Asymptotic analysis for a nonlinear viscoelastic problem with infinite history under a wider class of relaxation functions

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Abstract

In this paper, we consider a nonlinear viscoelastic problem with infinite history and a nonlinear feedback localized on the domain and a relaxation function satisfying

$$g'(t) \leq -\xi(t)G(g(t)).$$

We establish explicit and general decay rate results, using the multiplier method and some properties of the convex functions. Our results are obtained without imposing any restrictive growth assumption on the damping term and without imposing any assumption on the boundedness of initial data used in many earlier papers in the literature.

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1 Introduction

In this paper, we consider the following nonlinear viscoelastic problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + \sigma(t)h(u_t) = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, & \text{on } \partial\Omega \times (0,\infty) \\ u(x,-t) = u_0(x,t), u_t(x,0) = u_1(x), & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.1}$$

where u denotes the transverse displacement of waves, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega$ and g, h, σ are specific functions. The viscoelastic problems with infinite-memory terms have been studied by several authors. Giorgi et al. [1] considered the following semilinear hyperbolic equation, in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f$$

with $K(0), K(\infty) > 0$ and $K' \leq 0$ and gave the existence of global attractors for the solutions. Conti and Pata [2] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^n$,

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \;\; \text{in } \Omega \times \mathbb{R}^+ \tag{1.2}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(\infty) > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of at most cubic growth satisfying some conditions and proved the existence of a regular global attractor. Pata [3] discussed the decay properties of the semigroup generated by the following equation

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)ds = 0$$

where A is a strictly positive self-adjoint linear operator and $\alpha > 0$, $\beta \geq 0$ and the memory kernel μ is a decreasing function satisfying some specific conditions. He established the necessary as well as the sufficient conditions for the exponential stability. Al-Mahdi and Al-Gharabli [4] considered the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + |u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,-t) = u_0(x,t), \ u_t(x,0) = u_1(x), & \text{in } \Omega \times (0, +\infty), \end{cases} \tag{1.3}$$

and they established decay results with using a relaxation function g, satisfying the condition

$$g'(t) \leq -\xi(t)g^p(t), \quad 1 \leq p < \frac{3}{2}, \tag{1.4}$$

and they obtained a better decay rate than the one of [5] and [6]. Mustafa [7] consider the following coupled quasilinear system

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(s)\Delta u(t-s)ds + f_1(u,v) = 0 \\ |v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + \int_0^\infty g_2(s)\Delta v(t-s)ds + f_2(u,v) = 0 \end{cases} \tag{1.5}$$

and established more general decay rate results where the relaxation functions satisfy $g_i'(t) \leq -H(g_i(t)), \ i = 1,2$. He provided more general decay rates for which the usual exponential and polynomial rates are only special cases. Al-Mahdi [8] consider the following viscoelastic plate problem with a velocity-dependent material density and a logarithmic nonlinearity:

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = ku \ln|u| \quad \text{in } \Omega \times (0, \infty), \tag{1.6}$$

where Ω is a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$. He established an explicit and general decay rate results with imposing a minimal condition on the relaxation function, that is,

$$g'(t) \leq -\xi(t)H(g(t)), \tag{1.7}$$

where the two functions ξ and H satisfy some conditions. Recently, Al-Mahdi [9] considered the following plate problem:

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = 0,$$

and proved that the stability of this problem holds for which the relaxation function g satisfies the condition (1.7). In the present work, we study the asymptotic behavior of the solution of (1.1), under a wider class of relaxation functions. In fact we intend to establish a three-fold objective:

(a) extend the work for the viscoelastic problems with finite memory discussed in the literature such as the ones in [10, 11, 12] to infinite memory.

(b) generalize the condition, $g'(t) \leq -\xi(t)g^p(t)$, $1 \leq p < \frac{3}{2}$, used in many papers in the literature such as the ones in [13],[14], [4] and the one used in [7] for different viscoelastic problems to the condition $g'(t) \leq -\xi(t)G(g(t))$ where G satisfies some properties see [$(A1)$].

(c) drop the boundedness assumptions on the history data considered in many earlier results in the literature such as the ones in [14], [13], [4] and [15].

The rest of our paper is organized as follows. In section 2, we present some material needed to prove our result. Some technical lemmas will be given in section 3. We state and prove our main decay results in Section 4. We also, in Section 4, provide some examples to illustrate our theoretical results.

2 Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant. We consider the following hypotheses:

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying, for some $\beta_0 > 0$,

$$-\beta_0 g(s) \leq g'(s), \quad g(t) > 0 \ \text{ and } \ 1 - \int_0^{+\infty} g(s)ds = \ell > 0, \tag{2.1}$$

and there exists a C^1 function $G : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r_1]$ for some $r_1 > 0$ with $G(0) = G'(0) = 0$, $\lim_{s\to+\infty} G'(s) = +\infty$, $s \mapsto sG'(s)$ and $s \mapsto s\left(G'\right)^{-1}(s)$ are convex on $(0, r_1]$. Moreover, there exists a positive nonincreasing differentiable function ξ such that

$$g'(t) \leq -\xi(t)G(g(t)), \quad \forall t \geq 0. \tag{2.2}$$

$(A2)$ $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$, with $h_0(0) = 0$, and positive constants c_1, c_2, ε such that

$$\begin{aligned} &h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|) \ \text{ for all } |s| \leq \varepsilon \\ &c_1|s| \leq |h(s)| \leq c_2|s| \qquad \text{for all } |s| \geq \varepsilon \end{aligned} \tag{2.3}$$

In addition, we assume that the function H, defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r_2]$, for some $r_2 > 0$, when h_0 is nonlinear.

$(A3)$ $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function.

Remark 2.1. *Hypothesis $(A2)$ implies that $sh(s) > 0$, for all $s \neq 0$.*

Remark 2.2. *[16] If G is a strictly increasing and strictly convex C^2 function on $(0, r_1]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. For instance, if $G(r_1) = a, G'(r_1) = b, G''(r_1) = C$, we can define $\overline{G}$, for $t > r_1$, by*

$$\overline{G}(t) = \frac{C}{2}t^2 + (b - Cr_1)t + \left(a + \frac{C}{2}{r_1}^2 - br_1\right). \tag{2.4}$$

The same remark can be established for $\overline{H}$. For simplicity, in the rest of this paper, we use G and H instead of $\overline{G}$ and $\overline{H}$ respectively.

Remark 2.3. *[16] Since G is strictly convex on $(0, r_1]$ and $G(0) = 0$, then*

$$G(\theta z) \leq \theta G(z),\ 0 \leq \theta \leq 1 \text{ and } z \in (0, r_1]. \tag{2.5}$$

Remark 2.4. *We establish our decay results when the function H and G are nonlinear. Because, the other cases were discussed in [17].*

We introduce the "modified" energy associated to problem (1.1)

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1-\ell}{2}||\ \nabla u||_2^2 + \frac{1}{2}(go\nabla u)(t) \tag{2.6}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds$$

Direct differentiation, using (1.1), leads to

$$E'(t) = \frac{1}{2}(g'o\nabla u)(t) - \sigma(t)\int_\Omega u_t h(u_t)dx \leq 0 \tag{2.7}$$

For completeness we state, without proof, the following standard existence and regularity result (see [18], [19]).

Proposition 2.5. *Let $(u_0(.,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume $(A1) - (A3)$ are satisfied, then problem (1.1) has a unique global (weak) solution*

$$u \in C(R^+, H_0^1(\Omega)) \cap C^1(R^+, L^2(\Omega)).$$

Moreover, if

$$(u_0(.,0), u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$$

then the solution satisfies

$$u \in L^\infty\left(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega)\right) \cap W^{1,\infty}\left(\mathbb{R}^+, H_0^1(\Omega)\right) \cap W^{2,\infty}\left(\mathbb{R}^+, L^2(\Omega)\right).$$

3 Technical Lemmas

In this section, we state and establish several lemmas needed for the proof of our main result.

Lemma 3.1. *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx \le (1-\ell) C_p^2 (go\nabla u)(t),$$

where C_p is the Poincaré constant.

Proof.

$$\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx = \int_\Omega \left(\int_0^{+\infty} \sqrt{g(s)}\sqrt{g(s)}(u(t) - u(t-s))ds \right)^2 dx.$$

By applying Cauchy-Schwarz' and Poincaré's inequalities, we can show that

$$\begin{aligned}
&\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx \\
&\quad \le \int_\Omega \left(\int_0^{+\infty} g(s)ds \right) \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))^2 ds \right) dx \\
&\quad \le (1-\ell) C_p^2 (go\nabla u)(t).
\end{aligned}$$

□

Lemma 3.2. *There exists a positive constant M such that*

$$\int_t^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds \le M \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds, \tag{3.1}$$

Proof. The proof is identical to the one in [20]. Indeed, we have

$$\begin{aligned}
&\int_t^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds \le 2||\nabla u(t)||^2 \int_t^{+\infty} g(s)ds + 2\int_t^{+\infty} g(s)||\nabla u(t-s)||^2 ds \\
&\le 2 \sup_{s\ge 0} ||\nabla u(s)||^2 \int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u(-s)||^2 ds \\
&\le \frac{4E(s)}{(1-\ell)} \int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u_0(s)||^2 ds \\
&\le \frac{4E(0)}{(1-\ell)} \int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u_0(s)||^2 ds \\
&\le M \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds.
\end{aligned} \tag{3.2}$$

where $M = \max\left\{2, \frac{4E(0)}{(1-\ell)}\right\}$. □

Lemma 3.3. *Under the assumptions $(A1) - (A3)$, the functional*

$$\psi(t) := \int_\Omega u u_t dx$$

satisfies, along the solution, the estimate

$$\psi'(t) \le -\frac{\ell}{2}|| \nabla u||_2^2 + ||u_t||_2^2 + c(go\nabla u)(t) + c\int_\Omega h^2(u_t)dx \tag{3.3}$$

Proof. Direct computations, using (1.1), yield

$$\begin{aligned}\psi'(t) = &\int_\Omega u_t^2 dx + \int_\Omega u\Delta u dx - \int_\Omega u \int_0^{+\infty} g(s)\Delta u(t-s) ds dx \\ &- \sigma(t)\int_\Omega uh(u_t)dx \\ = &\int_\Omega u_t^2 dx - \ell\int_\Omega |\nabla u|^2 dx - \sigma(t)\int_\Omega uh(u_t)dx \\ &+ \int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx.\end{aligned} \tag{3.4}$$

Using Young's inequality and Lemma 3.1, we obtain

$$\begin{aligned}&\int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta}\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| ds\right)^2 dx \\ &\le \delta\int_\Omega |\nabla u|^2 dx + \frac{c}{\delta}(go\nabla u)(t).\end{aligned} \tag{3.5}$$

Also, the use of Young's and Poincaré's inequalities gives

$$\begin{aligned}-\sigma(t)\int_\Omega uh(u_t)dx &\le c\delta\int_\Omega u^2 dx + \frac{c}{4\delta}\int_\Omega h^2(u_t)dx \\ &\le c\delta\int_\Omega |\nabla u|^2 dx + \frac{c}{4\delta}\int_\Omega h^2(u_t)dx.\end{aligned} \tag{3.6}$$

Combining (3.4)-(3.6) and choosing δ small enough give (3.3). □

Lemma 3.4. *Under the assumptions* $(A1)-(A3)$*, the functional*

$$\chi(t) := -\int_\Omega u_t \int_0^{+\infty} g(s)(u(t) - u(t-s)) ds dx \tag{3.7}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\chi'(t) \le &\frac{\ell}{4}||\ \nabla u||_2^2 - (1-\ell-\frac{\ell}{4})||u_t||_2^2 + \frac{4c}{\ell}(go\nabla u)(t) \\ &- \frac{4c}{\ell}(g'o\nabla u)(t) + c\int_\Omega h^2(u_t)dx\end{aligned} \tag{3.8}$$

Proof. By differentiating (3.7), using (1.1), and performing integration by parts, we arrive at

$$\begin{aligned}
\chi'(t) =& \int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
& - \int_\Omega \left(\int_0^{+\infty} g(s) \nabla u(t-s) ds \right) . \left(\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds \right) dx \\
& + \int_\Omega \left(\int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \\
& - \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx - (1-\ell) \int_\Omega {u_t}^2 dx \\
& = \ell \int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
& + \int_\Omega \left| \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds \right|^2 dx \\
& + \int_\Omega \left(\int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \\
& - \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx - (1-\ell) \int_\Omega u_t^2 dx.
\end{aligned}$$

Using Young's inequality and Lemma 3.1, we obtain

$$\ell \int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \leq \delta \int_\Omega |\nabla u|^2 dx + \frac{c}{\delta} (go\nabla u)(t)$$

$$\int_\Omega \left(\int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \leq c(go\nabla u)(t) + c \int_\Omega h^2(u_t) dx$$

and

$$- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx \leq \delta \int_\Omega u_t^2 dx - \frac{c}{\delta} (g'o\nabla u)(t).$$

Combining all the above estimates and putting $\delta = \frac{\ell}{4}$, (3.8) is established. □

Lemma 3.5. *Assume that* $(A1)-(A3)$ *hold. Then there exist constants* $M_1, M_2, m, c > 0$ *such that the functional*

$$L(t) = M_1 E(t) + M_2 \chi(t) + \psi(t)$$

satisfies, for all $t \in \mathbb{R}^+$,

$$L'(t) \leq -mE(t) + c(go\nabla u)(t) + c \int_\Omega h^2(u_t) dx \tag{3.9}$$

Proof. By using (2.7), (3.3), (3.8), we easily see that

$$\begin{aligned}
L'(t) \leq& -\frac{\ell}{4} || \nabla u||_2^2 - \left(M_2 \left(1 - \ell - \frac{\ell}{4} \right) - 1 \right) ||u_t||_2^2 + \left(\frac{4c}{\ell} M_2^2 + c \right) (go\nabla u)(t) \\
& + \left(\frac{1}{2} M_1 - \frac{4c}{\ell} M_2^2 \right) (g'o\nabla u)(t) + (cM_2 + c) \int_\Omega h^2(u_t) dx.
\end{aligned}$$

At this point, we choose M_2 large enough so that

$$\alpha := M_2\left(1-\ell-\frac{\ell}{4}\right)-1>0,$$

and then M_1 large enough that

$$\frac{1}{2}M_1-\frac{4c}{\ell}M_2^2>0.$$

So, we arrive at

$$L'(t)\leq -\frac{\ell}{4}||\ \nabla u||_2^2-\alpha||u_t||_2^2+c(g'o\nabla u)(t)+c\int_\Omega h^2(u_t)dx \tag{3.10}$$

Therefore, (3.10) reduces to (3.9) for two positive constants m and c. On the other hand (see [21]), we can choose M_1 even larger (if needed)
so that

$$L\sim E \tag{3.11}$$

□

Lemma 3.6. *Under the assumptions (A2) and (A3), then we have*

$$\sigma(t)\int_\Omega h^2(u_t)dx\leq cH^{-1}(J(t))-cE'(t),\qquad \text{if } H \text{ is nonlinear} \tag{3.12}$$

where

$$J(t):=\frac{1}{|\Omega_2|}\int_{\Omega_2}u_t h(u_t)dx\leq -cE'(t) \tag{3.13}$$

and

$$\Omega_2=\{x\in\Omega:|u_t|\leq\varepsilon_1\}.$$

Proof. In case H is nonlinear on $[0,\varepsilon]$. We assume that $\max\{r_2,h_0(r_2)\}<\varepsilon$; otherwise we take r_2 smaller. Let $\varepsilon_1=\min\{r_2,h_0(r_2)\}$. Now, using $(A2)$, we have, for $\varepsilon_1\leq|s|\leq\varepsilon$,

$$|h(s)|\leq\frac{h_0^{-1}(|s|)}{|s|}|s|\leq\frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s|$$

and

$$|h(s)|\geq\frac{h_0(|s|)}{|s|}|s|\geq\frac{h_0(|\varepsilon_1|)}{|\varepsilon|}|s|$$

So, we deduce that

$$\begin{cases} h_0(|s|)\leq|h(s)|\leq h_0^{-1}(|s|) \ \text{ for all } \ |s|<\varepsilon_1 \\ c_1'|s|\leq|h(s)|\leq c_2'|s| \ \text{ for all } \ |s|\geq\varepsilon_1 \end{cases} \tag{3.14}$$

Using (3.14), we get for all $|s|\leq\varepsilon_1$

$$H(h^2(s))=|h(s)|h_0(|h(s)|)\leq sh(s)$$

which gives

$$h^2(s)\leq H^{-1}(sh(s)) \ \text{ for all } \ |s|\leq\varepsilon_1 \tag{3.15}$$

Now, we define the following partition which was introduced by Komornik [22]:

$$\Omega_1 = \{x \in \Omega : |u_t| > \varepsilon_1\}, \quad \Omega_2 = \{x \in \Omega : |u_t| \le \varepsilon_1\}$$

Using (3.14), we get on Ω_2

$$u_t h(u_t) \le \varepsilon_1 h_0^{-1}(\varepsilon_1) \le h_0(r_2) r_2 = H(r_2^2) \tag{3.16}$$

Then, Jensen's Inequality gives

$$H^{-1}(J(t)) \ge c \int_{\Omega_2} H^{-1}(u_t h(u_t)) dx \tag{3.17}$$

Thus, combining (2.7), (3.14) and (3.17), we arrive at

$$\begin{aligned}
\sigma(t) \int_\Omega h^2(u_t) dx &= \sigma(t) \int_{\Omega_2} h^2(u_t) dx + \sigma(t) \int_{\Omega_1} h^2(u_t) dx \\
&\le \sigma(t) \int_{\Omega_2} H^{-1}(u_t h(u_t)) dx + \sigma(t) \int_{\Omega_1} h^2(u_t) dx \\
&\le c H^{-1}(J(t)) - c E'(t).
\end{aligned} \tag{3.18}$$

This finishes the proof of (3.12). □

Lemma 3.7. *If (A1) and (A3) are satisfied, then we have, for all $t > 0$, the following estimate*

$$\int_0^t g(s) ||\nabla(t) - \nabla(t-s)||_2^2 ds \le \frac{(t+1)}{q_0} G^{-1}\left(\frac{q_0 \mu(t)}{(t+1)\xi(t)}\right) \tag{3.19}$$

where q_0 small positive number, G is defined in Remark (2.4) and

$$\mu(t) := -\int_0^t g'(s) ||\nabla(t) - \nabla(t-s)|_2^2 ds \le -cE'(t), \tag{3.20}$$

.

Proof. To establish (3.19), we introduce the following functional

$$\lambda(t) := \frac{q_0}{t+1} \int_0^t ||\nabla(t) - \nabla(t-s)||_2^2 ds. \tag{3.21}$$

Then, using the fact that E is nonincreasing and (2.6) to get

$$\begin{aligned}
\lambda(t) &\le \frac{2q_0}{t+1}\Big(\int_0^t ||\nabla(t)||_2^2 + \int_0^t ||\nabla(t-s)||_2^2 ds\Big). \\
&\le \frac{4q_0}{(1-\ell)(t+1)}\Big(\int_0^t \big(E(t) + E(t-s)\big) ds\Big) \\
&\le \frac{8q_0}{(1-\ell)(t+1)} \int_0^t E(s) ds \\
&\le \frac{8q_0}{(1-\ell)(t+1)} \int_0^t E(0) ds \\
&< +\infty.
\end{aligned} \tag{3.22}$$

Thus, q_0 can be chosen so small so that, for all $t > 0$,

$$\lambda(t) < 1. \tag{3.23}$$

Without loss of the generality, for all $t > 0$, we assume that $\lambda(t) > 0$, otherwise we get an exponential decay from (3.9). The use of Jensen's inequality and using (2.5), (3.20) and (3.23) gives

$$\begin{aligned}
\mu(t) &= \frac{1}{q_0\lambda(t)} \int_0^t \lambda(t)(-g'(s)) \int_\Omega q_0 |\nabla(t) - \nabla(t-s)|^2 dx ds \\
&\geq \frac{1}{q_0\lambda(t)} \int_0^t \lambda(t)\xi(s)G(g(s)) \int_\Omega q_0 |\nabla(t) - \nabla(t-s)|^2 dx ds \\
&\geq \frac{\xi(t)}{q_0\lambda(t)} \int_0^t G(\lambda(t)g(s)) \int_\Omega q_0 |\nabla(t) - \nabla(t-s)|^2 dx ds \\
&\geq \frac{(t+1)\xi(t)}{q_0} G\left(\frac{q_0}{(t+1)} \int_0^t g(s) \int_\Omega |\nabla(t) - \nabla(t-s)|^2 dx ds\right) \\
&= \frac{(t+1)\xi(t)}{q_0} \overline{G}\left(\frac{q_0}{(t+1)} \int_0^t g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds\right),
\end{aligned} \tag{3.24}$$

hence (3.19) is established. □

4 Stability result

In this section, we state and prove a new general decay result. For this purpose, we introduce the following functions:

$$G_1(t) := \int_t^1 \frac{1}{sW'(s)} ds, \tag{4.1}$$

$$G_2(t) = tW'(t), \quad G_3(t) = t(W')^{-1}(t), \quad G_4(t) = G_3^*(t). \tag{4.2}$$

where $W = (G^{-1} + H^{-1})^{-1}$. Further, we introduce the class S of functions $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfying for fixed $c_1, c_2 > 0$ (should be selected carefully in (4.24)):

$$\chi \in C^1(\mathbb{R}_+), \quad \chi \leq 1, \quad \chi' \leq 0, \tag{4.3}$$

and

$$c_2 G_4 \left[\frac{c}{d} q(t) f_0(t)\right] \leq c_1 \left(G_2\left(\frac{G_5(t)}{\chi(t)}\right) - \frac{G_2(G_5(t))}{\chi(t)}\right), \tag{4.4}$$

where $d > 0$, c is a generic positive constant which may change from line to line, f_0 and q will be defined in the proof of our main theorem and

$$G_5(t) = G_1^{-1}\left(c_1 \int_0^t \xi(s) ds\right). \tag{4.5}$$

Remark 4.1. *Thanks to (A1), G_2 is convex increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, G_1 is decreasing defines a bijection from $(0,1]$ to $\mathbb{R}_+$, and G_3 and G_4 are convex and increasing functions on $(0,r]$. Then the set S is not empty because it contains $\chi(s) = \varepsilon G_5(s)$ for any $0 < \varepsilon \leq 1$ small enough. Indeed, (4.3) is satisfied (since (4.1) and (4.5)).*

On the other hand, we have $q(t)f_0(t)$ is nonincreasing, $0 < G_5 \leq 1$, and G' and G_4 are increasing, then (4.4) *is satisfied if*

$$c_2 G_4\left[\frac{c}{d} q_0 f_0(0)\right] \leq \frac{c_1}{\varepsilon}\left(G'\left(\frac{1}{\varepsilon}\right) - G'(1)\right)$$

which holds, for $0 < \varepsilon \leq 1$ small enough, since $\lim_{t\to+\infty} G'(t) = +\infty$. But with the choice $\chi = \varepsilon G_5$, (4.6) *(below) does not lead to any stability estimate. The idea is to choose χ satisfy* (4.3) *and* (4.4) *such that* (4.6) *gives the best possible decay rate for E.*

Theorem 4.2. *Assume that $(A1) - (A3)$ hold and both $H(t)$ and $G(t)$ are non-linear functions, then there exists a strictly positive constant C such that, for any χ satisfying* (4.3) *and* (4.4)*, the solution of* (1.1) *satisfies, for all $t \geq 0$,*

$$E(t) \leq \frac{CG_5(t)}{\chi(t)q(t)}. \tag{4.6}$$

Proof. Using (3.9), (3.12) and (3.19), we obtain

$$\begin{aligned} L'(t) \leq & -mE(t) + c\left(\frac{t+1}{q_0}\right)G^{-1}\left(\frac{q_0\mu(t)}{(t+1)\xi(t)}\right)(t) + c\int_0^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||^2\right) ds \\ & + cH^{-1}(J(t)) - cE'(t). \end{aligned} \tag{4.7}$$

Since $\frac{1}{t+1} < 1$ whenever $t > 0$. Combining this with the strictly increasing and strictly convex properties of H, setting $\theta = \frac{1}{t+1} < 1$ and using Remark (2.4), we obtain

$$\begin{aligned} L'(t) \leq & -mE(t) + c\left(\frac{t+1}{q_0}\right)G^{-1}\left(\frac{q_0\mu(t)}{(t+1)\xi(t)}\right)(t) + c\int_0^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||^2\right) ds \\ & + c\frac{(t+1)}{q_0}H^{-1}\left(\frac{J(t)}{t+1}\right) - cE'(t). \end{aligned} \tag{4.8}$$

Let $L_1(t) = L(t) + cE(t)\ E$, then (4.8) becomes

$$\begin{aligned} L_1'(t) \leq & -mE(t) + c\left(\frac{t+1}{q_0}\right)G^{-1}\left(\frac{q_0\mu(t)}{(t+1)\xi(t)}\right)(t) + c\int_0^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||^2\right) ds \\ & + c\frac{(t+1)}{q_0}H^{-1}\left(\frac{J(t)}{t+1}\right). \end{aligned} \tag{4.9}$$

Let $r_0 = \min\{r_1, r_2\}$, $\chi(t) = \max\{\frac{q_0\mu(t)}{(t+1)\xi(t)}, \frac{J(t)}{t+1}\}$ and $W = (G^{-1} + H^{-1})^{-1}$. Then, we get

$$L_1'(t) \leq -mE(t) + c\left(\frac{t+1}{q_0}\right)W^{-1}(\chi(t)) + c\int_0^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||^2\right) ds. \tag{4.10}$$

Now, for $\varepsilon_0 < r_0$ and the fact that $E' \leq 0$, $W' > 0, W'' > 0$ on $(0, r_0]$, we find that the functional L_2, defined by

$$L_2(t) := W'\left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)}\right) L_1(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$.

$$\alpha_1 L_2(t) \leq E(t) \leq \alpha_2 L_2(t) \tag{4.11}$$

and

$$
\begin{aligned}
L_2'(t) &= \left(\frac{-\varepsilon_0}{(t+1)^2} \frac{E(t)}{E(0)} + \frac{\varepsilon_0}{(t+1)} \frac{E'(t)}{E(0)} \right) W'' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) L_1(t) \\
&\quad + W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) {L_1}'(t) \\
&\leq -mE(t)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c(t+1)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) W^{-1}(\chi(t)) \\
&\quad + cW' \left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)} \right) \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds.
\end{aligned}
\tag{4.12}
$$

Let W^* be the convex conjugate of W in the sense of Young (see [23]), then

$$
W^*(s) = s(W')^{-1}(s) - W\left[(W')^{-1}(s)\right], \quad \text{if } s \in (0, W'(r_0)] \tag{4.13}
$$

and W^* satisfies the following Young inequality

$$
AB \leq W^*(A) + W(B), \quad \text{if } A \in (0, W'(r_0)], \ B \in (0, r_0]. \tag{4.14}
$$

With $A = W'\left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)}\right)$ and $B = W^{-1}(\chi(t))$, using (2.7) and (4.12)-(4.14), we arrive at

$$
\begin{aligned}
L_2'(t) &\leq -mE(t)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c(t+1)W^* \left(W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) \right) + c(t+1)\chi(t) \\
&\quad + cW' \left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)} \right) \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds \\
&\leq -mE(t)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c(t+1)\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c(t+1)\chi(t) \\
&\quad + cW' \left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)} \right) \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds.
\end{aligned}
\tag{4.15}
$$

Using (3.13) and (3.20), we observe that

$$
\begin{aligned}
(t+1)\xi(t)\chi(t) &\leq q_0\mu(t) + \xi(t)J(t) \\
&\leq q_0\mu(t) + \xi(0)J(t) \\
&\leq -cE'(t) - cE'(t) \\
&\leq -cE'(t).
\end{aligned}
\tag{4.16}
$$

So, multiplying (4.15) by $\xi(t)$ and using the fact that, $\varepsilon_0 \frac{E(t)}{E(0)} < r_0$, give

$$
\begin{aligned}
\xi(t)L_2'(t) &\leq -m\xi(t)E(t)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon_0\xi(t) \cdot \frac{E(t)}{E(0)} W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) \\
&\quad - cE'(t) + c\xi(t)W' \left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)} \right) \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds.
\end{aligned}
\tag{4.17}
$$

Using the non-increasing property of ξ, we obtain, for all $t \geq 0$,

$$
\begin{aligned}
(\xi(t)L_2 + cE)'(t) &\leq -m\xi(t)E(t)W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) + c\xi(t) \cdot \frac{E(t)}{E(0)} W' \left(\frac{\varepsilon_0}{t+1} \cdot \frac{E(t)}{E(0)} \right) \\
&\quad + c\xi(t)W' \left(\frac{\varepsilon_0}{(t+1)} \cdot \frac{E(t)}{E(0)} \right) \int_0^{+\infty} g(t+s) \left(1 + ||\nabla u_0(s)||^2\right) ds.
\end{aligned}
\tag{4.18}
$$

Therefor, by setting $L_3 := \xi(t)L_2 + cE$. Then, for some $\alpha_3, \alpha_4 > 0$, we have

$$\alpha_3 L_3(t) \leq E(t) \leq \alpha_4 L_3(t). \tag{4.19}$$

Therefor, we get

$$\begin{aligned}
L_3'(t) &\leq -m\xi(t)E(t)W'\left(\frac{\varepsilon_0}{t+1}\cdot\frac{E(t)}{E(0)}\right) + c\xi(t)\frac{\varepsilon_0 E(t)}{E(0)}W'\left(\frac{\varepsilon_0}{t+1}\cdot\frac{E(t)}{E(0)}\right)\\
&\quad + c\xi(t)W'\left(\frac{\varepsilon_0}{(t+1)}\cdot\frac{E(t)}{E(0)}\right)\int_0^{+\infty} g(t+s)\left(1+||\nabla u_0(s)||^2\right)ds\\
&\leq \varepsilon_0(\frac{mE(0)}{\varepsilon_0} - c)\frac{\xi(t)}{(t+1)}\frac{E(t)}{E(0)}W'\left(\frac{\varepsilon_0}{t+1}\cdot\frac{E(t)}{E(0)}\right)\\
&\quad + c\xi(t)W'\left(\frac{\varepsilon_0}{(t+1)}\cdot\frac{E(t)}{E(0)}\right)\int_0^{+\infty} g(t+s)\left(1+||\nabla u_0(s)||^2\right)ds.
\end{aligned} \tag{4.20}$$

For simplicity, let $f_0(t) = \int_0^{+\infty} g(t+s)\left(1+||\nabla u_0(s)||^2\right)ds$ and $q(t) = \frac{1}{t+1}$ then by recalling the definition of G_2 and selecting ε_0 small enough

$$L_3'(t) \leq -\frac{k\xi(t)}{q(t)}G_2\left(\frac{\varepsilon_0}{(t+1)}\cdot\frac{E(t)}{E(0)}\right) + c\xi(t)f_0(t)W'\left(\frac{\varepsilon_0}{(t+1)}\cdot\frac{E(t)}{E(0)}\right). \tag{4.21}$$

Since $G_2'(t) = W'(t) + tW''(t)$, then, using the strict convexity of G on $(0, r_0]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, r_0]$. Using the general Young inequality (4.14) on the last term in (4.21) with $A = W'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)$ and $B = [\frac{c}{d}f_0(t)]$, we have for $d > 0$,

$$\begin{aligned}
cf_0(t)W'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) &= \frac{d}{q(t)}\left[\frac{c}{d}q(t)f_0(t)\right]\left(W'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right)\\
&\leq \frac{d}{q(t)}G_3\left(W'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_3^*\left[\frac{c}{d}q(t)f_0(t)\right]\\
&\leq \frac{d}{q(t)}\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\left(W'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)f_0(t)\right]\\
&\leq \frac{d}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)}G_4\left[\frac{c}{d}q(t)f_0(t)\right].
\end{aligned} \tag{4.22}$$

Now, combining (4.21) and (4.22) and choosing d small enough $k_1 = (k-d) > 0$, we arrive at

$$\begin{aligned}
L_3'(t) &\leq -k\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)f_0(t)\right]\\
&\leq -k_1\frac{\xi(t)}{q(t)}G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}G_4\left[\frac{c}{d}q(t)f_0(t)\right].
\end{aligned} \tag{4.23}$$

Using the equivalent property in (4.19) and the increasing of G_2, we have, for some $d_0 = \frac{\alpha_3}{E(0)} > 0$,

$$G_2\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right) \geq G_2\left(d_0 L_3(t)q(t)\right).$$

Letting $\mathcal{F}(t) := d_0 L_3(t) q(t)$ and recalling $q' \leq 0$, then for some constant $c_1 = d_0 k_1 > 0$ and $c_2 = d_0 d > 0$, we arrive at

$$\mathcal{F}'(t) \leq -c_1 \xi(t) G_2(\mathcal{F}(t)) + c_2 \xi(t) G_4 \left[\frac{c}{d} q(t) f_0(t)\right]. \tag{4.24}$$

Since $d_0 q(t)$ is is nonincreasing. Using the equivalent property $\mathcal{F} \sim E$ implies that there exists $b_0 > 0$ such that $\mathcal{F}(t) \geq b_0 E(t) q(t)$. Let $t \in \mathbb{R}_+$ and $\chi(t)$ satisfying (4.3) and (4.4). If $b_0 q(t) E(t) \leq 2\frac{G_5(t)}{\chi(t)}$, then, we have

$$E(t) \leq \frac{2}{b_0} \frac{G_5(t)}{\chi(t) q(t)}. \tag{4.25}$$

If $b_0 q(t) E(t) > 2\frac{G_5(t)}{\chi(t)}$. Then, for any $0 \leq s \leq t$, we get

$$b_0 q(s) E(s) > 2\frac{G_5(t)}{\chi(t)}, \tag{4.26}$$

since, $q(t)E(t)$ is nonincreasing function. Therefore, we have for any $0 \leq s \leq t$,

$$\mathcal{F}(s) > 2\frac{G_5(t)}{\chi(t)}. \tag{4.27}$$

Using (2.5), $0 < \chi \leq 1$ and the fact that G_2 is convex, we have, for any $0 < \epsilon_1 \leq 1$,

$$\begin{aligned} G_2\Big(\epsilon_1 \chi(s)\mathcal{F}(s) - \epsilon_1 G_5(s)\Big) &= G_2\Big(\epsilon_1 \chi(s)\mathcal{F}(s) - \frac{\epsilon_1 \chi(s) G_5(s)}{\chi(s)}\Big) \\ &\leq \epsilon_1 \chi(s) G_2\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big). \end{aligned} \tag{4.28}$$

Recalling the definition of G_2, that is $G_2(t) = tW'(t)$, we have

$$\begin{aligned} &G_2\Big(\epsilon_1 \chi(s)\mathcal{F}(s) - \epsilon_1 G_5(s)\Big) \leq \epsilon_1 \chi(s)\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big) W'\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big) \\ &\leq \epsilon_1 \chi(s)\mathcal{F}(s) W'\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big) - \epsilon_1 \chi(s) \frac{G_5(s)}{\chi(s)} W'\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big). \end{aligned} \tag{4.29}$$

Now, using (4.27) and the fact that W' is increasing, for any $0 \leq s \leq t$, we have

$$W'\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big) < W'\Big(\mathcal{F}(s)\Big), \quad W'\Big(\mathcal{F}(s) - \frac{G_5(s)}{\chi(s)}\Big) > W'\Big(\frac{G_5(s)}{\chi(s)}\Big). \tag{4.30}$$

Therefor, we have

$$G_2\Big(\epsilon_1 \chi(s)\mathcal{F}(s) - \epsilon_1 G_5(s)\Big) \leq \epsilon_1 \chi(s)\mathcal{F}(s) W'\Big(\mathcal{F}(s)\Big) - \epsilon_1 \chi(s) \frac{G_5(s)}{\chi(s)} W'\Big(\frac{G_5(s)}{\chi(s)}\Big). \tag{4.31}$$

Now, we let

$$\mathcal{F}_3(s) = \epsilon_1 \chi(s)\mathcal{F}(s) - \epsilon_1 G_5(s), \tag{4.32}$$

where ϵ_1 small enough so that $\mathcal{F}_3(0) \leq 1$. Using the definition of G_2; that is $G_2(t) = tW'(t)$. Then (4.31) becomes, for any $0 \leq s \leq t$,

$$G_2\bigg(\mathcal{F}_3(s)\bigg) \leq \epsilon_1\chi(s)G_2\bigg(\mathcal{F}(s)\bigg) - \epsilon_1\chi(s)G_2\bigg(\frac{G_5(s)}{\chi(s)}\bigg). \tag{4.33}$$

Further, we have

$$\mathcal{F}_3'(t) = \epsilon_1\chi'(t)\mathcal{F}(t) + \epsilon_1\chi(s)\mathcal{F}_2'(t) - \epsilon_1 G_5'(t). \tag{4.34}$$

Since $\chi' \leq 0$ and using (4.24), then for any $0 \leq s \leq t$, $0 < \epsilon_1 \leq 1$, we obtain

$$\begin{aligned}\mathcal{F}_3'(t) &\leq \epsilon_1\chi(t)\mathcal{F}_2'(t) - \epsilon_1 G_5'(t)\\ &\leq -c_1\epsilon_1\xi(t)\chi(t)G_2(\mathcal{F}(t)) + c_2\epsilon_1\xi(t)\chi(s)G_4\left[\frac{c}{d}q(t)f_0(t)\right] - \epsilon_1 G_5'(t).\end{aligned} \tag{4.35}$$

Then, using (4.33), we get

$$\begin{aligned}\mathcal{F}_3'(t) &\leq -c_1\xi(t)G_2(\mathcal{F}_3(t)) + c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)f_0(t)\right]\\ &- c_1\epsilon_1\xi(t)\chi(t)G_2\bigg(\frac{G_5(t)}{\chi(t)}\bigg) - \epsilon_1 G_5'(t).\end{aligned} \tag{4.36}$$

From the definition of G_1 and G_5, we have

$$G_1\left(G_5(s)\right) = c_1\int_0^s \xi(\tau)d\tau,$$

hence,

$$G_5'(s) = -c_1\xi(s)G_2\left(G_5(s)\right). \tag{4.37}$$

Now, we have

$$\begin{aligned}& c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)f_0(t)\right] - c_1\epsilon_1\xi(t)\chi(t)G_2\bigg(\frac{G_5(t)}{\chi(t)}\bigg) - \epsilon_1 G_5'(t)\\ &= c_2\epsilon_1\xi(t)\chi(t)G_4\left[\frac{c}{d}q(t)f_0(t)\right] - c_1\epsilon_1\xi(t)\chi(t)G_2\bigg(\frac{G_5(t)}{\chi(t)}\bigg) + c_1\epsilon_1\xi(t)G_2\left(G_5(t)\right)\\ &= \epsilon_1\xi(t)\chi(t)\bigg(c_2G_4\left[\frac{c}{d}q(t)f_0(t)\right] - c_1G_2\bigg(\frac{G_5(t)}{\chi(t)}\bigg) + c_1\frac{G_2\left(G_5(t)\right)}{\chi(t)}\bigg).\end{aligned} \tag{4.38}$$

Then, according to (4.4), we get

$$\epsilon_1\xi(t)\chi(t)\bigg(c_2G_4\left[\frac{c}{d}q(t)f_0(t)\right] - c_1G_2\bigg(\frac{G_5(t)}{\chi(t)}\bigg)\bigg) - c_1\frac{G_2\left(G_5(t)\right)}{\chi(t)}\bigg) \leq 0$$

Then (4.36) gives

$$\mathcal{F}_3'(t) \leq -c_1\xi(t)G_2(\mathcal{F}_3(t)). \tag{4.39}$$

Thus from (4.39) and the definition of G_1 and G_2 in (4.1) and (4.2), we obtain

$$\bigg(G_1\left(\mathcal{F}_3(t)\right)\bigg)' \geq c_1\xi(t). \tag{4.40}$$

Integrating (4.40) over $[0, t]$, we get

$$G_1(\mathcal{F}_3(t)) \geq c_1 \int_0^t \xi(s)ds + G_1(\mathcal{F}_3(0)). \tag{4.41}$$

Since G_1 is decreasing, $\mathcal{F}_3(0) \leq 1$ and $G_1(1) = 0$, then

$$\mathcal{F}_3(t) \leq G_1^{-1}\left(c_1 \int_0^t \xi(s)ds\right) = G_5(t). \tag{4.42}$$

Recalling that $\mathcal{F}_3(t) = \epsilon_1 \chi(t)\mathcal{F}(t) - \epsilon_1 G_5(t)$, we have

$$\mathcal{F}(t) \leq \frac{(1+\epsilon_1)}{\epsilon_1}\frac{G_5(t)}{\chi(t)}, \tag{4.43}$$

Similarly, recall that $\mathcal{F}(t) := d_0 L_3(t) q(t)$, then

$$\mathcal{F}_1(t) \leq \frac{(1+\epsilon_1)}{d_0\epsilon_1}\frac{G_5(t)}{\chi(t)q(t)}, \tag{4.44}$$

Since $L_3 \sim E$, then for some $b > 0$, we have $E(t) \leq b\mathcal{F}_1$; which gives

$$E(t) \leq \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\frac{G_5(t)}{\chi(t)q(t)}, \tag{4.45}$$

From (4.25) and (4.45), we obtain the following estimate

$$E(t) \leq c_3\left(\frac{G_5(t)}{\chi(t)q(t)}\right), \tag{4.46}$$

where $c_3 = \max\{\frac{2}{b_0}, \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\}$. □

Example 1 [20]: Let $g(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $0 < a < \nu - 1$ so that $(A1)$ is satisfied. In this case $\xi(t) = \nu a^{\frac{-1}{\nu}}$ and $G(t) = t^{\frac{\nu+1}{\nu}}$. For the fractional damping, let $h_0 = ct^\nu$ and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct^{\frac{\nu+1}{\nu}}$. We will discuss two cases:
Case 1: if

$$m_0(1+t)^r \leq 1 + ||\nabla u_0||^2 \leq m_1(1+t)^r \tag{4.47}$$

where $0 < r < \nu - 1$ and $m_0, m_1 > 0$. We recall the definition of the functions $G_i's$ and for simplicity, we choose $\nu = 2$ and then $0 < r < 1$, then we find that

$$W(t) = (\overline{G}^{-1} + \overline{H}^{-1})^{-1}(t) = ct^{\frac{3}{2}}, \quad G_2(t) = tW'(t) = ct^{\frac{3}{2}}$$

$$G_3(t) = ct^3, \quad G_4(t) = ct^{\frac{3}{2}}, \quad G_1(t) = ct^{\frac{-1}{2}}, \quad G_5(t) = ct^{-2}$$

Therefore, we have, for some positive constants a_i depending only on a, m_0, m_1, r, the following

$$a_1(1+t)^{-1+r} \leq f_0(t) \leq a_2(1+t)^{-1+r}, \tag{4.48}$$

$$a_3(1+t)^r \leq q(t) \leq a_4(1+t)^r, \tag{4.49}$$

We notice that condition (4.4) is satisfied if

$$(t+1)^2 q(t) f_0(t)\chi(t) \leq a_5\left(1-(\chi)^{\frac{1}{2}}\right)^{\frac{2}{3}}. \tag{4.50}$$

where $a_5 > 0$ depending only on a. Choosing $\chi(t)$ as the following

$$\chi(t) = \lambda(1+t)^{-1},\ 0 < r < 1, \tag{4.51}$$

so that (4.3) is valid. Moreover, using (4.48) and (4.49), we see that (4.50) is satisfied if $0 < \lambda \leq 1$ is small enough, and then (4.4) is satisfied. Hence (4.6) implies that, for any $t \in \mathbb{R}_+$

$$E(t) \leq c(1+t)^{-(r+1)},\quad 0 < r < 1. \tag{4.52}$$

Thus, the estimate (4.52) gives $\lim_{t\to+\infty} E(T) = 0$. Case 2: if $m_0 \leq 1 + ||\nabla u_0||^2 \leq m_1$. That is $r = 0$ in (4.47) (as it was assumed in [5], [17], [6] and [24]), then (4.52) holds with $r = 0$.

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