N-soliton solutions and the Hirota conditions in (1+1)-dimensions

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Abstract

We discuss N-soliton solutions and analyze the Hirota N-soliton conditions, starting from Hirota bilinear forms. An algorithm to verify the Hirota conditions is proposed by factoring out common factors for the terms in the conditions and comparing degrees of the involved polynomials containing the common factors. Applications to a class of generalized KdV equations and a class of generalized higher-order KdV equations are made, together with all proofs of the existence of N-soliton solutions to each equation in two classes.

Keywords: N-soliton solution, Hirota N-soliton condition, (1+1)-dimensional integrable equations

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1 Introduction

N-soliton solutions are typical exact solutions to integrable equations [1, 2, 3], and breather, lump and rogue wave solutions are their special situations. The Hirota bilinear method is a powerful approach to N-soliton solutions [4]. The concept of bilinear derivatives is the key tool in the theory, and Hirota bilinear forms are crucial in furnishing N-soliton solutions.

Hirota bilinear derivatives are defined by [5]:

$$D_x f\cdot g = f_x g - f g_x,\ D_x^2 f\cdot g = f_{xx}g - 2f_x g_x + f g_{xx},\ \cdots,$$
$$D_x^m f\cdot g = \sum_{i=1}^{m} (-1)^{m-i}\binom{m}{i}(\partial_x^i f)(\partial_x^{m-i} g),\ m\ge 1,$$

and more generally, bilinear partial derivatives with multiple variables are similarly defined:

$$(D_x^m D_t^n f\cdot g)(x,t) = (\partial_x - \partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t},\ m,n\ge 0,\ m+n\ge 1. \quad (1.1)$$

When $f=g$, we get Hirota bilinear expressions:

$$D_x f\cdot f = 0,\ D_x^2 f\cdot f = 2(f_{xx}f - f_x^2),\ \cdots,$$
$$D_x^{2m-1} f\cdot f = 0,\ D_x^{2m} f\cdot f = \sum_{i=1}^{2m} (-1)^{2m-i}\binom{2m}{i}(\partial_x^i f)(\partial_x^{2m-i} f),\ m\ge 1,$$

and similarly, bilinear partial derivative expressions:

$$D_x^m D_t^n f\cdot f = \sum_{i=1}^{m}\sum_{j=1}^{n} (-1)^{m+n-i-j}\binom{m}{i}\binom{n}{j}(\partial_x^i\partial_t^j f)(\partial_x^{m-i}\partial_t^{n-j} f),\ m,n\ge 0,\ m+n\ge 1. \quad (1.2)$$

In terms of Hirota bilinear expressions, we can define Hirota bilinear equations. Take an even polynomial $P(x_1,x_2,\cdots,x_M)$ in M variables, with no constant term, i.e., $P(\mathbf{0}) = P(0,0,\cdots,0)=0$. The associated Hirota bilinear equation is defined by

$$P(D_{x_1}, D_{x_2},\cdots, D_{x_M}) f\cdot f = 0, \quad (1.3)$$

each term of which is a Hirota bilinear expression.

For example, the bilinear KdV equation associated with $P(x,t)=x^4+xt$ reads

$$B(f) := (D_x^4 + D_xD_t) f\cdot f = 2(f_{4x}f - 4f_{3x}f_x + 3f_{xx}^2 + f_{xt}f - f_xf_t) = 0, \quad (1.4)$$

which gives the standard KdV equation

$$N(u) := u_t + 6uu_x + u_{xxx} = 0, \quad (1.5)$$

upon taking $u=2(\ln f)_{xx}$. The link is $N(u)=(B(f)/f^2)_x$. The bilinear Boussinesq equations associated with $P(x,t)=t^2\pm x^4$ are

$$B(f):=(D_t^2\pm D_x^4)f\cdot f=2[f_{tt}f-f_t^2\pm(f_{4x}f-4f_{3x}f_x+3f_{xx}^2)]=0, \tag{1.6}$$

which are transformed into the standard Boussinesq equations

$$N(u):=u_{tt}+(u^2)_{xx}\pm u_{4x}=0, \tag{1.7}$$

through the transformation $u=\pm 6(\ln f)_{xx}$, and the links are $N(u)=\pm 3(B(f)/f^2)_{xx}$ [6].

We would like to discuss N-soliton solutions and derive the corresponding Hirota conditions. An algorithm will be proposed for verifying the Hirota N-soliton conditions by figuring out common factors for the terms in the conditions and comparing degrees of the involved polynomials containing common factors. Applications will be made to a class of generalized KdV equations associated with

$$P(x,t)=ax^4+bx^3t+cx^2+dxt, \tag{1.8}$$

where a,b,c,d are arbitrary constants satisfying $b^2+d^2\neq 0$, and a class of generalized higher-order KdV equations associated with

$$P(x,t)=ax^6+bx^4+cx^2+xt, \tag{1.9}$$

where a,b,c are arbitrary constants. Our analysis implies that each equation in the two classes possesses N-soliton solutions.

2 N-soliton conditions

An N-soliton solution to a Hirota bilinear equation (1.3) is given by [7]:

$$f=\sum_{\mu=0,1}\exp\Big(\sum_{i=1}^N\mu_i\eta_i+\sum_{i<j}a_{ij}\mu_i\mu_j\Big), \tag{2.1}$$

where $\mu=(\mu_1,\mu_2,\cdots,\mu_N)$, $\mu=0,1$ means that each μ_i takes 0 or 1, and

$$\eta_i=k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M+\eta_{i,0},\ 1\le i\le N, \tag{2.2}$$

$$\mathrm{e}^{a_{ij}}=A_{ij}:=-\frac{P(\mathbf{k}_i-\mathbf{k}_j)}{P(\mathbf{k}_i+\mathbf{k}_j)},\ 1\le i<j\le N, \tag{2.3}$$

$\eta_{i,0}$'s being arbitrary phase shifts, under the dispersion relations

$$P(\mathbf{k}_i)=0,\ \mathbf{k}_i=(k_{1,i},k_{2,i},\cdots,k_{M,i}),\ 1\le i\le N. \tag{2.4}$$

We will show that a Hirota bilinear equation (1.3) has an N-soliton solution (2.1) iff

$$H(\mathbf{k}_1,\cdots,\mathbf{k}_n):=\sum_{\sigma=\pm1}P(\sum_{i=1}^{n}\sigma_i\mathbf{k}_i)\prod_{1\le i<j\le n}P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)\sigma_i\sigma_j=0,\ 1\le n\le N, \tag{2.5}$$

where $\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_n)$, and $\sigma=\pm1$ means that each σ_i takes 1 or -1. This is called the Hirota condition for an N-soliton solution, or simply, the N-soliton condition [8], and there are very few studies on this Hirota N-soliton condition [9, 10, 11, 12], due to its complexity [8].

The one-soliton condition is just the dispersion relation: $P(\mathbf{k}_1)=0$, which means that $f=1+\mathrm{e}^{\eta_1}$ is a solution. Besides the dispersion relations, the two-soliton condition is

$$2(P(\mathbf{k}_1+\mathbf{k}_2)P(\mathbf{k}_1-\mathbf{k}_2)-P(\mathbf{k}_1-\mathbf{k}_2)P(\mathbf{k}_1+\mathbf{k}_2))=0, \tag{2.6}$$

which is an identity. Therefore, there always exists a two-soliton solution:

$$f=1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+A_{12}\mathrm{e}^{\eta_1+\eta_2}. \tag{2.7}$$

Taking $N=3$, we obtain the three-soliton condition [13, 14]:

$$\begin{aligned}\sum_{\sigma_1,\sigma_2,\sigma_3=\pm1}&P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2)\\&\times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S}&P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2)\\&\times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0,\end{aligned} \tag{2.8}$$

where $S=\{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\}$. The three-soliton solution is given by

$$\begin{aligned}f=&1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+A_{12}\mathrm{e}^{\eta_1+\eta_2}+A_{13}\mathrm{e}^{\eta_1+\eta_3}\\&+A_{23}\mathrm{e}^{\eta_2+\eta_3}+A_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3},\ A_{123}=A_{12}A_{13}A_{23}.\end{aligned} \tag{2.9}$$

It is a direct computation that the three-soliton condition is satisfied for both the KdV equation and the Boussinesq equations. There is a conjecture that the three-soliton condition implies the N-soliton condition. No counterexample has been found, indeed.

If we require a sufficient Hirota N-soliton condition [15]:

$$P(\mathbf{k}_i-\mathbf{k}_j)=0,\ 1\le i<j\le N, \tag{2.10}$$

we obtain a resonant N-soliton solution

$$f = 1 + c_1 \mathrm{e}^{\eta_1} + c_2 \mathrm{e}^{\eta_2} + \cdots + c_N \mathrm{e}^{\eta_N}, \tag{2.11}$$

where c_i's are arbitrary constants. All wave vectors $\mathbf{k}_i$'s associated with resonant solutions form an affine space in $\mathbb{R}^M$ [16].

Note that we have

$$P(D_{x_1}, \cdots, D_{x_M}) \mathrm{e}^{\eta_i} \cdot \mathrm{e}^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j) \mathrm{e}^{\eta_i + \eta_j}, \tag{2.12}$$

and

$$P(D_{x_1}, \cdots, D_{x_M}) \mathrm{e}^{\eta_n} f \cdot \mathrm{e}^{\eta_n} g = \mathrm{e}^{2\eta_n} P(D_{x_1}, \cdots, D_{x_M}) f \cdot g, \tag{2.13}$$

where η_i, η_j and η_n are arbitrary linear functions.

Theorem 2.1 *Let f be defined by* (2.1)*. Then we have*

$$\begin{aligned} & P(D_{x_1}, \cdots, D_{x_M}) f \cdot f \\ = & (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_N)}{\prod_{1 \le i < j \le N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ + & \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1 \le i_1 < \cdots < i_n \le N} \frac{H(\mathbf{k}_1, \cdots, \hat{\mathbf{k}}_{i_1}, \cdots, \hat{\mathbf{k}}_{i_n}, \cdots, \mathbf{k}_N)}{\prod_{\substack{1 \le i < j \le N \\ i,j \notin \{i_1, \cdots, i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \cdots + \hat{\eta}_{i_1} + \cdots + \hat{\eta}_{i_n} + \cdots + \eta_N}, \end{aligned} \tag{2.14}$$

where $\hat{\mathbf{k}}_{i_1}, \cdots, \hat{\mathbf{k}}_{i_n}$ (or $\hat{\eta}_{i_1}, \cdots, \hat{\eta}_{i_n}$) mean that $\mathbf{k}_{i_1}, \cdots, \mathbf{k}_{i_n}$ (or $\eta_{i_1}, \cdots, \eta_{i_n}$) are not included.

Proof: Note that we have the properties (2.12) and (2.13), and so, we can expand all terms in $P(D_{x_1}, \cdots, D_{x_M}) f \cdot f$. Let us compute the terms which involve $\mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_N}$. For example, we have the following term of such a type:

$$\begin{aligned} & P(D_{x_1}, \cdots, D_{x_M})(A_{12\cdots(N-1)} \mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_{N-1}} \cdot \mathrm{e}^{\eta_N}) \\ = & A_{12\cdots(N-1)} P(D_{x_1}, \cdots, D_{x_M})(\mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_{N-1}} \cdot \mathrm{e}^{\eta_N}) \\ = & A_{12\cdots(N-1)} P(\mathbf{k}_1 + \cdots + \mathbf{k}_{N-1} - \mathbf{k}_N) \mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ = & (-1)^{\frac{1}{2}(N-1)(N-2)} \prod_{1 \le i < j \le N-1} \frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)} P(\mathbf{k}_1 + \cdots + \mathbf{k}_{N-1} - \mathbf{k}_N) \mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ = & (-1)^{\frac{1}{2}N(N-1)} \frac{P(\sigma_1 \mathbf{k}_1 + \cdots + \sigma_N \mathbf{k}_N) \prod_{1 \le i < j \le N} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j}{\prod_{1 \le i < j \le N} P(\mathbf{k}_i + \mathbf{k}_j)} \mathrm{e}^{\eta_1 + \eta_2 + \cdots + \eta_N}, \end{aligned}$$

where $\sigma = (\sigma_1, \cdots, \sigma_{N-1}, \sigma_N) = (1, \cdots, 1, -1)$ and $A_{12\cdots(N-1)} = \prod_{1 \le i < j \le N-1} A_{ij}$. Taking all possibilities of $\sigma_i = \pm 1,\ 1 \le i \le N$, we obtain the first sum determined by $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$

in (2.14). The other sums determined by $H(\mathbf{k}_1, \cdots, \hat{\mathbf{k}}_{i_1}, \cdots, \hat{\mathbf{k}}_{i_n}, \cdots, \mathbf{k}_N)$, $1 \le i_1 < i_2 < \cdots < i_n \le N$, can be similarly obtained. The proof is finished. $\square$

Based on this theorem, we see that the Hirota condition is a necessary and sufficient condition for a Hirota bilinear equation to have an N-soliton solution, which is summarized in the following corollary.

Corollary 2.1 *Let f be given by* (2.1). *Then f presents an N-soliton solution to a Hirota bilinear equation* (1.3) *iff the Hirota condition in* (2.5) *is satisfied.*

In order to figure out as more common factors out of the Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ as possible, we will use the following result, which is an automatic consequence of the definition of the Hirota function.

Theorem 2.2 *The Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ defined by* (2.5) *is a symmetric and even function in $\mathbf{k}_i$, $1 \le i \le N$.*

Taking $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, we have

$$P(\sigma_i \mathbf{k}_i - \mathbf{k}_{N-1}) P(\sigma_i \mathbf{k}_i \pm \mathbf{k}_N) = P(\mathbf{k}_i - \mathbf{k}_N) P(\mathbf{k}_i + \mathbf{k}_N) \tag{2.15}$$

in any case of $\sigma_i = \pm 1$, due to the even property of the polynomial P. Therefore, we can obtain the following consequence.

Theorem 2.3 *If $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, then we have*

$$H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 2H(\mathbf{k}_1, \cdots, \mathbf{k}_{N-2}) P(2\mathbf{k}_N) \prod_{i=1}^{N-2} P(\mathbf{k}_i - \mathbf{k}_N) P(\mathbf{k}_i + \mathbf{k}_N). \tag{2.16}$$

Proof: When $\mathbf{k}_{N-1} = \pm \mathbf{k}_N$, we can compute that

$$\begin{aligned} H(\mathbf{k}_1, \cdots, \mathbf{k}_N) &= \sum_{\sigma = \pm 1} P(\sigma_1 \mathbf{k}_1 + \cdots + \sigma_N \mathbf{k}_N) \prod_{1 \le i < j \le N} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j \\ &= \sum_{\sigma = \pm 1} P(\sigma_1 \mathbf{k}_1 + \cdots + \sigma_N \mathbf{k}_N) \prod_{1 \le i < j \le N-2} P(\sigma_i \mathbf{k}_i - \sigma_j \mathbf{k}_j) \sigma_i \sigma_j \\ &\quad \times \prod_{i=1}^{N-2} P(\sigma_i \mathbf{k}_i - \sigma_{N-1} \mathbf{k}_{N-1}) \sigma_i \sigma_{N-1} \prod_{i=1}^{N-1} P(\sigma_i \mathbf{k}_i - \sigma_N \mathbf{k}_N) \sigma_i \sigma_N \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{\sigma=\pm 1} P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_{N-2}) \prod_{1\le i<j\le N-2} P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)\sigma_i\sigma_j \\
&\quad \times \prod_{i=1}^{N-2} P(\sigma_i\mathbf{k}_i-\mathbf{k}_{N-1}) \prod_{i=1}^{N-2} P(\sigma_i\mathbf{k}_i\pm\mathbf{k}_N)P(2\mathbf{k}_N) \\
&= 2H(\mathbf{k}_1,\cdots,\mathbf{k}_{N-2})P(2\mathbf{k}_N)\prod_{i=1}^{N-2} P(\mathbf{k}_i-\mathbf{k}_N)P(\mathbf{k}_i+\mathbf{k}_N),
\end{aligned}
$$

where the last step is due to (2.15), but the last but one step follows from the fact that the two cases $(1,\mp 1)$ and $(-1,\pm 1)$ of (σ_{N-1},σ_N) are left and the other two cases lead to a zero factor owing to $P(\mathbf{0})=0$. Therefore, the proof of the theorem is finished. $\square$

This theorem will be used to factor out common factors out of the Hirota function H while proving the Hirota condition.

3 Applications to (1+1)-dimensional equations

3.1 A general algorithm

In the (1+1)-dimensional case, the wave vectors can be expressed as

$$\mathbf{k}_i = (k_i, -\omega_i),\ 1\le i\le N. \tag{3.1}$$

We assume that the dispersion relations (2.4) determine all frequencies $\omega_i=\omega(k_i),\ 1\le i\le N$. Therefore, $P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)$ are functions of k_i and k_j,

On one hand, we further assume that $P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)$ and $P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_N)$ can be simplified into rational functions as follows:

$$P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)=\frac{\sigma_i\sigma_j k_i k_j Q_1(k_i,k_j,\sigma_i,\sigma_j)}{Q_2(k_i,k_j)}, \tag{3.2}$$

where Q_1 and Q_2 are polynomial functions, and

$$P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_N)=\frac{Q_3(k_1,\cdots,k_N,\sigma_1,\cdots,\sigma_N)}{Q_4(k_1,\cdots,k_N)}, \tag{3.3}$$

where Q_3 and Q_4 are polynomial functions. Let us set the polynomial

$$\tilde{H}=H(\mathbf{k}_1,\cdots,\mathbf{k}_N)Q_4(k_1,\cdots,k_N)\prod_{1\le i<j\le N} Q_2(k_i,k_j), \tag{3.4}$$

for convenience of discussion.

On the other hand, Theorem 2.3 tells that under the induction assumption, the Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ will be zero, if two of the wave vectors satisfy $\mathbf{k}_i = \pm\mathbf{k}_j,\ 1 \le i < j \le N$. Based on the even property of H and P, we know that $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is still even with respect to the wave numbers k_i $1 \le i \le N$. Therefore, from the symmetric property in Theorem 2.2, we can factor out a factor $(k_i^2 - k_j^2)^2$ out of the polynomial $\tilde{H}$:

$$\tilde{H} = (k_i^2 - k_j^2)^2 g_{ij},\ \text{for any pair } 1 \le i < j \le N, \tag{3.5}$$

where g_{ij} is a polynomial of $k_n,\ 1 \le n \le N$.

Finally, it follows from the characteristic property of P in (3.2) that the Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ can be written as

$$H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = \frac{\prod_{1\le i<j\le N} k_i^2 k_j^2 \prod_{1\le i<j\le N} (k_i^2 - k_j^2)^2 g}{Q_4(k_1, \cdots, k_N) \prod_{1\le i<j\le N} Q_2(k_i, k_j)} \tag{3.6}$$

where g is another polynomial of $k_n,\ 1 \le n \le N$. Then, we can see that the degree of the polynomial

$$\tilde{H} = \prod_{1\le i<j\le N} k_i^2 k_j^2 \prod_{1\le i<j\le N} (k_i^2 - k_j^2)^2 g$$

is at least $2N(N-1) + 2N(N-1) = 4N(N-1)$.

Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \ne 0$, the degree of the polynomial $\tilde{H}$ defined by (3.4), which also equals

$$\tilde{H} = \sum_{\sigma=\pm1} Q_3(k_1, \cdots, k_N, \sigma_1, \cdots, \sigma_N) \prod_{1\le i<j\le N} k_i k_j Q_1(k_i, k_j, \sigma_i, \sigma_j), \tag{3.7}$$

should not be less than $4N(N-1)$. Otherwise, we will have $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0$, which is what we need to prove for the existence of N-soliton solutions. Thus, the problem for verifying the Hirota condition becomes quite simple, and one basically just needs to compute the degree of the polynomial in (3.7).

3.2 Applications

3.2.1 Generalized KdV equations

Let us consider a class of generalized KdV equations, which are associated with

$$P(x,t) = ax^4 + bx^3t + cx^2 + dxt \tag{3.8}$$

where a, b, c, d are arbitrary constants satisfying $b^2 + d^2 \neq 0$, which guarantees we will have a partial differential equation. The corresponding bilinear generalized KdV equations read

$$\begin{aligned}
&B(f) := (aD_x^4 + bD_x^3D_t + cD_x^2 + dD_xD_t)f \cdot f \\
&= 2[a(f_{4x}f - 4f_{3x}f_x + 3f_{xx}^2) + b(f_{3x,t}f - 3f_{xxt}f_x + 3f_{xt}f_{xx} - f_tf_{3x}) \\
&\quad +c(f_{xx}f - f_x^2) + d(f_{yt}f - f_yf_t)] = 0.
\end{aligned} \tag{3.9}$$

They are equivalent to the following generalized KdV equations:

$$N(u) := a(6u_xu_{xx} + u_{4x}) + b[3(u_xu_t)_x + u_{3x,t}] + cu_{xx} + du_{xt} = 0, \tag{3.10}$$

under the transformation $u = 2(\ln f)_x$. The link is $N(u) = (B(f)/f^2)_x$. If $b = 0$, then we get the KdV equation, and if $a = 0$, we get the Hirota-Satsuma equation [17].

Let us now set

$$\Delta = ad - bc. \tag{3.11}$$

It is direct to compute that

$$\omega_i = \omega_i(k_i) = \frac{ak_i^3 + ck_i}{bk_i^2 + d},\ 1 \leq i \leq N, \tag{3.12}$$

and

$$P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j) = -\frac{\sigma_i\sigma_jk_ik_j\Delta(\sigma_ik_i - \sigma_jk_j)^2[b(k_i^2 - \sigma_i\sigma_jk_ik_j + k_j^2) + 3d]}{(bk_i^2 + d)(bk_j^2 + d)}. \tag{3.13}$$

Case 1. $\Delta = 0$:

In this case, we have $P(\mathbf{k}_i \pm \mathbf{k}_j) = 0,\ 1 \leq i < j \leq N$, and thus, the Hirota N-soliton condition is automatically satisfied. This implies that we have a set of resonant solutions:

$$f = 1 + c_1\mathrm{e}^{\eta_1} + \cdots + c_N\mathrm{e}^{\eta_N},\ \eta_i = k_ix - \omega_i(k_i)t,\ 1 \leq i \leq N, \tag{3.14}$$

where c_i's and k_i's are arbitrary constants.

Case 2. $\Delta \neq 0$:

Sub-case 2.1. $d = 0$:

In this subcase, we have $c \neq 0$ and directly obtain

$$\begin{cases}
P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j) = \dfrac{R_1}{R_2},\ R_1 = c\sigma_i\sigma_j(\sigma_ik_i - \sigma_jk_j)^2(k_i^2 - \sigma_i\sigma_jk_ik_j + k_j^2),\ R_2 = k_ik_j, \\[2ex]
P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N) = \dfrac{R_3}{R_4},\quad \deg R_3 = N + 2,\ R_4 = \displaystyle\prod_{i=1}^N k_i.
\end{cases} \tag{3.15}$$

Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, let us check the degree of the polynomial

$$
\begin{aligned}
& H(\mathbf{k}_1, \cdots, \mathbf{k}_N) R_4(k_1, \cdots, k_N) \prod_{1 \leq i < j \leq N} R_2(k_i, k_j) \\
= & R_3(k_1, \cdots, k_N, \sigma_1, \cdots, \sigma_N) \prod_{1 \leq i < j \leq N} R_1(k_i, k_j, \sigma_i, \sigma_j) \sigma_i \sigma_j.
\end{aligned}
$$

We apply the same idea as in the general algorithm. On one hand, based on the expression on the right hand side, the degree is $(N+2) + 2N(N-1) = 2N^2 - N + 2$. But on the other hand, since $HR_4 \sum_{i<j} R_2$ can have a factor $\sum_{i<j}(k_i^2 - k_j^2)^2$ as explained before, based on the expression on the left hand side, the degree is at least $2N(N-1) + N + N(N-1) = 3N^2 - 2N$. Those two numbers could not be equal, when $N \geq 3$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0,\ N \geq 1$.

Sub-case 2.2. $d \neq 0$:

Sub-subcase 2.2.1. $b = 0$:

This is the KdV case. It is easy to work out

$$
Q_1 = -3a(\sigma_i k_i - \sigma_j k_j)^2,\ \deg Q_3 = 4,\ Q_2 = 1,\ Q_4 = 1. \tag{3.16}
$$

Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is $2N(N-1) + 4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N-1)$ when $N \geq 3$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0,\ N \geq 1$, and the KdV equation has N-soliton solutions, as shown in [7].

Sub-case 2.2.2. $b \neq 0$:

It is direct to get

$$
\begin{aligned}
& Q_1 = \Delta[b(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) + 3d](\sigma_i k_i - \sigma_j k_j)^2, \\
& \deg Q_3 = 2(N+1),\ Q_2 = (bk_i^2 + d)(bk_j^2 + d),\ Q_4 = \prod_{i=1}^{N}(bk_i^2 + d).
\end{aligned} \tag{3.17}
$$

Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial

$$
\begin{aligned}
\tilde{H} = & H(\mathbf{k}_1, \cdots, \mathbf{k}_N) Q_4(k_1, \cdots, k_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, , k_j) \\
= & Q_3(k_1, \cdots, k_N, \sigma_1, \cdots, \sigma_N) \prod_{1 \leq i < j \leq N} k_i k_j Q_1(k_i, k_j, \sigma_i, \sigma_j).
\end{aligned}
$$

is $2(N+1) + 3N(N-1) = 3N^2 - N + 2$ (from the second expression of $\tilde{H}$), which could not be greater than $4N(N-1) + 2N + 2N(N-1) = 6N^2 - 4N$ (from the first expression of $\tilde{H}$) when $N \geq 2$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0,\ N \geq 1$.

We remark that the three-soliton condition is also satisfied for all bilinear equations associated with

$$P = ax^4 + bx^3t + cx^2 + dxt + et^2,\ e \neq 0, \tag{3.18}$$

where a, b, c, d, e are arbitrary constants. This leads to a class of generalized Boussinesq equations, and the case of $b = c = d = 0$ corresponds to the Boussinesq equations. But we need a more general argument to verify the Hirota N-soliton condition, since the frequency functions involve square roots.

3.2.2 Generalized higher-order KdV equations

Let us consider a class of higher-order generalized higher-order KdV equations associated with

$$P(x, t) = ax^6 + bx^4 + cx^2 + xt, \tag{3.19}$$

where a, b, c are arbitrary constants. This class of polynomials generates the following bilinear generalized higher-order KdV equations:

$$\begin{aligned} B(f) &:= (aD_x^6 + bD_x^4 + cD_x^2 + D_xD_t)f \cdot f \\ &= 2[a(f_{6x}f - 6f_{5x}f_x + 15f_{4x}f_{xx} - 10f_{3x}^3) \\ &+ b(f_{4x}f - 4f_{3x}f_x + 3f_x^2) + c(f_{xx}f - f_x^2) + f_{xt}f - f_xf_t] = 0. \end{aligned} \tag{3.20}$$

The corresponding generalized higher-order KdV equations read as follows:

$$N(u) := a(15u_x^3 + 15u_xu_{3x} + u_{5x})_x + b(6u_xu_{xx} + u_{4x}) + cu_{xx} + du_{xt} = 0. \tag{3.21}$$

The transformation is $u = 2(\ln f)_x$ and the link is $N(u) = (B(f)/f^2)_x$. The case of $b = c = 0$ leads to the Sawada-Kotera equation [9] or the Caudrey-Dodd-Gibbon equation [10].

Using the dispersion relations, we can directly obtain

$$\omega_i = \omega_i(k_i) = ak_i^5 + bk_i^3 + ck_i,\ 1 \leq i \leq N, \tag{3.22}$$

and

$$P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j) = -\sigma_i\sigma_jk_ik_j(\sigma_ik_i - \sigma_jk_j)^2[5a(k_i^2 - \sigma_i\sigma_jk_ik_j + k_j^2) + 3b]. \tag{3.23}$$

Therefore, it is easy to find that

$$Q_1 = -(\sigma_ik_i - \sigma_jk_j)^2[5a(k_i^2 - \sigma_i\sigma_jk_ik_j + k_j^2) + 3b],\ \deg Q_3 = 6,\ Q_2 = 1,\ Q_4 = 1. \tag{3.24}$$

Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial $\tilde{H}$ $(= H)$ is at most $3N(N-1) + 6 = 3N^2 - 3N + 6$, which could not be greater than $4N(N-1)$ when $N \geq 4$. Another direct computation can show that the three-soliton condition holds for all generalized higher-order KdV equations in (3.20). Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0,\ N \geq 1$, and each of the generalized higher-order KdV equations in (3.20) possesses N-soliton solutions.

This class is different from the fifth-order KdV equations studied in the literature [18]. It has also been proved [12] that the higher-order KdV equations associated with

$$P(x,t) = x^{2n} + xt,\ n \geq 4, \tag{3.25}$$

does not pass the three-soliton test. A direct computation can show that all generalized higher-order KdV equations associate with

$$P(x,t) = x^6 + ax^4 + bx^2 + cxt + dt^2,\ d \neq 0, \tag{3.26}$$

do not possess three-soliton solutions, either.

4 Concluding remarks

We have analyzed the Hirota N-soliton conditions for bilinear differential equations and shown the existence of N-soliton solutions to two classes of generalized KdV and higher-order KdV equations. Definitely, there should be more bilinear equations which could possess N-soliton solutions. In the case of even higher-oder differential equations, the involved computations would be much more complicated. New ideas are needed to prove the existence of N-soliton solutions.

There are generalized bilinear derivatives, and particularly, we have the $D_{p,x}$-operators [19]:

$$D_{p,x}^m D_{p,t}^n f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f)(\partial_x^i \partial_t^j g),\ m,n \geq 0,\ m+n \geq 1, \tag{4.1}$$

where the powers of α_p are determined by

$$\alpha_p^i = (-1)^{r(i)},\ i = r(i) \bmod p,\ i \geq 0, \tag{4.2}$$

with $0 \leq r(i) < p$. The patters of those powers for $i = 1, 2, 3, \cdots$ read

$$p = 3:\ -, +, +, -, +, +, \cdots;$$

$$p = 5:\ -, +, -, +, +, -, +, -, +, +, \cdots;$$

$$p = 7:\ -, +, -, +, -, +, +, -, +, -, +, -, +, +, \cdots.$$

Particularly, we have $D_{3,x}$ and $D_{5,x}$ associated with the two odd prime numbers: $p = 3, 5$. There exist new characteristic properties of the corresponding generalized bilinear derivatives. For example, we have

$$D_{3,x}^3 f \cdot f = 2 f_{xxx} f,\ D_{3,x}^4 f \cdot f = 6 f_{xx}^2, \tag{4.3}$$

which is different from the Hirota case (i.e., $p = 2$). Of course, we can have other generalized bilinear derivatives: $D_{6,x},\ D_{9,x},\ \ldots$.

The corresponding generalized bilinear equations [20, 21] or trilinear equations [22] can possess resonant N-solitons. A generalized bilinear equation in $(1 + 1)$-dimensions:

$$P(D_{p,x}, D_{p,t}) f \cdot f = 0 \tag{4.4}$$

possesses a resonant N-soliton [20, 21]:

$$f = 1 + c_1 \mathrm{e}^{\eta_1} + c_2 \mathrm{e}^{\eta_2} + \cdots + c_N \mathrm{e}^{\eta_N} \tag{4.5}$$

where c_i's are arbitrary constants and and $\eta_i = k_i x - \omega_i t,\ 1 \le i \le N$, iff

$$P(\mathbf{k}_i + \alpha_p \mathbf{k}_j) + P(\mathbf{k}_j + \alpha_p \mathbf{k}_i) = 0,\ 1 \le i \le j \le N, \tag{4.6}$$

where $\mathbf{k}_i = (k_i, -\omega_i),\ 1 \le i \le N$.

However, we do not have any example of generalized bilinear equations which have N-soliton solutions. There are many interesting questions that we need to answer first. For example, what is the generalized N-soliton condition, i.e., the N-soliton condition for generalized bilinear equations? How to formulate generalized bilinear equations, for example,

$$P(D_{3,x}, D_{3,t}) = 0,$$

even in (1+1)-dimensions, which possess N-soliton solutions?

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References

[1] Ablowitz MJ, Segur H. *Solitons and the Inverse Scattering Transform*. Philadelphia: SIAM; 1981.

[2] Calogero F, Degasperis A. *Solitons and Spectral Transform I.* Amsterdam: North-Holland; 1982.

[3] Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. *Theory of Solitons, the Inverse Scattering Method*. New York: Consultants Bureau; 1984.

[4] Hirota R. *Direct Method in Soliton Theory.* Cambridge: Cambridge University Press; 2004.

[5] Hirota R. A new form of Bäcklund transformations and its relation to the inverse scattering problem. *Prog. Theor. Phys.* 1974;52:1498-1512.

[6] Ma WX, Li CX, He JS. A second Wronskian formulation of the Boussinesq equation. *Nonlinear Anal.: TMA* 2009;70:4245-4258.

[7] Hirota R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* 1971;27:1192-1194.

[8] Hirota R. Direct methods in soliton theory. in: *Solitons*, eds. Bullough RK, Caudrey P, Berlin, Heidelberg: Springer-Verlag; 1980, 157-176.

[9] Sawada K, Kotera T. A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation. *Prog. Theoret. Phys.* 1974;51:1355-1367.

[10] Caudrey PJ, Dodd RK, Gibbon JD. A new hierarchy of Korteweg-de Vries equations. *Proc. R. Soc. Lond. A* 1976;351:407-422.

[11] Newell AC, Zeng YB. The Hirota conditions *J. Math. Phys.* 1986;27:2016-2021.

[12] Guo FK. A simple approach to negating the Hirota condition. *Acta Math. Appl. Sinica* 1991;14:111-114.

[13] Hietarinta J. A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations. *J. Math. Phys.* 1987;28:1732-1742.

[14] Hietarinta J. Introduction to the Hirota bilinear method. In: *Integrability of Nonlinear Systems*, eds. Kosmann-Schwarzbach Y, Grammaticos B, Tamizhmani KM, Lecture Notes in Phys., Vol 495. Berlin: Springer; 1997, 95-103.

[15] Ma WX, Fan EG. Linear superposition principle applying to Hirota bilinear equations. *Comput. Math. Appl.* 2011;61:950-959.

[16] Ma WX, Zhang Y, Tang YN, Tu JY. Hirota bilinear equations with linear subspaces of solutions. *Appl. Math. Comput.* 2012;218:7174-7183.

[17] Hirota R, Satsuma J. N-soliton solutions of model equations for shallow water waves. *J. Phys. Soc. Jpn.* 1976;40:611-612.

[18] Wazwaz AM. A fifth-order Korteweg-de Vries equation for shallow water with surface tension: multiple soliton solutions. *Acta Phys. Pol. A* 2016;130:679-682.

[19] Ma WX. Generalized bilinear differential equations. *Stud. Nonlinear Sci.* 2011;2:140-144.

[20] Ma WX. Bilinear equations, Bell polynomials and linear superposition principle. *J. Phys.: Conf. Ser.* 2013;411:012021.

[21] Ma WX. Bilinear equations and resonant solutions characterized by Bell polynomials. *Rep. Math. Phys.* 2013;72:41-56.

[22] Ma WX. Trilinear equations, Bell polynomials, and resonant solutions. *Front. Math. China* 2013;8:1139-1156.