MILD SOLUTIONS FOR A MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATION VIA RESOLVENT OPERATORS

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Abstract. This paper is concerned with multi-term fractional differential equations. With the help of the theory of fractional resolvent families, we establish the existence of mild solutions to a multi-term fractional differential equation.

1. Introduction

In the last two decades, differential equations involving fractional derivatives, have been used in many mathematical models to describe a wide variety of phenomena, including problems in viscoelasticity, signal and image processing, engineering, economics, epidemiology and among others, and the study of this kind of equations has been a topic of interest in recent years. See [10, 16, 19, 26, 36, 40] and the references therein.

In this paper, we consider the following multi-term fractional differential equations

$$\partial^{\alpha}u(t) = Au(t) + \partial^{\alpha-\beta}f(t,u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$\partial_t^{\alpha}u(t) = Au(t) + \partial_t^{\alpha-\beta}f(t,u(t)), \quad t \in [0,T], \tag{1.2}$$

where A is a closed linear operator defined in a Banach space X, $1 < \alpha, \beta < 2$, $T > 0$, and f is a suitable continuous function. Here, for $\gamma > 0$ the derivatives $\partial^{\gamma}u$ and $\partial_t^{\gamma}u$, denote the Weyl and Caputo fractional derivatives, respectively.

Although the definition of the fractional derivatives in the sense of Weyl (defined on $\mathbb{R}$) and Caputo (defined on $[0,\infty)$) are different, we notice that the mild solution to equations (1.1) and (1.2) can be written in terms of the same resolvent family. In fact, if A is the generator of the fractional resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ (see its definition in Section 2) then the *mild* solutions to Equations (1.1) and (1.2) are defined, respectively, by

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s,u(s))ds, \quad t \in \mathbb{R},$$

and

$$u(t) = S_{\alpha,1}(t)x + S_{\alpha,2}(t)y + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \quad t > 0,$$

where $x = u(0)$ and $y = u'(0)$ are the initial conditions in equation (1.2), and the families $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, and $\{S_{\alpha,2}(t)\}_{t\geq 0}$, are given respectively by

$$S_{\alpha,\beta}(t) = (g_{\beta-1} * S_{\alpha,1})(t), \quad \text{and} \quad S_{\alpha,2}(t) = (g_1 * S_{\alpha,1})(t).$$

Here, the $*$ denotes the usual finite convolution and for $\gamma > 0$ the function g_{γ} is defined by $g_{\gamma}(t) := t^{\gamma-1}/\Gamma(\gamma)$, where $\Gamma(\cdot)$ is the Gamma function. The fractional resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$

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is defined by

$$S_{\alpha,1}(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda, \quad t \geq 0,$$

where Γ is a suitable complex path where the resolvent operator $(\lambda^\alpha - A)^{-1}$ is well defined. By the uniqueness of the Laplace transform it is easy to see that

$$S_{\alpha,2}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-2}(\lambda^\alpha - A)^{-1}d\lambda \quad \text{and} \quad S_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda,$$

for all $t \geq 0$. The existence of mild solutions to equation (1.1) in case $\beta = 1$ has been widely studied in the last years, see for instance [5, 13, 14, 25] and references therein. In these mentioned papers, the operator A is assumed to be an ω-sectorial operator of angle θ (see definition in Section 2). In this case, A generates a resolvent family $\{E_\alpha(t)\}_{t\geq 0}$ (see [12, 29]) which satisfies

$$\|E_\alpha(t)\| \leq \frac{C}{1+|\omega|t^\alpha}, \qquad \text{for all } t \geq 0,$$

where C is a positive constant depending only on α and θ. This decay of $\{E_\alpha(t)\}_{t\geq 0}$ provides also some tools to obtain many and interesting consequences in the study of qualitative properties of solutions to fractional (and integral) differential (and difference) equations. See for instance [5, 8, 9, 30, 32, 39] and the references therein for further details. We notice that, by the uniqueness of the Laplace transform, the resolvent families $\{E_\alpha(t)\}_{t\geq 0}$ and $\{S_{\alpha,1}(t)\}_{t\geq 0}$ are the same for $1 < \alpha < 2$.

On the other hand, the existence of mild solutions to fractional differential equations with nonlocal conditions has been studied by several authors in the last years. The concept of nonlocal initial condition was introduced by L. Byszewski [7] to extend the study of classical initial value problems. This notion results more suitable to describe more precisely several phenomena in applied sciences, because it considers additional information in the initial data. More concretely, the nonlocal conditions have the form $u(0)+g(u) = u_0$ instead $u(0) = u_0$, where g is an appropriate function that represents the additional information in the system and provides a better description of the initial state of the system than the classical initial value problem. The theory of nonlocal Cauchy problems has been developed rapidly and has been studied widely in the last years, see for instance [1, 4, 20, 33, 37, 38] and the references therein for more details.

There exists a wide recent literature on the existence of mild solutions to fractional differential equations with nonlocal initial conditions. More specifically, the problem

$$\text{(1.3)} \qquad \begin{cases} \partial_t^\alpha u(t) &= Au(t) + f(t, u(t)), \quad t \in [0, T] \\ u(0) + g(u) &= u_0, \end{cases}$$

where $T > 0$, A is a closed linear operator defined in a Banach space X, $0 < \alpha \leq 1$, $u_0 \in X$, f is a suitable semilinear continuous function has been studied extensively in recent years. See for instance [2, 3, 11, 27, 31, 35]. Since the fractional derivative ∂_t^α for $\alpha = 1$ is the usual derivative $\frac{d}{dt}$, the case $\alpha = 1$ in (1.3) corresponds precisely to the semilinear Cauchy problem introduced in the seminal paper [7] and the theory of C_0-semigroups of linear operators is the main tool to obtain the existence of solutions in this case. Similarly, for $\alpha > 0$ the theory of fractional resolvent families represents one of the main tools to study the existence of mild solutions to (1.3). Indeed, if $0 < \alpha \leq 1$ and A generates a resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, then the mild solution to (1.3) is given by

$$\text{(1.4)} \qquad u(t) = S_{\alpha,1}(u_0 - g(u)) + \int_0^t S_{\alpha,\alpha}(t-s)f(s, u(s))ds$$

where $S_{\alpha,1}(t) := (g_{1-\alpha} * S_{\alpha,\alpha})(t)$, see for instance [31]. We notice that the variation of constant formula (1.4) coincides with the case $\alpha = 1$ introduced in [7, Section 3]. Similarly, for $1 < \alpha < 2$ and $\beta = 1$ or $\beta = \alpha$, the equation (1.2) subject to the nonlocal conditions $u(0) + g(u) = u_0$, and $u'(0) + h(u) = u_1$, where $g, h : C(I,X) \to C(I,X)$ are continuous and u_0, u_1 belong to X, ($I := [0,T]$) has been considered by several authors in the last years. See for instance [3, 23] for the case $\beta = 1$ and [33, 34] in case $\beta = \alpha$.

In this paper, our concern is the study of existence of mild solutions to the fractional differential equations (1.1) and (1.2). Here, we assume certain conditions on the operator A and on the parameters α and β in order to ensure that A is the generator of a fractional resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$.

More specifically, in equation (1.1) we consider the Weyl fractional derivative, because it is defined for functions on $\mathbb{R}$. More precisely, we show that if the function f in (1.1) is an almost periodic or an almost automorphic (among others) vector-valued function, then the equation (1.1) has a unique almost periodic or almost automorphic function mild solution, respectively, which is given in terms of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$.

On the other hand, in equation (1.2) the derivative is taken in the sense of Caputo, because it is defined on the positive real axis $[0,\infty)$. Under the the nonlocal conditions $u(0)+g(u) = u_0$, and $u'(0) + h(u) = u_1$ we prove that (1.2) has at least one mild solution. Here, the properties of the fractional resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ are again an important tool to obtain the result.

This paper is organized as follows. The Section 2 gives the preliminaries on fractional calculus, sectorial operators, fractional resolvent families and some subspaces of bounded and continuous functions. Section 3 is devoted to the existence of mild solutions to (1.1). Finally, in Section 4 is studied the existence of mild solutions to the nonlocal problem (1.2).

2. Preliminaries

For a Banach space $(X, \|\cdot\|)$, the space of all bounded and linear operators form X into X is denoted by $\mathcal{B}(X)$. If A is a closed linear operator defined on X we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ its resolvent operator, which is defined for all $\lambda \in \rho(A)$. For $1 \leq p < \infty$, $L^p(\mathbb{R}_+, X)$ denotes the space of all Bochner measurable functions $g : \mathbb{R}_+ \to X$ such that

$$\|g\|_p := \left(\int_0^\infty \|g(t)\|^p dt\right)^{1/p} < \infty.$$

We recall that a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be exponentially bounded if there exist two constants $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$ for all $t > 0$.

A closed and densely defined operator A, defined on a Banach space $(X, \|\cdot\|)$, is said to be *ω-sectorial of angle ϕ*, if there exist $\phi \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector $\omega + \Sigma_\phi := \left\{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \phi\right\} \setminus \{\omega\}$ and $\|R(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}$ for all $\lambda \in \omega + \Sigma_\phi$. See [17] and [18] for further details.

Now, we review some results on fractional calculus. We recall that for $\gamma > 0$, the function g_γ is defined by $g_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}$ for all $t \geq 0$. For $\gamma > 0$, $\lceil\gamma\rceil$ denotes the smallest integer greater than or equal to γ, and $[\gamma]$ denotes the integer part of γ. As usual, the finite convolution of f and g is defined by $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Definition 2.1. *Let $\alpha > 0$ and $n = \lceil\alpha\rceil$. The Caputo fractional derivative of order α of a function $u : [0,\infty) \to X$ is defined by*

$$\partial_t^\alpha u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds.$$

Definition 2.2. *Let $\alpha > 0$ and $n = [\alpha]+1$. The Weyl fractional derivative of order α of a function $u : \mathbb{R} \to X$ is defined by*

$$\partial^\alpha u(t) := \frac{d^n}{dt^n}\partial^{-(n-\alpha)}u(t),$$

where for $\gamma > 0$, $\partial^{-\gamma}u(t) := \int_{-\infty}^{t} g_\gamma(t-s)u(s)ds$ *for all* $t \in \mathbb{R}$.

It is a well known fact that if $\alpha \in \mathbb{N}$, then $\partial_t^n = \partial^n = \frac{d^n}{dt^n}$, that is, the Caputo and Weyl fractional derivatives coincide with the usual derivative if $\alpha \in \mathbb{N}$. Moreover, if $\alpha, \beta \in \mathbb{R}$, then $\partial^\alpha \partial^\beta u = \partial^\beta \partial^\alpha u = \partial^{\alpha+\beta} u$. See [26] for more details on fractional differential calculus.

Now, we recall the resolvent families of operators generated by an operator A.

Definition 2.3. *Let A be closed linear operator with domain $D(A)$, defined on a Banach space X, $1 \leq \alpha \leq 2$ and $0 < \beta \leq 2$. We say that A is the generator of an (α, β)-resolvent family, if there exists $\nu \geq 0$ and a strongly continuous and exponentially bounded function $S_{\alpha,\beta} : [0, \infty) \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \nu\} \subset \rho(A)$, and for all $x \in X$,*

$$\lambda^{\alpha-\beta} (\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x dt, \ \ \mathrm{Re}\lambda > \nu.$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t \geq 0}$ is called the (α, β)-resolvent family *generated by A.*

If we compare Definition 2.3 with the notion of (a, k)-regularized families introduced in [22], then we notice that $t \mapsto S_{\alpha,\beta}(t)$, is a (g_α, g_β)-regularized family. Moreover, the family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is well known in some cases. For example, $S_{1,1}(t)$ is a C_0-semigroup, $S_{2,1}(t)$, corresponds to a cosine family and $S_{2,2}(t)$ is a sine family. In the scalar case, that is, when $A = \varrho\mathcal{I}$, where $\varrho \in \mathbb{C}$ and $\mathcal{I}$ denotes the identity operator, then by the uniqueness of the Laplace transform, $S_{\alpha,\beta}(t)$ corresponds to the function $t^{\beta-1}E_{\alpha,\beta}(\varrho t^\alpha)$, where for $z \in \mathbb{C}$ the generalized Mittag-Leffler function is defined by $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}$.

We have also the following result. Its proof follows similarly as in [21, Proposition 3.7].

Proposition 2.4. *Let $1 \leq \alpha, \beta \leq 2$. Let $S_{\alpha,\beta}(t)$ be the (α, β)-resolvent family generated by A. Then:*

1. *$S_{\alpha,\beta}(t)x \in D(A)$ and $S_{\alpha,\beta}(t)Ax = AS_{\alpha,\beta}(t)x$ for all $x \in D(A)$ and $t \geq 0$.*
2. *If $x \in D(A)$ and $t \geq 0$, then*

$$S_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AS_{\alpha,\beta}(s)xds \tag{2.1}$$

3. *If $x \in X, t \geq 0$, then $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and $S_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds$.*

In particular, $S_{\alpha,\beta}(0) = g_\beta(0)\mathcal{I}$.

The next result gives sufficient conditions on α, β and A to obtain generators of (α, β)-resolvent families.

Theorem 2.5. [29] *Let $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Assume that A is ω-sectorial of angle $\frac{(\alpha-1)\pi}{2}$, where $\omega < 0$. Then A generates an exponentially bounded (α, β)-resolvent family.*

Theorem 2.6. [29] *Let $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Assume that A is ω-sectorial of angle $\frac{(\alpha-1)}{2}\pi$, where $\omega < 0$. Then, there exists a constant $C > 0$, depending only on α and β, such that*

$$\|S_{\alpha,\beta}(t)\| \leq \frac{Ct^{\beta-1}}{1 + |\omega|t^\alpha}, \quad \text{for all } t > 0. \tag{2.2}$$

Finally, we recall some spaces of functions. For a given Banach space $(X, \|\cdot\|)$, let $BC(X) := \{f : \mathbb{R} \to X : \|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\| < \infty\}$ be the Banach space of all bounded and continuous functions. For $T > 0$ fixed, $P_T(X)$ denotes the space of all vector-valued periodic functions, that is, $P_T(X) := \{f \in BC(X) : f(t + T) = f(t), \text{ for all } t \in \mathbb{R}\}$. We denote by $AP(X)$ to the space of all almost periodic functions (in the sense of Bohr), which consists of all $f \in BC(X)$ such that for every $\varepsilon > 0$ there exists $l > 0$ such that for every subinterval of $\mathbb{R}$ of length l contains at least one point τ such that $\|f(t+\tau) - f(t)\|_\infty \le \varepsilon$. A function $f \in BC(X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \qquad \text{for each } t \in \mathbb{R}.$$

We denote by $AA(X)$ the Banach space of all almost automorphic functions.

On the other hand, the space of compact almost automorphic functions is the space of all functions $f \in BC(X)$ such that for all sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} g(t-s_n)$ uniformly over compact subsets of $\mathbb{R}$.

We notice that $P_T(X), AP(X), AA(X)$ and $AA_c(X)$ are Banach spaces under the norm $||\cdot||_\infty$ and

$$P_T(X) \subset AP(X) \subset AA(X) \subset AA_c(X) \subset BC(X).$$

We notice that all these inclusions are proper. Now we consider the set $C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$, and define the space of asymptotically periodic functions as $AP_T(X) := P_T(X) \oplus C_0(X)$. Analogously, we define the space of asymptotically almost periodic functions,

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions,

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural proper inclusions

$$AP_T(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

For more details on this function spaces, we refer to reader to [24, 28].

Throughout, we will use the notation $\mathcal{N}(X)$ to denote any of the function spaces $AP_T(X)$, $AAP(X)$, $AAA_c(X)$ and $AAA(X)$ defined above. Finally, we define the set $\mathcal{N}(\mathbb{R}\times X; X)$ which consists of all functions $f : \mathbb{R}\times X \to X$ such that $f(\cdot, x) \in \mathcal{N}(X)$ uniformly for each $x \in K$, where K is any bounded subset of X. Moreover, we have the following result.

Theorem 2.7. [24] *Let $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ be a strongly continuous and uniformly* 1*-integrable family, that is $\int_0^\infty \|S(t)\|dt < \infty$. If $f \in \mathcal{N}(X)$, then the function $u : \mathbb{R} \to X$ defined by*

$$u(t) := \int_{-\infty}^t S(t-s)f(s)ds,$$

belongs to $\mathcal{N}(X)$.

3. Bounded mild solutions to Equation (1.1)

Let $1 < \alpha < 2$ and $\beta \ge 1$. In this section, we first consider the linear version of the equation (1.1), that is,

$$\partial^\alpha u(t) = Au(t) + \partial^{\alpha-\beta} f(t), \quad t \in \mathbb{R}. \tag{3.3}$$

Definition 3.8. *A function $u \in C(\mathbb{R}, X)$ is called a* mild solution *to equation (3.3) if the function $s \mapsto S_{\alpha,\beta}(t-s)f(s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.4}$$

We notice that (3.3) can be considered as the limiting equation of the following integro-differential equation with singular kernels

$$\begin{cases} v'(t) &= \displaystyle\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s) + \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} f(s)ds, \quad t \geq 0 \\ v(0) &= v_0, \quad v_0 \in X, \end{cases} \tag{3.5}$$

in the sense that the mild solution to equation (3.5) converges to the mild solution of (3.3) as $t \to \infty$. In fact, if $\omega < 0$ and A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$, then taking Laplace transform in (3.5) we obtain

$$\lambda\hat{v}(\lambda) - v(0) = A\hat{v}(\lambda) + \frac{1}{\lambda^{\beta-1}}\hat{f}(\lambda), \quad \text{Re}\lambda > 0,$$

which is equivalent to

$$(\lambda^\alpha - A)\hat{v}(\lambda) = \lambda^{\alpha-1}v(0) + \lambda^{\alpha-\beta}\hat{f}(\lambda), \quad \text{Re}\lambda > 0.$$

Therefore the solution of problem (3.5) can be written as

$$v(t) = S_{\alpha,1}(t)v_0 + \int_0^t S_{\alpha,\beta}(t-s)f(s)ds, \quad t \geq 0, \tag{3.6}$$

where $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is the family of operators given by

$$S_{\alpha,\beta}(t) := (g_{\beta-1} * S_{\alpha,1})(t).$$

On the other hand, by [29, Corollary 3.9] the function $t \mapsto S_{\alpha,\beta}(t)$ is uniformly 1-integrable and therefore if f is a bounded continuous function (for example, f belongs to $\mathcal{N}(X)$), then the mild solution to equation (1.1) is given by

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds.$$

Since

$$v(t) - u(t) = S_{\alpha,1}(t)v_0 - \int_t^\infty S_{\alpha,\beta}(s)f(t-s)ds,$$

we conclude by [29, Corollary 3.8], that $v(t) - u(t) \to 0$ as $t \to \infty$.

Let $1 < \alpha < 2$, $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$, $\omega < 0$ and assume that A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$. By Theorem 2.5, the operator A generates a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. Take a bounded and continuous function $f : \mathbb{R} \to X$, (for example, we can take $f \in \mathcal{N}(X)$). Define the function $\phi(t)$ by

$$\phi(t) := \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.7}$$

By Theorem 2.6 we have $||\phi||_\infty \leq ||S_{\alpha,\beta}||_1\, ||f||_\infty$. If $f(t) \in D(A)$ for all $t \in \mathbb{R}$, then $\phi(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [6, Proposition 1.1.7]). Assume that $\partial^\alpha \phi$ exists. The Proposition 2.4 and Fubini's theorem imply that

$$\begin{aligned}
\partial^{\alpha}\phi(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\phi(s)ds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} S_{\alpha,\beta}(s-r)f(r)drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} \Big[g_{\beta}(s-r)f(r)+(g_{\alpha}*AS_{\alpha,\beta})(s-r)f(r)\Big]drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\partial^{-\beta}f(s)ds + \\
&\qquad \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{0}^{s-r} g_{\alpha}(s-r-v)AS_{\alpha,\beta}(v)f(r)dvdrds \\
&= \partial^{\alpha-\beta}f(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{r}^{s} g_{\alpha}(s-w)AS_{\alpha,\beta}(w-r)f(r)dwdrds \\
&= \partial^{\alpha-\beta}f(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{-\infty}^{w} g_{\alpha}(s-w)AS_{\alpha,\beta}(w-r)f(r)drdwds \\
&= \partial^{\alpha-\beta}f(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} g_{\alpha}(s-w)A\phi(w)dwds \\
&= \partial^{\alpha-\beta}f(t)+A\phi(t),
\end{aligned}$$

for all $t \in \mathbb{R}$. This means that, ϕ is a (strong) solution to Equation (3.3). We recall that a function $u \in C(\mathbb{R}, X)$ is called a strong solution of (3.3) on $\mathbb{R}$ if $u \in C(\mathbb{R}, D(A))$, the fractional derivative of u, $\partial^{\alpha}u$, exists and (3.3) holds for all $t \in \mathbb{R}$. If merely $u(t)$ belongs to X instead of the $D(A)$, then u is a mild solution to the equation (3.3) according to Definition 3.8. As consequence of the above computation we have the following result.

Theorem 3.9. *Let $1 < \alpha < 2$, $1 \le \beta \le \alpha$ and $\omega < 0$. Assume that A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$. Then for each $f \in \mathcal{N}(X)$ there is a unique mild solution $u \in \mathcal{N}(X)$ of equation* (3.3) *which is given by*

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Proof. By Theorem 2.5, the operator A generates a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$ and by [29, Corollary 3.9] the function $t \mapsto S_{\alpha,\beta}(t)$ is uniformly 1-integrable. By Theorem 2.7 the function $u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds$ belongs to $\mathcal{N}(X)$ and it is the mild solution to (3.3). □

Next, we consider the semilinear equation (1.1).

Definition 3.10. *A function $u \in C(\mathbb{R}, X)$ is called a* mild solution *to equation* (1.1) *if the function $s \mapsto S_{\alpha,\beta}(t-s)f(s,u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s,u(s))ds, \quad t \in \mathbb{R}. \tag{3.8}$$

Theorem 3.11. *Let $1 < \alpha < 2$, $1 \le \beta < \alpha$, $\omega < 0$ and A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$. If $f \in \mathcal{N}(\mathbb{R} \times X, X)$ satisfies*

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X, \tag{3.9}$$

where $L < \frac{\alpha}{C}|\omega|^{\beta/\alpha}B\left(\frac{\beta}{\alpha}, 1-\frac{\beta}{\alpha}\right)^{-1}$, *and* C *is the constant given in Theorem 2.6, and* $B(\cdot,\cdot)$ *denotes the Beta function, then the equation* (1.1) *has a unique mild solution* $u \in \mathcal{N}(X)$.

Proof. Define the operator $F : \mathcal{N}(X) \to \mathcal{N}(X)$ by

$$(F\phi)(t) := \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s,\phi(s))\,ds, \quad t \in \mathbb{R}. \tag{3.10}$$

By [29, Corollary 3.9] we have

$$\int_0^\infty \|S_{\alpha,\beta}(t)\|dt \le \frac{C}{\alpha}|\omega|^{-\beta/\alpha}B\left(\frac{\beta}{\alpha}, 1-\frac{\beta}{\alpha}\right) < \infty, \tag{3.11}$$

and [24, Theorems 3.3 and 4.1], F is well defined, that is, $F\phi \in \mathcal{N}(X)$ for all $\phi \in \mathcal{N}(X)$. For $\phi_1, \phi_2 \in \mathcal{N}(X)$ and $t \in \mathbb{R}$, by (3.11), we have:

$$\begin{aligned}
\|(F\phi_1)(t) - (F\phi_2)(t)\| &\le \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]\|ds \\
&\le \int_{-\infty}^{t} L\|S_{\alpha,\beta}(t-s)\| \cdot \|\phi_1(s) - \phi_2(s)\|ds \\
&\le L\|\phi_1 - \phi_2\|_\infty \int_0^\infty \|S_{\alpha,\beta}(r)\|dr \\
&\le \frac{LC}{\alpha}|\omega|^{-\beta/\alpha}B\left(\frac{\beta}{\alpha}, 1-\frac{\beta}{\alpha}\right)\|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in \mathcal{N}(X)$ such that $Fu = u$. □

Theorem 3.12. *Let* $1 < \alpha < 2$, $1 \le \beta < \alpha$, $\omega < 0$ *and* A *is an* ω*-sectorial operator of angle* $\theta = \frac{(\alpha-1)}{2}\pi$. *If* $f \in \mathcal{N}(\mathbb{R} \times X, X)$ *satisfies*

$$\|f(t,u) - f(t,v)\| \le \mathfrak{L}(t)\|u - v\|, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X,$$

where $\mathfrak{L}(\cdot) \in L^1(\mathbb{R}, \mathbb{R}_+)$, *then the equation* (1.1) *admits a unique mild solution* $u \in \mathcal{N}(X)$.

Proof. It easily follows by Theorem 2.6 that $\|S_{\alpha,\beta}(t)\| \le \widetilde{C} := \max\left\{C, \frac{C}{|\omega|}\right\}$. Define the operator F as (3.10). For $u, v \in \mathcal{N}(X)$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
||(Fu)(t) - (Fv)(t)|| &\le \int_{-\infty}^{t} ||S_{\alpha,\beta}(t-s)[f(s,u(s)) - f(s,v(s))]||ds \\
&\le \widetilde{C}\|u - v\|_\infty \int_0^\infty \mathfrak{L}(t-\xi)d\xi \\
&= \widetilde{C}\|u - v\|_\infty \int_{-\infty}^{t} \mathfrak{L}(s)ds.
\end{aligned}$$

Generally, we have

$$\begin{aligned}
\|(F^n u)(t) - (F^n v)(t)\| &\le \|u - v\|_\infty \frac{(\widetilde{C})^n}{(n-1)!}\left(\int_{-\infty}^{t} \mathfrak{L}(s)\left(\int_{-\infty}^{s} \mathfrak{L}(\xi)d\xi\right)^{n-1} ds\right) \\
&\le \|u - v\|_\infty \frac{(\widetilde{C})^n}{n!}\left(\int_{-\infty}^{t} \mathfrak{L}(s)ds\right)^n \\
&\le \|u - v\|_\infty \frac{(\|\mathfrak{L}\|_1\widetilde{C})^n}{n!}.
\end{aligned}$$

Since $\dfrac{(\|\mathfrak{L}\|_1\widetilde{C})^n}{n!} < 1$ for sufficiently large n, by the contraction principle F admits a unique fixed point $u \in \mathcal{N}(X)$. $\square$

4. Mild solutions to Equation (1.2) with nonlocal conditioins

Assume that A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$. By Theorem 2.5 the operator A generates a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. If $h : C(I,X) \to C(I,X)$ is a continuous function and $u_1 \in X$, then it is well known that the mild solution to problem

$$\left\{\begin{array}{lcl} \partial_t^\alpha u(t) & = & Au(t) + \partial_t^{\alpha-\beta} f(t,u(t)), \quad 0 \leq t \leq T \\ u(0) & = & 0, \\ u'(0) + h(u) & = & u_1, \end{array}\right. \tag{4.12}$$

is given by means of the variation-of-constant formula

$$u(t) = S_{\alpha,2}(t)[u_1 - h(u)] + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \quad t \in [0,T].$$

We assume the following

- **H1.** The function f satisfies the Carathéodory condition, that is $f(\cdot,u)$ is strongly measurable for each $u \in X$ and $f(t,\cdot)$ is continuous for each $t \in I := [0,T]$.
- **H2.** There exists a continuous function $\mu : I \to \mathbb{R}_+$ such that
$$\|f(t,u)\| \leq \mu(t)\|u\|, \quad \forall\, t \in I,\ u \in C(I,X).$$
- **H3.** The function $h : C(I,X) \to C(I,X)$ is continuous and there exists $L_h > 0$ such that
$$\|h(u) - h(v)\| < L_h\|u - v\|,\ \forall u,v \in C(I,X).$$
- **H4.** The set $\mathcal{K} = \{S_{\alpha,\beta}(t-s)f(s,u(s)) : u \in C(I,X), 0 \leq s \leq t\}$ is relatively compact for each $t \in I$.

Proposition 4.13. *Let $1 < \alpha < 2$ and $1 < \beta \leq 2$ such that $\alpha - \beta + 1 > 0$. If A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$, where $\omega < 0$, then the function $t \mapsto S_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(X)$ for all $t > 0$.*

Proof. It proof follows similarly to [31, Proposition 11]. We omit the details. $\square$

We recall the following results.

Lemma 4.14 (Mazur's Theorem). *If K is a compact subset of a Banach space X, then its convex closure $\overline{\text{conv}(K)}$ is compact.*

Lemma 4.15 (Leray-Schauder Alternative Theorem). *Let C be a convex subset of a Banach space X. Suppose that $0 \in C$. If $F : C \to C$ is a completely continuous map, then either F has a fixed point, or the set $\{x \in C : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded.*

Lemma 4.16 (Krasnoselskii Theorem). *Let C be a closed convex and nonempty subset of a Banach space X. Let Q_1 and Q_2 be two operators such that*

i) *If $u,v \in C$, then $Q_1u + Q_2v \in C$.*
ii) *Q_1 is a mapping contraction.*
iii) *Q_2 is compact and continuous.*

Then, there exists $z \in C$ such that $z = Q_1z + Q_2z$.

We have the following existence theorem.

Theorem 4.17. *Let $1 < \alpha < 2$ and $1 < \beta < 2$ such that $\alpha - \beta + 1 > 0$. Assume that A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$, where $\omega < 0$. Under assumptions H1-H4, the problem (4.12) has at least one mild solution.*

Proof. By Theorem 2.5, the operator A generates a resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. By the uniqueness of the Laplace transform we have $S_{\alpha,2}(t) = (g_1 * S_{\alpha,1})(t)$ and $S_{\alpha,\beta}(t) = (g_{\beta-1} * S_{\alpha,1})(t)$ for all $t \geq 0$. Moreover, by Theorem 2.6 there exists a constant $M > 0$ such that $\|S_{\alpha,2}(t)\| \leq M$ and $\|S_{\alpha,\beta}(t)\| \leq M$ for all $t \geq 0$. Now, we define the operator $\Gamma : C(I,X) \to C(I,X)$ by

$$(\Gamma u)(t) := S_{\alpha,2}(t)[u_1 - h(u)] + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \quad t \in [0,T].$$

Let $B_r := \{u \in C(I,X) : \|u\| \leq r\}$, where $r > 0$. We shall prove that Γ has at least one fixed point by the Leray-Schauder fixed point theorem. We will consider several steps in the proof.

Step 1. The operator Γ sends bounded sets of $C(I,X)$ into bounded sets of $C(I,X)$. In fact, take $u \in B_r$ and $G := \sup_{u\in B_r} \|h(u)\|$. Then

$$\begin{aligned}
\|\Gamma u(t)\| &\leq \|S_{\alpha,2}(t)\|(\|u_1\| + \|h(u)\|) + \int_0^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u(s))\|ds \\
&\leq M(\|u_1\| + G) + M\int_0^t \mu(s)\|u(s)\|ds \\
&\leq M(\|u_1\| + G) + Mr\int_0^t \mu(s)ds \\
&\leq M(\|u_1\| + G) + Mr\|\mu\|_\infty T := R.
\end{aligned}$$

Therefore $\Gamma B_r \subset B_R$.

Step 2. Γ is a continuous operator.

Let $u_n, u \in B_r$ such that $u_n \to u$ in $C(I,X)$. Then we have

$$\begin{aligned}
\|\Gamma u_n(t) - \Gamma u(t)\| &\leq \|S_{\alpha,2}(t)\|(\|h(u_n) - h(u)\|) + \int_0^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u_n(s)) - f(s,u(s))\|ds \\
&\leq ML_h\|u_n - u\| + M\int_0^t \|f(s,u_n(s)) - f(s,u(s))\|ds \\
&\leq ML_h\|u_n - u\| + M\int_0^t \mu(s)(\|u_n(s)\| + \|u(s)\|)ds \\
&\leq ML_h\|u_n - u\| + 2rM\int_0^t \mu(s)ds.
\end{aligned}$$

We notice that the function $s \mapsto \mu(s)$ is integrable on I. By the Lebesgue's Dominated Convergence Theorem, $\int_0^t \|f(s,u_n(s)) - f(s,u(s))\|ds \to 0$ as $n \to \infty$. Since $u_n \to u$ we obtain that Γ is continuous in $C(I,X)$.

Step 3 The operator Γ sends bounded sets of $C(I,X)$ into equicontinuous sets of $C(I,X)$.

In fact, let $u \in B_r$, with $r > 0$ and take $t_1, t_2 \in I$ with $t_2 < t_1$. Then we have

$$\begin{aligned}
\|\Gamma u(t_1) - \Gamma u(t_2)\| &\leq \|(S_{\alpha,2}(t_1) - S_{\alpha,2}(t_2))(u_1 - h(u))\| + \int_{t_2}^{t_1} \|S_{\alpha,\beta}(t_1-s)f(s,u(s))\|ds \\
&+ \int_0^{t_2} \|(S_{\alpha,\beta}(t_1-s) - S_{\alpha,\beta}(t_2-s))f(s,u(s))\|ds \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Observe that

$$I_1 \leq \|(S_{\alpha,2}(t_1) - S_{\alpha,2}(t_2))\|\|(u_1 - h(u))\|.$$

Using the norm continuity of $t \mapsto S_{\alpha,2}(t)$ (see Proposition 4.13) we obtain that $\lim_{t_1 \to t_2} I_1 = 0$.

On the other hand,

$$I_2 \leq M \int_{t_2}^{t_1} \mu(s)\|u(s)\|ds \leq rM\|\mu\|_\infty (t_1 - t_2),$$

and therefore $\lim_{t_1 \to t_2} I_2 = 0$. Finally, for I_3 we have

$$\begin{aligned} I_3 &\leq \int_0^{t_2} \|S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\|\|f(s, u(s))\|ds \\ &\leq \int_0^{t_2} \|S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\|\mu(s)\|u(s)\|ds \\ &\leq r\int_0^{t_2} \|S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\|\mu(s)ds. \end{aligned}$$

Since

$$\|S_{\alpha,\beta}(t_1 - \cdot) - S_{\alpha,\beta}(t_2 - \cdot)\|\mu(\cdot) \leq 2M\mu(\cdot) \in L^1(I, \mathbb{R}),$$

and $S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s) \to 0$ in $\mathcal{B}(X)$, as $t_1 \to t_2$ (see Proposition 4.13) we obtain by the Lebesgue's dominated convergence theorem that $\lim_{t_1 \to t_2} I_3 = 0$. The proof of the claim is finished.

Step 4. The function Γ maps B_r into relatively compact sets in X.

The hypothesis and Lemma 4.14 imply that $\overline{\mathrm{conv}(\mathcal{K})}$ is compact. Moreover, for $u \in B_r$, by the Mean-Value Theorem for the Bochner integral (see [15, Corollary 8, p. 48]), we get

$$\Gamma(u(t)) \in t\,\overline{\mathrm{conv}(\mathcal{K})},$$

for all $t \in [0, T]$. Thus the set $\overline{\{\Gamma u(t); u \in B_r\}}$ is compact in X for every $t \in [0, T]$.

We conclude from Steps 1,2, 3 and 4, that Γ is continuous and compact by the Arzela-Ascoli's theorem, which means that the function Γ is completely continuous.

Step 5. The set $\Omega := \{u \in B_r : u = \lambda\Gamma u,\ 0 < \lambda < 1\}$ is bounded. In fact, since $0 \in \Omega$ we obtain that $\Omega \neq \emptyset$. For $u \in \Omega$ we have

$$\begin{aligned} \|u(t)\| &\leq \lambda[M(\|u_1\| + \|h(u)\|) + M\int_0^t \|f(s, u(s)\|ds] \\ &\leq \lambda[M(\|u_1\| + G) + M\int_0^t \mu(s)\|u(s)\|ds] \\ &\leq [M(\|u_1\| + G) + Mr\|\mu\|_\infty T], \end{aligned}$$

for all $t \in [0, T]$, which means that Ω is a bounded set.

Therefore, by Lemma 4.15 we conclude that Γ has a fixed point, and the proof of the Theorem is finished. $\square$

Now, we consider the problem

$$\left\{ \begin{array}{lcl} \partial_t^\alpha u(t) &=& Au(t) + \partial_t^{\alpha-\beta} f(t, u(t)), \quad 0 \leq t \leq T \\ u(0) + g(u) &=& u_0, \\ u'(0) + h(u) &=& u_1, \end{array} \right. \tag{4.13}$$

where $g, h : C(I, X) \to C(I, X)$ are continuous and $u_0, u_1 \in X$. By (2.2) in Theorem 2.6, there exists a constant $M > 0$ such that

$$\|S_{\alpha,1}(t)\| \leq \frac{M}{1 + |\omega|t^\alpha}, \quad \|S_{\alpha,2}(t)\| \leq \frac{Mt}{1 + |\omega|t^\alpha}, \quad \|S_{\alpha,\beta}(t)\| \leq \frac{Mt^{\beta-1}}{1 + |\omega|t^\alpha}, \quad t \geq 0. \tag{4.14}$$

Thus

$$\|S_{\alpha,1}(t)\| \leq M, \quad \|S_{\alpha,2}(t)\| \leq MT, \quad \|S_{\alpha,\beta}(t)\| \leq MT^{\beta-1}, \quad t \in [0, T]. \tag{4.15}$$

Under the same assumptions H1-H3 and

- **H3'.** The function $g : C(I,X) \to C(I,X)$ is continuous and there exists $L_g > 0$ such that
$$\|g(u) - g(v)\| < L_g \|u - v\|, \ \forall u, v \in C(I,X).$$

we have the following result.

Theorem 4.18. *Let $1 < \alpha < 2$ and $1 < \beta < 2$ such that $\alpha - \beta + 1 > 0$. Assume that A is an ω-sectorial operator of angle $\theta = \frac{(\alpha-1)}{2}\pi$, where $\omega < 0$. Suppose that $M\|\mu\|_\infty T^\beta < 1$ and $M(L_g + TL_h) < 1$, where M is the constant in* (4.15)*. Assume that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \nu^{1/\alpha}$, where ν is a positive constant. Under assumptions H1-H3 and H3', the problem* (4.13) *has at least one mild solution.*

Proof. By Theorem 2.5, the operator A generates the resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, and $S_{\alpha,2}(t) = (g_1 * S_{\alpha,1})(t)$ and $S_{\alpha,\beta}(t) = (g_{\beta-1} * S_{\alpha,1})(t)$ for all $t \geq 0$. Then, the mild solution to problem (4.13) is given by

$$u(t) = S_{\alpha,1}(t)[u_0 - g(u)] + S_{\alpha,2}(t)[u_1 - h(u)] + \int_0^t S_{\alpha,\beta}(t-s) f(s, u(s)) ds, \quad t \in [0,T].$$

Let $B_r := \{u \in C(I,X) : \|u\| \leq r\}$, where

$$r := \frac{M(\|u_0\| + \|g(u)\|) + MT(\|u_1\| + \|h(u)\|)}{1 - M\|\mu\|_\infty T^\beta}.$$

On B_r we define the operators Γ_1, Γ_2 by

$$\begin{aligned} (\Gamma_1 u)(t) : &= S_{\alpha,1}(t)[u_0 - g(u)] + S_{\alpha,2}(t)(u_1 - h(u)) \quad t \in [0,T] \\ (\Gamma_2 u)(t) : &= \int_0^t S_{\alpha,\beta}(t-s) f(s, u(s)) ds, \quad t \in [0,T], \end{aligned}$$

where $u \in B_r$. We claim that $\Gamma := \Gamma_1 + \Gamma_2$ has at least one fixed. To prove this, we will consider several steps.

Step 1. We claim that if $u, v \in B_r$, then $\Gamma_1 u + \Gamma_2 v \in B_r$. In fact,

$$\begin{aligned} &\|(\Gamma_1 u)(t) + (\Gamma_2 v)(t)\| \leq \\ &\leq \|S_{\alpha,1}(t)\| \, \|u_0 - g(u)\| + \|S_{\alpha,2}(t)\| \, \|u_1 - h(u)\| + \int_0^t \|S_{\alpha,\beta}(t-s)\| \, \|f(s, v(s))\| ds \\ &\leq M(\|u_0\| + \|g(u)\|) + MT(\|u_1\| + \|h(u)\|) + M\int_0^t (t-s)^{\beta-1} \mu(s) \|v(s)\| ds \\ &\leq M(\|u_0\| + \|g(u)\|) + MT(\|u_1\| + \|h(u)\|) + MT^\beta \|\mu\|_\infty r = r. \end{aligned}$$

Thus $\Gamma_1 u + \Gamma_2 v \in B_r$ for all $u, v \in B_r$.

Step 2. Γ_1 is a contraction on B_r. In fact, if $u, v \in B_r$, then

$$\|\Gamma_1 u(t) - \Gamma_1 v(t)\| \leq \|S_{\alpha,1}(t)\| \, \|g(u) - g(v)\| + \|S_{\alpha,2}(t)\| \, \|h(u) - h(v)\| \leq (ML_g + MTL_h)\|u - v\|$$

Since $M(L_g + TL_h) < 1$, we get that Γ_1 is a contraction.

Step 3. Γ_2 is completely continuous.

Firstly, we prove that Γ_2 is a continuous operator on B_r. Let $u_n, u \in B_r$ such that $u_n \to u$ in B_r. We notice that by (4.14)

$$\|\Gamma_2 u_n(t) - \Gamma_2 u(t)\| \leq \int_0^t \|S_{\alpha,\beta}(t-s)\| \|f(s, u_n(s)) - f(s, u(s))\| ds \leq 2MrT^\beta \int_0^t \mu(s) ds.$$

Moreover, the function $s \mapsto \mu(s)$ is integrable on $[0,T]$. The Lebesgue's Dominated Convergence Theorem implies that $\int_0^t \|f(s, u_n(s)) - f(s, u(s))\| ds \to 0$ as $n \to \infty$. Since $u_n \to u$ we obtain that Γ_2 is continuous in B_r.

Now, we prove that $\{\Gamma_2 u : u \in B_r\}$ is a relatively compact set. In fact, by the Ascoli-Arzela theorem we only need to prove that the family $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded and equicontinuous, and the set $\{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$. For each $u \in B_r$ we have $\|\Gamma_2 u\| \leq MT^{\beta} r \|\mu\|_{\infty}$, which implies that $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded.

Next, we prove the equicontinuity. For $u \in B_r$ and $0 \leq t_2 < t_1 \leq T$ we have

$$\begin{aligned} \|\Gamma_2 u(t_1) - \Gamma_2 u(t_2)\| &\leq \int_{t_2}^{t_1} \|S_{\alpha,\beta}(t_1 - s) f(s,u(s))\| ds \\ &+ \int_0^{t_2} \| \left(S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\right) f(s,u(s))\| ds =: I_1 + I_2. \end{aligned}$$

Observe that for I_1, by (4.14) we have $I_1 \leq MT^{\beta} \int_{t_2}^{t_1} \mu(s)\|u(s)\| ds \leq MT^{\beta} r \|\mu\|_{\infty}(t_1 - t_2)$, and thus $\lim_{t_1 \to t_2} I_1 = 0$. On the other hand, for I_2 we have

$$I_2 \leq \int_0^{t_2} \|S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\| \|f(s,u(s))\| ds \leq r \int_0^{t_2} \mu(s) \|S_{\alpha,\beta}(t_1 - s) - S_{\alpha,\beta}(t_2 - s)\| ds.$$

By (4.15) we have $\mu(\cdot)\|S_{\alpha,\beta}(t_1 - \cdot) - S_{\alpha,\beta}(t_2 - \cdot)\| \leq 2T^{\beta-1} M \mu(\cdot) \in L^1([0,T],\mathbb{R})$, and by Proposition 4.13 the function $t \mapsto S_{\alpha,\beta}(t)$ is norm continuous. This implies that if $t_1 \to t_2$, then $S_{\alpha,\beta})(t_1 - s) - S_{\alpha,\beta})(t_2 - s) \to 0$ in $\mathcal{B}(X)$. By the Lebesgue's dominated convergence theorem we conclude that $\lim_{t_1 \to t_2} I_2 = 0$. Therefore, $\{\Gamma_2 u : u \in B_r\}$ is an equicontinuous family.

Finally, we prove that $H(t) := \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$. Clearly, $H(0)$ is relatively compact in X. Now, we take $t > 0$. For $0 < \varepsilon < t$ we define on B_r the operator

$$(\Gamma_2^{\varepsilon} u)(t) : \;\; = \int_0^{t-\varepsilon} S_{\alpha,\beta}(t-s) f(s,u(s)) ds.$$

By [31, Theorem 14] we have that $S_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$. Thus $\mathcal{K}_{\varepsilon} := \{S_{\alpha,\beta}(t-s) f(s,u(s)) : u \in B_r, 0 \leq s \leq t - \varepsilon\}$ is a compact set for all $\varepsilon > 0$. By Lemma 4.14, $\overline{\text{conv}(\mathcal{K}_{\varepsilon})}$ is also a compact set. The Mean-Value Theorem for the Bochner integrals (see [15, Corollary 8, p. 48]), implies that $(\Gamma_2^{\varepsilon} u)(t) \in t\overline{\text{conv}(\mathcal{K}_{\varepsilon})}$, for all $t \in [0,T]$. Therefore, the set $H_{\varepsilon}(t) := \{(\Gamma_2^{\varepsilon} u)(t) : u \in B_r\}$ is relatively compact in X for all $\varepsilon > 0$. Since

$$\|(\Gamma_2 u)(t) - (\Gamma_2^{\varepsilon} u)(t)\| \leq \int_{t-\varepsilon}^{t} \|S_{\alpha,\beta}(t-s) f(s,u(s))\| ds \leq MT^{\beta-1} r \int_{t-\varepsilon}^{t} \mu(s) ds$$

and the function $s \mapsto \mu(s)$ belongs to $L^1([t-\varepsilon,t],\mathbb{R}_+)$ we conclude by the Lebesgue dominated convergence Theorem that $\lim_{\varepsilon \to 0} \|(\Gamma_2 u)(t) - (\Gamma_2^{\varepsilon} u)(t)\| = 0$. Therefore the set $\{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in (0,T]$. The Ascoli-Arzela theorem implies that the set $\{\Gamma_2 u : u \in B_r\}$ is relatively compact. We conclude that Γ_2 is a completely continuous operator. By Lemma 4.16 we have that $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r, and therefore the problem (4.13) has a mild solution. □

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