A generalized fractional (q,h)-Gronwall inequality and its applications to nonlinear fractional delay (q,h)-difference systems

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Abstract. In this paper, a generalized fractional (q,h)-Gronwall inequality is investigated. Based on this inequality, we derive the uniqueness theorem and the finite-time stability criterion of nonlinear fractional delay (q,h)-difference systems. Several examples are given to illustrate our theoretical result.
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1. Introduction

In the past decades, fractional calculus has become an active field. Although numerous papers of fractional difference systems are already published [**3**–**5**, **9**, **11**, **12**, **15**, **17**, **29**]. The investigation of a qualitative theory for fractional difference systems is still in its infancy due to the memory features of fractional operators.

It is well known that fractional Gronwall inequalities are the lifeblood of fractional differential/difference systems. In 2007, Ye et al. [**30**] derived a fractional Gronwall inequality. Since then several different versions of discrete Gronwall inequality are constantly emerging. In 2012, Atici et al. [**2**] proposed a nabla discrete Gronwall inequality on $\mathbb{T}=\mathbb{N}_a$. In 2016, Abdeljawad et al. [**25**] gave a nabla discrete Gronwall inequality on $\mathbb{T}=\mathbb{T}_q$.

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In 2018, Wu et al. [**28**] presented a delta discrete Gronwall inequality on $\mathbb{T} = \mathbb{N}_a$. In 2018, Alzabut et al. [**1**] discussed a nabla discrete Gronwall inequality on $\mathbb{T} = \mathbb{N}_a$. In 2019, Liu et al. [**14**] developed a delta discrete Gronwall inequality on $\mathbb{T} = (h\mathbb{N})_{a+h}^{a+Th}$. In 2019, Chen et al.[**8**] obtained a nabla discrete Gronwall inequality on $\mathbb{T} = \mathbb{N}_{a+1}$. In 2020, Makhlouf et al. [**19**] developed some HenryCGronwall type q-fractional integral inequalities. However, there are some flaws in the proof (see Section 3 for details) in [**1**, **18**, **25**]. We give a rigorous proof in this paper.

(q, h)-calculus [**6**, **7**] was introduced as an extension of the fundamental conceptions of discrete fractional calculus. It can be reduced to q-difference calculus ($h = 0$) and ordinary difference calculus ($h = q = 1$). Some outstanding research papers about fractional (q, h)-calculus can be seen in [**22**–**24**].

In the latest years, there has been increasing attention in mathematical tools to investigate stability of fractional (q, h)-difference systems, for example, comparison theorem and inequality techniques [**10**], Liapunov functional [**16**, **27**], Lyapunov-Krasovskii functional [**13**].

Based on the reason presented in [**10**], we know that (q, h)-Mittag-Leffler function (when $q > 1$ and $h > 0$) can't be used directly to study asymptotic behavior of fractional (q, h)-difference systems. However, we can use (q, h)-Mittag-Leffler function to investigate the finite-time stability of fractional (q, h)-difference systems. To the best of our knowledge, there is no paper that has dealt with this problem. Motivated by [**21**], [**26**] and [**28**], we develop a generalized fractional (q, h)-Gronwall inequality to study the following nonlinear fractional delay (q, h)-difference systems:

$$\begin{cases} {}_a^C\nabla_{(q,h)}^{\alpha} x(t) = Ax(t) + Bx(\rho^m(t)) + f(t, x(t), x(\rho^m(t))), t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}, \\ x(t) = \varphi(t),\ t \in [\rho^{m-1}(a), a]_{\mathbb{T}}. \end{cases} \tag{1.1}$$

and give the following Theorem A.

Theorem A. For given positive numbers δ, ϵ, H, $(\nu(t))^{\alpha} b < 1, t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, the system (4.1) is finite-time stability w.r.t (δ, ϵ, H) if

$$E_{\alpha,1}^{a,b}(t) \leq \epsilon/\delta,\ \forall t \in [\sigma(a), H]_{\mathbb{T}}. \tag{1.2}$$

2. Preliminaries

Definition 2.1. ([**6**, **7**]) The (q, h)-time scale is introduced as:

$$\mathbb{T}_{(q,h)}^{t_0} = \{[k]_q h + t_0 q^k, k \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, q \geq 1, t_0 > 0, h \geq 0, q + h > 1.$$

Let $a \in \mathbb{T}_{(q,h)}^{t_0}, a > h/(1-q)$ be fixed. Then we present restrictions of the time scale $\mathbb{T}_{(q,h)}^{t_0}$ by relation $[a, b]_{\mathbb{T}} = \{t : a \leq t \leq b, a, b, t \in \mathbb{T}_{(q,h)}^{t_0}\}$ and $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)} = \{t \in \mathbb{T}_{(q,h)}^{t_0} : t \geq \sigma^i(a), i \in \mathbb{N}_0\}$, where for any $c \in \mathbb{R}$, $\mathbb{N}_c = \{c, c+1, c+2, \cdots, \}$.

Definition 2.2 ([**6**]). q-Gamma function $\Gamma_{\tilde{q}}(t)$ is given by

$$\Gamma_{\tilde{q}}(t)=\frac{(1-\tilde{q})^{1-t}(\tilde{q},\tilde{q})_\infty}{(\tilde{q}^t,\tilde{q})_\infty},\quad 0<\tilde{q}=1/q<1, \tag{2.1}$$

where $t\in\mathbb{R}\setminus\{0,-1,-2,\cdots\}$ and $(b,\tilde{q})_\infty=\prod_{j=0}^{\infty}(1-b\tilde{q}^j)$.

Definition 2.3. The q-binomial coefficient is defined as

$$\begin{bmatrix}\xi\\ j\end{bmatrix}_{\tilde{q}}=\frac{\Gamma_{\tilde{q}}(\xi+1)}{\Gamma_{\tilde{q}}(j+1)\Gamma_{\tilde{q}}(\xi-j+1)},\quad \xi\in\mathbb{R},\quad j\in\mathbb{Z}. \tag{2.2}$$

Definition 2.4 ([**6**]). The backward and forward jump operator are defined as

$$\rho(t)=(t-h)q^{-1}$$

and

$$\sigma(t)=h+qt$$

respectively.

Definition 2.5 ([**6**]). The backward and forward graininess are defined as

$$\nu(t)=(1-q^{-1})t+hq^{-1}$$

and

$$\mu(t)=h+(q-1)t,$$

respectively.

Lemma 2.6 ([**6**]). *For any $t\in\mathbb{T}_{(q,h)}^{t_0}$, $\beta\in\mathbb{R}$ and $q\in(0,1)\cup(1,+\infty)$, let $[\beta]_q=\frac{q^\beta-1}{q-1}$. Then we have that*

$$\sigma^k(t)=[k]_q+hq^kt$$

and

$$\rho^k(t)=(t-[k]_qh)q^{-k},$$

where $k\in\mathbb{N}_0$, $\sigma^0(a)=a$, $\sigma^i(a)=\sigma(\sigma^{i-1}(a))$, $\rho^0(a)=a$ and $\rho^i(a)=\rho(\rho^{i-1}(a))$ for $i\in\mathbb{N}_1$.

It is easy to proof the following lemma, so we omit the proof here.

Lemma 2.7. *The relation*

$$\nu(\sigma^k(t))=q^{k-1}((q-1)t+h)$$

holds for $t\in\mathbb{T}_{(q,h)}^{t_0}$.

Definition 2.8 ([**6**]). Assume $x:\mathbb{T}_{(q,h)}^{t_0}\to\mathbb{R}$, its 1-th order nabla (q,h)-derivative is defined as

$$\nabla_{(q,h)}x(t)=\frac{x(t)-x(\tilde{q}(t-h))}{(1-\tilde{q})t+\tilde{q}h}.$$

LEMMA 2.9 ([**6**]). *The nabla fractional monomial of order α on $\mathbb{T}_{(q,h)}^{t_0}$ can be written as*

$$\hat{h}_\alpha(t,s) = \begin{bmatrix} \alpha+k-1 \\ k-1 \end{bmatrix}_{\tilde{q}} (\nu(t))^\alpha, \tag{2.3}$$

where $\alpha \in \mathbb{R}$, $s,t \in \mathbb{T}_{(q,h)}^{t_0}$, $t = \sigma^k(s)$, $k \in \mathbb{N}$.

DEFINITION 2.10 ([**6**]). Let $\beta, \lambda, \alpha \in \mathbb{R}$. We define (q,h)-Mittag-Leffler function $E_{\alpha,\beta}^{s,\lambda}(t)$ as

$$E_{\alpha,\beta}^{s,\lambda}(t) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{\alpha k+\beta-1}(t,s),$$

where $t,s \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ and $s \le t$.

DEFINITION 2.11 ([**6**]). Assume $x : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$ and $t = \sigma^n(a)$, $n \ge 1$. Then the nabla fractional (q,h)-integral of order $\alpha > 0$ is defined as

$${}_a\nabla_{(q,h)}^{-\alpha} x(t) = \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau)) x(\tau) \nabla\tau, \tag{2.4}$$

where by convention ${}_a\nabla_{(q,h)}^{0} x(t) = x(t)$ and ${}_a\nabla_{(q,h)}^{-\alpha} x(a) = 0$.

DEFINITION 2.12 ([**6**]). The nabla Caputo-like fractional (q,h)-difference of order $0 < \alpha < 1$ is defined as

$${}_a^C\nabla_{(q,h)}^{\alpha} x(t) = {}_a\nabla_{(q,h)}^{-(1-\alpha)} \nabla_{(q,h)} x(t).$$

DEFINITION 2.13 ([**6**]). The nabla Riemann-Liouville-like (q,h)-fractional difference of order $0 < \alpha < 1$ is defined as

$${}_a\nabla_{(q,h)}^{\alpha} x(t) = \nabla_{(q,h)}\, {}_a\nabla_{(q,h)}^{-(1-\alpha)} x(t).$$

Similar the proof of [**12**, Theorem 3.109], we obtain the following lemma.

LEMMA 2.14. *Assume $x : \tilde{\mathbb{T}}_{(q,h)}^{a} \to \mathbb{R}$ and $\alpha, \beta > 0$. Then*

$${}_a\nabla_{(q,h)}^{\alpha}\, {}_a\nabla_{(q,h)}^{-\beta} x(t) = {}_a\nabla_{(q,h)}^{\alpha-\beta} x(t). \tag{2.5}$$

3. A generalized fractional (q,h)-Gronwall inequality

LEMMA 3.1. *Assume $x \in \mathbb{R}, b > a$ and $b - a \in \mathbb{N}_1$. Then*

$$\lim_{x\to\infty} \frac{\Gamma_{\tilde{q}}(x+b)}{\Gamma_{\tilde{q}}(x+a)} = (1-\tilde{q})^{a-b}. \tag{3.1}$$

PROOF.

$$\begin{aligned}\lim_{x\to\infty}\frac{\Gamma_{\tilde{q}}(x+b)}{\Gamma_{\tilde{q}}(x+a)} &= \lim_{x\to\infty}\frac{(1-\tilde{q})^{1-(x+b)}(\tilde{q},\tilde{q})_\infty}{(\tilde{q}^{x+b},\tilde{q})_\infty}\cdot\frac{(\tilde{q}^{x+a},\tilde{q})_\infty}{(1-\tilde{q})^{1-(x+a)}(\tilde{q},\tilde{q})_\infty}\\ &= \lim_{x\to\infty}\frac{(1-\tilde{q})^{a-b}\prod_{j=0}^{b-a-1}(1-\tilde{q}^{x+a+j})\prod_{j=b-a}^{\infty}(1-\tilde{q}^{x+a+j})}{\prod_{j=0}^{\infty}(1-\tilde{q}^{x+b+j})}\\ &= (1-\tilde{q})^{a-b}.\end{aligned}$$

□

REMARK 3.2. Note that

$$\begin{aligned}\lim_{x\to\infty}\frac{\Gamma_{\tilde{q}}(x+b)}{\Gamma_{\tilde{q}}(x+a)} &= (1-\tilde{q})^{a-b}\\ &= \lim_{x\to\infty}\left(\frac{1-\tilde{q}^x}{1-\tilde{q}}\right)^{b-a}\\ &= \lim_{x\to\infty}([x]_{\tilde{q}})^{b-a}.\end{aligned}$$

It follows that

$$\frac{\Gamma_{\tilde{q}}(x+b)}{\Gamma_{\tilde{q}}(x+a)}\sim([x]_{\tilde{q}})^{b-a}\qquad(x\to\infty),$$

which can be regarded as the q-analogue of the formula

$$\frac{\Gamma(x+b)}{\Gamma(x+a)}\sim x^{b-a}\quad(x\to\infty),\quad\text{see [20, (1.5.15)]}.$$

LEMMA 3.3 ([24]). *Assume all $\beta,\alpha\in\mathbb{R}$ and $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. Then*

$$\int_a^t\hat{h}_{\beta-1}(t,\rho(\tau))\hat{h}_{\alpha-1}(\tau,a)\nabla\tau=\hat{h}_{\beta+\alpha-1}(t,a).\tag{3.2}$$

Now we present a generalized fractional (q,h)-Gronwall inequality.

THEOREM 3.4. *Assume $\alpha>0$, $x(t)$ is a nonnegative function on $\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, $f(t),g(t)$ are nonnegative, nondecreasing functions on $\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ and $g(t)\le M$, where $M>0$ and $(\nu(t))^\alpha M<1$ for any $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. If*

$$x(t)\le f(t)+g(t)\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s,\tag{3.3}$$

then

$$x(t)\le f(t)E_{\alpha,1}^{a,g(t)}(t),\ t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}.\tag{3.4}$$

PROOF. Define

$$B\phi(t)=g(t)\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))\phi(s)\nabla s.$$

It follows that

$$x(t)\le f(t)+Bx(t),$$

which implies that

$$x(t) \le \sum_{k=0}^{n-1} B^k f(t) + B^n x(t), \tag{3.5}$$

where $B^0 x(t) = x(t)$.

Let us prove that

$$B^n x(t) \le (g(t))^n \int_a^t \hat{h}_{n\alpha-1}(t, \rho(s)) x(s) \nabla s \tag{3.6}$$

and

$$\lim_{n\to\infty} B^n x(t) = 0. \tag{3.7}$$

It is easy to see that (3.6) is true for $n = 1$. Assume that (3.6) is true for some $n = k$, namely,

$$B^k x(t) \le (g(t))^k \int_a^t \hat{h}_{k\alpha-1}(t, \rho(s)) x(s) \nabla s.$$

If $n = k + 1$, then

$$\begin{aligned}
B^{k+1} x(t) &= B(B^k x(t)) \\
&\le (g(t))^{k+1} \int_a^t \hat{h}_{\alpha-1}(t, \rho(s)) \int_a^s \hat{h}_{k\alpha-1}(s, \rho(r)) x(r) \nabla r \nabla s \\
&\overset{[\mathbf{10}, \text{ Lem. } 4.7]}{=\!=\!=} (g(t))^{k+1} \int_a^t \int_{\rho(r)}^t \hat{h}_{\alpha-1}(t, \rho(s)) \hat{h}_{k\alpha-1}(s, \rho(r)) x(r) \nabla s \nabla r \\
&= (g(t))^{k+1} \int_a^t x(r) \int_{\rho(r)}^t \hat{h}_{\alpha-1}(t, \rho(s)) \hat{h}_{k\alpha-1}(s, \rho(r)) \nabla s \nabla r \\
&\overset{(3.2)}{=} (g(t))^{k+1} \int_a^t \hat{h}_{(k+1)\alpha-1}(t, \rho(r)) x(r) \nabla r.
\end{aligned}$$

The relation (3.6) is proved.

Let $t = \sigma^m(a)$, $m \in \mathbb{N}_1$, notice that

$$\begin{aligned}
B^n x(t) &\le (g(t))^n \int_a^t \hat{h}_{n\alpha-1}(t, \rho(s)) x(s) \nabla s \\
&\le (M)^n K \int_a^t \hat{h}_{n\alpha-1}(t, \rho(s)) \nabla s \\
&\overset{[\mathbf{10}, \text{ Lem. } 3.4]}{=\!=\!=} (M)^n K \hat{h}_{n\alpha}(t, s)\big|_{s=t}^{a} \\
&= (M)^n K \hat{h}_{n\alpha}(t, a) \\
&= (M)^n K (\nu(t))^{n\alpha} \begin{bmatrix} n\alpha + m - 1 \\ m - 1 \end{bmatrix}_{\tilde{q}} \\
&= (M(\nu(t))^\alpha)^n K \frac{\Gamma_{\tilde{q}}(n\alpha + m)}{\Gamma_{\tilde{q}}(n\alpha + 1)\Gamma_{\tilde{q}}(m)},
\end{aligned}$$

where we use $\hat{h}_{n\alpha}(t,t)=0$ and $K=\max_{s\in[\sigma(a),t]_{\mathbb{T}}} x(s)$.

From Lemma 3.1, we have

$$\lim_{n\to\infty}(M(\nu(t))^{\alpha})^{n}K\frac{\Gamma_{\tilde{q}}(n\alpha+m)}{\Gamma_{\tilde{q}}(n\alpha+1)\Gamma_{\tilde{q}}(m)}\overset{(3.1)}{=}\lim_{n\to\infty}\frac{(M(\nu(t))^{\alpha})^{n}K(1-\tilde{q})^{1-m}}{\Gamma_{\tilde{q}}(m)}=0.$$

So we obtain (3.7).

Taking the limit on both side of (3.5) gives

$$\begin{aligned} x(t)&=\lim_{n\to\infty}x(t)\\ &\leq\lim_{n\to\infty}\left(\sum_{k=0}^{n-1}B^{k}f(t)+B^{n}x(t)\right)\\ &\overset{(3.7)}{\leq}\lim_{n\to\infty}\left(f(t)+\sum_{k=1}^{n-1}B^{k}f(t)\right)\\ &\overset{(3.6)}{\leq} f(t)+\sum_{n=1}^{\infty}(g(t))^{n}\int_{a}^{t}\hat{h}_{n\alpha-1}(t,\rho(s))f(s)\nabla s. \end{aligned}\tag{3.8}$$

Since $f(t)$ is a nondecreasing function on $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, we can rewrite (3.8) as

$$\begin{aligned} x(t)&\leq f(t)\left[1+\sum_{n=1}^{\infty}(g(t))^{n}\int_{a}^{t}\hat{h}_{n\alpha-1}(t,\rho(s))\nabla s\right]\\ &=f(t)\left[1+\sum_{n=1}^{\infty}(g(t))^{n}\hat{h}_{n\alpha}(t,a)\right]\\ &=f(t)\sum_{n=0}^{\infty}(g(t))^{n}\hat{h}_{n\alpha}(t,a)\\ &=f(t)E_{\alpha,1}^{a,g(t)}(t). \end{aligned}$$

The theorem is proved. □

Remark 3.5. Let $t=\sigma^{k}(a)$, $q>1$, $k\geq 1$, from Lemma 2.7 we know that $(\nu(t))^{\alpha}M<1$ if and only if $k<1+\log_{q}\frac{M^{-1/\alpha}}{(q-1)a+h}$.

Remark 3.6. From the proof of Theorem 3.4, we can see that the condition $(\nu(t))^{\alpha}M<1$ is necessary to the theorem, which was ignored in [**1**, **18**, **25**].

Remark 3.7. The generalized Gronwall inequality obtained in [**8**] is a special case of our result. That is to say, letting $\alpha=\nu+1$ in Theorem 3.4, we can get the Theorem 3.1 in [**8**].

4. Applications to nonlinear fractional delay (q,h)-difference systems

Let R^{n} be the n-dimensional Euclidean space, $\|x\|_{L}$ be any Euclidean norm ($L=1,2,\infty$) of vector $x\in\mathbb{R}$, $\|A\|_{L}$ be the matrix norm $A\in\mathbb{R}^{n\times n}$

$(L = 1, 2, \infty)$ induced by the vector. If $x \in \mathbb{R}^n$, then $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. If $A \in \mathbb{R}^{n\times n}$, then the induced norm $\|\cdot\|_1$ is defined as $\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^n |a_{i,j}|$. $\lambda(A)$ denotes the set of all eigenvalues of $A \in R^{n\times n}$, $\lambda_{max}(A) = \max\{Re(\lambda) : \lambda \in \lambda(A)\}$. The induced norm $\|\cdot\|_2$ of the matrix A is defined as $\|A\|_2 = \sqrt{\lambda_{max}(A^TA)}$.

Consider the following nonlinear fractional delay (q, h)-difference systems:

(4.1)
$$\begin{cases} {}_a^C\nabla^\alpha_{(q,h)}x(t) = Ax(t) + Bx(\rho^m(t)) + f(t, x(t), x(\rho^m(t))), t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \\ x(t) = \varphi(t),\ t \in [\rho^{m-1}(a), a]_{\mathbb{T}}, \end{cases}$$

where $x : \tilde{\mathbb{T}}^{\rho^{m-1}(a)}_{(q,h)} \to \mathbb{R}^n$, $m \in \mathbb{N}_1$ is a fixed constant, $\alpha \in (0, 1)$, the constant matrices $A, B \in R^{n\times n}$, ${}_a^C\nabla^\alpha_{(q,h)}$ denotes the Caputo fractional (q, h)-difference, the nonlinear term $f : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the initial function $\varphi : [\rho^{m-1}(a), a]_{\mathbb{T}} \to \mathbb{R}^n$.

Throughout this paper, we make the following assumptions:

(H1) For $f(t, x(t), x(\rho^m(t))) \in C([\sigma(a), H]_{\mathbb{T}}, R^n)$, there exists a constant L such that

$$\begin{aligned}\|f(t, x_1(t), x_1(\rho^m(t))) - f(t, x_2(t), x_2(\rho^m(t)))\| \le{}& L(\|x_1(t) - x_2(t)\| \\ &+ \|x_1(\rho^m(t)) - x_2(\rho^m(t))\|)\end{aligned}$$

for any $t \in [\sigma(a), H]_{\mathbb{T}}$.

(H2) $f(t, 0, 0) = [\underbrace{0, 0, \cdots, 0}_{n}]^T$.

The following Theorem 4.1 provides an alternative representation for the solutions of system (4.1).

Theorem 4.1. *$x(t)$ is a solution of system (4.1) if and only if*

(4.2)
$$\begin{cases} x(t) = \varphi(a) + \int_a^t \hat{h}_{\alpha-1}(t, \rho(s))[Ax(s) + Bx(\rho^m(s)) + f(t, x(s), x(\rho^m(s)))]\nabla s, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \\ x(t) = \varphi(t),\ t \in [\rho^{m-1}(a), a]_{\mathbb{T}}, \end{cases}$$

Proof. For $t \in [\rho^{m-1}(a), a]_{\mathbb{T}}$, $x(t) = \varphi(t)$ is clearly the solution to the system (4.1). For $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, applying ${}_a\nabla^\alpha_{(q,h)}$ on both sides of (4.2), we have

$${}_a\nabla^\alpha_{(q,h)}x(t) = \varphi(a)\hat{h}_{-\alpha}(t, a) + Ax(s) + Bx(\rho^m(t)) + f(t, x(t), x(\rho^m(t))),$$

where we use ${}_a\nabla^\alpha_{(q,h)}{}_a\nabla^{-\alpha}_{(q,h)}x(t) = x(t)$ (see (2.5)).

By use of the relationship

$${}_a^C\nabla^\alpha_{(q,h)}x(t) = {}_a\nabla^\alpha_{(q,h)}x(t) + x(a)\hat{h}_{-\alpha}(t, a) \quad \text{(see [\textbf{24}, (15)])},$$

we obtain

$${}_a^C\nabla^\alpha_{(q,h)}x(t) = Ax(t) + Bx(\rho^m(t)) + f(t, x(t), x(\rho^m(t))), t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}.$$

Conversely, from system (4.1), we get that $x(t)=\varphi(t)$ for $t\in[\rho^{m-1}(a),a]_{\mathbb{T}}$. For $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, Taking ${}_a\nabla_{(q,h)}^{-\alpha}$ on both sides of (4.1), we get

$$\begin{aligned}{}_a\nabla_{(q,h)}^{-\alpha}\,{}_a^C\nabla_{(q,h)}^{\alpha}x(t)=\int_a^t&\hat{h}_{\alpha-1}(t,\rho(s))[Ax(s)+Bx(\rho^m(s))\\&+f(t,x(s),x(\rho^m(s)))]\nabla s.\end{aligned}$$

Applying the relationship

$${}_a\nabla_{(q,h)}^{-\alpha}\,{}_a^C\nabla_{(q,h)}^{\alpha}x(t)=x(t)-x(a)\quad(\text{see } [\mathbf{24},\ (14)]),$$

we have

$$x(t)=\varphi(a)+\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))[Ax(s)+Bx(\rho^m(s))+f(t,x(s),x(\rho^m(s)))]\nabla s.$$

This completes the proof. □

In [**25**], let $\tilde{z}(t)=\sup_{\theta\in\mathbb{I}_\tau}\|z(\theta t)\|$, where $t=q^d\in\mathbb{T}_q, d\in\mathbb{N}_0$, $\mathbb{I}_\tau=\{\tau a,q^{-1}\tau,q^{-2}\tau a,\cdots,a\}$. In [**1**], let $\tilde{z}(t)=\sup_{\theta\in\mathbb{I}_\tau}\|z(t+\theta)\|$, where $\mathbb{I}_\tau=\{-\tau,-\tau+1,\cdots,0\}$. The authors used the monotonicity of fractional integral function of $\tilde{z}(t)$ in the proof of [**25**, Theorem 5,6] and [**1**, Theorem 3,4]. The following examples show that when order $\alpha\in(0,1)$, the fractional integral function of a nonnegative function is not always increasing on $\mathbb{N}_a$, $\mathbb{T}_q$, $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)}$, respectively.

Let us denotes $\mathbb{N}_a^b=\{a,a+1,a+2,\cdots,b\}$, where $b,a\in\mathbb{R}$ and $b-a\in\mathbb{N}_1$.

Example 4.2. For $a=0$, $x(t)=\frac{1}{t+2}$, $h=q=1$, $\alpha=0.6$.

$$X(t)=\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s=\sum_{s=1}^{t}\binom{-0.4+t-s}{t-s}\frac{1}{s+2}.$$

According to Figure 1,we can see easily that the function $X(t)$ is not always an increasing function.

Example 4.3. For $a=1$, $x(t)=\frac{1}{t+2}$, $q=1.2$, $h=0$, $\alpha=0.7$.

$$\begin{aligned}x_k&=X(\sigma^k(a))\\&=\int_a^t\hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s\\&=\sum_{j=1}^{k}(\nu(\sigma^k(1)))^{-0.3}\begin{bmatrix}-0.3+k-j\\k-j\end{bmatrix}_{\tilde{q}}\frac{1}{\sigma^j(1)+2}\nu(\sigma^j(1))\\&=\sum_{j=1}^{k}(1.2^{k-1}\cdot 0.2)^{-0.3}\begin{bmatrix}-0.3+k-j\\k-j\end{bmatrix}_{\tilde{q}}\frac{1.2^{j-1}\cdot 0.2}{1.2^j+2}.\end{aligned}$$

According to Figure 2,we can see easily that the function x_k is not always an increasing function.

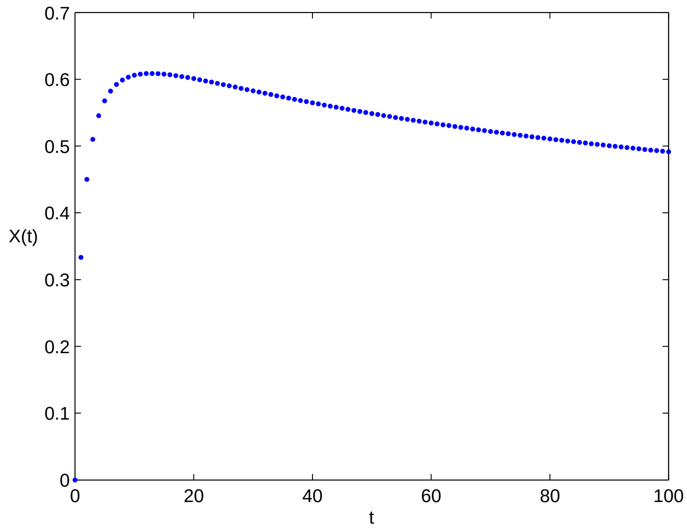

FIGURE 1. The values of $X(t)$ for $t \in \mathbb{N}_0^{100}, a = 0, q = h = 1$ and $\alpha = 0.6$ in Example 4.2.

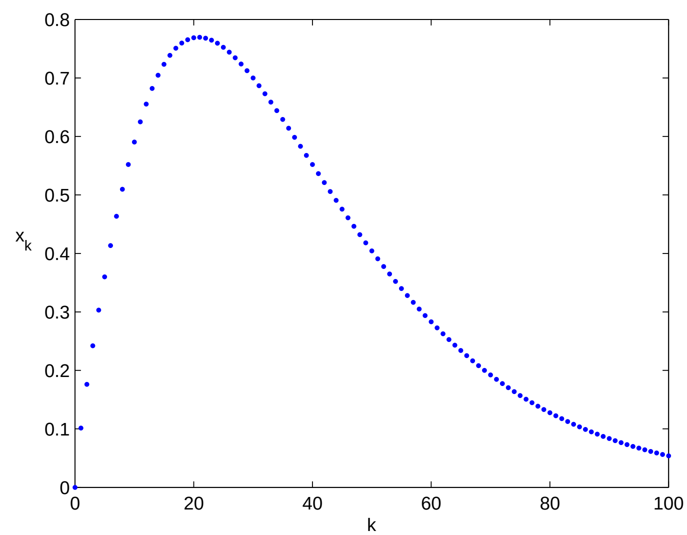

FIGURE 2. The values of x_k for $k \in \mathbb{N}_0^{100}$, $a = 1$, $q = 1.2$, $h = 0$ and $\alpha = 0.7$ in Example 4.3.

EXAMPLE 4.4. For $a=1$, $x(t)=\frac{1}{t+2}$, $q=1.5$, $h=2$, $\alpha=0.8$.

$$\begin{aligned}
x_k &= X(\sigma^k(a)) \\
&= \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s \\
&= \sum_{j=1}^{k}(\nu(\sigma^k(1)))^{-0.2}\begin{bmatrix}-0.2+k-j\\ k-j\end{bmatrix}_{\tilde{q}}\frac{1}{\sigma^j(1)+2}\nu(\sigma^j(1)) \\
&= \sum_{j=1}^{k}(5\cdot 1.5^k-4)^{-0.2}\begin{bmatrix}-0.2+k-j\\ k-j\end{bmatrix}_{\tilde{q}}\frac{2.5\cdot 1.5^{j-1}}{5\cdot 1.5^j-2}.
\end{aligned}$$

According to Figure 3,we can see easily that the function x_k is not always an increasing function.

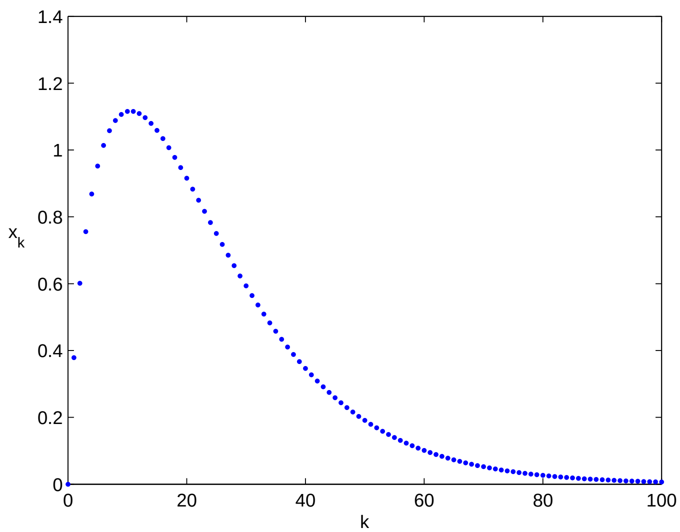

FIGURE 3. The values of x_k for $k\in\mathbb{N}_0^{100}$, $a=1$, $q=1.5$, $h=2$ and $\alpha=0.8$ in Example 4.4.

Example 4.2 and Example 4.3 shows that the proofs in [**25**, Theorem 5,6] and [**1**, Theorem 3,4] are not complete. For completeness, inspired by [**21**], we give rigourous proofs to them.

REMARK 4.5. It is worth mentioning that Phat et al. [**21**] shows that for a nondecreasing, nonnegative function $x(t)$, its fractional integral function $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds$ is increasing with respect to t, but using the idea (variable substitution) in [**21**, Theorem 1], we can obtain the similar result if $\mathbb{T}=\mathbb{N}_a$, but we can't obtain the one if $\mathbb{T}=\mathbb{T}_q$ or $\mathbb{T}=\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)}$. The method in the following lemma is successful to overcome this fault. That is to say, our method is essentially new and has its own merit.

LEMMA 4.6. *Assume* $x : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \to \mathbb{R}$, $x(t) \geq 0$ *and* $\nabla x(t) \geq 0$. *Then fractional integral function* $\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s$ *is increasing with respect to* t.

PROOF. Taking $t = \sigma^k(a)$, $s = \sigma^i(a), 1 \leq i \leq k$.
Note that

$$\begin{aligned}\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s &= -\int_a^t \nabla_s \hat{h}_\alpha(t,s)x(s)\nabla s \\ &= -\hat{h}_\alpha(t,s)x(s)|_{s=a}^t + \int_a^t \hat{h}_\alpha(t,\rho(s))\nabla x(s)\nabla s \\ &= \hat{h}_\alpha(t,a)x(a) + \int_a^t \hat{h}_\alpha(t,\rho(s))\nabla x(s)\nabla s,\end{aligned}$$

we have

$$\begin{aligned}&{}_t\nabla_{(q,h)}\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s \\ &\xlongequal{[\mathbf{16}, \text{ Lem. 2.4}]} \hat{h}_{\alpha-1}(t,a)x(a) + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\nabla x(s)\nabla s \\ &\quad + \hat{h}_\alpha(\rho(t),\rho(t))\nabla x(t) \\ &= \hat{h}_{\alpha-1}(t,a)x(a) + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\nabla x(s)\nabla s \\ &> 0,\end{aligned}$$

where we use

$$\hat{h}_\alpha(\rho(t),\rho(t)) = 0,$$

$$\begin{aligned}\hat{h}_{\alpha-1}(t,a) &= (\nu(t))^{\alpha-1}\begin{bmatrix}\alpha-1+k-1 \\ k-1\end{bmatrix}_{\tilde{q}} \\ &= (\nu(t))^{\alpha-1}\frac{\Gamma_{\tilde{q}}(\alpha-1+k)}{\Gamma_{\tilde{q}}(k)\Gamma_{\tilde{q}}(\alpha)} \\ &> 0\end{aligned}$$

and

$$\begin{aligned}\hat{h}_{\alpha-1}(t,\rho(s)) &= (\nu(t))^{\alpha-1}\begin{bmatrix}\alpha-1+k-i \\ k-i\end{bmatrix}_{\tilde{q}} \\ &= (\nu(t))^{\alpha-1}\frac{\Gamma_{\tilde{q}}(\alpha+k-i)}{\Gamma_{\tilde{q}}(k-i+1)\Gamma_{\tilde{q}}(\alpha)} \\ &> 0.\end{aligned}$$

So $\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))x(s)\nabla s$ is increasing with respect to t. $\square$

Let us denotes $\|A\| + \|B\| + 2L = b$.

THEOREM 4.7. *Assume that* $(\nu(t))^{\alpha} b < 1, t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ *holds. Then the solution of system* (4.1) *is unique.*

PROOF. Let $x(t)$ and $\tilde{x}(t)$ be any two different solutions of system (4.1). Let $z(t) = x(t) - \tilde{x}(t)$, we can obtain $z(t) = 0$ for $t \in [\rho^{m-1}(a), a]_{\mathbb{T}}$.

If $t \in [\sigma(a), H]_{\mathbb{T}}$, from Theorem 4.1 we have

$$\begin{aligned} z(t) = \int_a^t & \hat{h}_{\alpha-1}(t,\rho(s))[Az(s) + Bz(\rho^m(s)) \\ & + f(t, x(s), x(\rho^m(s))) - f(t, \tilde{x}(s), \tilde{x}(\rho^m(s)))]\nabla s. \end{aligned} \tag{4.3}$$

Taking the norm on both sides of (4.3), it follows that

$$\begin{aligned} \|z(t)\| \leq & \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))(\|A\|\|z(t)\| + \|B\|\|z(\rho^m(t))\| \\ & + \|f(s, x(s), x(\rho^m(s))) - f(s, \tilde{x}(s), \tilde{x}(\rho^m(s)))\|\nabla s \\ \leq & \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\Big[(\|A\| + L)\|z(s)\| + (\|B\| + L)\|z(\rho^m(s))\|\Big]\nabla s. \end{aligned} \tag{4.4}$$

Let $z^*(t) = \max_{\theta \in [\rho^{m-1}(a), t]_{\mathbb{T}}} \|z(\theta)\|$ for $t \in [\sigma(a), H]_{\mathbb{T}}$, it is obvious that $z^*(t)$ is a increasing function and we have

$$\|z(\rho^m(t))\| \leq z^*(t), \quad \forall t \in [\sigma(a), H]_{\mathbb{T}}$$

and

$$\|z(t)\| \leq z^*(t), \quad \forall t \in [\sigma(a), H]_{\mathbb{T}}.$$

It follows from (4.4) that

$$\|z(t)\| \leq \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\Big[(\|A\| + \|B\| + 2L)\|z^*(s)\|\Big]\nabla s$$

Note that for all $\theta \in [\sigma(a), t]_{\mathbb{T}}$, we have

$$\|z(\theta)\| \leq \int_a^\theta \hat{h}_{\alpha-1}(\theta,\rho(s))\Big[(\|A\| + \|B\| + 2L)\|z^*(s)\|\Big]\nabla s$$

From Lemma 4.6, we obtain

$$\|z(\theta)\| \leq (\|A\| + \|B\| + 2L)\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|z^*(s)\|\nabla s,\ \theta \in [\sigma(a), t]_{\mathbb{T}}.$$

Therefore, we have

$$
\begin{aligned}
z^*(t) &= \max_{\theta\in[\rho^{m-1}(a),t]_{\mathbb{T}}} \|z(\theta)\| \\
&\leq \max\left\{ \max_{\theta\in[\rho^{m-1}(a),a]_{\mathbb{T}}} \|z(\theta)\|, \max_{\theta\in[\sigma(a),t]_{\mathbb{T}}} \|z(\theta)\| \right\} \\
&= \max\left\{0, (\|A\|+\|B\|+2L)\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|z^*(s)\|\nabla s\right\} \\
&= (\|A\|+\|B\|+2L)\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|z^*(s)\|\nabla s.
\end{aligned}
\tag{4.5}
$$

Applying Theorem 3.4 on (4.5), it follows that

$$
\|z(t)\| \leq z^*(t) \leq 0 \cdot E^{a,b}_{\alpha,1}(t).
$$

Therefore, we obtain $x(t) = \tilde{x}(t)$ for $t \in [\sigma(a), H]_{\mathbb{T}}$. □

Let us denotes $\|\varphi\|_c = \max_{t\in[\rho^{m-1}(a),a]_{\mathbb{T}}} \|\varphi(t)\|$.

Definition 4.8. The system (4.1) is finite-time stable w.r.t. $\{\delta, \epsilon, H\}$, $\delta < \epsilon$ if and only if $\|\varphi\|_c < \delta$ implies $\|x(t)\| < \epsilon, \forall t \in [\sigma(a), H]_{\mathbb{T}}$.

In the following theorem, we give a finite-time stability criterion of the solution.

Theorem 4.9. *For given positive numbers δ, ϵ, H, $(\nu(t))^\alpha b < 1, t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, the system (4.1) is finite-time stability w.r.t (δ, ϵ, H) if*

$$
E^{a,b}_{\alpha,1}(t) \leq \epsilon/\delta,\ \forall t \in [\sigma(a), H]_{\mathbb{T}}. \tag{4.6}
$$

Proof. we have for all $t \in [\sigma(a), H]_{\mathbb{T}}$:

$$
\begin{aligned}
\|x(t)\| \leq& \|x(a)\| + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))(\|A\|\|x(t)\| + \|B\|\|x(\rho^m(t))\| \\
&+ \|f(s,x(s),x(\rho^m(s)))\|)\nabla s \\
\leq& \|\varphi\|_c + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\Big[(\|A\|+L)\|x(s)\| \\
&+ (\|B\|+L)\|x(\rho^m(s))\|\Big]\nabla s.
\end{aligned}
$$

Let $x^*(t) = \max_{\theta\in[\rho^{m-1}(a),t]_{\mathbb{T}}} \|x(\theta)\|$ for $t \in [\sigma(a), H]_{\mathbb{T}}$, similar to the proof of Theorem 4.7, we have

$$
\|x(t)\| \leq \|\varphi\|_c + \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\Big[(\|A\|+\|B\|+2L)\|x^*(s)\|\Big]\nabla s.
$$

Note that for all $\theta \in [\sigma(a), t]_{\mathbb{T}}$, we have

$$
\|x(\theta)\| \leq \|\varphi\|_c + \int_a^\theta \hat{h}_{\alpha-1}(\theta,\rho(s))\Big[(\|A\|+\|B\|+2L)\|x^*(s)\|\Big]\nabla s
$$

From Lemma 4.6, we obtain

$$\|x(\theta)\| \le \|\varphi\|_c + (\|A\| + \|B\| + 2L)\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|x^*(s)\|\nabla s,\ \theta \in [\sigma(a), t]_{\mathbb{T}}.$$

Therefore, we have

$$\begin{aligned} x^*(t) &= \max_{\theta\in[\rho^{m-1}(a),t]_{\mathbb{T}}} \|x(\theta)\| \\ &\le \max\left\{ \max_{\theta\in[\rho^{m-1}(a),a]_{\mathbb{T}}} \|x(\theta)\|, \max_{\theta\in[\sigma(a),t]_{\mathbb{T}}} \|x(\theta)\| \right\} \\ &= \max\left\{ \|\varphi\|_c, \|\varphi\|_c + b\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|x^*(s)\|\nabla s \right\} \\ &= \|\varphi\|_c + b\int_a^t \hat{h}_{\alpha-1}(t,\rho(s))\|x^*(s)\|\nabla s. \end{aligned} \tag{4.7}$$

From Theorem 3.4, we have

$$\|x(t)\| \le x^*(t) \le \|\varphi\|_c E_{\alpha,1}^{a,b}(t).$$

Therefore, from (4.6) we obtain that

$$\|x(t)\| \le \delta\cdot\epsilon/\delta \le \epsilon.$$

□

THEOREM 4.10. *Assume* $f(t,x(t),x(\rho^m(t))) = [\underbrace{0,0,\cdots,0}_{n}]^T$ *and* $t = \sigma^l(a),\ l \ge 1$. *If* $(\nu(\sigma^l(a)))^{-\alpha}I - A$ *is a invertible matrix, which is denoted as* $A_2 = (\nu(\sigma^l(a))^{-\alpha}I - A)^{-1}$, *then the solution of system* (4.1) *can be rewritten as*

$$\begin{aligned} x(\sigma^l(a)) &= \begin{bmatrix} -\alpha+l-1 \\ l-1 \end{bmatrix}_{\tilde{q}} (\nu(\sigma^l(a)))^{-\alpha}A_2x(a) + A_2Bx(\sigma^{l-n}(a)) \\ &\quad - \sum_{j=1}^{l-1}\nu(\sigma^l(a))^{-\alpha-1}\begin{bmatrix} -\alpha-j+l-1 \\ l-j \end{bmatrix}_{\tilde{q}}\nu(\sigma^j(a))A_2x(\sigma^j(a)), \end{aligned} \tag{4.8}$$

where $l \ge 1$ *and* $\sigma^{l-n}(a) = \rho^{n-l}(a)$.

PROOF. From [**10**, Lemma 3.5], we have

$$\begin{aligned} {}_a^C\nabla_{(q,h)}^{\alpha}x(t) &= -\hat{h}_{-\alpha}(\sigma^l(a),a)x(a) + x(\sigma^l(a))(\nu(\sigma^l(a)))^{-\alpha} \\ &\quad + \sum_{j=1}^{l-1}\hat{h}_{-\alpha-1}(\sigma^l(a),\sigma^{j-1}(a))x(\sigma^j(a))\nu(\sigma^j(a)). \end{aligned} \tag{4.9}$$

Applying (4.9) to (4.1), we have

(4.10)

$$(\nu(\sigma^l(a))^{-\alpha}I - A)x(\sigma^l(a)) = \hat{h}_{-\alpha}(\sigma^l(a), a)x(a) + Bx(\sigma^{l-n}(a)) - \sum_{j=1}^{l-1} \hat{h}_{-\alpha-1}(\sigma^l(a), \sigma^{j-1}(a))x(\sigma^j(a))\nu(\sigma^j(a)).$$

As a result, we can update (4.10) as

(4.11)

$$\begin{aligned} x(\sigma^l(a)) =& \hat{h}_{-\alpha}(\sigma^l(a), a)(\nu(\sigma^l(a))^{-\alpha}I - A)^{-1}x(a) \\ &+ (\nu(\sigma^l(a))^{-\alpha}I - A)^{-1}Bx(\sigma^{l-n}(a)) \\ &- \sum_{j=1}^{l-1} \hat{h}_{-\alpha-1}(\sigma^l(a), \sigma^{j-1}(a))\nu(\sigma^j(a))(\nu(\sigma^l(a))^{-\alpha}I - A)^{-1}x(\sigma^j(a)) \\ =& \nu(\sigma^l(a))^{-\alpha} \begin{bmatrix} -\alpha + l - 1 \\ l - 1 \end{bmatrix}_{\tilde{q}} A_2x(a) + A_2Bx(\sigma^{l-n}(a)) \\ &- \sum_{j=1}^{l-1} \nu(\sigma^l(a))^{-\alpha-1} \begin{bmatrix} -\alpha - j + l - 1 \\ l - j \end{bmatrix}_{\tilde{q}} \nu(\sigma^j(a))A_2x(\sigma^j(a)). \end{aligned}$$

This complete the proof. □

REMARK 4.11. Letting $x_l = x(\sigma^l(a)), \nu_l = \nu(\sigma^l(a))$, from Theorem 4.10 we can derive the explicit numerical formulae for fractional difference systems (4.1)

$$\begin{aligned} x_l =& \nu_l^{-\alpha} \begin{bmatrix} -\alpha + l - 1 \\ l - 1 \end{bmatrix}_{\tilde{q}} A_2x_0 + A_2Bx_{l-n} \\ &- \sum_{j=1}^{l-1} \nu_l^{-\alpha-1} \begin{bmatrix} -\alpha - j + l - 1 \\ l - j \end{bmatrix}_{\tilde{q}} \nu_jA_2x_j, \end{aligned}$$

where $l \geq 1$.

5. Examples

EXAMPLE 5.1.

(5.1)

$$\begin{cases} {}_0^C\nabla_{(1,0.5)}^{0.7}x(t) = \begin{pmatrix} 0.1 & 0.1 \\ 0.05 & 0.15 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} x(\rho^3(t)), t \in \tilde{\mathbb{T}}_{(1,0.5)}^{\sigma(0)}, \\ x(t) = (0.05, 0.05)^T, \ t \in [\rho^2(0), 0]_{\mathbb{T}}, \end{cases}$$

where $\alpha = 0.7$, $\|\phi\|_c = \max_{t\in[\rho^2(0),0]_{\mathbb{T}}} \|x(t)\|_2 = 0.1$, $\|A\|_2 = 0.2065$, $\|B\|_2 = 0.2414$, $b = 0.4479, a = 0$, $q = 1$, $h = 0.5$, $(\nu(t))^{\alpha} b = h^{\alpha} b = 0.5^{0.7} * 0.4479 = 0.2757 < 1$. We use $\delta = 0.11$ and $\epsilon = 0.6$.

From [**24**, Corollary 4.4] we know that for fractional difference equation

$$ {}_a^C\nabla^{\alpha}_{(q,h)} x(t) = bx(t), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)},\ 0 < \alpha < 1, \quad x(a) = 1, \tag{5.2}$$

its solution is $x(t) = E^{a,b}_{\alpha,1}(t)$. Let $t = \sigma^k(a), k \in \mathbb{N}_0$, from [**10**, Remark 3.9] we have exact values of $x(t)$ can be written as

$$ x(\sigma^k(a)) = \frac{x(a) + bh^{\alpha} \sum_{i=1}^{k-1} \binom{\alpha - 1 + k - i}{k - i} x(\sigma^i(a))}{1 - h^{\alpha} b}, \tag{5.3}$$

for $k \geq 1$.

TABLE 1. The values of $\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$ in Example 5.1.

k	$\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$	k	$\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$	k	$\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$	k	$\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$
1	0.1519	3	0.2375	5	0.3507	7	0.5074
2	0.1923	4	0.2897	6	0.4226	8	0.6079

From Table 1, we can have the finite-time $H = \sigma^7(0) = 3.5s$ for $m = 3$. From Fig 4, we can see that for $\|\varphi\|_c = 0.1 < 0.11 = \delta$, the 2-norm of the solution $\|x(t)\|_2 \leq \epsilon = 0.6$ for $[0, \sigma^7(0)]_{\mathbb{T}}$, which support Theorem 3.4 numerically.

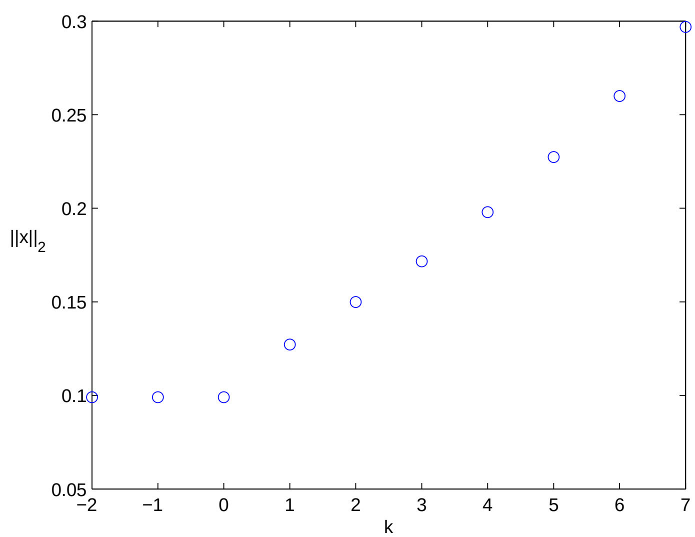

FIGURE 4. Solution's norm $\|x\|_2$ within $H = 3.5s$: $\alpha = 0.7, q = 1, h = 0.5$ and $m = 3$.

Example 5.2.

$$\begin{cases} {}_{1}^{C}\nabla_{(1.2,0.3)}^{0.5}x(t) = \begin{pmatrix} 0.1 & 0.1 \\ 0.05 & 0.15 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} x(\rho^3(t)), t \in \tilde{\mathbb{T}}_{(1.2,0.3)}^{\sigma(1)}, \\ \\ x(t) = (0.05, 0.05)^T,\ t \in [\rho^2(1), 1]_{\mathbb{T}}, \end{cases} \tag{5.4}$$

where $q = 1.2, h = 0.3, a = 1, \alpha = 0.5, \|\phi\|_c = \max_{t\in[\rho^2(1),1]_{\mathbb{T}}} \|x(t)\|_2 = 0.1$, $\|A\|_2 = 0.2065,\ \|B\|_2 = 0.2414,\ b = 0.4479,\ (\nu(t))^\alpha b = (\nu(\sigma^5(a)))^\alpha b < 1$. We use $\delta = 0.11$ and $\epsilon = 0.6$.

From [**24**, Corollary 4.4] we know that for fractional (q, h)-difference equation

$$ {}_{a}^{C}\nabla_{(q,h)}^{\alpha}x(t) = bx(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)},\ 0 < \alpha < 1, \quad x(a) = 1, \tag{5.5}$$

its solution is $x(t) = E_{\alpha,1}^{a,b}(t)$. Let $t = \sigma^k(a), k \in \mathbb{N}_0$, from [**10**, Remark 3.9] we have exact values of $x(t)$ can be written as

$$x(\sigma^k(a)) = \frac{x(a) + \sum_{j=1}^{k-1}(\nu(\sigma^k(a)))^{\alpha-1} \begin{bmatrix} \alpha - 1 + k - j \\ k - j \end{bmatrix}_{\tilde{q}} b\nu\big(\sigma^j(a)\big)x(\sigma^j(a))}{1 - (\nu(\sigma^k(a)))^\alpha b}, \tag{5.6}$$

for $k \geq 1$.

Table 2. The values of $\delta E_{\alpha,1}^{a,b}(\sigma^k(a))$ in Example 5.2.

k	1	2	3	4	5	6
$\delta E_{\alpha,1}^{a,b}(\sigma^k(a))$	0.1661	0.2172	0.2818	0.3727	0.5111	0.7388

From Table 2, we can have the finite-time $H = \sigma^5(1)s = 4.7208s$ for $m = 3$. From Fig 5, we can see that for $\|\varphi\|_c = 0.1 < 0.11 = \delta$, the 2-norm of the solution $\|x(t)\|_2 \leq \epsilon = 0.6$ for $t \in [1, \sigma^5(1)]_{\mathbb{T}}$, which support Theorem 3.4 numerically.

Example 5.3.

$$\begin{cases} {}_{1}^{C}\nabla_{(1.2,0.3)}^{0.5}x(t) = \begin{pmatrix} 0.1 & 0.1 \\ 0.05 & 0.15 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} x(\rho^3(t)) + 0.1 \begin{pmatrix} \cos x(t) \\ \cos x(\rho^3(t)) \end{pmatrix}, \\ \qquad t \in \tilde{\mathbb{T}}_{(1.2,0.3)}^{\sigma(1)}, \\ \\ x(t) = (0.03, 0.03)^T,\ t \in [\rho^2(1), 1]_{\mathbb{T}}, \end{cases} \tag{5.7}$$

where $f(t, x(t), x(\rho^3(t))) = 0.1 \begin{pmatrix} \cos x_2(t) \\ \cos x_1(\rho^3(t)) \end{pmatrix}$, $q = 1.2, h = 0.3, a = 1, \alpha = 0.5, \|\phi\|_c = \max_{t\in[\rho^2(1),1]_{\mathbb{T}}} \|x(t)\|_1 = 0.1,\ \|A\|_1 = 0.2500,\ \|B\|_1 = 0.3000$, $b = 0.6500$.

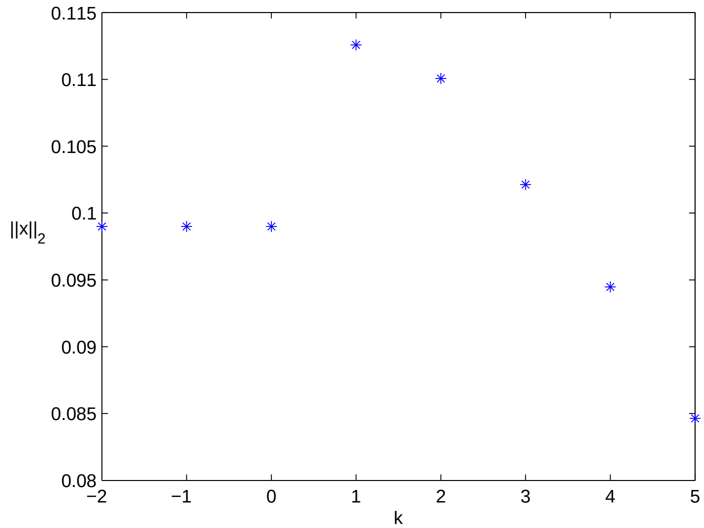

FIGURE 5. Solution's norm $\|x\|_2$ within $H = 4.7208s$: $\alpha = 0.5, q = 1.2, h = 0.3$ and $m = 3$.

Since

$$\begin{aligned}
&\|f(t, x(t), x(\rho^3(t))) - f(t, \tilde{x}(t), \tilde{x}(\rho^3(t)))\|_1 \\
=&|\cos x_2(t) - \cos \tilde{x}_2(t)| + |\cos x_1(\rho^3(t)) - \cos \tilde{x}_1(\rho^3(t))| \\
\leq&|x_2(t) - \tilde{x}_2(t)| + |x_1(\rho^3(t)) - \tilde{x}_1(\rho^3(t))| \\
\leq&\|x(t) - \tilde{x}(t)\|_1 + \|x(\rho^3(t)) - \tilde{x}(\rho^3(t))\|_1.
\end{aligned}$$

Thus, condition (H1) holds with $L = 0.1$. $(\nu(t))^\alpha b = (\nu(\sigma^4(a)))^\alpha b < 1$. We use $\delta = 0.11$ and $\epsilon = 0.9$.

TABLE 3. The values of $\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$ in Example 5.3.

k	1	2	3	4	5
$\delta E^{a,b}_{\alpha,1}(\sigma^k(a))$	0.2036	0.3115	0.4833	0.7984	1.4628

From Table 3 we can have the finite-time $H = \sigma^4(1)s = 3.6840s$ for $m = 3$ and the 1-norm of the solution $\|x(t)\|_1 \leq \epsilon = 0.9$ for $t \in [1, \sigma^4(1)]_{\mathbb{T}}$.

6. Conclusions

In this paper, we establish a fractional (q,h)-Gronwall inequality, which is the generalization of the some existing works. With the help of this inequality, we prove the uniqueness and give the finite-time stability criterion for the solution of nonlinear fractional delay (q,h)-difference systems. Several numerical examples are given to show the validity and effectiveness of the obtained results. In addition, using three counterexamples, we point out the problems which may be ignored easily when we deal with the fractional delay difference systems.

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