases} \tag{14}$$

Let j_0 be defined by

$$2^{j_0} \ge 2m - 1, \tag{15}$$

which is a minimum to contain one complete wavelet function in $[0, b]$.

Definition 6. For $j \ge j_0$, the mth order scaling functions of space $V_j^{[0,b]}$ be defined by

$$\phi_{i,m,j}(x) = \begin{cases} B_{i,m,j_0}(2^{j-j_0}x), & i = -m+1, \cdots, -1, \\ B_{2^j-m-i,m,j_0}(b/2 - 2^{j-j_0}x), & i = 2^j - m + 1, \cdots, 2^j - 1, \\ B_{0,m,j_0}(2^{j-j_0}x - 2^{-j_0}i), & i = 0, \cdots, 2^j - 1. \end{cases} \tag{16}$$

Definition 7. For $j \ge j_0$, the wavelet subspace $W_j^{[0,b]}$ is spanned by inner wavelets

$$\psi_{j,i}(x) = 1/2^{m-1+(j+1)^m} \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1) B_{2i+k,2m,j+1}^{(m)}(x), \tag{17}$$

$i = 0, \cdots, 2^j - 2m + 1$, the boundary wavelets for 0,

$$\begin{aligned}\psi_{j,i}(x) =& 1/2^{m-1+(j+1)^m} \sum_{k=-m+1}^{-1} (\tilde{B}^{-1} r_i)_k B_{k,2m,j+1}^{(m)}(x) \\ &+ 1/2^{m-1+(j+1)^m} \sum_{k=0}^{2m-2+2i} (-1)^k N_{2m}(k+1-2i) B_{k,2m,j+1}^{(m)}(x),\end{aligned} \tag{18}$$

$i = -m+1, \cdots, 1$, and the boundary wavelets for b,

$$\psi_{j,2^j-2m+1-i}(x) = \psi_{j,i}(x), \tag{19}$$

$i = -m+1, \cdots, 1$.

3.2 | Fractional integral and derivative of semiorthogonal B-spline wavelets

From the Definition 5, 6 and 7, the fractional integral and derivative of SOBW can be reduced to solving fractional integral and derivative of function $(ax-b)_+^k$, here $k > 0$, $a > 0$ and $b > 0$.

Lemma 3. For $\beta > 0$, $k > 0$, $a > 0$, $b > 0$, we can get

$$I^\beta (ax-b)_+^k = a^{-\beta} \frac{\Gamma(1+k)}{\Gamma(1+k+\beta)} (ax-b)_+^{k+\beta}. \tag{20}$$

Proof. According to Lemma 1 and Definition 3,

$$\begin{aligned}\mathcal{L}\left[I^\beta (ax-b)_+^k\right] &= s^{-\beta} \mathcal{L}\left[(ax-b)_+^k\right] \\ &= s^{-\beta-1-k} a^k e^{-\frac{b}{a}s} \Gamma(1+k) \\ &= a^k \frac{\Gamma(1+k)}{\Gamma(1+k+\beta)} \mathcal{L}\left[(x-\frac{b}{a})_+^{k+\beta}\right] \\ &= a^{-\beta} \frac{\Gamma(1+k)}{\Gamma(1+k+\beta)} \mathcal{L}\left[(ax-b)_+^{k+\beta}\right],\end{aligned} \tag{21}$$

then utilizing the properties of Laplace transform, we get

$$I^\beta (ax-b)_+^k = a^{-\beta} \frac{\Gamma(1+k)}{\Gamma(1+k+\beta)} (ax-b)_+^{k+\beta}. \tag{22}$$

□

Lemma 4. For $n-1 < \alpha < n$, $n \in \mathbb{N}_+$, $k > 0$, $a > 0$, $b > 0$, if $\alpha \leq k$, we have

$$D^\alpha (ax-b)_+^k = a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} (ax-b)_+^{k-\alpha}. \tag{23}$$

Proof. See [28]. □

4 | FUNCTION APPROXIMATION

The first step is to apply the quasilineaeization approach[29] to approximate $u(x)$ in the problem I. To write concisely, we note $L^{(\alpha)} \equiv \sum_{i=1}^{q} a_i(x) D^{\alpha_i} u(x) + a_{q+1}(x) I^\beta u(x)$, so problem I is transformed into problem II as

$$\begin{aligned}L^{(\alpha)} u_{r+1}(x) =& f\left(x, u_r(x), \cdots, u_r^{(p)}(x)\right) \\ &+ \sum_{l=0}^{p} \left(u_{r+1}^{(l)}(x) - u_r^{(l)}(x)\right) f_{u^{(l)}}\left(x, u_r(x), \cdots, u_r^{(p)}(x)\right),\end{aligned} \tag{24}$$

with linearized nonlinear boundary conditions

$$\sum_{l=0}^{p} \left(u_{r+1}^{(l)}(0) - u_r^{(l)}(0)\right) g_{ku^{(l)}}\left(0, u_r(0), \cdots, u_r^{(p)}(0)\right) = 0, \quad k = 1, \cdots, j-1, \tag{25}$$

and

$$\sum_{l=0}^{p}\left(u_{r+1}^{(l)}(b)-u_{r}^{(l)}(b)\right)g_{ku^{(l)}}\left(b,u_r(b),\cdots,u_r^{(p)}(b)\right)=0,\quad k=j,\cdots,p+1, \tag{26}$$

here $u_r^{(0)}(x) = u_r(x)$, $f_u^{(l)}(x,u(x),\cdots,u^{(p)}(x)) = \partial f(x,u(x),\cdots,u^{(p)}(x))/\partial u^{(l)}(x)$ and $g_{ku^{(l)}}(x,u(x),\cdots,u^{(p)}(x)) = \partial g_k(x,u(x),\cdots,u^{(p)}(x))/\partial u^{(l)}(x)$, $l=0,1,\cdots,p$. The initial value $u_0(x)$ can be selected from mathematical or physical conditions, and $u_{r+1}(x)$ is further obtained by the iteration.

The Equations (24)-(26) are linear equations about $u_{r+1}(x)$, therefore problem II is of multi-term linear fractional order equations which can be addressed efficiently by SOBWCM. For simplicity, problem II is sorted into equivalent problem III as

$$\sum_{i=1}^{p+q+1}h_i(x)D^{\alpha_i}u_{r+1}(x)+h_{p+q+2}(x)I^{\beta}u_{r+1}(x)=b(x), \tag{27}$$

with the boundary consitions

$$\sum_{l=0}^{p}h_k^{b_1}(l)D^l u_{r+1}(0)=0,\quad k=1,\cdots,j-1, \tag{28}$$

$$\sum_{l=0}^{p}h_k^{b_2}(l)D^l u_{r+1}(b)=0,\quad k=j,\cdots,p+1. \tag{29}$$

here $h_i(x)$, $h_k^{b_1}(l)$, $h_k^{b_2}(l)$, $b(x)$ are related to functions and values of rth iterative by quasilineaeization approach.

A function $u_{r+1}(x)$ in $L^2[0,b]$ can be expanded by the semiorthogonal B-spline scaling functions and wavelets[27] as

$$u_{r+1}(x)=\sum_{k=-m+1}^{2^{j_0}-1}c_{j_0,k}\phi_{j_0,k}(x)+\sum_{j=j_0}^{\infty}\sum_{k=-m+1}^{2^j-m}d_{j,k}\psi_{j,k}(x). \tag{30}$$

To meet the needs of practical application, the higher frequency components are cut off, so that the infinite series in Equation (30) is truncated at M as

$$u_{r+1}(x)\approx\sum_{k=-m+1}^{2^{j_0}-1}c_{j_0,k}\phi_{j_0,k}(x)+\sum_{j=j_0}^{M}\sum_{k=-m+1}^{2^j-m}d_{j,k}\psi_{j,k}(x)=C^{\mathrm{T}}\Psi(x), \tag{31}$$

here C and Ψ are the $(2^{M+1}+m-1)\times 1$ vectors:

$$C=\left[c_{j_0,-m+1},\cdots,c_{j_0,2^{j_0}-1},d_{j_0,-m+1},\cdots,d_{M,2^j-m}\right]^{\mathrm{T}},$$

$$\Psi=\left[\phi_{j_0,-m+1},\cdots,\phi_{j_0,2^{j_0}-1},\psi_{j_0,-m+1},\cdots,\psi_{M,2^j-m}\right]^{\mathrm{T}}.$$

Substituting $u_{r+1}(x)$ of Equation (31) into Equations (25-29) of problem III, we obtain:

$$\sum_{i=1}^{p+q+1}h_i(x)C^{\mathrm{T}}D^{\alpha_i}\Psi(x)+h_{p+q+2}(x)C^{\mathrm{T}}I^{\beta}\Psi(x)=b(x), \tag{32}$$

with the boundary consitions

$$\sum_{l=0}^{p}h_k^{b_1}(l)C^{\mathrm{T}}D^l\Psi(0)=0,\quad k=1,\cdots,j-1, \tag{33}$$

$$\sum_{l=0}^{p}h_k^{b_2}(l)C^{\mathrm{T}}D^l\Psi(b)=0,\quad k=j,\cdots,p+1. \tag{34}$$

In order to increase the computation efficiency, Equation (32) is rewritten in matrix form as:

$$C^{\mathrm{T}}\sum_{i=1}^{p+q+1}D^{\alpha_i}\Psi H_i+C^{\mathrm{T}}I^{\beta}\Psi H_{p+q+2}=B, \tag{35}$$

here

$$H_i=\begin{bmatrix} h_i(x_1) & 0 & \cdots & 0\\ 0 & h_i(x_2) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & h_i(x_n)\end{bmatrix} \tag{36}$$

and

$$B=\left(b(x_1),b(x_2),\cdots,b(x_n)\right). \tag{37}$$

The $D^{\alpha_i}\Psi$ and $I^{\beta}\Psi$ can be implemented on computer based on Lemma 3 and Lemma 4. Therefore, problem III is converted into the solution of system of linear equations which consist of Equations (33)-(35).

5 | CONVERGENCE ANALYSIS

The QSOBWM is a hybrid numerical method which combines quasilinearization method and SOBWCM. Thus we respectively analyze the convergence of the two basic methods first, then the convergence of the method in this paper is obtained by the process of function approximation.

Theorem 1. Suppose $u(x) \in C^{p+1}[0,b]$, and $f_{u^{(l)}u^{(n)}}\left(x, u(x), \ldots, u^{(p)}(x)\right) \in L^2[0,b]$, $l, n = 0, 1, \cdots, p$, the difference function $\delta u_{r+1}(x) = u_{r+1}(x) - u_r(x)$ is considered in quasilinearization method for problem II, then there exists a positive constant C_M, such that

$$\|\delta u_{r+1}\| \leq C_M \|\delta u_r\|^2 \tag{38}$$

here $\|\delta u_r\|$ is maximum value of any of $\|\delta u_r^{(l)}\|$ in $[0,b]$, $l = 0, 1, \cdots, p$.

Proof. From iterations of Equations (24)-(26), we have

$$\begin{aligned} L^{(\alpha)}\delta u_{r+1}(x) =& f\left(x, u_r(x), \ldots, u_r^{(p)}(x)\right) - f\left(x, u_{r-1}(x), \ldots, u_{r-1}^{(p)}(x)\right) \\ &+ \sum_{l=0}^{p}\left[\delta u_{r+1}^{(l)}(x) f_{u^{(l)}}\left(x, u_r(x), \ldots, u_r^{(l)}(x)\right)\right. \\ &\left. -\delta u_r^{(l)}(x) f_{u^{(l)}}\left(x, u_{r-1}(x), \ldots, u_{r-1}^{(l)}(x)\right)\right], \end{aligned} \tag{39}$$

with the corresponding boundary conditions:

$$\begin{aligned} &\sum_{l=0}^{p}\left[\delta u_{r+1}^{(l)}(0) g_{ku^{(l)}}\left(0, u_r(0), \ldots, u_r^{(p)}(0)\right)\right. \\ &\qquad\left. -\delta u_r^{(l)}(0) g_{ku^{(l)}}\left(0, u_{r-1}(0), \ldots, u_{r-1}^{(p)}(0)\right)\right] = 0, \\ &k = 1, \ldots, j-1, \end{aligned} \tag{40}$$

and

$$\begin{aligned} &\sum_{l=0}^{p}\left[\delta u_{r+1}^{(l)}(b) g_{ku^{(l)}}\left(b, u_r(b), \ldots, u_r^{(p)}(b)\right)\right. \\ &\qquad\left. -\delta u_r^{(l)}(b) g_{ku^{(l)}}\left(b, u_{r-1}(b), \ldots, u_{r-1}^{(p)}(b)\right)\right] = 0, \\ &k = j, \ldots, p+1. \end{aligned} \tag{41}$$

According to the mean value theorem from[30]

$$\begin{aligned} & f\left(x, u_r(x), \ldots, u_r^{(p)}(x)\right) - f\left(x, u_{r-1}(x), \ldots, u_{r-1}^{(p)}(x)\right) \\ =& \sum_{l=0}^{p} \delta u_r^{(l)}(x) f_{u^{(l)}}\left(x, u_{r-1}(x), \ldots, u_{r-1}^{(p)}(x)\right) \\ &+ \frac{1}{2}\sum_{l,n=0}^{p} \delta u_r^{(l)}(x)\delta u_r^{(n)}(x) f_{u^{(l)}u^{(n)}}\left(x, \bar{u}_{r-1}(x), \ldots, \bar{u}_{r-1}^{(p)}(x)\right), \end{aligned} \tag{42}$$

here $\bar{u}_{r-1}^{(l)}(x)$ is between $u_{r-1}^{(l)}(x)$ and $u_r^{(l)}(x)$, $k = 0, 1, \ldots, p$. Substituting Equation (42) into Equation (39) yields

$$\begin{aligned} & L^{(\alpha)}\delta u_{r+1}(x) - \sum_{l=0}^{p} \delta u_{r+1}^{(l)}(x) f_{x,u^{(l)}}\left(x, u_r(x), \ldots, u_r^{(p)}(x)\right) \\ =& \frac{1}{2}\sum_{l,n=0}^{p} \delta u_r^{(l)}(x)\delta u_r^{(n)}(x) f_{u^{(l)}u^{(n)}}\left(x, \bar{u}_{r-1}(x), \ldots, \bar{u}_{r-1}^{(p)}(x)\right). \end{aligned} \tag{43}$$

In the view of n-term linear fractional Green's function properties in[26], there is a Green function $G(x, y)$ such that

$$\begin{aligned}\delta u_{r+1}(x) =&\frac{1}{2}\int_0^b G_r^{(n)}(x, y)\\&\cdot\sum_{l,n=0}^{p}\delta u_r^{(l)}(y)\delta u_r^{(n)}(y)f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)\mathrm{d}y.\end{aligned} \tag{44}$$

Therefore

$$\begin{aligned}\left|\delta u_{r+1}(x)\right| =&\frac{1}{2}\int_0^b |G(x, y)|\\&\cdot\sum_{l,n=0}^{p}\left|\delta u_r^{(l)}(y)\right|\left|\delta u_r^{(n)}(y)\right|\left|f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)\right|\mathrm{d}y\\ \leq&\frac{\|\delta u_r\|^2}{2}\int_0^b |G(x, y)|\sum_{l,n=0}^{p}\left|f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)\right|\mathrm{d}y\\ \leq&\frac{\|\delta u_r\|^2}{2}\sum_{l,n=0}^{p}\|G(x, y)\|_2\left\|f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)\right\|_2,\end{aligned} \tag{45}$$

here $\|\cdot\|$ is 2-norm. As the properties of Green functions and the boundedness of $f_{u^{(l)}u^{(n)}}\left(y, u(y), \dots, u^{(p)}(y)\right)$, there exist a positive constant C_M depends on $G(x, y)$ and $f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)$ and p, such that

$$C_M \geq \frac{1}{2}\sum_{l,n=0}^{p}\|G(x, y)\|_2\left\|f_{u^{(l)}u^{(n)}}\left(y, \bar{u}_{r-1}(y), \dots, \bar{u}_{r-1}^{(p)}(y)\right)\right\|_2. \tag{46}$$

Hence

$$\left\|\delta u_{r+1}\right\| \leq C_M\|\delta u_r\|^2. \tag{47}$$

□

Convergence of the quasilinearization method is shown in Theorem 1. Then, to estimate the error of SOBWCM, we use the following theorem from[20].

Theorem 2. Suppose $\tilde{u}_{r+1}(x) \in C^m[0, b]$ is approximation by SOBWCM of order m in problem III, the truncation error for $j = M$ is

$$\left|u_{r+1}(x) - \tilde{u}_{r+1}(x)\right| = O(2^{-mM}). \tag{48}$$

Theorem 2 implies that $\left|u_{r+1}(x) - \tilde{u}_{r+1}(x)\right| \to 0$ when $M \to \infty$, reflects the SOBWCM is convergent. As both each iteration from $u_r(x)$ to $u_{r+1}(x)$ and the process of the approximate of $u_{r+1}(x)$ are convergent, QSOBWM in this paper is convergent.

6 | NUMERICAL EXAMPLES

This section presents some numerical examples to demonstrate the validity of the proposed scheme. We denote u as the exact solution, and $\tilde{u}$ as the numerical solution. To assess the performance of the method, we calculate the absolute error, L^2-error and L^∞-error.

The absolute error in $[0, b]$ is

$$E(i) = |u(x_i) - \tilde{u}(x_i)|, \quad k = 1, \cdots, N_x, \tag{49}$$

the L^2-error is defined asÂčÂż

$$\|u - \tilde{u}\|_{L^2} = \sqrt{\frac{1}{N_x}\sum_{i=1}^{N_x}\left[u\left(x_i\right) - \tilde{u}\left(x_i\right)\right]^2}, \tag{50}$$

and the L^∞-error is defined asÂčÂż

$$\|u - \tilde{u}\|_{L^\infty} = \max_{1\leq i\leq N_x} E(i), \tag{51}$$

where N_x is the number of collocation points in $[0, b]$.

Example 1. Consider the fractional integro-differential equation with weakly singular kernel[8,9]:

$$D^{\frac{1}{3}}u(x) = g(x) + p(x)u(x) + \Gamma\left(\frac{1}{2}\right) I^{\frac{1}{2}}u(x), \quad 0 < x < 1, \tag{52}$$

with the initial condition

$$u(0) = 0,$$

here $g(x) = \frac{6x^{8/3}}{\Gamma\left(\frac{11}{3}\right)} + \left(\frac{32}{35} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\right) x^{\frac{11}{6}} + \Gamma\left(\frac{7}{3}\right) x$, $p(x) = -\frac{32}{35}x^{\frac{1}{2}}$. The exact solution of the problem is $u(x) = x^{\frac{4}{3}} + x^3$.

TABLE 1 Absolute errors of QSOBWM of Example 1.

x	$M = 3$	$M = 4$	$M = 5$	$M = 6$
0.1	1.0243×10^{-4}	2.5517×10^{-5}	6.2542×10^{-6}	1.5552×10^{-6}
0.2	8.5807×10^{-5}	2.1381×10^{-5}	5.3596×10^{-6}	1.3452×10^{-6}
0.3	8.2395×10^{-5}	2.0939×10^{-5}	5.2812×10^{-6}	1.3263×10^{-6}
0.4	8.4718×10^{-5}	2.1482×10^{-5}	5.4225×10^{-6}	1.3631×10^{-6}
0.5	8.8306×10^{-5}	2.2454×10^{-5}	5.6693×10^{-6}	1.4254×10^{-6}
0.6	9.2957×10^{-5}	2.3665×10^{-5}	5.9766×10^{-6}	1.5029×10^{-6}
0.7	9.8386×10^{-5}	2.5041×10^{-4}	6.3258×10^{-6}	1.5909×10^{-6}
0.8	1.0422×10^{-4}	2.6547×10^{-5}	6.7080×10^{-6}	1.6871×10^{-6}
0.9	1.1058×10^{-4}	2.8165×10^{-5}	7.1174×10^{-6}	1.7902×10^{-6}

TABLE 2 Example 1: L^2-errors of QSOBWM, SKCPM in[8] and FEFM in[9].

N	SKCPM	FEFsM($\alpha = 1$)	FEFsM($\alpha = 1.5$)	M	QSOBWM
2	1.3426×10^{-2}	1.0502×10^{-2}	1.7511×10^{-4}	3	9.6130×10^{-5}
4	5.1379×10^{-4}	2.6252×10^{-4}	2.4436×10^{-4}	4	2.5255×10^{-5}
6	1.5578×10^{-4}	6.6216×10^{-5}	3.0232×10^{-5}	5	6.5901×10^{-6}
8	6.4612×10^{-5}	4.8349×10^{-4}	9.4123×10^{-5}	6	1.7168×10^{-6}

We adopt the QSOBWM with $m = 4$ for several truncation values M, and the absolute errors for each case are exhibited in Table 1 , respectively. The numerical results of the proposed scheme in Table 1 illustrate that the absolute errors decrease when the value of truncation M increases. More intuitively, we describe the absolute errors of the present scheme for various values of M in Figure 1 . The results accord with the convergence analysis of the present scheme.

In order to compare with the SKCPM in[8] and FEFsM in[9], we compute the L^2-errors of the present scheme with various values of M and list the results in Table 2 . N denotes the maximal degree of the polynomials in the space spanned by all polynomials for SKCPM and FEFM, and M indicates the truncation of the present scheme. The degree of all polynomials for the present scheme is not more than three when $m = 4$. Table 2 displays that L^2-errors of the present scheme with $m = 4$ are relatively smaller.

Example 2. Consider the following fractional Langevin equation[14]:

$$D^{\alpha_1}u(x) + D^{\alpha_2}u(x) + I^{\beta}u(x) = f(x), \quad 0 < x < 1, \tag{53}$$

with the initial condition

$$u(0) = 0,$$

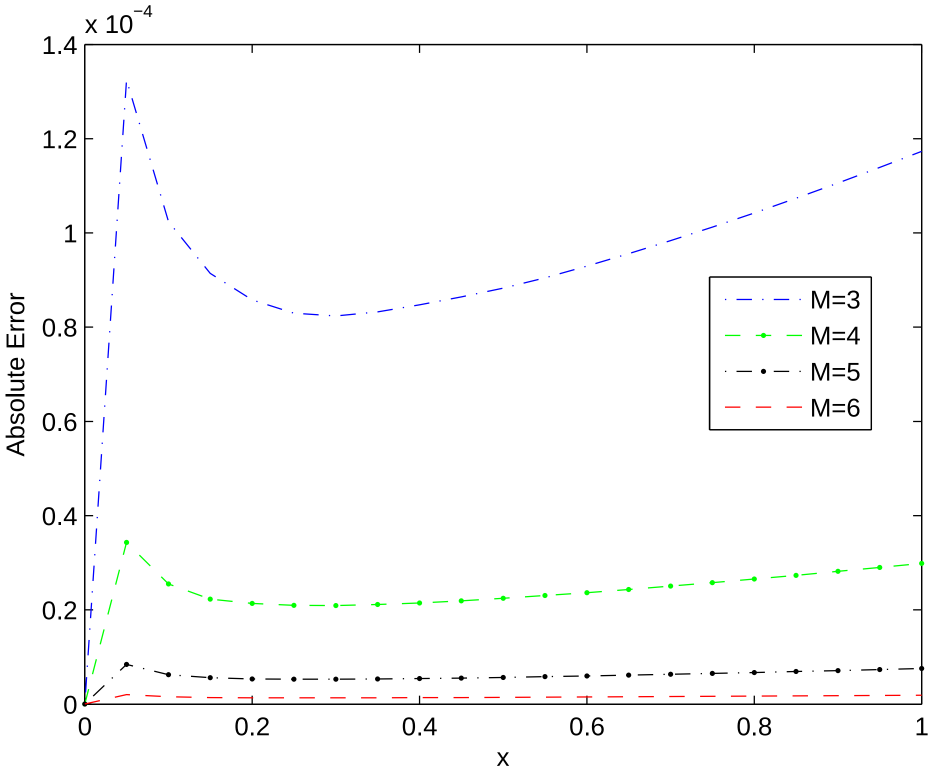

FIGURE 1 The absolute errors of Example 1 for QSOBWM with $M = 3, 4, 5, 6$.

here $f(x) = \left(\frac{2.0x^{2-\alpha_1}}{\Gamma(3.0-\alpha_1)} - \frac{x^{1-\alpha_1}}{\Gamma(2.0-\alpha_1)}\right) + \left(\frac{2.0x^{2-\alpha_2}}{\Gamma(3.0-\alpha_2)} - \frac{x^{1-\alpha_2}}{\Gamma(2.0-\alpha_2)}\right) + \left(\frac{2.0x^{2+\beta}}{\Gamma(3.0+\beta)} - \frac{x^{1+\beta}}{\Gamma(2.0+\beta)}\right)$ and $\alpha_1, \alpha_2, \beta \in (0, 1)$. The exact solution is given by $u(x) = x^2 - x$.

We solve this problem by using the QSOBWM with $m = 4$, $M = 3$ for fixed step size $h = \frac{1}{20}$. Table 3 exhibits the absolute errors, which demonstrates that the absolute errors of various values of α_1, α_2 and β are less than 10^{-13} and the computation only takes 4.10 seconds. Also, the results confirm the effectiveness of this method. Figure 2 and 3 depict the absolute errors of the present scheme for different α_1 and β, respectively. Figure 2 presents absolute errors of the present scheme with $\alpha_2 = 0.8$, $\beta = 0.3$ for various values of α_1, and Figure 3 displays absolute errors of the present scheme with $\alpha_1 = 0.5$, $\alpha_2 = 0.8$ for several values of β. These graphs demonstrate that the approximate solutions are highly agree with the analytical result for various values of α_1 and β, and absolute errors are less than 10^{-13}.

In order to verify the performance of QSOBWM, in Table 4 , we calculate L^2-error and L^∞-error of the present scheme and method in [14] for $\alpha_1 = 0.5, \alpha_2 = 0.8, \beta = 0.3$ with different sizes of grid. The results demonstrate that the approximate results for the present scheme are closer to the analytical solutions than the method in [14].

Example 3. Consider the following fractional Duffing-Holmes model for a nonlinear oscillator[31]:

$$D^2u(x) + 0.5D^\mu u(x) - u(x) + u^3(x) = g(x), \quad 0 < x < 1 \tag{54}$$

with the initial value:

$$u(0) = u'(0) = 0.$$

From[31], $u(x) = x^{\mu+2} \sin x$ is the given exact solution.

We approximate the solution using QSOBWM with $m = 4$ and different M, where $M = 3, 4, 5, 6$. This problem is also solved by the spectral collocation method (SCM) in[31] with $\mu = 0.5$, $N = 5, 6, 7, 8$, where N denotes the maximal degree of polynomials in the space formed by the basis of method. In Table 5 , we compare the L^2-errors and L^∞-errors of the above two methods. N in the present scheme is three. L^2-errors and L^∞-errors of the present scheme are much smaller than those of the

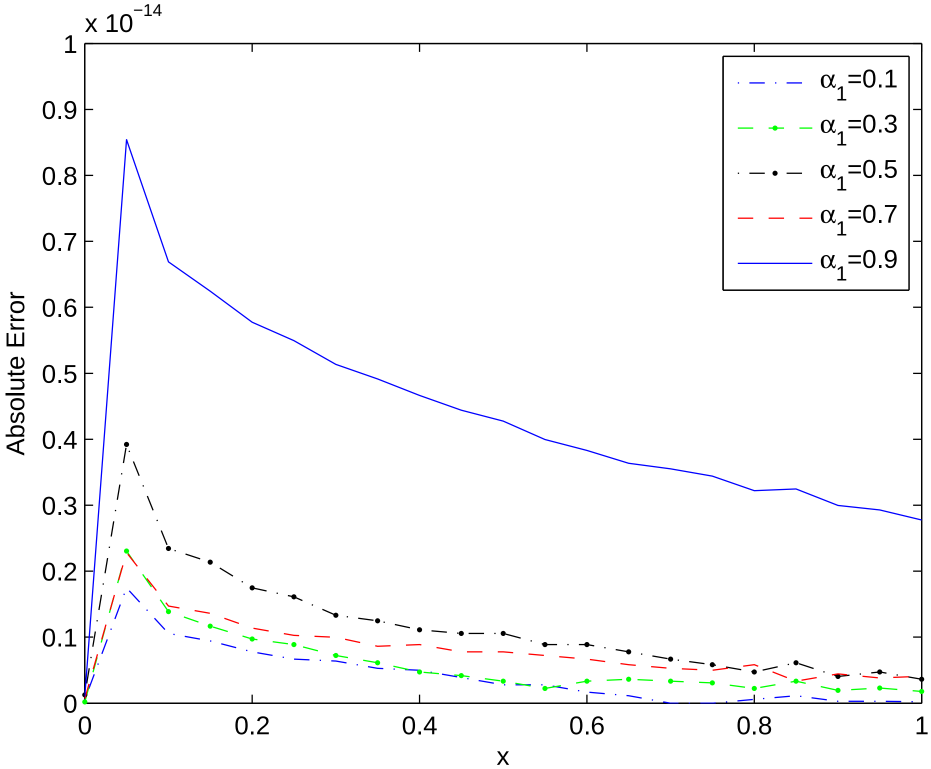

FIGURE 2 Absolute errors of Example 2 for QSOBWM with $\alpha_1 = 0.1, 0.3, 0.5, 0.7, 0.9$.

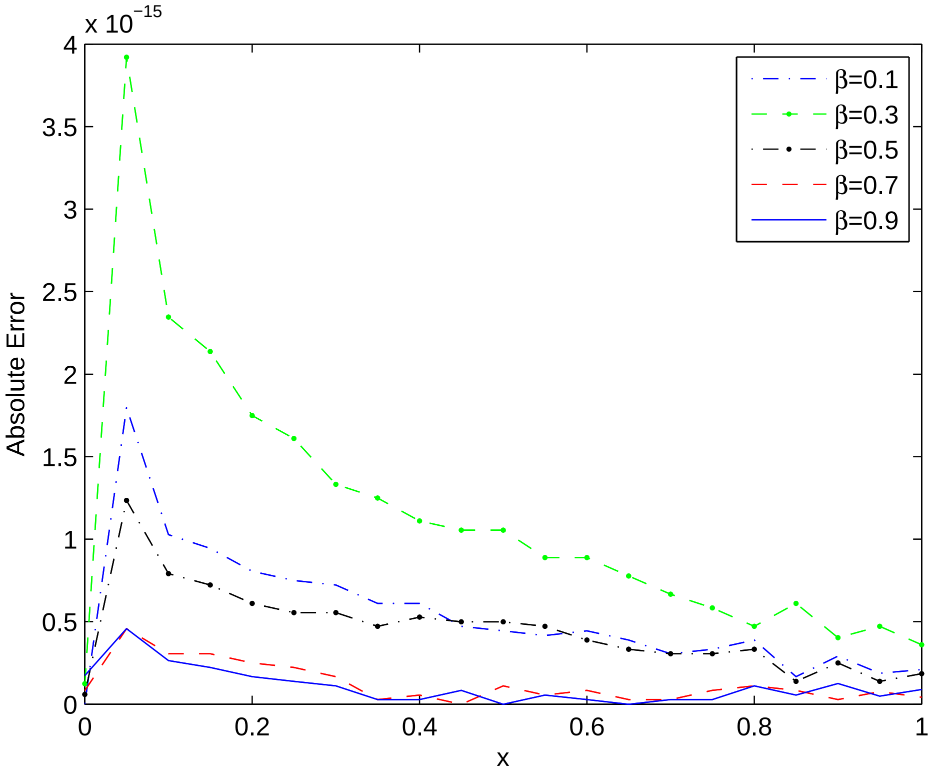

FIGURE 3 Absolute errors of Example 2 for QSOBWM with $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$.

TABLE 3 Absolute errors of QSOBWM when $h = \frac{1}{20}$ for Example 2.

x	$\alpha_1 = 0.99, \alpha_2 = 0.6, \beta = 0.5$	$\alpha_1 = 0.5, \alpha_2 = 0.8, \beta = 0.3$	$\alpha_1 = 0.1, \alpha_2 = 0.4, \beta = 0.99$
0.1	1.9582×10^{-14}	2.0539×10^{-15}	4.1633×10^{-17}
0.2	1.6986×10^{-14}	1.5266×10^{-15}	2.7756×10^{-17}
0.3	1.4932×10^{-14}	1.1657×10^{-15}	1.1102×10^{-16}
0.4	1.3323×10^{-14}	9.7145×10^{-16}	1.6653×10^{-16}
0.5	1.2046×10^{-14}	8.8818×10^{-16}	2.7756×10^{-17}
0.6	1.0880×10^{-14}	7.2164×10^{-16}	5.5511×10^{-17}
0.7	9.7700×10^{-15}	5.2736×10^{-16}	2.7756×10^{-17}
0.8	8.6042×10^{-15}	3.6082×10^{-16}	8.3267×10^{-17}
0.9	7.6328×10^{-15}	2.9143×10^{-16}	9.7145×10^{-17}

TABLE 4 Comparison of the errors of QSOBWM and the method in [14] of Example 2.

	Method of [14]			QSOBWM	
Errors	$h = \frac{1}{40}$	$h = \frac{1}{80}$	$h = \frac{1}{160}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$
$\|u - \tilde{u}\|_{L^2}$	2.3198×10^{-3}	8.3037×10^{-4}	9.9886×10^{-6}	1.2270×10^{-15}	5.7478×10^{-16}
$\|u - \tilde{u}\|_{L^\infty}$	7.2983×10^{-3}	2.7903×10^{-3}	2.3399×10^{-5}	3.4556×10^{-15}	1.1102×10^{-17}

SCM with the minimal degree from Table 5 . Moreover, the L^2-errors and L^∞-errors of the present scheme decrease when M increases. Thus, it accords with the convergence analysis discussed in the previous section.

Figure 4 depicts the analytical and numerical results of the present scheme with $M = 3$ for $\mu = 0.01, 0.3, 0.5, 0.7, 0.99$. This figure demonstrates that the numerical results tend to the analytical solution for different values of μ.

TABLE 5 Example 3: L^2-errors and L^∞-errors of $\mu = 0.5$ for SCM and QSOBWM.

Methods	$\|u - \tilde{u}\|_{L^2}$	$\|u - \tilde{u}\|_{L^\infty}$
SCM		
$N = 5$	8.237×10^{-5}	2.204×10^{-5}
$N = 6$	2.291×10^{-5}	8.859×10^{-6}
$N = 7$	1.467×10^{-5}	4.763×10^{-6}
$N = 8$	6.885×10^{-6}	2.714×10^{-6}
QSOBWM		
$M = 3$	1.310×10^{-5}	5.950×10^{-5}
$M = 4$	1.284×10^{-6}	5.115×10^{-7}
$M = 5$	1.119×10^{-7}	4.511×10^{-8}
$M = 6$	1.077×10^{-8}	4.026×10^{-9}

Example 4. Consider the following multi-term fractional nonlinear boundary value problem [11,12]:

$$D^2 u(x) + \Gamma\left(\frac{4}{5}\right) \sqrt[5]{x^6} D^{\frac{6}{5}} u(x) + \frac{11}{9} \Gamma\left(\frac{6}{5}\right) \sqrt[6]{x} D^{\frac{1}{6}} u(x) - (Du(x))^2 = 2 + \frac{1}{10} x^2, \tag{55}$$

with boundary conditions:

$$u(0) = 1, \quad u(1) = 2.$$

The exact solution of this problem is $u(x) = x^2 + 1$.

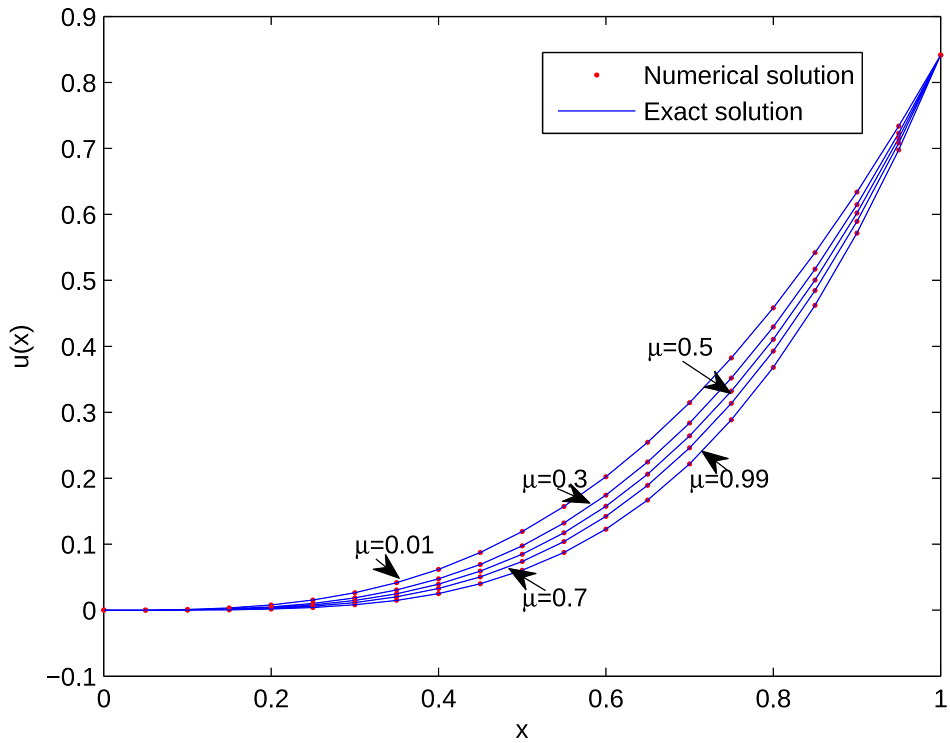

FIGURE 4 Analytical and approximate results of QSOBWM for $M = 3$ with $\mu = 0.01, 0.3, 0.5, 0.7, 0.99$ for Example 3.

TABLE 6 The L^2-errors and L^∞-errors of QSOBWM, BSOMM in [12] and BOMM in [11] of Example 4.

	BSOMM		BOMM		QSOBWM	
Errors	$M = 5$	$M = 7$	$N = 16$	$N = 20$	$M = 3$	$M = 4$
$\|\|u - \tilde{u}\|\|_{L^2}$	1.2×10^{-4}	7.6×10^{-6}	2.3×10^{-8}	7.9×10^{-9}	7.315×10^{-14}	1.335×10^{-14}
$\|\|u - \tilde{u}\|\|_{L^\infty}$	3.3×10^{-4}	2.1×10^{-5}	5.5×10^{-8}	1.9×10^{-8}	1.112×10^{-13}	2.043×10^{-14}

The problem was resolved by the B-spline operational matrix method (BSOMM)[12], and the Bernstein operational matrix method (BOMM)[11], respectively. We solve the example by applying the present scheme with $m = 4$, and list the values of L^2-errors and L^∞-errors of the three methods in Table 6 . The L^2-errors of the present scheme can achieve 10^{-14} within 7.98 seconds. As showing in Table 6 , the present scheme is most accurate.

7 | CONCLUSION

In the article, the quasilinearized semiorthogonal B-spline wavelets method is proposed to approximate the multi-term nonlinear fractional order equations contained fractional integral and derivatives. The present scheme significantly reduced the computational complexity of solving the nonlinear fractional order equations by the semiorogonal B-spline wavelets collocation method. The solution procedure has been described for approximating the nonlinear fractional equations. Furthermore, we discussed the convergence property of the present scheme. As the initial and boundary conditions have been both considered during the process of function approximation, the method can be applied for solving initial and boundary value problems of fractional order. Illustrative examples and comparison results testified the efficiency and accuracy of the present scheme.

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