The effect of stochastically environmental variability on transmission dynamics of echinococcosis

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Abstract

Echinococcosis, one of the most serious zoonotic diseases, has a severe impact on the human health and economic development. This paper mainly investigates the effect of s-tochastically environmental variability on transmission dynamics of echinococcosis. Firstly, sufficient conditions of the extinction in the mean for the disease are obtained. In addition, by constructing a suitable stochastic Lyapunov function, the existence of the unique ergodic stationary distribution is established. Lastly, numerical simulations have been performed to not only verify our analytical results but also display that noise intensities would affect the dynamical behaviors of this model, (i) these noise intensities for three subgroups all have significantly negative impact on the extinction time for $I_H(t)$, in particular, when the noise intensity for the livestock σ_L increases, the extinction time for $I_H(t)$ decreases; (ii) these noise intensities for three subgroups have the influence on the skewness and kurtosis of the stationary distribution for $I_H(t)$, where the effect of the noise intensity for humans σ_H on the alteration of the distribution shape for $I_H(t)$ is obvious, from skyscraping to pyknic and gradually migrating towards left as σ_H increasing.

Keywords: stochastic echinococcosis model; stochastic Lyapunov function; extinction; stationary distribution; noise intensity.

1. Introduction

Echinococcosis is also called the hydatid disease, which is a parasitic disease caused by the larvas of *Echinococcus* infesting in human body. It has become a global problem in the field of public health and animal husbandry with its wide epidemic range. Echinococcosis (especially cystic echinococcosis) almost spreads all over every continent. The spreading range in China is about 4200,000 square kilometers, which makes up 44% of the total area

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of China. The main endemic areas with high risk of infection of echinococcosis to both animals and humans center on northern part of pastoral regions or farming-pastoral regions, including Xinjiang, Ningxia, Tibet, Qinghai and Inner Mongolia *et al*. [1–3].

Two species of *Echinococcus* that cause human hydatid disease are *Echinococcus* granulosus and *Echinococcus* multilocularis, where the *Echinococcus* granulosus is the most common. The mature *Echinococcus* granulosus lay eggs in the small intestine of the definitive host (e.g. dogs, wolves, and other canines *et al*). Their eggs are passed out with stool and released to the environment. The intermediate host (e.g. sheep, goat, hog, cattle, horses *et al*) accidentally contact and devour these eggs. After digestion, the larva becomes oncosphere as soon as it emerges from behind its shell. Then the oncosphere develops into metacestode cysts in the host's organs, producing protoscolices which will develop into adult worms when they are eaten by a definitive host [4]. A life cycle of *Echinococcus* granulosus is accomplished. It is noted that humans are almost always dead-end intermediate hosts. The life cycle of *Echinococcus* multilocularis is similar with granulosus.

There has been various mathematical models to study echinococcosis transmission (see [5–16]) since these seminal work of Gemmell *et al*. [5–8]. For example, Torgerson [12] illustrated some of the potential uses of such modelling techniques to gain insights into the epidemiology of parasite transmission. In [13], a deterministic model which described the dog-livestock-egg-human life-cycle of *Echinococcus* granulosus was proposed to study the transmission dynamics of echinococcosis in Xinjiang, and it was shown that the dynamics of the model is completely determined by the basic reproductive number. In [14], the baseline model and intervention model were applied to study the transmission dynamics of echinococcosis in Qinghai. Wang *et al*. [15] proposed a novel spreading model for echinococcosis with distributed time delays and obtained a threshold type result that the disease dies out when the basic reproductive number $R_0 < 1$ and the disease persists if $R_0 > 1$. Rong *et al*. in [16] proposed a deterministic model distinguishing stray dogs from domestic dogs to explore the special role of stray dogs and potential effects of disposing stray dogs for the eradication of echinococcosis infection in Inner Mongolia.

Thanks to the insightful work in [15], we propose the following deterministic model describing the dog-livestock-human life-cycle of *Echinococcus* granulosus

$$\left\{\begin{array}{lcl} \dot{S}_D &=& A_1 - \beta_1 S_D I_L - d_1 S_D + \delta I_D, \\ \dot{I}_D &=& \beta_1 S_D I_L - (d_1+\delta) I_D, \\ \dot{S}_L &=& A_2 - \beta_2 S_L I_D - d_2 S_L, \\ \dot{I}_L &=& \beta_2 S_L I_D - d_2 I_L, \\ \dot{S}_H &=& A_3 - \beta_3 S_H I_D - d_3 S_H + \gamma I_H, \\ \dot{E}_H &=& \beta_3 S_H I_D - (d_3+\omega) E_H, \\ \dot{I}_H &=& \omega E_H - (d_3+\mu+\gamma) I_H. \end{array}\right. \tag{1.1}$$

For dogs, the definitive hosts, includes susceptible dogs (S_D) and infectious dogs (I_D). A_1 is the recruitment rate of dogs. $1/d_1$ is the average lifespan of dogs. δ denotes the recovery rate of transition from infected to noninfected dogs, including natural recovery rate and recovery due to anthelmintic treatment. $\beta_1 S_D I_L$ describes the transmission of echinococcosis between susceptible dogs and infectious livestock after the ingestion of cyst-containing organs of the infected livestock. For the livestock, the intermediate hosts, includes susceptible individuals (S_L) and infectious individuals (I_L). A_2 is the recruitment rate of the livestock. $1/d_2$ is the average lifespan of livestock. $\beta_2 S_L I_D$ describes the transmission of echinococcosis to livestock by the ingestion of *Echinococcus* eggs in the environment which are emitted by dogs. For humans, the accidental intermediate hosts, includes susceptible individuals (S_H), exposed individuals (E_H) and infectious individuals (I_H). A_3 is the recruitment rate of humans. $1/d_3$ is the average lifespan of humans. μ is the disease-related death rate for humans. $\beta_3 S_H I_D$ describes the transmission of echinococcosis to humans by the ingestion of *Echinococcus* eggs in the environment which are emitted by dogs. The incubation period of infected individuals is $1/\omega$, and γ denotes the recovery rate. All parameters are positive. It's not hard to verify model (1.1) has the following property:

- Model (1.1) always has a unique disease-free equilibrium $P_0 = (\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, \frac{A_3}{d_3}, 0, 0)$ and a unique endemic equilibrium if and only if the basic reproduction number $R_0 \triangleq \sqrt{\frac{A_1 A_2 \beta_1 \beta_2}{d_1(d_1+\delta)d_2^2}} > 1.$

However, in fact, the spread mechanism of infectious diseases is probabilistic in nature. In real world, on the development and propagation of an epidemic, environmental random factors have a significant affect in different various degree. The variability and randomness of environment are fed through the state of the epidemic [17–19]. The intrinsic fluctuation or noise occurs in the time evolution of the number in each compartment for the transmission of echinococcosis. For example, the livestock, as the intermediate hosts in the process of echinococcosis transmission, has access to the *Echinococcus* eggs in the environment which are emitted by dogs, therefore, the livestock is more affected by stochastic fluctuations of the environment, (the temperature, humidity etc). On the other hand, deterministic models may ignore the stochastic nature of the rare events that lead to initial resistance generation and spread while the stochastic approach could capture the intrinsic fluctuations when the size of population is small. Hence, in the epidemic dynamics, stochastically differential equation models could be more appropriate way of modeling epidemics under various circumstances [20–22].

In this paper, by incorporating the effect of environmental random fluctuations into deterministic model (1.1), it is assumed that during the transmission of echinococcosis, the mobility for subpopulations S_D, I_D, S_L, I_L, S_H, E_H and I_H will fluctuate around some average values due to continuous fluctuations in the environment and they are different due to infection risks. In this case, d_i change to three random variable $\tilde{d}_i$ ($i = 1, 2, 3$). More precisely, in a small time interval $[t, t + dt)$,

$$-\tilde{d}_i dt = -d_i dt + \sigma(X_i) dB_i(t) \text{ for i=1,\ldots,7},$$

here $(X_1, \ldots, X_7) = (S_D, I_D, S_L, I_L, S_H, E_H, I_H)$ and $dB_i(t) = B_i(t + dt) - B_i(t)$ are the increments of standard Brownian motion $B_i(t)$ for $i = 1, \ldots, 7$. For simplicity, we adopt $\sigma_i := \sigma_i(X_i)$ which are all real constants and known as the intensities of environmental fluctuation for subpopulations S_D, I_D, S_L, I_L, S_H, E_H and I_H, respectively. Thus, in $[t, t+dt)$, $-\tilde{d}_i dt$ is normally distributed with mean $-d_i dt$ and variance $\sigma_i^2 dt$. Hence, $E(-\tilde{d}_i dt) = -d_i dt$ and $Var(-\tilde{d}_i dt) = \sigma_i^2 dt$. Owing to $Var(-\tilde{d}_i dt) \to 0$ as $dt \to 0$, this is a biologically reasonable model. Therefore, we replace $-d_i dt$ in model (1.1) by $-\tilde{d}_i dt = -d_i dt + \sigma_i dB_i(t)$ to get the following stochastic version corresponding to model (1.1)

$$\begin{cases} dS_D &= [A_1 - \beta_1 S_D I_L - d_1 S_D + \delta I_D]dt + \sigma_1 S_D dB_1(t), \\ dI_D &= [\beta_1 S_D I_L - (d_1 + \delta) I_D]dt + \sigma_2 I_D dB_2(t), \\ dS_L &= [A_2 - \beta_2 S_L I_D - d_2 S_L]dt + \sigma_3 S_L dB_3(t), \\ dI_L &= [\beta_2 S_L I_D - d_2 I_L]dt + \sigma_4 I_L dB_4(t), \\ dS_H &= [A_3 - \beta_3 S_H I_D - d_3 S_H + \gamma I_H]dt + \sigma_5 S_H dB_5(t), \\ dE_H &= [\beta_3 S_H I_D - (d_3 + \omega) E_H]dt + \sigma_6 E_H dB_6(t), \\ dI_H &= [\omega E_H - (d_3 + \mu + \gamma) I_H]dt + \sigma_7 I_H dB_7(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, \cdots, 7)$ are mutually independent standard Brownian motions with $B_i(0) = 0$, and σ_i^2 denotes the intensities of $B_i(t)$, respectively. Other parameters are the same with model (1.1).

The main concern of this paper is how environmental fluctuations affect the transmission dynamics of echinococcosis, and more specifically, the noise intensity of which population (human, dog and livestock) has the greatest impact on the stochastic disease-free and endemic dynamics? The rest of this paper is organized as follows. In the next section, some useful lemmas are presented. In Section 3, we obtain sufficient conditions of the extinction in the mean for the disease. Particularly, by the comparison theorem of stochastically differential equations and the law of large numbers, the more precise conditions for the extinction of the disease are given for a special case. In Section 4, the existence of the unique ergodic stationary distribution is derived by constructing a suitable Lyapunov function. In Section 5, we provide some numerical examples to show the effect of noise intensities on the dynamical behaviors of the model. A brief discussion is given in Section 6.

2. Preliminaries

Denote $\mathbb{R}_+^n = \{x = (x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$, and $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ for all $a, b \in \mathbb{R}_+ = [0, +\infty)$. Let $f(t)$ be an integrable function on $\mathbb{R}_+$, we denote $\langle f \rangle = \frac{1}{t}\int_0^t f(s)ds$. Through this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. And we assume that model (1.2) is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. From the biological background of model (1.2), we assume that the initial value for any solution of model (1.2) $X_0 := (S_D(0), I_D(0), S_L(0), I_L(0), S_H(0), E_H(0), I_H(0)) \in \mathbb{R}_+^7$.

In the following, we introduce some useful lemmas which will be used in the proof of the main results in this paper. Firstly, the following lemma reveals the existence of positive solution for model (1.2).

Lemma 2.1 *For any initial value X_0, the solution $(S_D(t), I_D(t), S_L(t), I_L(t), S_H(t), E_H(t), I_H(t))$ of model (1.2) uniquely exists and remains in $\mathbb{R}^7_+$ with probability one for any $t \geq 0$.*

In fact, this lemma can be proved by using the following Lyapunov function

$$\begin{aligned} V(S_D, I_D, S_L, I_L, S_H, E_H, I_H) = \ & (S_D - \frac{d_2}{\beta_1} - \frac{d_2}{\beta_1}\ln S_D) + (I_D - 1 - \ln I_D) + (S_L - \frac{d_1}{\beta_2+\beta_3} \\ & - \frac{d_1}{\beta_2+\beta_3}\ln S_L) + (I_L - 1 - \ln I_L) + (S_H - \frac{d_1}{\beta_2+\beta_3} \\ & - \frac{d_1}{\beta_2+\beta_3}\ln S_H) + (E_H - 1 - \ln E_H) + (I_H - 1 - \ln I_H). \end{aligned}$$

Hence we omit it here.

These following two lemmas will be helpful for us to investigate the extinction of the disease for model (1.2), which can be proved by the similar methods given in [23]. Here we omit it.

Lemma 2.2 *For any initial value X_0, the solution $(S_D(t), I_D(t), S_L(t), I_L(t), S_H(t), E_H(t), I_H(t))$ of model (1.2) has the following property:*

$$\lim_{t\to\infty}\frac{S_D(t)+I_D(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{S_L(t)+I_L(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{S_H(t)+E_H(t)+I_H(t)}{t} = 0, \quad a.s.$$

Lemma 2.3 *Let $(S_D(t), I_D(t), S_L(t), I_L(t), S_H(t), E_H(t), I_H(t))$ be the solution of model (1.2) with initial value X_0, if $d_1 > \frac{1}{2}(\sigma_1^2 \wedge \sigma_2^2)$, then*

$$\lim_{t\to\infty}\frac{\int_0^t S_D(r)dB_1(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t I_D(r)dB_2(r)}{t} = 0, \quad a.s. \tag{2.1}$$

if $d_2 > \frac{1}{2}(\sigma_3^2 \wedge \sigma_4^2)$, then

$$\lim_{t\to\infty}\frac{\int_0^t S_L(r)dB_3(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t I_L(r)dB_4(r)}{t} = 0, \quad a.s. \tag{2.2}$$

if $d_3 > \frac{1}{2}(\sigma_5^2 \wedge \sigma_6^2 \wedge \sigma_7^2)$, then

$$\lim_{t\to\infty}\frac{\int_0^t S_H(r)dB_5(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t E_H(r)dB_6(r)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t I_H(r)dB_7(r)}{t} = 0, \quad a.s. \tag{2.3}$$

3. Extinction and persistence of the disease

For convenience, we define the constant as follows

$$\widehat{\mathcal{R}}_0 = \frac{2(A_1\beta_1 d_2 + A_2\beta_2 d_1)}{d_1 d_2[(d_1+\delta+\frac{\sigma_2^2}{2}) \wedge (d_2+\frac{\sigma_4^2}{2})]}.$$

Firstly, we have the following result about the extinction and persistence of the disease.

Theorem 3.1 Assume that $\widehat{\mathcal{R}}_0 < 1$, and the conditions $d_1 > \frac{1}{2}(\sigma_1^2 \wedge \sigma_2^2)$, $d_2 > \frac{1}{2}(\sigma_3^2 \wedge \sigma_4^2)$ and $d_3 > \frac{1}{2}(\sigma_5^2 \wedge \sigma_6^2 \wedge \sigma_7^2)$ hold, then for any initial value $X_0 \in \mathbb{R}_+^7$, the solution $(S_D(t), I_D(t), S_L(t), I_L(t), S_H(t), E_H(t), I_H(t))$ of model (1.2) has the following property:

$$\lim_{t\to\infty} \langle S_D(t)\rangle = \frac{A_1}{d_1}, \quad \lim_{t\to\infty} \langle S_L(t)\rangle = \frac{A_2}{d_2}, \quad \lim_{t\to\infty} \langle S_H(t)\rangle = \frac{A_3}{d_3} \quad a.s.$$

and

$$\lim_{t\to\infty} I_D(t) = 0, \quad \lim_{t\to\infty} I_L(t) = 0, \quad \lim_{t\to\infty} E_H(t) = 0, \quad \lim_{t\to\infty} I_H(t) = 0 \quad a.s..$$

Proof. From model (1.2), one can separately obtain

$$d(S_D(t) + I_D(t)) = [A_1 - d_1(S_D(t) + I_D(t))]dt + \sigma_1 S_D(t)dB_1(t) + \sigma_2 I_D(t)dB_2(t), \tag{3.1}$$

$$d(S_L(t) + I_L(t)) = [A_2 - d_2(S_L(t) + I_L(t))]dt + \sigma_3 S_L(t)dB_3(t) + \sigma_4 I_L(t)dB_4(t) \tag{3.2}$$

and

$$\begin{aligned} d(S_H(t) + E_H(t) + I_H(t)) = \ & [A_3 - d_3(S_H(t) + E_H(t) + I_H(t))]dt \\ & + \sigma_5 S_H(t)dB_5(t) + \sigma_6 E_H(t)dB_6(t) + \sigma_7 I_H(t)dB_7(t). \end{aligned} \tag{3.3}$$

Integrating (3.1) from 0 to t and then dividing by t on both sides, we have the following formula by using lemma 2.2 and (2.1),

$$\limsup_{t\to\infty} \langle S_D(t) + I_D(t)\rangle \le \frac{A_1}{d_1}. \tag{3.4}$$

Similarly, by lemma 2.2 and the formulas (2.2), (2.3), (3.2) and (3.3), we separately obtain

$$\limsup_{t\to\infty} \langle S_L(t) + I_L(t)\rangle \le \frac{A_2}{d_2}, \tag{3.5}$$

and

$$\limsup_{t\to\infty} \langle S_H(t) + E_H(t) + I_H(t)\rangle \le \frac{A_3}{d_3}. \tag{3.6}$$

Furthermore, let $\mathcal{U}(t) = I_D(t) + I_L(t)$. Using Itô's formula to $\ln \mathcal{U}(t)$, we have

$$\begin{aligned}
d(\ln \mathcal{U}) = \quad & \{\frac{\beta_1 S_D I_L + \beta_2 S_L I_D}{I_D + I_L} - \frac{(d_1+\delta)I_D + d_2 I_L}{I_D + I_L} - \frac{\sigma_2^2 I_D^2 + \sigma_4^2 I_L^2}{2(I_D+I_L)^2}\}dt \\
& + \frac{\sigma_2 I_D}{I_D + I_L} dB_2(t) + \frac{\sigma_4 I_L}{I_D + I_L} dB_4(t) \\
\leq \quad & (\beta_1 S_D + \beta_2 S_L)dt - \{\frac{(d_1+\delta)I_D + d_2 I_L}{I_D + I_L} + \frac{\sigma_2^2 I_D^2 + \sigma_4^2 I_L^2}{2(I_D+I_L)^2}\}dt \\
& + \frac{\sigma_2 I_D}{I_D + I_L} dB_2(t) + \frac{\sigma_4 I_L}{I_D + I_L} dB_4(t) \\
\leq \quad & (\beta_1 S_D + \beta_2 S_L)dt - \frac{1}{(I_D+I_L)^2}\{(d_1 + \delta + \frac{\sigma_2^2}{2})I_D^2 + (d_2 + \frac{\sigma_4^2}{2})I_L^2\}dt \\
& + \frac{\sigma_2 I_D}{I_D + I_L} dB_2(t) + \frac{\sigma_4 I_L}{I_D + I_L} dB_4(t) \\
\leq \quad & (\beta_1 S_D + \beta_2 S_L)dt - \frac{1}{(I_D+I_L)^2}[(d_1 + \delta + \frac{\sigma_2^2}{2}) \wedge (d_2 + \frac{\sigma_4^2}{2})](I_D^2 + I_L^2)dt \\
& + \frac{\sigma_2 I_D}{I_D + I_L} dB_2(t) + \frac{\sigma_4 I_L}{I_D + I_L} dB_4(t) \\
\leq \quad & (\beta_1 S_D + \beta_2 S_L)dt - \frac{(d_1 + \delta + \frac{\sigma_2^2}{2}) \wedge (d_2 + \frac{\sigma_4^2}{2})}{2}dt + \frac{\sigma_2 I_D}{I_D + I_L} dB_2(t) \\
& + \frac{\sigma_4 I_L}{I_D + I_L} dB_4(t).
\end{aligned}$$

Integrating the above inequality from 0 to t and then dividing by t on both sides, we have

$$\begin{aligned}
\frac{\ln \mathcal{U}(t)}{t} - \frac{\ln \mathcal{U}(0)}{t} \leq \quad & \frac{\beta_1}{t}\int_0^t S_D(s)ds + \frac{\beta_2}{t}\int_0^t S_L(s)ds - \frac{(d_1 + \delta + \frac{\sigma_2^2}{2}) \wedge (d_2 + \frac{\sigma_4^2}{2})}{2} \\
& + \frac{\beta_1}{t}\int_0^t \frac{\sigma_2 I_D(s)}{I_D(s) + I_L(s)} dB_2(s)ds + \frac{\beta_1}{t}\int_0^t \frac{\sigma_4 I_L(s)}{I_D(s) + I_L(s)} dB_4(s)ds,
\end{aligned}$$

hence, by (3.4) and (3.5), we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\ln \mathcal{U}(t)}{t} \quad & \leq (\frac{A_1\beta_1}{d_1} + \frac{A_2\beta_2}{d_2}) - \frac{(d_1 + \delta + \frac{\sigma_2^2}{2}) \wedge (d_2 + \frac{\sigma_4^2}{2})}{2} \\
& = \frac{(d_1 + \delta + \frac{\sigma_2^2}{2}) \wedge (d_2 + \frac{\sigma_4^2}{2})}{2}(\widehat{\mathcal{R}}_0 - 1) \quad a.s..
\end{aligned}$$

When $\widehat{\mathcal{R}}_0 < 1$, it can be obtained that $\lim_{t\to\infty} \mathcal{U}(t) = \lim_{t\to\infty}(I_D(t) + I_L(t)) = 0 \quad a.s.$, which is shown that

$$\lim_{t\to\infty} I_D(t) = 0 \ \text{ and } \ \lim_{t\to\infty} I_L(t) = 0 \ \ a.s..$$

Therefore, it is easily obtained by model (1.2) that $\lim_{t\to\infty} E_H(t) = 0$ a.s., which leads to $\lim_{t\to\infty} I_H(t) = 0$. Then, we have by (3.4),(3.5) and (3.6)

$$\lim_{t\to\infty}\langle S_D(t)\rangle = \frac{A_1}{d_1}, \ \ \lim_{t\to\infty}\langle S_L(t)\rangle = \frac{A_2}{d_2}, \ \ \lim_{t\to\infty}\langle S_H(t)\rangle = \frac{A_3}{d_3} \ \ a.s..$$

This completes the proof. $\square$

Remark 3.1 Theorem 3.1 shows that the dynamics of disease-free situation for the stochastic model (1.2), that is, if $\widehat{\mathcal{R}}_0 < 1$ and the certain conditions about the environmental fluctuations are satisfied, $I_D(t)$, $I_L(t)$, $E_H(t)$ and $I_H(t)$ in model (1.2) goes to extinction with probability one, respectively. In the same time, $S_D(t)$, $S_L(t)$ and $S_H(t)$ are weak persistence in the mean with probability one, more precisely, they separately tend to their own steady states.

Now, we will establish the following more precise results on the extinction of the disease in a special case $\delta = \gamma = 0$ for model (1.2).

For convenience, we define the matrix

$$\Phi = \begin{pmatrix} 0 & \dfrac{A_1\beta_1}{d_1(d_1+\delta)} \\ \dfrac{A_2\beta_2}{d_2^2} & 0 \end{pmatrix}.$$

Obviously, by Theorem 1.4 in [24], there must be a left eigenvector (ω_1, ω_2) of matrix Φ corresponding to the value R_0, which is denoted as $(\omega_1, \omega_2) = (\frac{A_2\beta_2}{d_2^2}, R_0)$ i.e.,

$$(\omega_1, \omega_2)\Phi = R_0(\omega_1, \omega_2). \tag{3.7}$$

Theorem 3.2 *If $\delta = \gamma = 0$, and conditions $d_1 > \frac{\sigma_1^2}{2}$ and $d_2 > \frac{\sigma_3^2}{2}$ hold, then for any given initial value X_0, the solution $(S_D(t), I_D(t), S_L(t), I_L(t), S_H(t), E_H(t), I_H(t))$ of model (1.2) obeys*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{\omega_1}{d_1+\delta} I_D + \frac{\omega_2}{d_2} I_L\right) \le m,$$

where

$$\begin{aligned} m = \ & \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}} \\ & + R_0[\sigma_1 d_2(2d_1-\sigma_1^2)^{-\frac{1}{2}} + \sigma_3(d_1+\delta)(2d_2-\sigma_3^2)^{-\frac{1}{2}}] - \tfrac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}. \end{aligned}$$

In particular, if $m < 0$, then

$$\lim_{t\to\infty} I_D(t) = 0, \quad \lim_{t\to\infty} I_L(t) = 0 \ \ \text{and} \ \ \lim_{t\to\infty} I_H(t) = 0, \quad a.s..$$

Namely, $I_D(t)$, $I_L(t)$ and $I_H(t)$ go to extinction with probability one.

Proof. By lemma 2.1, from the first equation of model (1.2), it is obvious that if $\delta = 0$ we obtain

$$dS_D \le \ (A_1 - d_1 S_D)dt + \sigma_1 S_D dB_1(t),$$

and from the third equation of model (1.2) we get

$$dS_L \le \ (A_2 - d_2 S_L)dt + \sigma_3 S_L dB_3(t).$$

Considering the following auxiliary systems

$$dX = \quad (A_1 - d_1 X)dt + \sigma_1 X dB_1(t) \tag{3.8}$$

and

$$dY = \quad (A_2 - d_2 Y)dt + \sigma_3 Y dB_3(t), \tag{3.9}$$

with the initial values $X(0) = S_D(0)$ and $Y(0) = S_L(0)$, respectively. It is easy to check by Theorem 1.16 in [25] that Eqs. (3.8) and (3.9) separately have a stationary solution with the densities $\pi_1(x)$ and $\pi_2(y)$. It then follows that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t X(s)ds = \int_0^\infty x\pi_1(x)dx \;\; a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Y(s)ds = \int_0^\infty y\pi_2(y)dy \;\; a.s..$$

Let $X(t)$ and $Y(t)$ be the solutions of systems (3.8) and (3.9) with the initial values $X(0) = S_D(0)$ and $Y(0) = S_L(0)$, respectively. By the comparison theorem of stochastically differential equations [27], we obtain separately

$$S_D(t) \le X(t) \quad \text{for all } \; t \ge 0, \quad a.s. \tag{3.10}$$

and

$$S_L(t) \le Y(t) \quad \text{for all } \; t \ge 0, \quad a.s.. \tag{3.11}$$

So, we obtain

$$\int_0^\infty (x - \frac{A_1}{d_1})^2 \pi_1(x)dx = \frac{A_1^2\sigma_1^2}{d_1^2(2d_1 - \sigma_1^2)}.$$

Similarly, we can obtain

$$\int_0^\infty (y - \frac{A_2}{d_2})^2 \pi_2(y)dy = \frac{A_2^2\sigma_3^2}{d_2^2(2d_2 - \sigma_3^2)}.$$

Define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_{+0}$ by

$$V(I_D, I_L) = \frac{\omega_1}{d_1 + \delta} I_D + \frac{\omega_2}{d_2} I_L,$$

where $(\omega_1, \omega_2) = (\frac{A_2\beta_2}{d_2^2}, R_0)$ is the left eigenvector of Φ corresponding the value R_0.

By using Itô formula, we obtain

$$d(\ln V) = \quad L(\ln V)dt + \frac{1}{V}(\frac{\omega_1\sigma_2}{d_1 + \delta} I_D dB_2(t) + \frac{\omega_2\sigma_4}{d_2} I_L dB_4(t)), \tag{3.12}$$

where

$$\begin{aligned}L(\ln V) = \;& \frac{\omega_1}{V(d_1+\delta)}(\beta S_D I_L - (d_1+\delta)I_D) + \frac{\omega_2}{Vd_2}(\beta_2 S_L I_D - d_2 I_L)\\ & -\frac{\omega_1\sigma_2^2 I_D^2}{2V^2(d_1+\delta)} - \frac{\omega_2\sigma_4^2 I_L^2}{2V^2 d_2}.\end{aligned}$$

Note that

$$\begin{aligned}V^2 = \;& (\frac{\omega_1}{d_1+\delta}I_D + \frac{\omega_2}{d_2}I_L)^2 = (\frac{\omega_1}{d_1+\delta}\sigma_2 I_D\cdot\frac{1}{\sigma_2} + \frac{\omega_2}{d_2}\sigma_4 I_L\cdot\frac{1}{\sigma_4})^2\\ & \leq (\frac{\omega_1^2\sigma_2^2}{(d_1+\delta)^2}I_D^2 + \frac{\omega_2^2\sigma_4^2}{d_2^2}I_L^2)(\sigma_2^{-2}+\sigma_4^{-2})^{-1}.\end{aligned} \tag{3.13}$$

Then

$$\begin{aligned}L(\ln V) \leq \;& [\frac{\omega_1}{V(d_1+\delta)}(\beta_1 S_D I_L - (d_1+\delta)I_D) + \frac{\omega_2}{Vd_2}(\beta_2 S_L I_D - d_2 I_L)] - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}\\ = \;& \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}(S_D - \frac{A_1}{d_1}) + \frac{\omega_2\beta_2 I_D}{Vd_2}(S_L - \frac{A_2}{d_2}) - \frac{\omega_1 I_D}{V} - \frac{\omega_2 I_L}{V}\\ & + \frac{\omega_1\beta_1 I_L A_1}{Vd_1(d_1+\delta)} + \frac{\omega_2\beta_2 I_D A_2}{Vd_2^2} - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}\\ = \;& \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}(S_D - \frac{A_1}{d_1}) + \frac{\omega_2\beta_2 I_D}{Vd_2}(S_L - \frac{A_2}{d_2}) + \frac{1}{V}[\frac{\omega_1}{d_1+\delta}(\frac{\beta_1 I_L A_1}{d_1} - (d_1+\delta)I_D)\\ & + \frac{\omega_2}{d_2}(\frac{\beta_2 I_D A_2}{d_2} - d_2 I_L)] - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}\\ \leq \;& \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}|S_D - \frac{A_1}{d_1}| + \frac{\omega_2\beta_2 I_D}{Vd_2}|S_L - \frac{A_2}{d_2}| + \frac{1}{V}[(\omega_1,\omega_2)(\Phi(I_D,I_L)^T - (I_D,I_L)^T)]\\ & - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}\\ = \;& \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}|S_D - \frac{A_1}{d_1}| + \frac{\omega_2\beta_2 I_D}{Vd_2}|S_L - \frac{A_2}{d_2}| + \frac{1}{V}(R_0-1)(\omega_1 I_D + \omega_2 I_L)\\ & - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}\\ \leq \;& \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\leq 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}}\\ & + \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}|S_D - \frac{A_1}{d_1}| + \frac{\omega_2\beta_2 I_D}{Vd_2}|S_L - \frac{A_2}{d_2}| - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}.\end{aligned}$$

By (3.10), (3.11) and (3.12), we have

$$\begin{aligned}d(\ln V) \leq \;& [\min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\leq 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}}\\ & + \frac{\omega_1\beta_1 I_L}{V(d_1+\delta)}|X(t) - \frac{A_1}{d_1}| + \frac{\omega_2\beta_2 I_D}{Vd_2}|Y(t) - \frac{A_2}{d_2}| - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1}]dt\\ & + \frac{\omega_1\sigma_2 I_D}{V(d_1+\delta)}dB_2(t) + \frac{\omega_2\sigma_4 I_L}{Vd_2}dB_4(t).\end{aligned}$$

Integrating the above inequality from 0 to t and then dividing t on both sides leads to

$$\begin{aligned}\frac{\ln V}{t} \leq\ & \frac{\ln V(0)}{t} + \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\leq 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}} \\ & + \frac{\omega_1\beta_1}{(d_1+\delta)t}\int_0^t |X(s)-\frac{A_1}{d_1}|\frac{I_L(s)}{V(s)}ds + \frac{\omega_2\beta_2}{d_2 t}\int_0^t |Y(s)-\frac{A_2}{d_2}|\frac{I_D(s)}{V(s)}ds - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1} \\ & + \frac{1}{t}\int_0^t \frac{\omega_1\sigma_2 I_D(s)}{V(s)(d_1+\delta)}dB_2(s) + \frac{1}{t}\int_0^t \frac{\omega_2\sigma_4 I_L(s)}{V(s)d_2}dB_4(s).\end{aligned}$$

Now, we denote $M_1(t) = \int_0^t \frac{\omega_1\sigma_2 I_D(s)}{V(s)(d_1+\delta)}dB_2(s)$ and $M_2(t) = \int_0^t \frac{\omega_2\sigma_4 I_L(s)}{V(s)d_2}dB_4(s)$ are two local martingales whose quadratic variations are $\langle M_1, M_1\rangle_t = \sigma_2^2\int_0^t (\frac{\omega_1 I_D(s)}{V(s)(d_1+\delta)})^2 ds \leq \sigma_2^2 t$ and $\langle M_2, M_2\rangle_t = \sigma_4^2\int_0^t (\frac{\omega_2 I_L(s)}{V(s)d_2})^2 ds \leq \sigma_4^2 t$, respectively. Applying the strong law of large numbers for martingale [26] yields

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = 0 \quad a.s., \quad i=1,2. \tag{3.14}$$

Then, we have

$$\begin{aligned}\frac{\ln V}{t} \leq\ & \frac{\ln V(0)}{t} + \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\leq 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}} \\ & + \frac{\omega_1\beta_1}{(d_1+\delta)t}\int_0^t |X(s)-\frac{A_1}{d_1}|\frac{I_L(s)}{V(s)}ds + \frac{\omega_2\beta_2}{d_2 t}\int_0^t |Y(s)-\frac{A_2}{d_2}|\frac{I_D(s)}{V(s)}ds - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1} \\ & + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}.\end{aligned} \tag{3.15}$$

In addition, since $X(t)$ is ergodic and $\int_0^\infty x\pi_1(x)dx < \infty$, we have

$$\begin{aligned} & \lim_{t\to\infty}\frac{1}{t}\int_0^\infty |X(s)-\frac{A_1}{d_1}|ds = \int_0^\infty |x-\frac{A_1}{d_1}|\pi_1(x)dx \\ \leq\ & (\int_0^\infty (x-\frac{A_1}{d_1})^2\pi_1(x)dx)^{\frac{1}{2}} = \frac{A_1\sigma_1}{d_1(2d_1-\sigma_1^2)^{\frac{1}{2}}}.\end{aligned} \tag{3.16}$$

Similarly,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^\infty |Y(s)-\frac{A_2}{d_2}|ds \leq \frac{A_2\sigma_3}{d_2(2d_2-\sigma_3^2)^{\frac{1}{2}}}. \tag{3.17}$$

Taking the superior limit on both sides in (3.15) and combining with (3.14), (3.16) and

(3.17), we obtain

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln V}{t} \le\ & \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}} \\
& + \limsup_{t\to\infty}\frac{\omega_1\beta_1 d_2}{\omega_2(d_1+\delta)t}\int_0^t |X(s)-\frac{A_1}{d_1}|ds \\
& + \limsup_{t\to\infty}\frac{\omega_2\beta_2(d_1+\delta)}{\omega_1 d_2 t}\int_0^t |Y(s)-\frac{A_2}{d_2}|ds - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1} \\
\le\ & \min\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0\le 1\}} + \max\{d_1+\delta, d_2\}(R_0-1)I_{\{R_0>1\}} \\
& + R_0[\sigma_1 d_2(2d_1-\sigma_1^2)^{-\frac{1}{2}} + \sigma_3(d_1+\delta)(2d_2-\sigma_3^2)^{-\frac{1}{2}}] - \frac{1}{2}(\sigma_2^{-2}+\sigma_4^{-2})^{-1} \\
=\ & m,
\end{aligned}
\tag{3.18}
$$

which is the required statement. If $m<0$, then it is concluded that

$$
\limsup_{t\to\infty}\frac{\ln I_D(t)}{t} < 0 \ \text{ and } \ \limsup_{t\to\infty}\frac{\ln I_L(t)}{t} < 0 \ \ a.s..
$$

Hence, there exists a constant $\varsigma > 0$ such that

$$
\limsup_{t\to\infty}\frac{\ln I_D(t)}{t} < -\varsigma.
$$

In other words, for an arbitrary small constant $0<\xi<\frac{\varsigma}{2}$, there exist $T_1 = T_1(\omega)$ and a set Ω_ξ such that $P(\Omega_\xi) \ge 1-\xi$ and $\ln I_D \le -\frac{\varsigma}{2}t$ for $t \ge T_1$ and $\omega \in \Omega_\xi$, and hence $I_D \le e^{-\frac{\varsigma}{2}t}$. It means that

$$
\limsup_{t\to\infty} I_D(t) = 0 \quad a.s.
$$

which combines with the positivity of the solution shows

$$
\lim_{t\to\infty} I_D(t) = 0 \quad a.s. \tag{3.19}
$$

Similarly, one can obtain $\lim_{t\to\infty} I_L(t) = 0$ a.s.. And from (3.19) and the sixth equation of model (1.2), one can easily see that $\lim_{t\to\infty} E_H(t) = 0$ a.s. and thus $\lim_{t\to\infty} I_H(t) = 0$ a.s. from the last equation of model (1.2). This finishes the proof.

□

Remark 3.2 *Theorem 3.2 discusses the case that only $I_D(t)$, $I_L(t)$ and $I_H(t)$ go to extinction almost surely when $m<0$. Therefore, a natural question arises: in this situation, what will be the limiting magnitude of the susceptible compartments $S_D(t)$, $S_L(t)$ and $S_H(t)$?*

In fact, by Theorem 3.2, it can be further obtained that the distributions of $S_D(t)$ and $S_L(t)$ weakly converge to the measures which have separately the densities given by

$$
\pi_1(x) = Q_1\sigma_1^{-2}x^{-2-\frac{2d_1}{\sigma_1^2}}e^{-\frac{2A_1}{\sigma_1^2 x}} \ \ \textit{for all } \ x \in (0,\infty)
$$

and

$$\pi_2(y) = Q_2\sigma_3^{-2}y^{-2-\frac{2d_2}{\sigma_3^2}}e^{-\frac{2A_2}{\sigma_3^2 y}} \text{ for all } y \in (0,\infty),$$

where $Q_1 = \sigma_1^2(\frac{2A_1}{\sigma_1^2})^{1+\frac{2d_1}{\sigma_1^2}}\Gamma^{-1}(1+\frac{2d_1}{\sigma_1^2})$ *and* $Q_2 = \sigma_3^2(\frac{2A_2}{\sigma_3^2})^{1+\frac{2d_2}{\sigma_3^2}}\Gamma^{-1}(1+\frac{2d_2}{\sigma_3^2})$ *are two constants satisfying* $\int_0^\infty \pi_1(x)dx = 1$ *and* $\int_0^\infty \pi_2(y)dy = 1$*, respectively.*

Furthermore, when $m < 0$*, by* (3.19) *one can obtain that for any small constant* $\kappa > 0$ *there exist* $T_2 = T_2(\omega)$ *and a set* Ω_κ *such that* $P(\Omega_\kappa) > 1-\kappa$ *and* $\beta_3 S_H I_D \le \kappa S_H$ *for* $t \ge T_2$ *and* $\omega \in \Omega_\kappa$*. Hence,*

$$dS_H \ge (A_3 - \kappa S_H - d_3 S_H)dt + \sigma_5 S_H dB_5(t). \tag{3.20}$$

On the other hand, when $\gamma = 0$*, one has*

$$dS_H \le [A_3 - d_3 S_H]dt + \sigma_5 S_H dB_5(t), \tag{3.21}$$

Consider the following stochastically differential equation

$$d\tilde{u}(t) = (A_3 - d_3\tilde{u}(t))dt + \sigma_5\tilde{u}(t)dB_5(t). \tag{3.22}$$

It is easy to check that Eq. (3.22) *has a stationary solution* $\hat{u}(t)$ *which has the density* $\pi_3(z) = Q_3\sigma_5^{-2}z^{-2-\frac{2d_3}{\sigma_5^2}}e^{-\frac{2A_3}{\sigma_5^2 z}}$ *where* $Q_3 = \sigma_5^2(\frac{2A_3}{\sigma_5^2})^{1+\frac{2d_3}{\sigma_5^2}}\Gamma^{-1}(1+\frac{2d_3}{\sigma_5^2})$*. From the ergodic theorem [25], it follows that*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat{u}(s)ds = \int_0^{+\infty} z\pi_3(z)dz \quad a.s. \tag{3.23}$$

By direct calculation, one can obtain from (3.22)

$$\int_0^{+\infty} z\pi_3(z)dz = \mathbb{E}(\hat{u}(t)) = \frac{A_3}{d_3} \quad a.s.$$

From (3.20)*,* (3.21)*,* (3.23) *and the comparison theorem for stochastic differential equation [27] and let* $\kappa \to 0$*, one can finally obtain that the distribution of the process* $S_H(t)$ *weakly converges to the measure with the density* $\pi_3(z)$*.*

4. Stationary distribution and ergodicity

For the sake of understanding of stochastic endemic dynamics of model (1.2), we now focus in the existence of stationary distribution.

To do this, we define

$$\widetilde{\mathcal{R}}_0 = \sqrt{\frac{A_1A_2\beta_1\beta_2}{(d_1+\frac{\sigma_1^2}{2})(d_1+\delta+\frac{\sigma_2^2}{2})(d_2+\frac{\sigma_3^2}{2})(d_2+\frac{\sigma_4^2}{2})}}.$$

Theorem 4.1 *Let* $(S_D, I_D, S_L, I_L, S_H, E_H, I_H)$ *be a solution of model* (1.2) *with any initial value* X_0*. If* $\widetilde{\mathcal{R}}_0 > 1$ *and* $(d_1 \wedge d_2 \wedge d_3) > \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2)$*, then there exists a unique stationary solution of model* (1.2) *and it is ergodic.*

Proof. In order to prove Theorem 4.1, it is sufficient to show the following two conditions:

(H_1) For any bounded domain $Z \subset \mathbb{R}^7_+$, there exists a positive constant Δ such that $\sum\limits_{i,j=1}^{7} a_{ij}(x)\xi_i\xi_j \geq \Delta|\xi|^2$ for all $x \in \bar{Z}$ and $\xi \in \mathbb{R}^7_+$. (For more details see [28] and [29]).

(H_2) There exist a neighborhood U and a nonnegative $\mathcal{C}^2$-function $\mathcal{V}$ such that $L\mathcal{V}$ is negative for any $\xi \in \mathbb{R}^7_+ \setminus U$. (For more details see [26] and [30]).

Firstly, we are going to show the condition (H_1) is satisfied. In fact, the diffusion matrix of model (1.2) is given by

$$A = \begin{pmatrix} \sigma_1^2 S_D^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 I_D^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^2 S_L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4^2 I_L^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5^2 S_H^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_6^2 E_H^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_7^2 I_H^2 \end{pmatrix}.$$

Let Z be any bounded domain in $\mathbb{R}^7_+$, then there exists a positive constant

$$\Delta = \min_{(S_D,I_D,S_L,I_L,S_H,E_H,I_H)\in\bar{Z}\subset\mathbb{R}^7_+} \{\sigma_1^2 S_D^2, \sigma_2^2 I_D^2, \sigma_3^2 S_L^2, \sigma_4^2 I_L^2, \sigma_5^2 S_H^2, \sigma_6^2 E_H^2, \sigma_7^2 I_H^2\}$$

such that

$$\begin{aligned} &\textstyle\sum_{i,j=1}^{7} a_{ij}(S_D, I_D, S_L, I_L, S_H, E_H, I_H)\xi_i\xi_j \\ &= \sigma_1^2 S_D^2 \xi_1^2 + \sigma_2^2 I_D^2 \xi_2^2 + \sigma_3^2 S_L^2 \xi_3^2 + \sigma_4^2 I_L^2 \xi_4^2 + \sigma_5^2 S_H^2 \xi_5^2 + \sigma_6^2 E_H^2 \xi_6^2 + \sigma_7^2 I_H^2 \xi_7^2 \geq \Delta|\xi|^2 \end{aligned}$$

for all $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \bar{Z}$, $\xi \in \mathbb{R}^7_+$. This shows condition (H_1) holds.

Next, we show that the validity of condition (H_2). By the condition $(d_1 \wedge d_2 \wedge d_3) > \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2)$, we can choose a sufficiently small constant $\theta > 0$ such that

$$\rho \triangleq (d_1 \wedge d_2 \wedge d_3) - \frac{\theta+1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2) > 0.$$

Define a C^2-function $\bar{\mathcal{V}} : \mathbb{R}^7_+ \to \mathbb{R}$ as follows

$$\bar{\mathcal{V}}(S_D, I_D, S_L, I_L, S_H, E_H, I_H) = M\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3,$$

where

$$\mathcal{V}_1(S_D, I_D, S_L, I_L) = -\ln S_D - p_1 \ln I_D - p_2 \ln S_L - p_3 \ln I_L + \frac{\beta_1 I_L}{d_2},$$

$$\mathcal{V}_2(S, V, E) = -\ln S_D - \ln I_D - \ln S_L - \ln S_H - \ln E_H - \ln I_H + \frac{\beta_1 I_L}{d_2},$$

$$\mathcal{V}_3(S_D, I_D, S_L, I_L, S_H, E_H, I_H) = \frac{1}{\theta+2}(S_D + I_D + S_L + I_L + S_H + E_H + I_H)^{\theta+2},$$

here

$$p_1 = \frac{d_1 + \frac{\sigma_1^2}{2}}{d_1 + \delta + \frac{\sigma_2^2}{2}}, \quad p_2 = \frac{d_1 + \frac{\sigma_1^2}{2}}{d_2 + \frac{\sigma_3^2}{2}}, \quad p_3 = \frac{d_1 + \frac{\sigma_1^2}{2}}{d_2 + \frac{\sigma_4^2}{2}},$$

and $M > 0$ is a constant satisfying the following condition

$$-4M(d_1 + \frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0} - 1) + G \leq -2$$

where

$$\begin{aligned} G = \sup_{S_L \in \mathbb{R}_+} & \{2\big(\frac{\beta_1\beta_2}{d_2}\big)^2 S_L^2 + 2d_1 + d_2 + 3d_3 + \delta + \omega + \mu + \gamma + D \\ & -\frac{\rho}{2} S_L^{\theta+2} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2) < \infty \end{aligned}$$

and

$$\begin{aligned} D = \sup_{x \in \mathbb{R}_+^7} & \{(A_1 + A_2 + A_3)(S_D + I_D + S_L + I_L + S_H + E_H + I_H)^{\theta+1} \\ & -\frac{\rho}{2}(S_D + I_D + S_L + I_L + S_H + E_H + I_H)^{\theta+2}\} < \infty, \end{aligned}$$

with $x = (S_D, I_D, S_L, I_L, S_H, E_H, I_H)$.

It is obvious that

$$\liminf_{\substack{k \to +\infty \\ (S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}_+^7 \setminus U_k}} \bar{\mathcal{V}}(S_D, I_D, S_L, I_L, S_H, E_H, I_H) = +\infty,$$

where $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$. Obviously, $\bar{\mathcal{V}}(S_D, I_D, S_L, I_L, S_H, E_H, I_H)$ is a continuous function, then $\bar{\mathcal{V}}(S_D, I_D, S_L, I_L, S_H, E_H, I_H)$ must have a minimum point $(\bar{S}_D, \bar{I}_D, \bar{S}_L, \bar{I}_L, \bar{S}_H, \bar{E}_H, \bar{I}_H) \in \mathbb{R}_+^7$.

Therefore, we construct a nonnegative $\mathcal{C}^2$-function $U : \mathbb{R}^7 \to \mathbb{R}_+$ in the following form

$$\mathcal{V}(S_D, I_D, S_L, I_L, S_H, E_H, I_H) = \bar{\mathcal{V}}(S_D, I_D, S_L, I_L, S_H, E_H, I_H) - \bar{\mathcal{V}}(\bar{S}_D, \bar{I}_D, \bar{S}_L, \bar{I}_L, \bar{S}_H, \bar{E}_H, \bar{I}_H).$$

Hence, $\mathcal{V}(S_D, I_D, S_L, I_L, S_H, E_H, I_H)$ is a nonnegative C^2-function.

Therefore, we can separately obtain by It$\hat{o}$ formula

$$\begin{aligned} L(\mathcal{V}_1) = & -\frac{A_1}{S_D} - \frac{p_1\beta_1 S_D I_L}{I_D} - \frac{p_2 A_2}{S_L} - \frac{p_3\beta_2 S_L I_D}{I_L} + p_1(d_1 + \delta + \frac{\sigma_2^2}{2}) \\ & + p_2(d_2 + \frac{\sigma_3^2}{2}) + p_3(d_2 + \frac{\sigma_4^2}{2}) + d_1 + \frac{\sigma_1^2}{2} + \frac{\beta_1\beta_2 S_L I_D}{d_2} - \frac{\delta I_D}{S_D} + p_2\beta_2 I_D \\ \leq & -4\sqrt[4]{A_1 A_2 \beta_1\beta_2 p_1 p_2 p_3} + p_1(d_1 + \delta + \frac{\sigma_2^2}{2}) + p_2(d_2 + \frac{\sigma_3^2}{2}) \\ & + p_3(d_2 + \frac{\sigma_4^2}{2}) + d_1 + \frac{\sigma_1^2}{2} + \frac{\beta_1\beta_2 S_L I_D}{d_2} + p_2\beta_2 I_D \\ = & -4(d_1 + \frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0} - 1) + p_2\beta_2 I_D + \frac{\beta_1\beta_2 S_L I_D}{d_2}, \end{aligned}$$

$$
\begin{aligned}
L(\mathcal{V}_2) = \ & -\frac{A_1}{S_D} - \frac{\beta_1 S_D I_L}{I_D} - \frac{A_2}{S_L} - \frac{A_3}{S_H} - \frac{\beta_3 S_H I_D}{E_H} - \frac{\omega E_H}{I_H} - \frac{\delta I_D}{S_D} - \frac{\gamma I_H}{S_H} + \frac{\beta_1\beta_2 S_L I_D}{d_2} \\
& +(\beta_2+\beta_3)I_D + 2d_1 + d_2 + 3d_3 + \delta + \omega + \mu + \gamma + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2+\sigma_6^2+\sigma_7^2) \\
\leq \ & -\frac{A_1}{S_D} - \frac{\beta_1 S_D I_L}{I_D} - \frac{A_2}{S_L} - \frac{A_3}{S_H} - \frac{\beta_3 S_H I_D}{E_H} - \frac{\omega E_H}{I_H} + (\frac{\beta_1\beta_2}{d_2})^2 S_L^2 + \frac{1}{4}I_D^2 \\
& +(\beta_2+\beta_3)I_D + 2d_1 + d_2 + 3d_3 + \delta + \omega + \mu + \gamma + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2+\sigma_6^2+\sigma_7^2).
\end{aligned}
$$

and

$$
\begin{aligned}
L(\mathcal{V}_3) = \ & (S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta+1}[A_1+A_2+A_3 \\
& -d_1(S_D+I_D) - d_2(S_L+I_L) - d_3(S_H+E_H+I_H) - \mu I_H] \\
& +\frac{\theta+1}{2}(S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta}(\sigma_1^2 S_D^2 + \sigma_2^2 I_D^2 \\
& +\sigma_3^2 S_L^2 + \sigma_4^2 I_D^2 + \sigma_5^2 S_H^2 + \sigma_6^2 I_H^2 + \sigma_7^2 E_H^2) \\
\leq \ & (S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta+1}[A_1+A_2+A_3 - (d_1 \wedge d_2 \wedge d_3) \\
& \times(S_D+I_D+S_L+I_L+S_H+E_H+I_H)] + \frac{\theta+1}{2}(S_D+I_D+S_L+I_L \\
& +S_H+E_H+I_H)^{\theta+2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2) \\
= \ & (A_1+A_2+A_3)(S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta+1} \\
& -\rho(S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta+2} \\
\leq \ & D - \frac{\rho}{2}(S_D+I_D+S_L+I_L+S_H+E_H+I_H)^{\theta+2} \\
\leq \ & D - \frac{\rho}{2}(S_D^{\theta+2}+I_D^{\theta+2}+S_L^{\theta+2}+I_L^{\theta+2}+S_H^{\theta+2}+E_H^{\theta+2}+I_H^{\theta+2})
\end{aligned}
$$

Therefore, one can obtain

$$
\begin{aligned}
L\mathcal{V} \leq \ & -4M(d_1+\frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0}-1) - \frac{A_1}{S_D} - \frac{\beta_1 S_D I_L}{I_D} - \frac{A_2}{S_L} - \frac{A_3}{S_H} - \frac{\beta_3 S_H I_D}{E_H} \\
& -\frac{\omega E_H}{I_H} + \frac{M\beta_1\beta_2 S_L I_D}{d_2} + \big(\frac{\beta_1\beta_2}{d_2}\big)^2 S_L^2 + \frac{1}{4}I_D^2 + (Mp_2\beta_2+\beta_2+\beta_3)I_D \\
& -\frac{\rho}{2}(S_D^{\theta+2}+I_D^{\theta+2}+S_L^{\theta+2}+I_L^{\theta+2}+S_H^{\theta+2}+E_H^{\theta+2}+I_H^{\theta+2}) + 2d_1 + d_2 \\
& +3d_3 + \delta + \omega + \mu + \gamma + D + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2+\sigma_6^2+\sigma_7^2) \\
\leq \ & -4M(d_1+\frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0}-1) - \frac{A_1}{S_D} - \frac{\beta_1 S_D I_L}{I_D} - \frac{A_2}{S_L} - \frac{A_3}{S_H} - \frac{\beta_3 S_H I_D}{E_H} \\
& -\frac{\omega E_H}{I_H} + 2\big(\frac{\beta_1\beta_2}{d_2}\big)^2 S_L^2 + \frac{M^2+1}{4}I_D^2 + (Mp_2\beta_2+\beta_2+\beta_3)I_D \\
& -\frac{\rho}{2}(S_D^{\theta+2}+I_D^{\theta+2}+S_L^{\theta+2}+I_L^{\theta+2}+S_H^{\theta+2}+E_H^{\theta+2}+I_H^{\theta+2}) + 2d_1 + d_2 \\
& +3d_3 + \delta + \omega + \mu + \gamma + D + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2+\sigma_6^2+\sigma_7^2)
\end{aligned}
$$

Define a bounded closed set

$$
\begin{aligned}
U_\varepsilon = \ & \big\{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : \varepsilon \le S_D \le \frac{1}{\varepsilon}, \varepsilon^3 \le I_D \le \frac{1}{\varepsilon^3}, \varepsilon \le S_L \le \frac{1}{\varepsilon}, \\
& \varepsilon \le I_L \le \frac{1}{\varepsilon}, \varepsilon \le S_H \le \frac{1}{\varepsilon}, \varepsilon^5 \le E_H \le \frac{1}{\varepsilon^5}, \varepsilon^6 \le I_H \le \frac{1}{\varepsilon^6}\big\},
\end{aligned}
$$

where $0 < \varepsilon < 1$ is a sufficiently small constant such that the following conditions hold

$$
\frac{1}{\varepsilon} \ge (1+F) \max\{\frac{1}{A_1}, \frac{1}{A_2}, \frac{1}{A_3}, \frac{1}{\beta_1}, \frac{1}{\beta_3}, \frac{1}{\omega}\}, \tag{4.1}
$$

$$
M(p_2\beta_2 + \beta_2 + \beta_3)\varepsilon^3 + \frac{M^2+1}{4}\varepsilon^6 \le 1, \tag{4.2}
$$

$$
-\frac{\rho}{4\varepsilon^{\theta+2}} + F \le -1, \tag{4.3}
$$

where

$$
\begin{aligned}
F = \sup_{(S_L, I_D) \in \mathbb{R}^2_+} & \big\{2\big(\frac{\beta_1\beta_2}{d_2}\big)^2 S_L^2 + \frac{M^2+1}{4} I_D^2 + (Mp_2\beta_2 + \beta_2 + \beta_3) I_D \\
& -\frac{\rho}{2}(S_L^{\theta+2} + I_D^{\theta+2}) + 2d_1 + d_2 + 3d_3 + \delta + \omega + \mu + \gamma + D \\
& +\frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2)\big\} < +\infty.
\end{aligned}
$$

For the sake of convenience, we divide $U_\varepsilon^C \triangleq \mathbb{R}^7_+ \setminus U_\varepsilon$ into fourteen domains,

$$
\begin{aligned}
U_1 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < S_D < \varepsilon\}, \\
U_2 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < I_D < \varepsilon^3\}, \\
U_3 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : S_D > \frac{1}{\varepsilon}\}, \\
U_4 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : I_D > \frac{1}{\varepsilon^3}\}, \\
U_5 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < S_L < \varepsilon\}, \\
U_6 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < I_L < \varepsilon, S_D > \varepsilon, I_D > \varepsilon^3\}, \\
U_7 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : S_L > \frac{1}{\varepsilon}\}, \\
U_8 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : I_L > \frac{1}{\varepsilon}\}, \\
U_9 =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < S_H < \varepsilon\}, \\
U_{10} =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < E_H < \varepsilon^5, S_H > \varepsilon, I_D > \varepsilon^3\}, \\
U_{11} =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : 0 < I_H < \varepsilon^6, E_H > \varepsilon^5\}, \\
U_{12} =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}^7_+ : S_H > \frac{1}{\varepsilon}\},
\end{aligned}
$$

$$
\begin{aligned}
U_{13} =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}_+^7 : E_H > \frac{1}{\varepsilon^5}\},\\
U_{14} =& \{(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in \mathbb{R}_+^7 : I_H > \frac{1}{\varepsilon^6}\}.
\end{aligned}
$$

Clearly, $U_\varepsilon^C = U_1 \cup U_2 \cup \cdots \cup U_{14}$. Next, we will show that

$$
LV(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \leq -1
$$

on U_ε^C, which is equivalent to show it on the above fourteen domains.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_1$, one can see that

$$
L\mathcal{V} \leq -\frac{A_1}{S_D} + F \leq -\frac{A_1}{\varepsilon} + F.
$$

By virtue of the condition (4.1), it can be obtained that $L\mathcal{V} \leq -1$ on U_1.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_2$, one can see that

$$
\begin{aligned}
L\mathcal{V} \leq & \quad -4M(d_1 + \frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0} - 1) + 2\big(\frac{\beta_1\beta_2}{d_2}\big)^2 S_L^2 + \frac{M^2+1}{4} I_D^2 \\
& \quad + (Mp_2\beta_2 + \beta_2 + \beta_3) I_D - \frac{\rho}{2} S_L^{\theta+2} + 2d_1 + d_2 + 3d_3 + \delta \\
& \quad + \omega + \mu + \gamma + D + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2) \\
\leq & \quad -4M(d_1 + \frac{\sigma_1^2}{2})(\sqrt{\widetilde{\mathcal{R}}_0} - 1) + (Mp_2\beta_2 + \beta_2 + \beta_3)\varepsilon^3 + \frac{M^2+1}{4}\varepsilon^6 + G \\
\leq & \quad -2 + \frac{M^2+1}{4}\varepsilon^6 + (Mp_2\beta_2 + \beta_2 + \beta_3)\varepsilon^3.
\end{aligned}
$$

By the condition (4.2), we can get $L\mathcal{V} \leq -1$ on U_2.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_3$, it yields

$$
L\mathcal{V} \leq \quad -\frac{\rho}{4} S_D^{\theta+2} + F \leq -\frac{\rho}{4}\frac{1}{\varepsilon^{\theta+2}} + F.
$$

In view of the condition (4.3), we have $L\mathcal{V} \leq -1$ on U_3.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_4$, it yields

$$
L\mathcal{V} \leq \quad -\frac{1}{4} I_D^{\theta+2} + F \leq -\frac{\rho}{4}\frac{1}{\varepsilon^{3(\theta+2)}} + F,
$$

which together with the condition (4.3), we can conclude that $L\mathcal{V} \leq -1$ on U_4.

when $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_5$, we can derive

$$
L\mathcal{V} \leq \quad -\frac{A_2}{S_L} + F \leq -\frac{A_2}{\varepsilon} + F.
$$

According to the condition (4.1), we can deduce that $L\mathcal{V} \leq -1$ on U_5.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_6$, it yields

$$L\mathcal{V} \leq -\frac{\beta_1 S_D I_L}{I_D} + F \leq -\frac{\beta_1}{\varepsilon} + F,$$

which follows from the condition (4.1) that $L\mathcal{V} \leq -1$ on U_6.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_7$, we have

$$L\mathcal{V} \leq \quad -\frac{\rho}{4} S_L^{\theta+2} + F \leq -\frac{\rho}{4}\frac{1}{\varepsilon^{\theta+2}} + F.$$

By the condition (4.3), we can conclude that $L\mathcal{V} \leq -1$ on U_7.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_8$, one can derive that

$$L\mathcal{V} \leq \quad -\frac{\rho}{4} I_L^{\theta+2} + F \leq -\frac{\rho}{4}\frac{1}{\varepsilon^{3(\theta+2)}} + F.$$

Combing with the condition (4.3), we can conclude that $L\mathcal{V} \leq -1$ on U_8.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_9$, we obtain

$$L\mathcal{V} \leq \quad -\frac{A_3}{S_H} + F \leq -\frac{A_3}{\varepsilon} + F.$$

In view of the condition (4.1), we have $L\mathcal{V} \leq -1$ on U_9.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_{10}$, we derive

$$L\mathcal{V} \leq \quad -\frac{\beta_3 S_H I_D}{E_H} + F \leq -\frac{\beta_3}{\varepsilon} + F,$$

which together with the condition (4.1), we have that $L\mathcal{V} \leq -1$ on U_{10}.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_{11}$, we get

$$L\mathcal{V} \leq \quad -\frac{\omega E_H}{I_H} + F \leq -\frac{\omega}{\varepsilon} + F.$$

It follows from the condition (4.1) that $L\mathcal{V} \leq -1$ on U_{11}.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_{12}$, we can derive

$$L\mathcal{V} \leq \quad -\frac{\rho}{4} S_H^{\theta+2} + F \leq \frac{1}{\varepsilon^{\theta+2}} + F.$$

By the condition (4.3), we can obtain that $L\mathcal{V} \leq -1$ on U_{12}.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_{13}$, we derive

$$L\mathcal{V} \leq \quad -\frac{\rho}{4} E_H^{\theta+2} \leq -\frac{\rho}{4}\frac{1}{\varepsilon^{5(\theta+2)}} + F,$$

which follows from the condition (4.3) that $L\mathcal{V} \leq -1$ on U_{13}.

When $(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_{14}$, it yields

$$L\mathcal{V} \leq \ -\frac{\rho}{4} I_H^{\theta+2} + F \leq -\frac{\rho}{4} \frac{1}{\varepsilon^{6(\theta+2)}} + F.$$

According to the condition (4.3), one can conclude that $L\mathcal{V} \leq -1$ on U_{14}.

Hence, one finally drive

$$L\mathcal{V}(S_D, I_D, S_L, I_L, S_H, E_H, I_H) \leq -1 \quad \text{for all} \quad (S_D, I_D, S_L, I_L, S_H, E_H, I_H) \in U_\varepsilon^C,$$

which shows that (H_2) is satisfied. Summarising the above discussion, one can obtain that there is a stationary distribution for model (1.2) and it is ergodic. This completes the proof. □

Remark 4.1 *From the process of proving Theorem 4.1, it is found that constructing a suitably stochastic Lyapunov function to prove the existence of the stationary distribution seems to be intricate, however, this method has an advantage in being more applicable for higher-dimensionally stochastic system and the permitting conditions are a little weaker. It is worth noting that when model (1.2) degenerated into the corresponding deterministic model (1.1) if $\sigma_i = 0$ $(i = 1, 2, \ldots, 7)$, $\widetilde{\mathcal{R}}_0$ in Theorem 4.1 is consistent with the basic reproduction number R_0 of the deterministic model (1.1).*

5. Numerical simulations

In this section, by utilizing Milsteins higher-order method in [31], we make simulations for model (1.2) to verify our analytical results and explore the effect of stochastically environmental variability on transmission dynamics of echinococcosis. Hereon, we mainly study which population is more affected by stochastic fluctuations of the environment (namely, which population is more susceptible to environmental fluctuations). So, for convenience, we assume that the noise intensities for each subgroup of a same population is same. For example, the noise intensities for subgroups S_D and I_D is same, with no loss of generality, we denote it by σ_D (that is $\sigma_1 = \sigma_2 \triangleq \sigma_D$). Similarly, it is assumed that the noise intensities for the livestock is same and denoted by σ_L, and the noise intensities for humans is same and denoted by σ_H.

5.1. Effect of environmental noises on the extinction for model (1.2)

Firstly, we consider the extinction of model (1.2). In fact, if the subgroups I_D and I_L tend to extinguish, it can be finally derived from model (1.2) that I_H goes to extinction. On the other hand, if the practical data is considered, people's major concern is the extinction for I_H. Therefore, we only pay attention to the extinction for I_H. Because model (1.2) is the one of a continuous time and continuous state space, the quantities for each population are not equal to zero, and we assume that 100,000 individuals are deemed to be 1 unit population approximately in the numerical experiments. In other words, if the value of I_H is less than 0.00001, then I_H can be deemed to become extinct [32].

Example 5.1 Adopting $A_1 = 0.51$, $A_2 = 0.32$, $A_3 = 1.85$, $d_1 = 0.7$, $d_2 = 0.5$, $d_3 = 0.4$, $\beta_1 = 0.14$, $\beta_2 = 0.21$, $\beta_3 = 0.8$, $\mu = 0.12$, $\gamma = 0.63$, $\omega = 0.45$, $\delta = 0.8$, $\sigma_1 = \sigma_2 \triangleq \sigma_D = 0.32$, $\sigma_3 = \sigma_4 \triangleq \sigma_L = 0.35$, $\sigma_5 = \sigma_6 = \sigma_7 \triangleq \sigma_H = 0.29$. By calculating, it can be obtained that $\widehat{\mathcal{R}}_0 \approx 0.8424$, $d_1 - \frac{1}{2}\sigma_D^2 \approx 0.6488$, $d_2 - \frac{1}{2}\sigma_L^2 \approx 0.4388$ and $d_3 - \frac{1}{2}\sigma_H^2 \approx 0.3580$, which means that the conditions of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, $S_D(t)$, $S_L(t)$ and $S_H(t)$ separately tend to its steady states in the mean with probability one, and the others finally go to extinction with probability one (see Fig. 1).

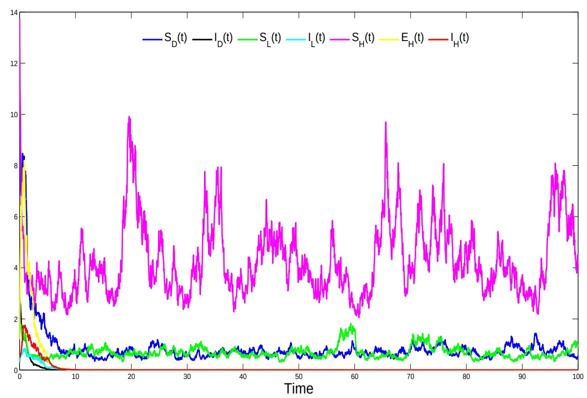


Fig. 1: The evolution of a single path of the solution for model (1.2) with initial values $S_D(0) = 10$, $I_D(0) = 3$, $S_L(0) = 2$, $I_L(0) = 0.3$, $S_H(0) = 15$, $E_H(0) = 0.16$, $I_H(0) = 0.5$.

Example 5.2 To further explore the effect of the noises intensities σ_D, σ_L and σ_H on the extinction for $I_H(t)$ in model (1.2), we mainly focus on the stochastic extinction time for $I_H(t)$. We fix the other parameter values from Example 5.1, except for σ_D, σ_L and σ_H. By using the Latin hypercube method [33] to generate 1,000 random samples, we perform the sensitivity analysis, as illustrated in Fig. 2 and Table 1, which indicate that PRCC values [34] of the extinction time for $I_H(t)$ with respect to three noise intensities. Therefore, we can find that these noise intensities have significantly negative impact for the extinction time of $I_H(t)$, moreover, the absolute value of the PRCC for σ_L is largest, which is suggested that σ_L is more sensitive than the others for the extinction of $I_H(t)$.

In the following, we discuss the influence of noise intensity σ_L on the extinction time of $I_H(t)$.

Example 5.3 We fix $\sigma_D = 0.32$ and $\sigma_H = 0.29$, the other parameter values taking as Example 5.1. With changing σ_L from 0.35 to 0.9324, we repeat 1,000 simulations and calculate the extinction time for $I_H(t)$ with different values of σ_L. The corresponding simulation results are shown in Fig. 3, which give us graphically depicting groups of numerical data through their statistical feature. It is easily seen that the extinction time for $I_H(t)$ decreases as σ_L increases.

Noise intensity	Extinction time for $I_H(t)$	
	PRCC	p value
σ_D	-0.3285	< 0.001
σ_L	-0.9839	< 0.001
σ_H	-0.8176	< 0.001

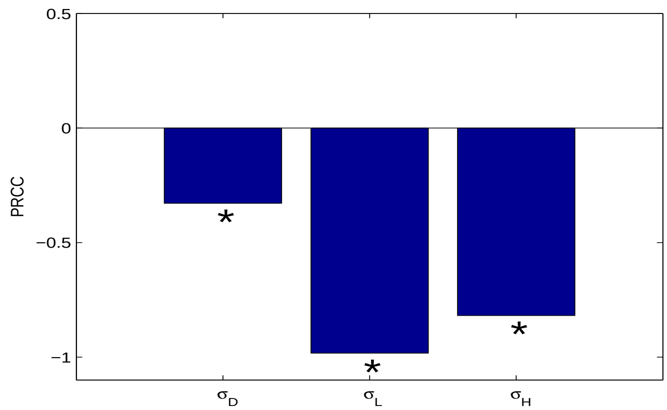
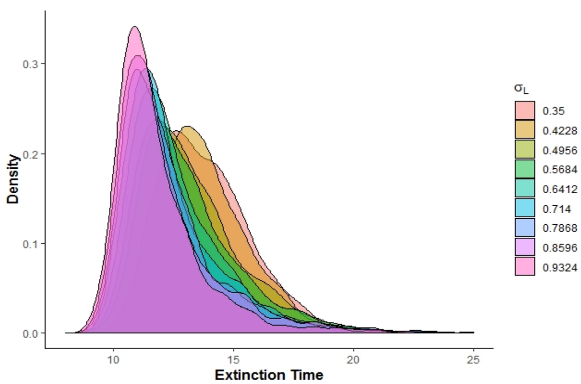


Fig. 2: The partial rank correlation coefficient (PRCC) of the stochastic extinction time for $I_H(t)$ in model (1.2) with respect to σ_D, σ_L and σ_H. * denotes the value of PRCC which is not zero significantly, where the significance level is 0.05.

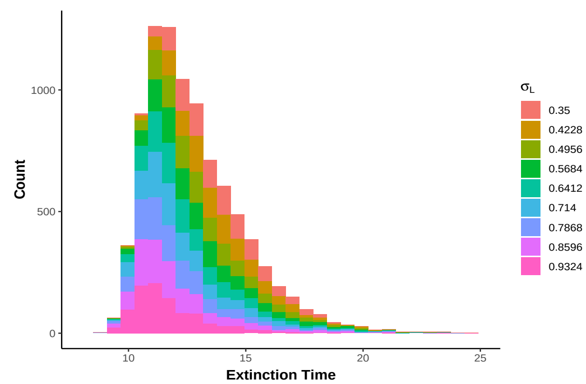
5.2. Effect of environmental noises on the distribution for model (1.2)

In Theorem 4.1, the existence of the unique ergodic stationary distribution is proved. Firstly, we provide the numerical simulation to support the theoretical result presented in Theorem 4.1. Next, we further explore the influence of different noise intensities on the stationary distribution for model (1.2).

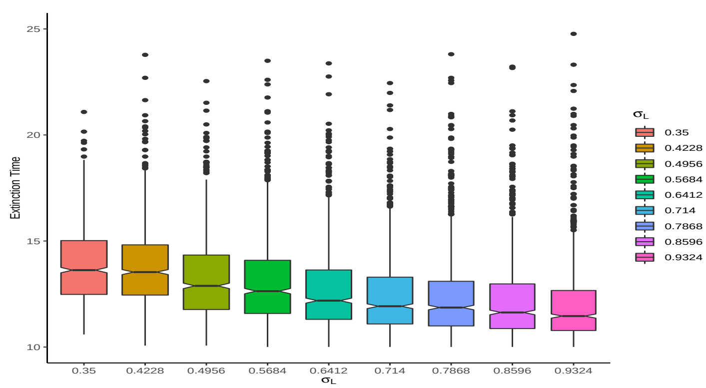
Example 5.4 Adopting $A_1 = 1.61$, $A_2 = 2.42$, $A_3 = 0.85$, $d_1 = 0.67$, $d_2 = 0.45$, $d_3 = 0.34$, $\beta_1 = 0.314$, $\beta_2 = 0.81$, $\beta_3 = 1.8$, $\mu = 0.12$, $\gamma = 0.73$, $\omega = 0.2$, $\delta = 0.29$, $\sigma_1 = 0.432$, $\sigma_2 = 0.43$, $\sigma_3 = 0.3$, $\sigma_4 = 0.15$, $\sigma_5 = 0.29$, $\sigma_6 = 0.39$ and $\sigma_7 = 0.49$. By calculating, it can be obtained that $\widetilde{\mathcal{R}}_0 \approx 2.3244$ and $(d_1 \wedge d_2 \wedge d_3) - \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2) \approx 0.22$, which are satisfied with the conditions of Theorem 4.1. Thus, by Theorem 4.1, there exists the unique ergodic stationary distribution for model (1.2). Based on the 1,000 sample paths, after iterating 10,000 times, 20,000 times and 30,000 times, respectively, we obtain three groups of density functions of the solution for model (1.2) with initial value $S_D(0) = 15$, $I_D(0) = 0.3$, $S_L(0) = 4.2$, $I_L(0) = 1.3$, $S_H(0) = 3.2$, $E_H(0) = 0.86$, $I_H(0) = 2.5$, (see Fig. 4.) In addition, based on the 1,000 sample paths, with different initial values (value 1: $S_D(0) = 5$, $I_D(0) = 3$, $S_L(0) = 0.2$, $I_L(0) = 0.3$, $S_H(0) = 4.2$, $E_H(0) = 4.16$, $I_H(0) = 7.5$; value 2: $S_D(0) = 15$, $I_D(0) = 0.3$, $S_L(0) = 4.2$, $I_L(0) = 1.3$, $S_H(0) = 3.2$, $E_H(0) = 0.86$, $I_H(0) = 2.5$; value 3: $S_D(0) = 0.5$, $I_D(0) = 1.28$, $S_L(0) = 2.7$, $I_L(0) = 0.64$, $S_H(0) = 1.82$, $E_H(0) = 2.3$, $I_H(0) = 0.9$;), respectively, we obtain three groups of density functions of the



(a)



(b)



(c)

Fig. 3: The extinction time for $I_H(t)$ of model (1.2) with different values of σ_L. (a) The probability density. (b) The histogram. (c) The box.

solution for model (1.2) after iterating 30,000 times, (see Fig. 5.) The two figures implied that no matter how many times we iterate and wherever the initial value start from, the density functions of the solution for model (1.2) seperately converge to the same functions, which indicates that there is a ergodic stationary distribution for model (1.2).

Furthermore, in order to investigate the effect of noise intensities on the stationary distribution of model (1.2), we only consider the change of noise intensities for three subgroups (dogs, the livestock and humans), for the sake of simplicity, (i.e., we fix $\sigma_1 = \sigma_2 \triangleq \sigma_D$, $\sigma_3 = \sigma_4 = \sigma_5 \triangleq \sigma_L$ and $\sigma_6 = \sigma_7 \triangleq \sigma_H$). More precisely, we only analysis the influence of the noise intensities σ_D, σ_L and σ_H on the distribution of $I_H(t)$ due to the importance of $I_H(t)$. We use the skewness and kurtosis to describe the shape for the distribution of $I_H(t)$ by the software **R**, where the skewness is a measure of the asymmetry of the probability distribution of the a real-valued random variable about its mean, and the kurtosis is a measure of the "tailedness" of the probability distribution of a real-valued random variable. We choose $A_1 = 0.61$, $A_2 = 1.42$, $A_3 = 1.8$, $d_1 = 0.15$, $d_2 = 0.27$, $d_3 = 0.34$, $\beta_1 = 0.1514$, $\beta_2 = 0.81$, $\beta_3 = 0.8$, $\mu = 0.52$, $\gamma = 0.26$, $\omega = 0.42$ and $\delta = 0.49$.

Example 5.5 Firstly, we fix $\sigma_L = 0.15$, $\sigma_H = 0.19$ and σ_D from 0.02 to 0.52 by step-size 0.0357, it is easy to be verified the conditions of Theorem 4.1 are satisfied. By repeating 1,0000 simulations at time $t = 100$, we obtain 15 groups 10,000-dimensional data, numerical solution of $I_H(t)$ for model (1.6), which can be used to study the distributions of $I_H(t)$. Fig. 6 (a) displays the skewness of the distribution $I_H(t)$ is positive all along, whereas the kurtosis is always negative with the different values of σ_D. This means that right data of the distribution for $I_H(t)$ is more scattered and the extreme data in bilateral sides is less, (the corresponding numerical simulations are given in Fig. 6 (b).) In short, the change of the shape for the distribution $I_H(t)$ is inconspicuous as σ_D increases, (see Fig. 6 (c)).

Example 5.6 In addition, we fix $\sigma_D = 0.432$, $\sigma_H = 0.19$ and σ_L from 0.05 to 0.4589 by step-size 0.0292, it is easy to be verified the conditions of Theorem 4.1 are satisfied. Similar as the example 5.5, we repeat 1,0000 simulations at time $t = 100$ and obtain the skewness and kurtosis functions of $I_H(t)$. From Fig. 7 (a), it is demonstrated that by changing σ_L from small to big, the skewness remains positive and early increases then appears to the decreasing trend in the main. The kurtosis changes sign from negative to positive and early increases then it is basically decreasing with the rise of σ_L. Fig. 7 (b) also reveals that the right data of the distribution for $I_H(t)$ is more scattered. With the increasing of σ_L, the shape of the distribution $I_H(t)$ becomes slowly shorter and more left-skewed, (see the simulation results (b) and (c) of Fig.7).

Example 5.7 Lastly, we fix $\sigma_D = 0.432$, $\sigma_L = 0.15$ and σ_H from 0.09 to 0.5233 by step-size 0.031, it is easy to be verified the conditions of Theorem 4.1 are satisfied. Similar as the example 5.5, we repeat 1,0000 simulations at time $t = 100$ and obtain the skewness and kurtosis functions of $I_H(t)$, which are both positive and decrease along with the rise of σ_H from the beginning and then become flat. (See Fig. 8 (a)). As shown in numerical simulations of Fig. 8 (b)-(c), the shape of the distribution of $I_H(t)$ changes from skyscraping to pyknic and gradually migrates towards left as the noise intensity σ_H for humans increases.

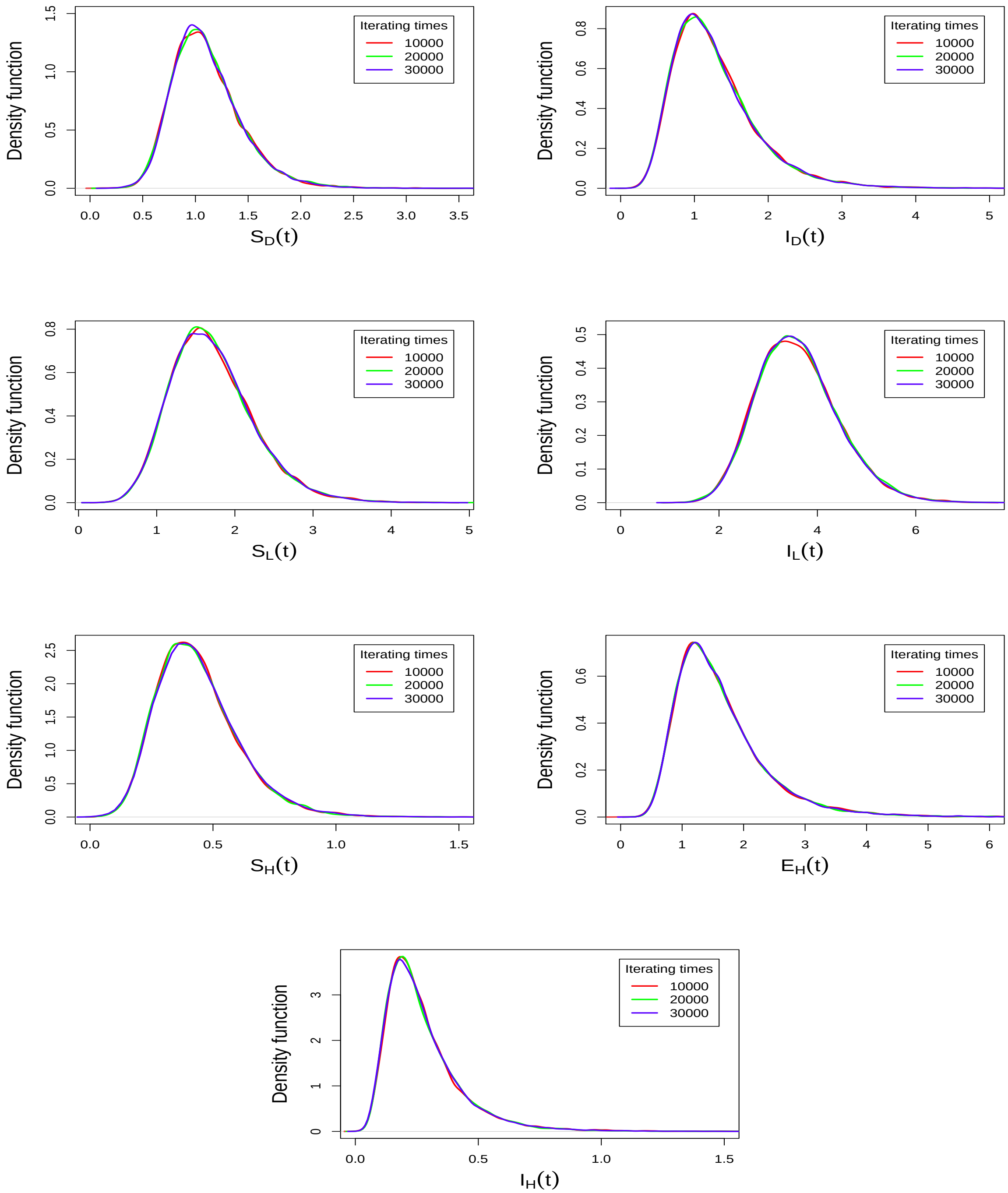


Fig. 4: The density functions of the solution for model (1.2) after iterating different times with initial value $S_D(0) = 15$, $I_D(0) = 0.3$, $S_L(0) = 4.2$, $I_L(0) = 1.3$, $S_H(0) = 3.2$, $E_H(0) = 0.86$, $I_H(0) = 2.5$.

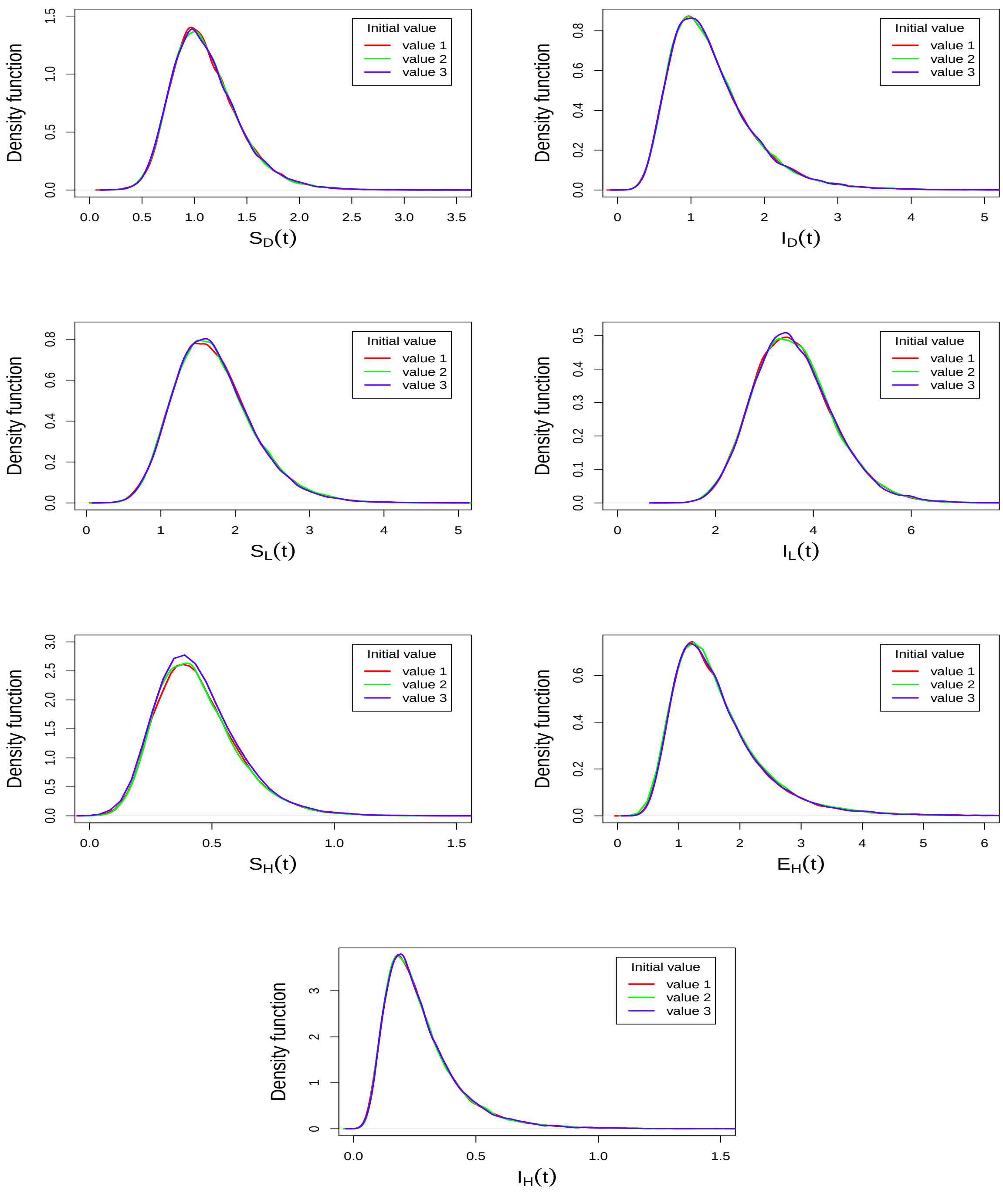


Fig. 5: The density functions of the solution for model (1.2) after iterating 30,000 times with different initial values.

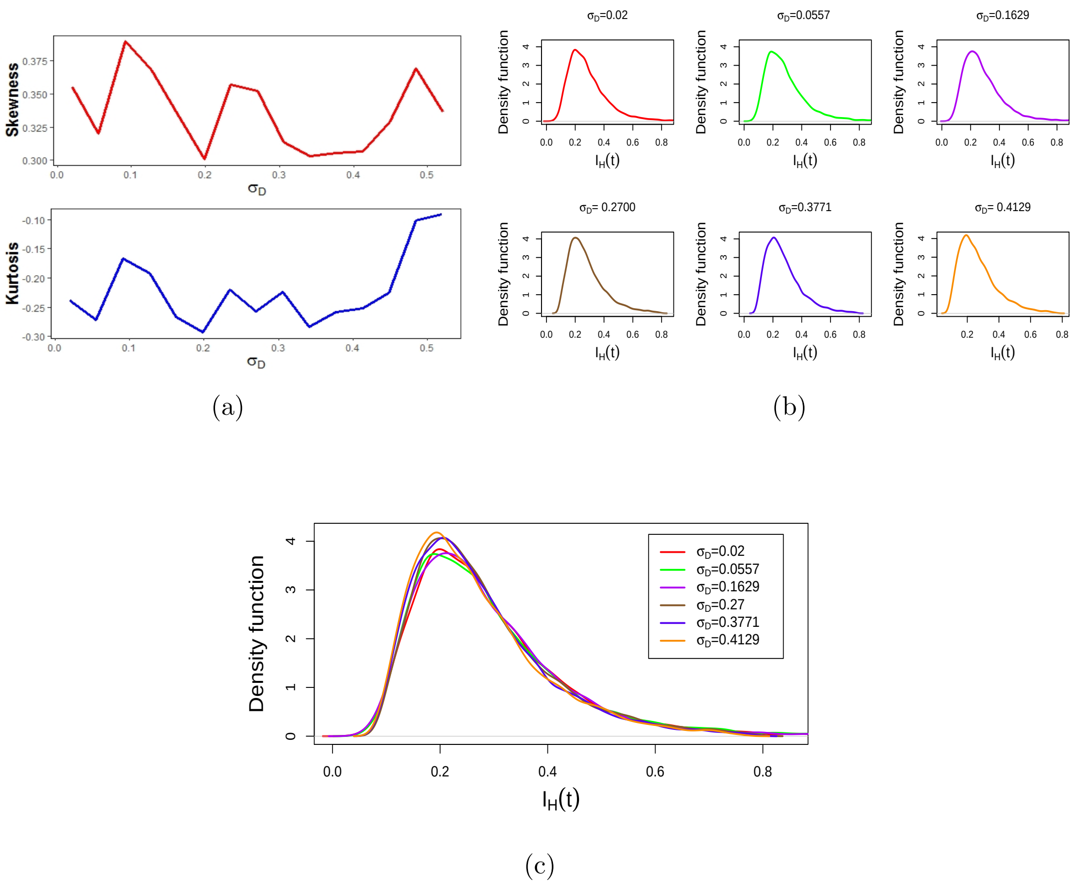
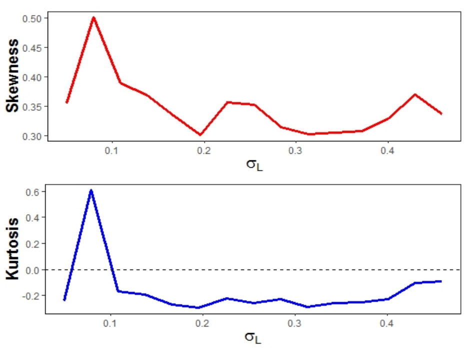
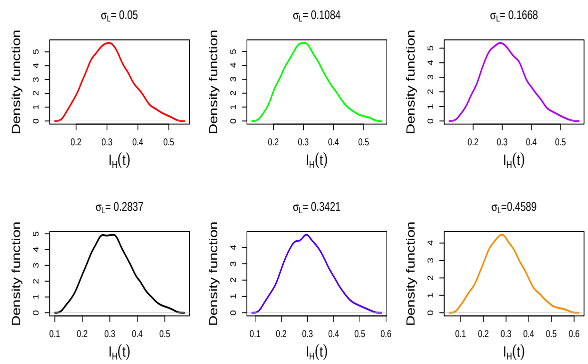


Fig. 6: Effect of the noise intensity σ_D on the distribution of $I_H(t)$, where $\sigma_L = 0.15$ and $\sigma_H = 0.19$. (a) The skewness and kurtosis for the distribution of $I_H(t)$ with respect to σ_D. (b) The density functions of $I_H(t)$ with different σ_D. (c) The relation between the distribution of $I_H(t)$ and σ_D.



(a)



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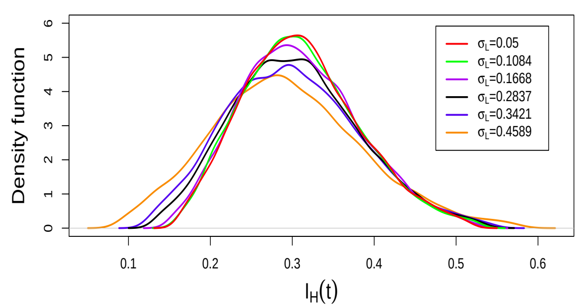
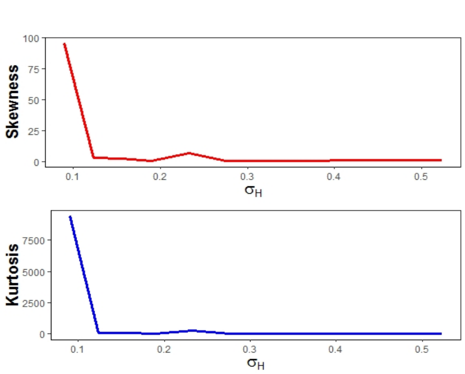
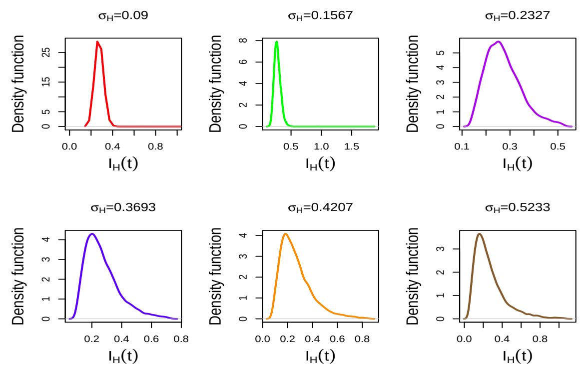


(c)

Fig. 7: Effect of the noise intensity σ_L on the distribution of $I_H(t)$, where $\sigma_D = 0.432$ and $\sigma_H = 0.19$. (a) The skewness and kurtosis for the distribution of $I_H(t)$ with respect to σ_L. (b) The density functions of $I_H(t)$ with different σ_L. (c) The relation between the distribution of $I_H(t)$ and σ_L.



(a)



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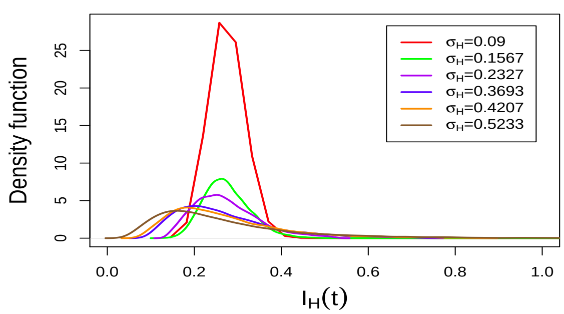

(c)

Fig. 8: Effect of the noise intensity σ_H on the distribution of $I_H(t)$, where $\sigma_D = 0.432$ and $\sigma_L = 0.15$. (a) The skewness and kurtosis for the distribution of $I_H(t)$ with respect to σ_H. (b) The density functions of $I_H(t)$ with different σ_H. (c) The relation between the distribution of $I_H(t)$ and σ_H.

6. Discussion

Considering many environmental fluctuations in the process of echinococcosis transmission (for instance, the change of season, temperature and humidity etc), in the paper, we propose the stochastic echinococcosis model incorporating environmental white noises to understand the effects of environmental driving forces on the stochastic disease-free and endemic dynamics. Sufficient conditions for the extinction of the disease and the existence of the unique ergodic stationary distribution are obtained. It is worth to notice that we could not obtain the threshold type result, that is, $\widehat{\mathcal{R}}_0$ in Theorem 3.1 and $\widetilde{\mathcal{R}}_0$ in Theorem 4.1 are unable to unite as one. This is an interesting topic deserving further consideration.

Epidemiologically, one of the highlights of this paper is to illustrate that stochastically environmental variability has a certain influence on dynamical behaviors of echinococcosis transmission by means of numerical simulations, as shown in the following. (i) in the respect of extinction: these noise intensities for three subgroups all have significantly negative impact on the extinction time for $I_H(t)$ of model (1.2), in particular, when the noise intensity σ_L for the livestock increases, the extinction time for $I_H(t)$ decreases; (ii)in the respect of stationary distribution: these noise intensities for three subgroups have the influence on the shape of the stationary distribution for $I_H(t)$ of model (1.2), through the describing of the skewness and kurtosis, where the effect of the noise intensity for humans σ_H on the alteration of the distribution shape for $I_H(t)$ is obvious, from skyscraping to pyknic and gradually migrating towards left as σ_H increasing. Certainly, although these conclusions are validated without basing on realistic parameters of echinococcosis transmission, it could still offer some help to understand the impaction of environmental noises on transmission dynamics of echinococcosis.

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References

[1] J.J. Chai, Epidemiological studies on cystic echinococcosis in ChinaCa review, Biomed. Environ. Sci. 1995, 8 (2): 122-136.

[2] T.Y. Li, J.M. Qiu, W. Yang, S.C. Philip, X.W. Chen, N. Xiao, I. Akira, G. Patrick, W. Mamuti, W. Yu, M.S. Peter, Echinococcosis in Tibetan populations, western Sichuan province, China, Emerg. Infect. Dis. 2015, 11 (12): 1866-1873.

[3] P. Liu, J.H. Li, Y. Li, J.L. Kang, Q. Dai, N. Zhao, X.D. Sun, The epidemic situation and causative analysis of Echinococcosis, China Anim. Health Inspection 1 2016, 33: 48-51.

[4] S. Wang, S. Ye, Text Book of Medical Microbiology and Parasitology, first ed., Science Press, Beijing, 2006.

[5] M.A. Gemmell, J.R. Lawson, M.G. Roberts, Population dynamics in echinococcosis and cysticercosis: biological parameters of Echinococcus granulosus in dogs and sheep, Parasitology, 1986, 92 (3): 599-620.

[6] M.G. Roberts, J.R. Lawson, M.A. Gemmell, Population dynamics in echinococcosis and cysticercosis: mathematical model of the life-cycle of Echinococcus granulosus, Parasitology 1986, 92 (3): 621-641.

[7] M.A. Gemmell, J.R. Lawson, M.G. Roberts, B.R. Kerin, C.J. Mason, Population dynamics in echinococcosis and cysticercosis: comparison of the response of Echinococcus granulosus, Taenia hydatigena and T. ovis to control, Parasitology 1986, 93(2): 357-369.

[8] M.A. Gemmell, J.R. Lawson, M.G. Roberts, Population dynamics in echinococcosis and cysticercosis: evaluation of the biological parameters of Taenia hydatigena and T. ovis and comparison with those of Echinococcus granulosus, Parasitology 1987, 93 (1): 161-180.

[9] P.R. Torgerson, B.S. Shaikenov, A.T. Rysmukhambetova, A.E. Ussenbayev, A.M. Abdybekova, K.K. Burtisurnov, Modelling the transmission dynamics of Echinococcus granulosus in dogs in rural Kazakhstan, Parasitology, 2003, 126 (5): 417-424.

[10] P.R. Torgerson, K.K. Burtisurnov, B. Shaikenov, A.T. Rysmukhambetova, A.M. Abdybekova, A.E. Ussenbayev, Modelling the transmission dynamics of Echinococcus granulosus in sheep and cattle in Kazakhstan, Vet. Parasitol, 2003, 114 (2): 143-153.

[11] S. Lahmar, H. Debbek, L.H. Zhang, D.P. McManus, A. Souissi, S. Chelly, P.R. Torgerson, Transmission dynamics of the Echinococcus granulosus sheep-dog strain (G1 genotype) in camels in Tunisia, Vet. Parasitol, 2004, 121 (1): 151-156.

[12] P.R. Torgerson, Mathematical models for the control of cystic echinococcosis, Parasitol. Int, 2006, 55: 253-258.

[13] K. Wang, X. Zhang, Z. Jin, H. Ma, Z. Teng, L. Wang, Modeling and analysis of the transmission of Echinococcosis with application to Xinjiang Uygur Autonomous Region of China, Journal of Theoretical Biology, 2013, 333: 78-90.

[14] L. Wu, B. Song, W. Du and J. Lou, Mathematical modelling and control of Echinococcus in Qinghai province, China, Math. Biosci. Eng., 2013, 10: 425-444.

[15] K. Wang , Z. Teng, X. Zhang. Dynamical behaviors of an Echinococcosis epidemic model with distributed delays. Mathematical Biosciences and Engineering, 2017, 14(5/6):1425-1445.

[16] X. Rong, M. Fan, X. Sun, Y. Wang, H. Zhu, Impact of disposing stray dogs on risk assessment and control of Echinococcosis in Inner Mongolia. Mathematical Biosciences, 2018, 299: 85-96.

[17] J. R. Beddington, R. M. May, Harvesting natural populations in a randomly fluctuating environment. Science, 1977, 197 (4302): 463-465.

[18] Allen L J S. An introduction to stochastic epidemic models. Lecture Notes in Mathematics Springer Verlag, 2008, 1945 (4): 81-130.

[19] X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics. Stochastic Processes and their Applications, 2002, 97 (1): 95-110.

[20] D.A. Rand, H.B. Wilson, Chaotic stochasticity: A ubiquitous source of unpredictability in epidemics, Proc. R. Soc. A 1991, 246: 179-184.

[21] L.J.S. Allen, An Introduction to Stochastic Processes with Applications to Biology, Prentice Hall, Upper Saddle River, NJ, 2003.

[22] M. Carletti, K. Burrage, P.M. Burrage, Numerical simulation of stochastic ordinary differential equations in biomathematical modelling, Math. Comput. Simul. 2004, 64: 271-277.

[23] Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, Applied Mathematics and Computation, 2014, 243:718-727.

[24] A. Berman, R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. Academic press, New York, 1979.

[25] Y. A. Kutoyants, Statistical Inference for Ergodic Diffusion Processes, Springer, London, UK, 2004.

[26] X. Mao. Stochastic differential equations and their applications. Horwood Publication, Chichester. 1997.

[27] N. Ikeda, S.Watanabe, A comparison theoremfor solutions of stochastic differential equations and its applications, Osaka Journal of Mathematics, 1977, 14 (3): 619-633.

[28] T. C. Gard, Introduction to Stochastic Differential Equations, 270 Madison Avenue, New York. 1988.

[29] G. Strang, Linear Algebra and Its Applications, Thomson Learning, Singapore. 1988.

[30] C. Zhu, G. Yin, Asymptotic properties of hybrid diffusion systems, SIAM Journal on Control and Optimization. 2007, 46: 1155-1179.

[31] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Review 2001, 43: 525-546.

[32] Y. Cai, Y. Kang, M. Banerjee, W. Wang. A stochastic SIRS epidemic model with infectious force under intervention strategies. *Journal of Differential Equations*, 2015, 259 (12): 7463-7502.

[33] R. L. Iman, J. C. Helton, J. E. Campbell, An approach to sensitivity analysis of computer models: I-introduction, input variable selection and preliminary variable assessment. Journal of Quality Technology, 1981, 13: 174-183.

[34] S.Marino, I. B. Hogue, C. J. Ray, A methodology for performing global uncertainty and sensitivity analysis in systems biology. Journal of Theoretical Biology, 2008, 254: 178-196.