Analytical Method for solving fractional order generalized KdV Equation by beta-fractional derivative

Majid Bagheri* and **Ali Khani**†

Faculty of Science, Department of Applied Mathematics,

Azarbaijan Shahid Madani University, Tabriz, Iran

(Dated: July 11, 2020)

The present work is related to solve the fractional generalized Korteweg-de Vries (gKdV) equation in fractional time derivative form of order α. Some exact solutions of the fractional-order gKdV equation is attained by employing the new powerful expansion approach by using the beta-fractional derivative which is used to get many solitary wave solutions by changing the various parameters. The obtained solutions include three classes of soliton wave solutions in terms of hyperbolic function, trigonometric function, and rational function solutions. The obtained solutions and the exact solutions are shown graphically, highlighting the effects of non-linearity. Many other such types of nonlinear equations arising in fluid dynamics and nonlinear phenomena.

Keywords: Fractional generalized Korteweg-de Vries (gKdV) equation; Beta-fractional derivative; New powerful expansion approach

PACS numbers:

I. INTRODUCTION

The differential equation of fractional order is the new form of classical integer order differential equations. Different types of differential equations of both ordinary differential equations (ODEs) and partial differential equations (PDEs) in the various fields of science like fluid, mechanics, biology etc, are expressed in fractional forms [1]. There is no any particular method for accessing the exact type solutions of fractional PDEs but some approximate solutions are determined by using the method Adomain decomposition approach, the homotopy perturbation approach and the homotopy analysis approach etc. [2–4]. The analytical method and its various forms are well known approach for determining solitons wave type solutions of the nonlinear PDEs. With the chronology, some investigators have utilized the new analytical approach on fractional type nonlinear PDEs for obtaining the solitary solutions. The work is related to the fractional order generalized Korteweg-de Vries (gKdV) equation [5], for this the fractional form of gKdV equation [5] is taken as

$$u_t^\alpha + F(u)u_x + u_{xxx} = 0, \quad 0 \leq \alpha < 1, \quad 0 < d < 4, \quad F(u) = \lambda u^d + \mu u^{2d}, \quad \lambda, \mu \in R. \tag{1}$$

Equation (1) is fractional form of classical generalized Korteweg-de Vries equation which exists by changing the Ist-order time derivatives by fractional derivatives.

In the past two decades, the fractional calculus theory gained a great attention and popularity in the various fields of science and engineering due to its demonstrated applications. These contributions to the fields of sciences and

* Email: majid.bagheri@azaruniv.ac.ir

† Corresponding author. Email: khani@azaruniv.ac.ir

engineering are based on the mathematical analysis. It covers the widely known classical fields such as Abel's integral equation and viscoelasticity. Also, including the analysis of feedback amplifiers, fractional-order Chua-Hartley systems, electrode-electrolyte interface models, fractional-order models of neurons, electric conductance of biological systems, generalized voltage dividers, fitting of experimental data, capacitor theory and the fields of special functions [6–9].

Several robust methods have been used to solve the FDEs, the fractional differential equations and dynamic systems containing fractional derivatives. Some of the most important methods are the Adomian's decomposition method [10–12], the exp-function method [13], the He's the variational iteration method [14, 15], the fractional sub-equation method [16], the first integral method [17], the homotopy analysis method [18], the (G'/G)-expansion method [19], the homotopy perturbation method [20, 21], the spectral methods [22], the transform methods [23]. In [24], the authors presented two methods, which are the exp(-$\phi(\xi)$)-expansion method and Kudryashov method. In [25], the authors demonstrated three methods, which are csch function method, tanh-coth method and modified simple equation method. In [26–29], the authors introduced the semi-inverse variational principle method, the extended Kudryashov's method, the modified simple equation method and the expanded trail equation method respectively. Moreover, the fractional differential equations have been studied by powerful authors and introduced the applications in sciences and engineering branches [30–32].

One of well-known equations is the ZK equation, first obtained as a description of weakly nonlinear ion-acoustic modes in a strongly magnetized plasma, is of particular interest as it is the simplest equation that admits cylindrical and spherical solitary wave solutions in addition to the planar KdV soliton solutions [33]. Another powerful analytical method is called the Exp-function method (EFM), which was first presented by He [34]. The EFM has successfully been applied to many situations. For example, He [34] solved the nonlinear wave equations via the EFM. Abdou [35] solved generalized solitonary and periodic solutions for nonlinear partial differential equations by the EFM. For further information refer to vigorous references therein ([35–40]). In the following years, this proposed method was improved by many researchers. Yang developed general fractional derivatives along with theory, methods and applications, to some nonlinear fractional differential equations [41]. Recently A new fractal nonlinear Burgers' equation in which arising in the acoustic signals propagation studied by Yang [42]. Also, Yang et al investigated fundamental solutions of anomalous equation and implemented with the decay exponential kernel [43]. A new integral transform operator for solving the heat-diffusion problem has been utilized by Yang [44]. In [45], Liu and co-workers probed the group analysis to the time fractional nonlinear wave equation and found the many exact solutions. Moreover, Liu and other collaborators worked on time-fractional nonlinear diffusion equation [46]. Same authors proposed the fractional symmetry group method for time fractional nonlinear heat conduction equation which usually appears in mathematics physics, integrable system, fluid mechanics and nonlinear areas [47].

The pattern of this article is summarized as: In Section-II, the detail of initial definitions is given, In Section-III is utilized an introduction of the direct truncation method. Also, for getting the exact solutions of the gKdV equation in section-IV. In section V, the numerical simulation and details of graph and some conclusions are given in the end in section VI.

II. INITIAL DEFINITIONS

Definition II.1 *Definition of β-derivative: Suppose $\varphi : [0;1) \rightarrow \mathbf{R}$, then the β derivative of φ of order α is defined as*

$$D_t^\alpha(\varphi)(t) = \lim_{\epsilon \to 0} \frac{\varphi\left[t + \epsilon\left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right] - \varphi(t)}{\epsilon}, \quad \alpha \in (0,1], \quad t > 0. \tag{2}$$

The properties and new theorems will used as follow:

Theorem II.2 *Suppose $\alpha \in (0,1]$; φ, ω be α-differentiable at a point t, therefore we will*

1. $D_t^\alpha(a\varphi(t) + b\omega(t)) = aD_t^\alpha(\varphi(t)) + bD_t^\alpha(\omega(t)), \quad for \quad a, b \in \mathbf{R}$.
2. $D_t^\alpha(c) = 0, \quad for \quad c \in \mathbf{R}$.
3. $D_t^\alpha(\varphi(t)\omega(t)) = \varphi(t)D_t^\alpha(\omega(t)) + \omega(t)D_t^\alpha(\varphi(t))$.
4. $D_t^\alpha(\frac{\varphi(t)}{\omega(t)}) = \frac{\varphi(t)D_t^\alpha(\omega(t)) - \omega(t)D_t^\alpha(\varphi(t))}{\omega^2(t)}$.
5. $D_t^\alpha\varphi(t) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\frac{d\varphi(t)}{dt}$.

Theorem II.3 *[48–50] Suppose $\varphi : [0;1) \rightarrow \mathbf{R}$; be a function such that φ is differentiable and also α-differentiable. Assume ω be a differentiable function defined in the range of φ. Therefore, we have*

$$D_t^\alpha(\varphi o \omega)(t) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\omega'(t)\chi'(\omega(t)), \tag{3}$$

where prime denotes the classical derivatives with respect to t.

III. METHODOLOGY

In this section, we give a description for the direct truncation method and introduce it for partial differential equation.

For a given partial differential equation

$$P(u, u_x, u_{xx}, ..., D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u, ...) = 0, \qquad 0 < \alpha \leq 1. \tag{4}$$

Using a transformation as

$$u(x,t) = u(\phi), \qquad \eta = kx + \frac{l}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha, \tag{5}$$

where k and l are constants to be determined later, we can rewrite equation Eq. (4) in the following nonlinear ODE

$$Q(u, ku', k^2u'', ..., lu', ...) = 0, \tag{6}$$

where the prime denotes derivative with respect to ϕ. If possible, integrate Eq. (6) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero. Suppose g has the following truncation form

$$g(\phi) = \frac{\sum_{j=0}^{\tau} a_j \xi(\phi)^j}{\zeta(\phi)^\tau}, \tag{7}$$

in which $\xi(\phi)$ and $\zeta(\phi)$ are introduced as below form

$$\xi(\phi) = p_1 F(\chi(\phi)) + q_1 G(\chi(\phi)) + r_1,$$

$$\zeta(\phi) = p_2 F(\chi(\phi)) + q_2 G(\chi(\phi)) + r_2,$$

$$u(\xi) = g(\phi) = \frac{\sum_{j=0}^{\tau} a_j \left(p_1 F(\chi(\phi)) + q_1 G(\chi(\phi)) + r_1\right)^j}{\left(p_2 F(\chi(\phi)) + q_2 G(\chi(\phi)) + r_2\right)^\tau}, \tag{8}$$

where $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ are constants to be determined, $\chi(\phi)$ is given and F, G are functions determined by an ordinary differential system, or F, G are functions given by direct ansatz such that their derivations are combinations of F and G, and $\chi(\phi)$ is determined by an ordinary differential equation

$$\frac{d\chi(\phi)}{d\phi} = H(\chi(\phi)) = LF(\chi(\phi)) + MG(\chi(\phi)) + N, \tag{9}$$

in which the function H is also given by a direct ansatz according to the context, the exponent τ is determined by utilizing homogeneous balance method in Eq. (4). The value τ is determined by equalizing the maximum order non-linear term and the maximum order partial derivative term appearing in (6). If τ is the rational, then the appropriate transformations can be applied to conquer these hurdles. Substituting (8), (9) into (7) leads to a polynomial in $F(\phi)$ and $G(\phi)$, then set the coefficients of $F^i(\phi)G^j(\phi)$ and the constant term to be zero to get a system of algebraic equations on the unknown parameters in H together with the unknown numbers $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ for $j = 0, 1, ..., \tau$, by solving the system one can get $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ and the unknown parameters in H, then solving Eq. (9) to get $\chi(\phi)$ and the solutions of Eq. (4) can be obtained.

IV. TRAVELING WAVE SOLUTION FRACTIONAL ORDER FORM OF GENERALIZED KDV EQUATION

The given section deals with application of new powerful expansion technique by determining the traveling wave form solutions of fractional order generalized KdV equation,

$$u_t^\alpha + F(u)u_x + u_{xxx} = 0, \quad F(u) = \lambda u^d + \mu u^{2d}, \quad \lambda, \mu \neq 0, \tag{10}$$

applying aforementioned method. By using the fractional beta complex transform $\eta = kx + \frac{l}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha$, Eq. (10) is reduced to an ODE as

$$lu' + (\lambda u^d + \mu u^{2d})ku' + k^3 u''' = 0. \tag{11}$$

Integrating Eq. (11) once and setting the constant of integration equal to zero, results in

$$lu + \frac{k\lambda}{d+1}u^{d+1} + \frac{k\mu}{d+1}u^{2d+1} + k^3 u'' = 0. \tag{12}$$

Balancing the u'' and u^{2d+1} by employing the homogenous principle, we get

$$M + 2 = (2d+1)M, \qquad \Rightarrow M = \frac{1}{d}. \tag{13}$$

To get a closed form solution, we use the transformation

$$u(\eta) = v(\eta)^{\frac{1}{d}}. \tag{14}$$

Substituting (14) into Eq. (12), we get

$$lv^2+\frac{k\lambda}{d+1}v^3+\frac{k\mu}{2d+1}v^4+\frac{k^3}{d}\left(\frac{1-d}{d}v'^2+vv''\right)=0. \tag{15}$$

Balancing the vv'' and v^4, we get

$$2M+2=4M, \qquad \Rightarrow M=1. \tag{16}$$

A. Case I:

Then the exact solution will be as

$$v(\eta)=\frac{\mathrm{e}^{2\,\chi(\phi)}a_1p_1+\mathrm{e}^{\chi(\phi)}a_1q_1+a_1r_1+a_0}{p_2\mathrm{e}^{2\,\chi(\phi)}+q_2\mathrm{e}^{\chi(\phi)}+r_2}. \tag{17}$$

Inserting (17) in to Eq. (15), we obtain

$$\left(\mathrm{d}^2(\mathrm{d}+1)(2\mathrm{d}+1)\left(\mathrm{p}_2\mathrm{e}^{2\,\chi(\phi)}+\mathrm{q}_2\mathrm{e}^{\chi(\phi)}+\mathrm{r}_2\right)^4\right)^{-1}\sum_{\mathrm{n}=0}^{11}\mathrm{C}_\mathrm{n}\exp(\mathrm{n}\chi\,(\phi))=0, \tag{18}$$

where $\mathrm{C}_\mathrm{n}(0\le \mathrm{n}\le 11)$ are polynomial statements in terms of $a_0,a_1,p_1,p_2,q_1,q_2,r_1$ and r_2. Hence, solving the resulting system $\mathrm{C}_\mathrm{n}=0(0\le \mathrm{n}\le 11)$ simultaneously, we acquire the below set of parameters of solutions

Set I:

$$L=0,\quad M=0,\quad N=\frac{1}{2}\,\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k},\quad k=k,\quad l=\frac{\lambda^2\,(2\,d+1)\,k}{\mu\,(d^3+5\,d^2+8\,d+4)},\quad a_0=-\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu}, \tag{19}$$

$$a_1=0,\quad p_1=p_1,\quad p_2=p_2,\quad q_1=q_1,\quad q_2=0,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_1(\phi)=\left(-\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu\,\left(p_2\mathrm{e}^{2\,\chi(\phi)}+r_2\right)}\right)^{\frac{1}{d}},\quad \chi\,(\phi)=\frac{1}{2}\,\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}(\phi+C), \tag{20}$$

in which

$$\phi=kx+\frac{\lambda^2\,(2\,d+1)\,k}{\mu\alpha\,(d^3+5\,d^2+8\,d+4)}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{21}$$

Set II:

$$L=0,\quad M=0,\quad N=\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k},\quad k=k,\quad l=\frac{\lambda^2\,(2\,d+1)\,k}{\mu\,(d^3+5\,d^2+8\,d+4)},\quad a_0=-\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu}, \tag{22}$$

$$a_1=0,\quad p_1=p_1,\quad p_2=0,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_2(\phi)=\left(-\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu\,\left(q_2\mathrm{e}^{\chi(\phi)}+r_2\right)}\right)^{\frac{1}{d}},\quad \chi\,(\phi)=\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}(\phi+C), \tag{23}$$

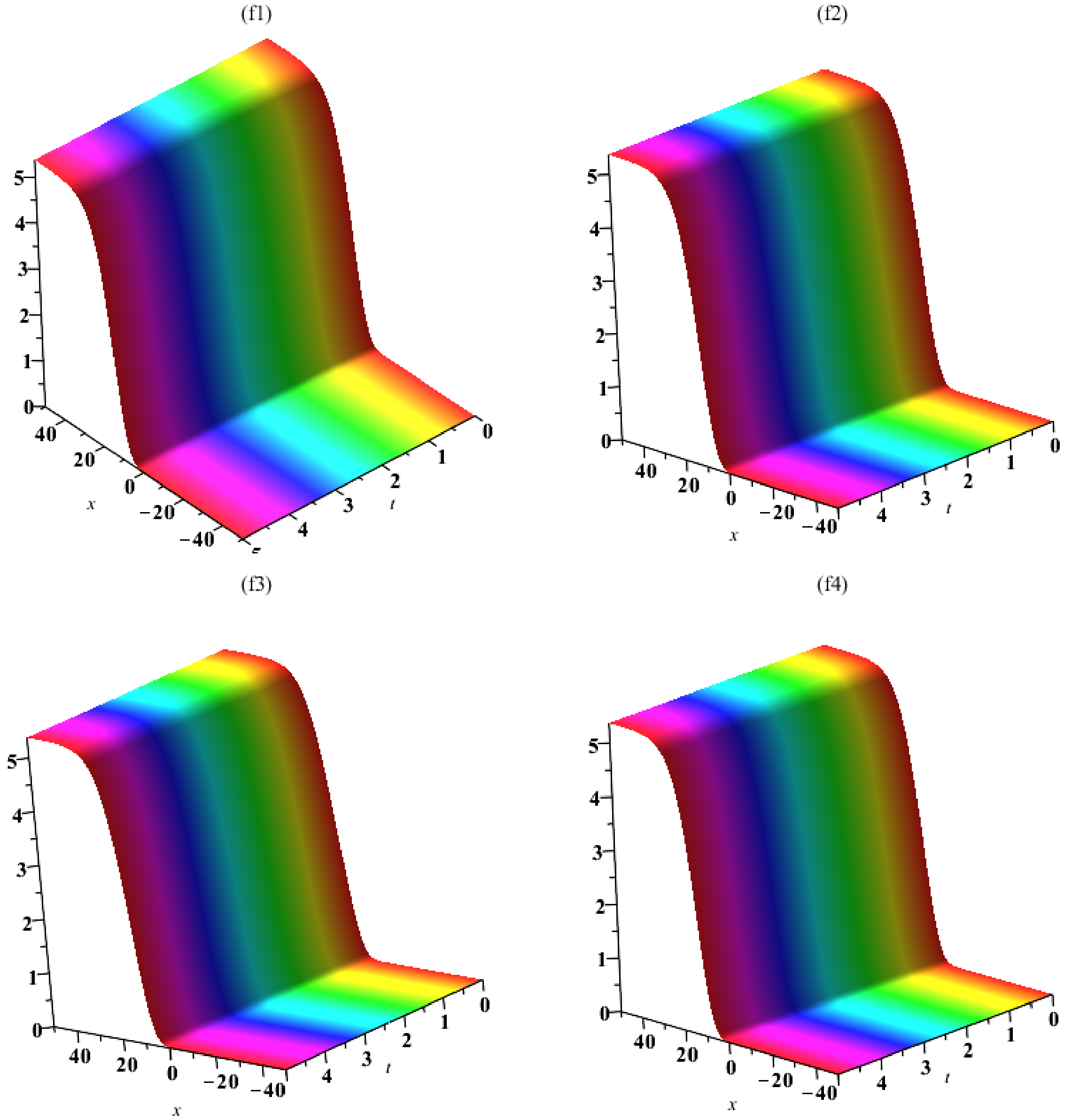

FIG. 1: The 3D plot of (20) at $d = 0.2, \mu = -1, p_2 = 1.5, r_2 = 2, \lambda = 2.2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

in which

$$\phi = kx + \frac{\lambda^2 (2d+1) k}{\mu\alpha (d^3 + 5d^2 + 8d + 4)} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{24}$$

Set III:

$$L = 0, \quad M = M, \quad N = N, \quad k = \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) N}, \quad l = \frac{\lambda^3 (2d+1) \sqrt{-\mu (d+1)(2d+1)} d}{\mu^2 (d+1)(d^3 + 5d^2 + 8d + 4)(d+2) N}, \tag{25}$$

$$a_0 = -\frac{(Mq_2 - Np_2)\lambda (2d+1) r_1}{N (d+2) \mu p_1}, \; a_1 = \frac{(Mq_2 - Np_2)\lambda (2d+1) r_1}{N (d+2) \mu p_1}, \; p_1 = p_1, \; p_2 = p_2, \; q_1 = 0, \; q_2 = q_2, \; r_1 = r_1, \; r_2 = 0.$$

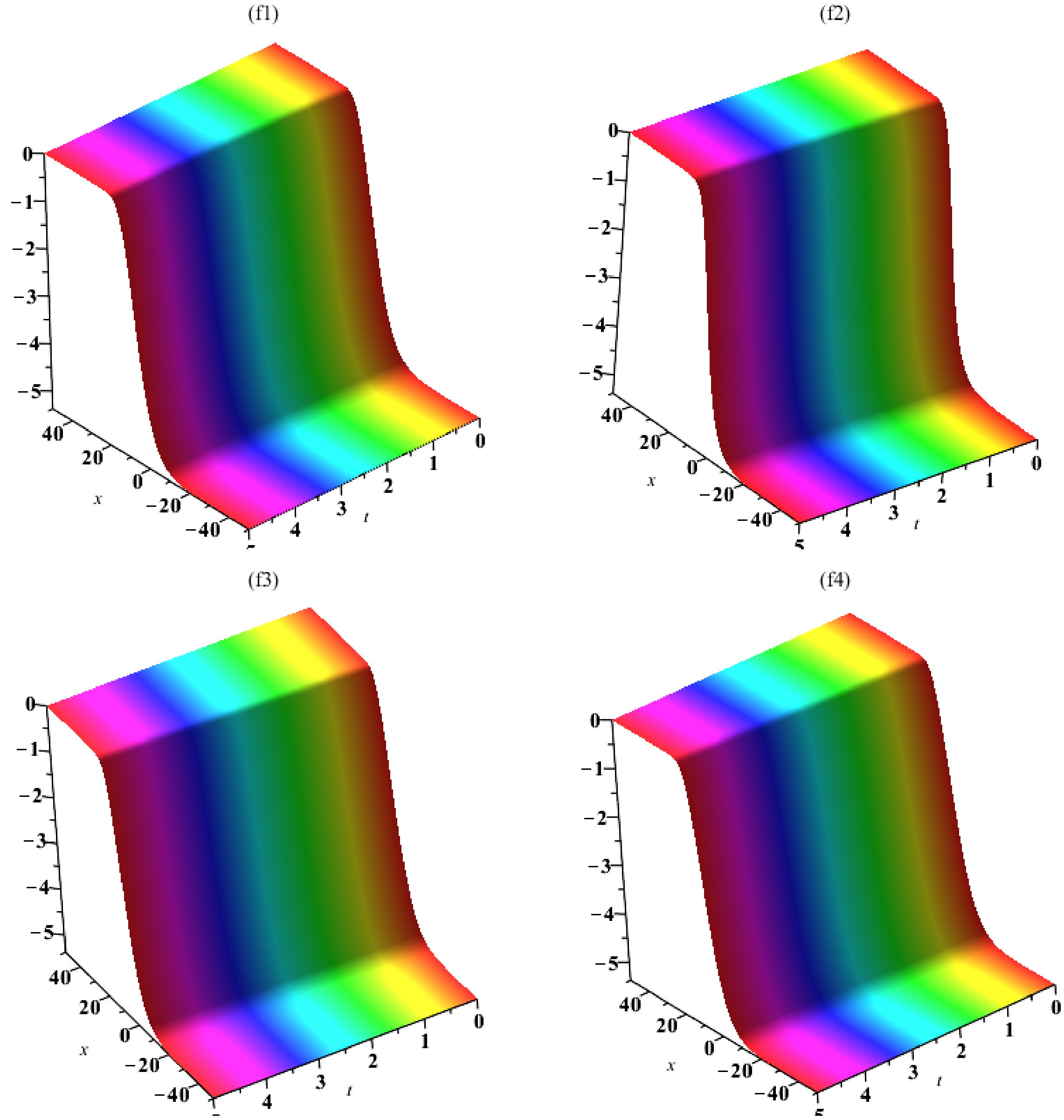

FIG. 2: The 3D plot of (26) at $d = 0.2, \mu = -1, p_2 = 1.5, q_2 = 2, \lambda = 2.2, M = -3, N = 2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

We, therefore, gained the following generalized solitary solution

$$u_3(\phi) = \left(\frac{(Mq_2 - Np_2)\,\lambda\,(2\,d+1)\,e^{\chi(\phi)}}{N\,(d+2)\,\mu\,\left(e^{\chi(\phi)}p_2 + q_2\right)} \right)^{\frac{1}{d}}, \quad \chi(\phi) = N(\phi + C) + \ln\left(\frac{N}{1 - M\exp(N(\phi + C))} \right), \tag{26}$$

in which

$$\phi = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,N}x + \frac{\lambda^3\,(2\,d+1)\,\sqrt{-\mu\,(d+1)\,(2\,d+1)}d}{\mu^2\alpha\,(d+1)\,(d^3+5\,d^2+8\,d+4)\,(d+2)\,N}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{27}$$

Set IV:

$$L = 0, \quad M = M, \quad N = \frac{1}{2}\,\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}, \quad k = k, \quad l = \frac{\lambda^2\,(2\,d+1)\,k}{\mu\,(d^3+5\,d^2+8\,d+4)}, \quad r_1 = r_1, \quad r_2 = r_2, \tag{28}$$

$$a_0=\frac{\left(4\,M^2d^3k^2\mu\,r_2+20\,M^2d^2k^2\mu\,r_2+32\,M^2dk^2\mu\,r_2+16\,M^2k^2\mu\,r_2+2\,d^3\lambda^2p_2+d^2\lambda^2p_2\right)r_1}{(d+2)\,\mu\,d^2\lambda\,p_1},\ \ p_1=p_1,\ \ p_2=p_2,\ \ q_1=0,$$

$$a_1=-\frac{4\,M^2d^3k^2\mu\,r_2+20\,M^2d^2k^2\mu\,r_2+32\,M^2dk^2\mu\,r_2+16\,M^2k^2\mu\,r_2+2\,d^3\lambda^2p_2+d^2\lambda^2p_2}{(d+2)\,\mu\,d^2\lambda\,p_1},\ \ q_2=4\,\frac{r_2M\,(d+2)\,k\mu\,\,(d+1)}{\sqrt{-\mu\,\,(d+1)\,(2\,d+1)}\lambda\,d}$$

We, therefore, gained the following generalized solitary solution

$$u_4(\phi)=-\frac{1}{2}\frac{\left(4\,M^2k^2\mu\,r_2\,(d+1)\,(d+2)^2+d^2\lambda^2p_2\,(2\,d+1)\right)\sqrt{-\mu\,\,(d+1)\,(2\,d+1)}\mathrm{e}^{2\,\chi(\phi)}}{\mu^2\,(d+1)\,d\,(d+2)\left(1/2\,\frac{\sqrt{-\mu\,(d+1)(2\,d+1)}\lambda\,d\left(p_2\mathrm{e}^{2\,\chi(\phi)}+r_2\right)}{\mu\,(d+1)}+2\,r_2Mk\mathrm{e}^{\chi(\phi)}\,(d+2)\right)},\tag{29}$$

in which

$$\chi\,(\phi)=N(\phi+C)+\ln\left(\frac{N}{1-M\exp(N(\phi+C))}\right),\qquad \phi=kx+\frac{\lambda^2\,(2\,d+1)\,k}{\alpha\mu\,\,(d^3+5\,d^2+8\,d+4)}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}.\tag{30}$$

Set V:

$$L=0,\ \ M=M,\ \ N=N,\ \ k=k,\ \ l=-\frac{N^2k^3}{d^2},\ a_0=\frac{\left(2\,Md^2r_2-Nd^2q_2+6\,Mdr_2-3\,Ndq_2+4\,Mr_2-2\,Nq_2\right)Nk^2r_1}{q_1d^2\lambda},\tag{31}$$

$$a_1=-\frac{\left(2\,Md^2r_2-Nd^2q_2+6\,Mdr_2-3\,Ndq_2+4\,Mr_2-2\,Nq_2\right)Nk^2}{q_1d^2\lambda},\ \ p_1=\frac{Mq_1}{N},\ q_1=q_1,\ q_2=q_2,\ r_1=r_1,\ r_2=r_2,$$

$$p_2=\frac{4\,Mk^2\mu\,r_2\,(d+1)\,(d+2)^2\,(Mr_2-Nq_2)+{q_2}^2\left(N^2d^3k^2\mu+5\,N^2d^2k^2\mu+8\,N^2dk^2\mu+4\,N^2k^2\mu+2\,d^3\lambda^2+d^2\lambda^2\right)}{4d^2\lambda^2r_2\,(2\,d+1)},$$

We, therefore, gained the following generalized solitary solution

$$u_5(\phi)=\Big\{-\left(4\,\,(d+2)\,(d+1)\,(2\,Mr_2-Nq_2)\,k^2\lambda\,\,(2\,d+1)\,r_2\mathrm{e}^{\chi(\phi)}\left(M\mathrm{e}^{\chi(\phi)}+N\right)\right)\tag{32}$$

$$/\ \mathrm{e}^{2\,\chi(\phi)}\left(4\,M^2d^3k^2\mu\,{r_2}^2-4\,MNd^3k^2\mu\,q_2r_2+N^2d^3k^2\mu\,{q_2}^2+20\,M^2d^2k^2\mu\,{r_2}^2-20\,MNd^2k^2\mu\,q_2r_2+5\,N^2d^2k^2\mu\,{q_2}^2+\right.$$

$$\left.32\,M^2dk^2\mu\,{r_2}^2-32\,MNdk^2\mu\,q_2r_2+8\,N^2dk^2\mu\,{q_2}^2+16\,M^2k^2\mu\,{r_2}^2-16\,MNk^2\mu\,q_2r_2+4\,N^2k^2\mu\,{q_2}^2+2\,d^3\lambda^2{q_2}^2+d^2\lambda^2{q_2}^2\right)+$$

$$4\,q_2\mathrm{e}^{\chi(\phi)}d^2\lambda^2r_2\,(2\,d+1)+4\,d^2\lambda^2{r_2}^2\,(2\,d+1)\Big\}^{\frac{1}{d}},$$

in which

$$\chi\,(\phi)=N(\phi+C)+\ln\left(\frac{N}{1-M\exp(N(\phi+C))}\right),\qquad \phi=kx-\frac{N^2k^3}{\alpha d^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}.\tag{33}$$

Set VI:

$$L=0,\ \ \ M=-\frac{(2\,d+1)\,d\lambda\,p_1}{q_1k\sqrt{-\mu\,\,(d+1)\,(2\,d+1)}\,(d+2)},\ \ \ N=\frac{\sqrt{-\mu\,\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,\,(d+1)\,(d+2)\,k},\ \ \ k=k,\ \ \ l=\frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,\,(d+2)^2},\tag{34}$$

$$a_0=\frac{\lambda\,q_2\,(2\,d+1)\,r_1}{\mu\,q_1\,(d+2)},\ \ \ a_1=-\frac{\lambda\,q_2\,(2\,d+1)}{\mu\,q_1\,(d+2)},\ \ p_1=p_1,\ \ p_2=p_2,\ \ q_1=q_1,\ \ q_2=q_2,\ \ r_1=r_1,\ \ r_2=0.$$

We, therefore, gained the following generalized solitary solution

$$u_6(\phi) = \left\{ -\frac{\lambda\, q_2\, (2\,d+1)\, \left(\mathrm{e}^{\chi(\phi)} p_1 + q_1\right)}{\mu\, q_1\, (d+2)\, \left(\mathrm{e}^{\chi(\phi)} p_2 + q_2\right)} \right\}^{\frac{1}{d}}, \tag{35}$$

in which

$$\chi\,(\phi) = N(\phi + C) + \ln\left(\frac{N}{1 - M\exp(N(\phi + C))}\right), \qquad \phi = kx + \frac{k\lambda^2\,(2\,d+1)}{\alpha\,(d+1)\,\mu\,(d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{36}$$

Set VII:

$$L = 0, \quad M = M, \quad N = N, \quad k = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\, d}{\mu\,(d+1)\,(d+2)\,N}, \quad l = \frac{\lambda^3\,(2\,d+1)\,\sqrt{-\mu\,(d+1)\,(2\,d+1)} d}{\mu^2\,(d+1)\,(d^3 + 5\,d^2 + 8\,d + 4)\,(d+2)\,N}, \tag{37}$$

$$a_0 = -\frac{(Mq_1r_2 + Np_1r_2 - Nq_1q_2)\,\lambda\,(2\,d+1)\,r_1}{N\,(d+2)\,\mu\, {q_1}^2}, \quad a_1 = \frac{(Mq_1r_2 + Np_1r_2 - Nq_1q_2)\,\lambda\,(2\,d+1)}{N\,(d+2)\,\mu\, {q_1}^2},$$

$$p_1 = p_1,\ p_2 = -\frac{p_1\,(p_1r_2 - q_1q_2)}{{q_1}^2},\ q_1 = q_1,\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_7(\phi) = \left\{ -\frac{(Mq_1r_2 + Np_1r_2 - Nq_1q_2)\,\lambda\,(2\,d+1)\,\mathrm{e}^{\chi(\phi)}}{\left(\mathrm{e}^{\chi(\phi)} p_1 r_2 - q_2 \mathrm{e}^{\chi(\phi)} q_1 - q_1 r_2\right) N\,(d+2)\,\mu} \right\}^{\frac{1}{d}}, \tag{38}$$

in which

$$\chi\,(\phi) = N(\phi + C) + \ln\left(\frac{N}{1 - M\exp(N(\phi + C))}\right), \tag{39}$$

$$\phi = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\, d}{\mu\,(d+1)\,(d+2)\,N} x + \frac{\lambda^3\,(2\,d+1)\,\sqrt{-\mu\,(d+1)\,(2\,d+1)} d}{\alpha\mu^2\,(d+1)\,(d^3 + 5\,d^2 + 8\,d + 4)\,(d+2)\,N}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}.$$

Set VIII:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\, d}{\mu\,(d+1)\,(d+2)\,k}, \quad k = k, \quad l = \frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,(d+2)^2}, \tag{40}$$

$$a_0 = -\frac{d\mu\, a_1 r_1 + 2\,d\lambda\, r_2 + 2\,\mu\, a_1 r_1 + \lambda\, r_2}{(d+2)\,\mu}, \quad a_1 = a_1, \quad p_1 = 0,\ p_2 = 0,\ q_1 = 0,\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_8(\phi) = \left\{ -\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu\,\left(q_2 \mathrm{e}^{\chi(\phi)} + r_2\right)} \right\}^{\frac{1}{d}}, \tag{41}$$

in which

$$\chi\,(\phi) = N(\phi + C), \qquad \phi = kx + \frac{k\lambda^2\,(2\,d+1)}{\alpha\,(d+1)\,\mu\,(d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{42}$$

Set IX:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\, d}{\mu\,(d+1)\,(d+2)\,k}, \quad k = k, \quad l = \frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,(d+2)^2}, \tag{43}$$

$$a_0 = -\frac{r_2\lambda\,(2\,d+1)}{(d+2)\,\mu}, \quad a_1 = 0, \quad p_1 = p_1,\ p_2 = 0,\ q_1 = q_1,\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_9(\phi) = \left\{\frac{a_1q_1\mathrm{e}^{\chi(\phi)}\mu\,d + 2\,a_1q_1\mathrm{e}^{\chi(\phi)}\mu - 2\,d\lambda\,r_2 - \lambda\,r_2}{(d+2)\,\mu\,r_2}\right\}^{\frac{1}{d}}, \tag{44}$$

in which

$$\chi\,(\phi) = N(\phi + C), \qquad \phi = kx + \frac{k\lambda^2\,(2\,d+1)}{\alpha\,(d+1)\,\mu\,(d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{45}$$

Set X:

$$L = 0,\ \ M = \frac{\sqrt{-\mu\,(2\,d^2+3\,d+1)}a_1q_1d}{(2\,d^2+3\,d+1)\,r_2k},\ \ N = \frac{d\lambda\,\left(2\,d^2+3\,d+1\right)}{(d+2)\,\sqrt{-\mu\,(2\,d^2+3\,d+1)}k\,(d+1)},\ \ k = k,\ \ l = \frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,(d+2)^2}, \tag{46}$$

$$a_0 = -\frac{d\mu\,a_1r_1 + 2\,d\lambda\,r_2 + 2\,\mu\,a_1r_1 + \lambda\,r_2}{(d+2)\,\mu}, \quad a_1 = a_1, \quad p_1 = 0,\ p_2 = 0,\ q_1 = q_1,\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{10}(\phi) = \left\{\frac{a_1q_1\mathrm{e}^{\chi(\phi)}\mu\,d + 2\,a_1q_1\mathrm{e}^{\chi(\phi)}\mu - 2\,d\lambda\,r_2 - \lambda\,r_2}{(d+2)\,\mu\,\left(q_2\mathrm{e}^{\chi(\phi)} + r_2\right)}\right\}^{\frac{1}{d}}, \tag{47}$$

in which

$$\chi\,(\phi) = N(\phi + C) + \ln\left(\frac{N}{1 - M\exp(N(\phi + C))}\right), \qquad \phi = kx + \frac{k\lambda^2\,(2\,d+1)}{\alpha\,(d+1)\,\mu\,(d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{48}$$

Set XI:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}, \quad k = k, \quad l = \frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,(d+2)^2}, \tag{49}$$

$$a_0 = -\frac{d\mu\,a_1r_1 + 2\,d\lambda\,r_2 + 2\,\mu\,a_1r_1 + \lambda\,r_2}{(d+2)\,\mu}, \quad a_1 = a_1, \quad p_1 = 0,\ p_2 = p_2,\ q_1 = q_1,$$

$$q_2 = -\frac{d^2\mu^2{a_1}^2{q_1}^2 + 4\,d\mu^2{a_1}^2{q_1}^2 + 4\,d^2\lambda^2p_2r_2 + 4\,\mu^2{a_1}^2{q_1}^2 + 4\,d\lambda^2p_2r_2 + \lambda^2p_2r_2}{\mu\,(2\,d^2+5\,d+2)\,a_1q_1\lambda},\ \ r_1 = r_1,\ \ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{11}(\phi) = \left\{\frac{\lambda\,q_1a_1\,(2\,d+1)}{2\,\mathrm{e}^{\chi(\phi)}d\lambda\,p_2 - d\mu\,a_1q_1 + \mathrm{e}^{\chi(\phi)}\lambda\,p_2 - 2\,\mu\,a_1q_1}\right\}^{\frac{1}{d}}, \tag{50}$$

in which

$$\chi\,(\phi) = N(\phi + C), \qquad \phi = kx + \frac{k\lambda^2\,(2\,d+1)}{\alpha\,(d+1)\,\mu\,(d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{51}$$

Set XII:

$$L = 0, \quad M = -\frac{1}{4}\frac{q_2d\lambda\,(2\,d+1)}{r_2k\sqrt{-\mu\,(d+1)\,(2\,d+1)}\,(d+2)}, \quad N = \frac{1}{2}\,\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}}{\mu\,(d+1)}, \tag{52}$$

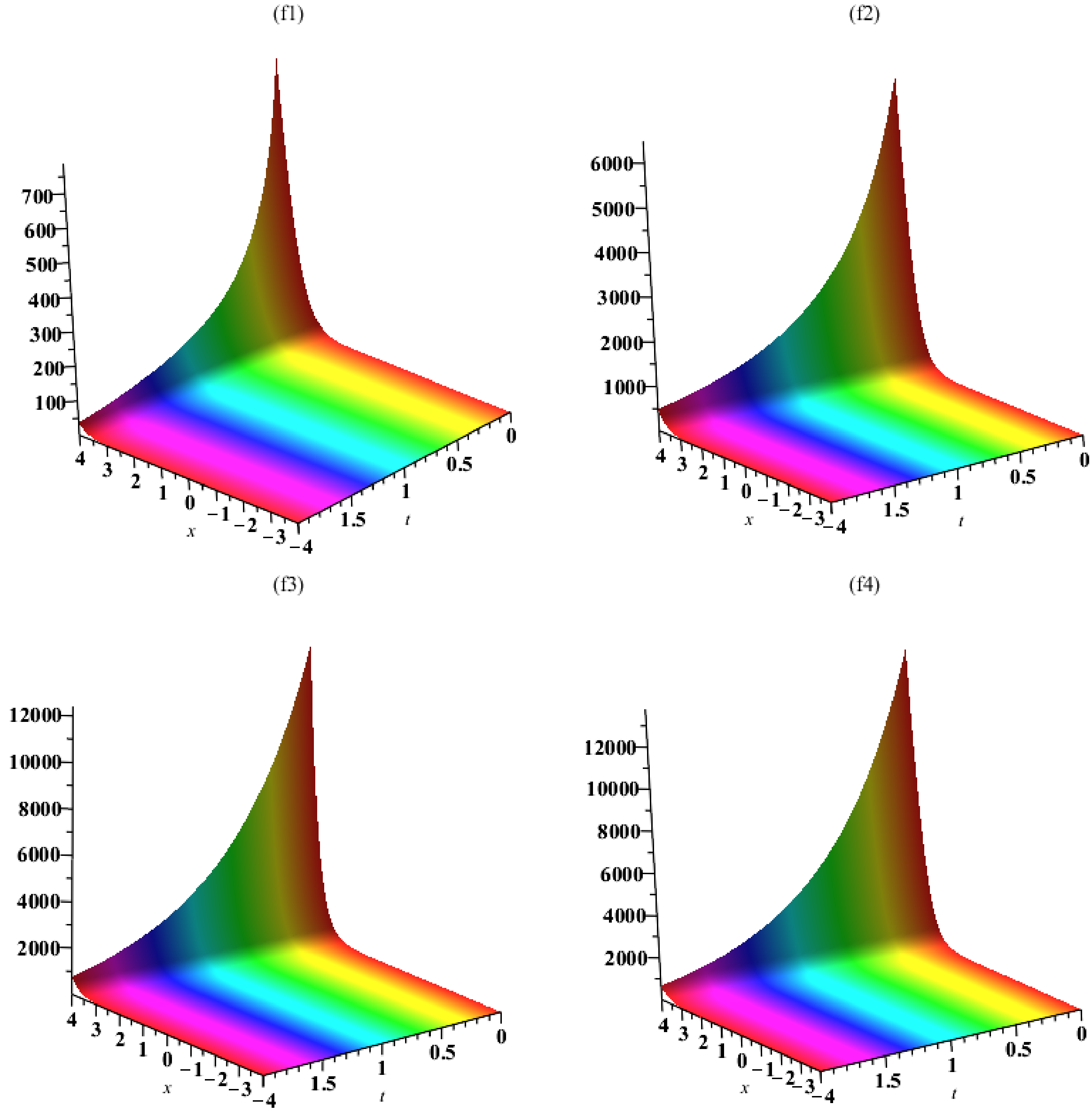

FIG. 3: The 3D plot of (44) at $d = 2, \mu = -1, a_1 = 1.5, q_1 = 0.2, r_2 = 2, \lambda = 2.2, M = -3, N = 2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

$$k = k,\quad l = \frac{k\lambda^2\,(2\,d+1)}{(d+1)\,\mu\,(d+2)^2},\quad a_0 = -\frac{d\mu\,a_1r_1 + 2\,d\lambda\,r_2 + 2\,\mu\,a_1r_1 + \lambda\,r_2}{(d+2)\,\mu},\quad a_1 = a_1,$$

$$p_1 = -\frac{1}{4}\,\frac{(2\,d+1)\,{q_2}^2\lambda}{(d+2)\,a_1r_2\mu},\ p_2 = p_2,\ q_1 = -\frac{q_2\lambda\,(2\,d+1)}{(d+2)\,a_1\mu},\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{12}(\phi) = \left\{ -\frac{1}{4}\,\frac{\lambda\,\left(2\,{q_2}^2\mathrm{e}^{2\,\chi(\phi)}d + {q_2}^2\mathrm{e}^{2\,\chi(\phi)} + 8\,q_2\mathrm{e}^{\chi(\phi)}r_2d + 4\,q_2\mathrm{e}^{\chi(\phi)}r_2 + 8\,d{r_2}^2 + 4\,{r_2}^2\right)}{\left(p_2\mathrm{e}^{2\,\chi(\phi)} + q_2\mathrm{e}^{\chi(\phi)} + r_2\right)(d+2)\,r_2\mu} \right\}^{\frac{1}{d}}, \tag{53}$$

in which

$$\chi(\phi) = N(\phi + C) + \ln\left(\frac{N}{1 - M\exp(N(\phi + C))}\right), \qquad \phi = kx + \frac{k\lambda^2 (2d+1)}{\alpha (d+1)\mu (d+2)^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{54}$$

B. Case II:

Then the exact solution will be as

$$v(\eta) = \frac{\sinh(\chi(\phi)) a_1 p_1 + \cosh(\chi(\phi)) a_1 q_1 + a_1 r_1 + a_0}{p_2 \sinh(\chi(\phi)) + q_2 \cosh(\chi(\phi)) + r_2}. \tag{55}$$

Inserting (55) in to Eq. (15), we obtain

$$\left(\mathrm{d}^2(\mathrm{d}+1)(2\mathrm{d}+1)\left(\mathrm{p}_2 \sinh(\chi(\phi)) + \mathrm{q}_2 \cosh(\chi(\phi)) + \mathrm{r}_2\right)^4\right)^{-1} \sum_{\mathrm{i+j=6}} \mathrm{C_{ij}} \sinh^{\mathrm{i}}(\chi(\phi)) \cosh^{\mathrm{j}}(\chi(\phi)) = 0, \tag{56}$$

where $\mathrm{C_{ij}}(\mathrm{i}+\mathrm{j}=6, 0 \le \mathrm{i}, \mathrm{j} \le 6)$ are polynomial statements in terms of $a_0, a_1, p_1, p_2, q_1, q_2, r_1$ and r_2. Hence, solving the resulting system $\mathrm{C_{ij}} = 0(\mathrm{i}+\mathrm{j}=6, 0 \le \mathrm{i}, \mathrm{j} \le 6)$ simultaneously, we acquire the below set of parameters of solutions
Set I:

$$L = 0, \quad M = 0, \quad N = \frac{1}{2}\frac{\sqrt{-\mu(d+1)(2d+1)}\lambda d}{\mu(d+1)(d+2)k}, \quad k = k, \quad l = \frac{\lambda^2(2d+1)k}{\mu(d^3+5d^2+8d+4)}, \quad a_0 = -\frac{r_2\lambda(2d+1)}{(d+2)\mu}, \tag{57}$$

$$a_1 = 0, \quad p_1 = p_1, \quad p_2 = q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_1(\phi) = \left(-\frac{\lambda(2d+1)r_2}{\mu(d+2)(q_2\sinh(\chi(\phi)) + q_2\cosh(\chi(\phi)) + r_2)}\right)^{\frac{1}{d}}, \quad \chi(\phi) = \frac{1}{2}\frac{\sqrt{-\mu(d+1)(2d+1)}\lambda d}{\mu(d+1)(d+2)k}(\phi + C), \tag{58}$$

in which

$$\phi = kx + \frac{\lambda^2(2d+1)k}{\mu\alpha(d^3+5d^2+8d+4)}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{59}$$

Set II:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\mu(d+1)(2d+1)}\lambda d}{\mu(d+1)(d+2)k}, \quad k = k, \quad l = \frac{\lambda^2(2d+1)k}{\mu(d^3+5d^2+8d+4)}, \quad a_0 = -\frac{r_2\lambda(2d+1)}{(d+2)\mu}, \tag{60}$$

$$a_1 = 0, \quad p_1 = p_1, \quad p_2 = -q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_2(\phi) = \left(\frac{\lambda(2d+1)r_2}{\mu(d+2)(q_2\sinh(\chi(\phi)) - q_2\cosh(\chi(\phi)) - r_2)}\right)^{\frac{1}{d}}, \quad \chi(\phi) = \frac{\sqrt{-\mu(d+1)(2d+1)}\lambda d}{\mu(d+1)(d+2)k}(\phi + C), \tag{61}$$

in which

$$\phi = kx + \frac{\lambda^2(2d+1)k}{\mu\alpha(d^3+5d^2+8d+4)}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{62}$$

Set III:

$$L = 0, \quad M = 0, \quad N = \frac{1}{2}\frac{\sqrt{-\mu(d+1)(2d+1)}\lambda d}{\mu(d+1)(d+2)k}, \quad k = k, \quad l = \frac{\lambda^2(2d+1)k}{\mu(d^3+5d^2+8d+4)}, \quad a_0 = \frac{1}{2}\frac{\lambda(2dp_2 + 2dq_2 + p_2 + q_2)r_1}{q_1(d+2)\mu}, \tag{63}$$

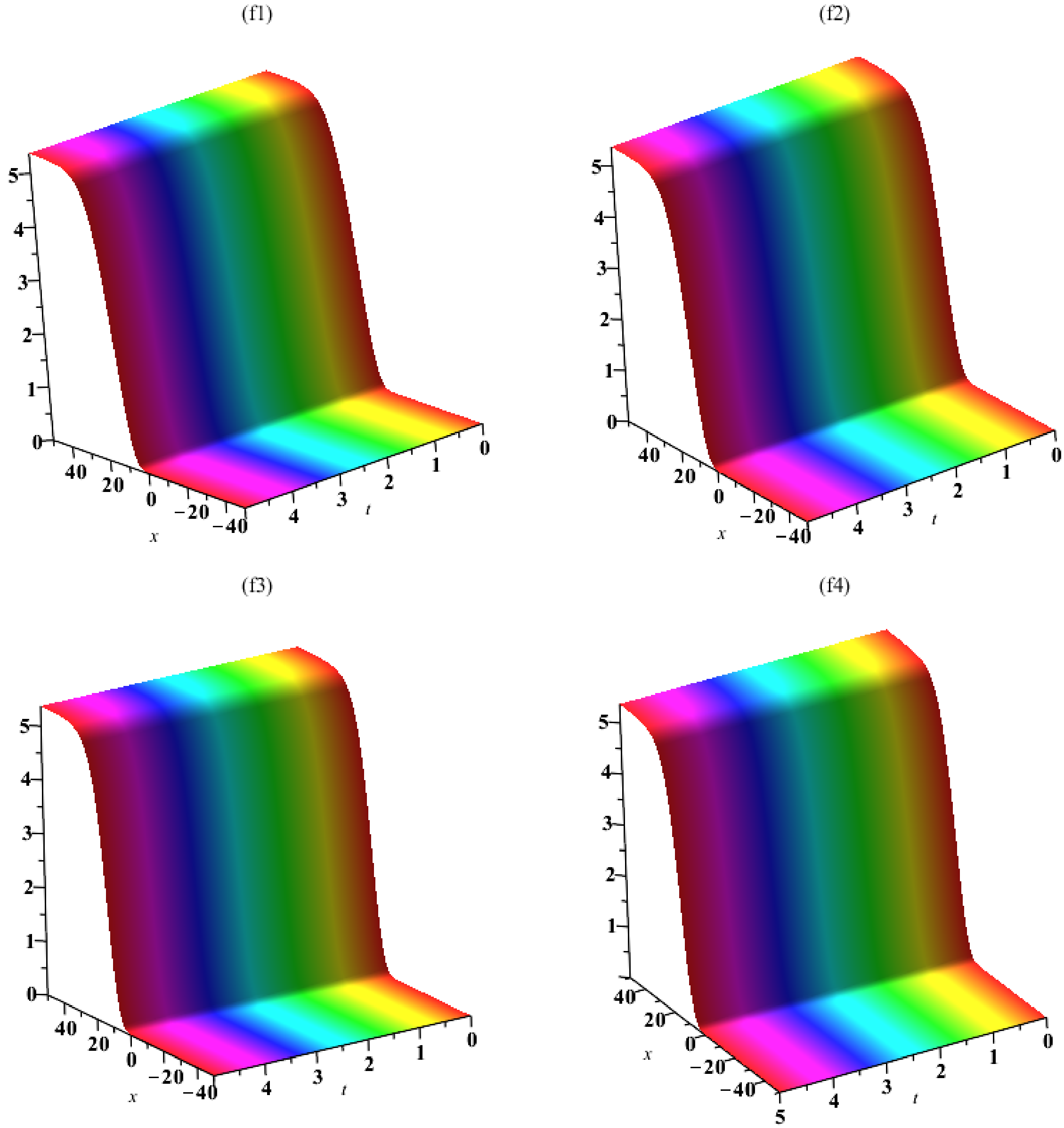

FIG. 4: The 3D plot of (58) at $d = 0.2, \mu = -1, q_2 = 1.5, r_2 = 2, \lambda = 2.2, M = 3, N = 2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

$$a_1 = -\frac{1}{2}\,\frac{\lambda\,(2\,dp_2 + 2\,dq_2 + p_2 + q_2)}{q_1\,(d+2)\,\mu}, \quad p_1 = q_1, \quad p_2 = p_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_3(\phi) = \left(-\frac{1}{2}\,\frac{\lambda\,(2\,dp_2 + 2\,dq_2 + p_2 + q_2)\,(\sinh(\chi(\phi)) + \cosh(\chi(\phi)))}{\mu\,(d+2)\,(p_2\sinh(\chi(\phi)) + q_2\cosh(\chi(\phi)))}\right)^{\frac{1}{d}}, \quad \chi(\phi) = \frac{1}{2}\,\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}(\phi + C), \tag{64}$$

in which

$$\phi = kx + \frac{\lambda^2\,(2\,d+1)\,k}{\mu\alpha\,(d^3 + 5\,d^2 + 8\,d + 4)}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{65}$$

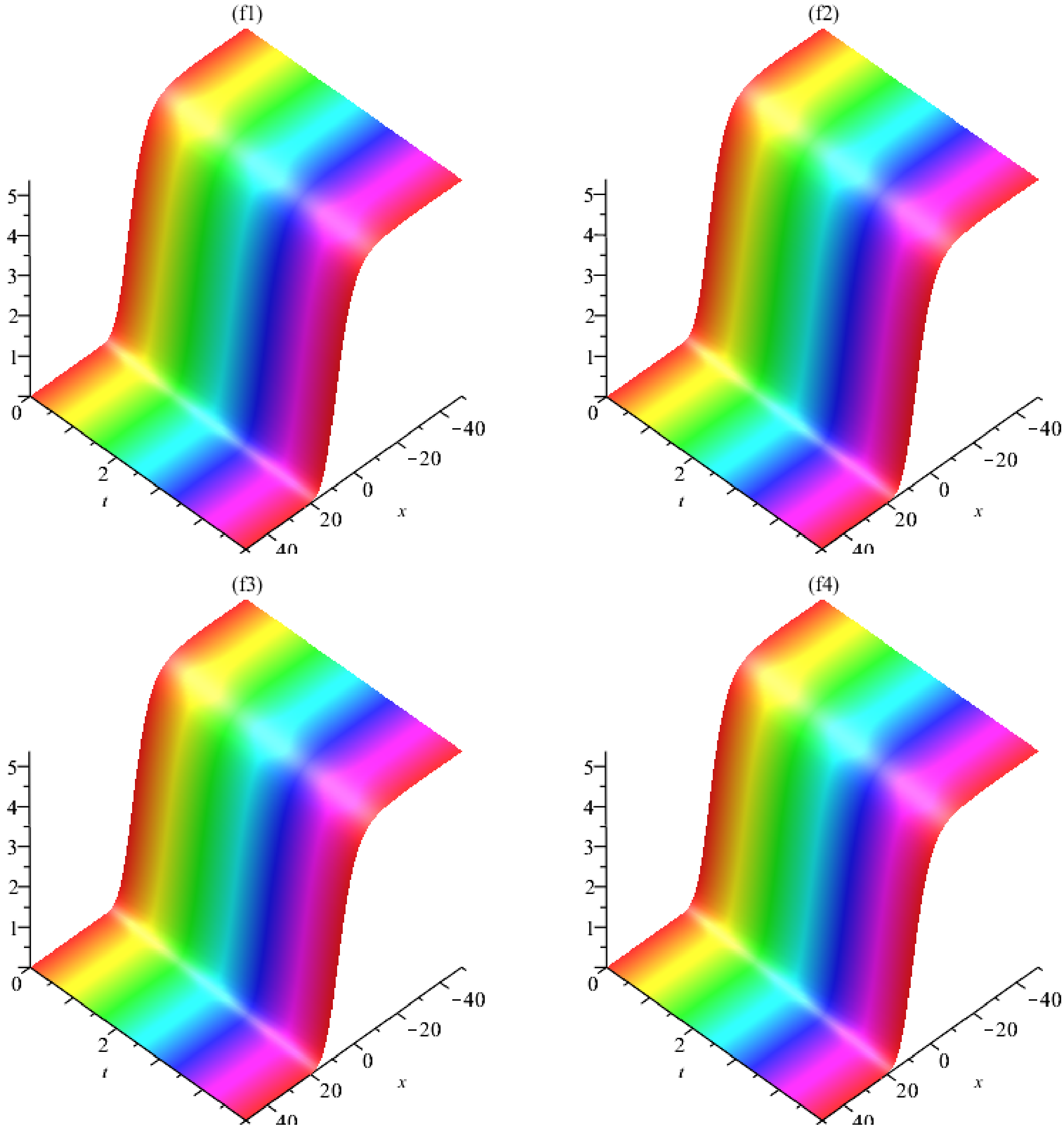

FIG. 5: The 3D plot of (64) at $d = 0.2, \mu = -1, p_2 = 1.5, q_2 = 2, \lambda = 2.2, M = 3, N = 2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

Set IV:

$$L = 0, \ \ M = 0, \ \ N = \frac{1}{2}\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}, \ \ k = k, \ \ l = \frac{\lambda^2\,(2\,d+1)\,k}{\mu\,(d^3+5\,d^2+8\,d+4)}, \ \ a_0 = \frac{1}{2}\frac{\lambda\,(2\,dp_2 - 2\,dq_2 + p_2 - q_2)\,r_1}{q_1\,(d+2)\,\mu}, \tag{66}$$

$$a_1 = -\frac{1}{2}\frac{\lambda\,(2\,dp_2 - 2\,dq_2 + p_2 - q_2)}{q_1\,(d+2)\,\mu}, \quad p_1 = -q_1, \quad p_2 = p_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_4(\phi) = \left(-\frac{1}{2}\frac{\lambda\,(2\,dp_2 - 2\,dq_2 + p_2 - q_2)\,(\sinh(\chi(\phi)) - \cosh(\chi(\phi)))}{\mu\,(d+2)\,(p_2\sinh(\chi(\phi)) + q_2\cosh(\chi(\phi)))}\right)^{\frac{1}{d}}, \ \ \chi(\phi) = \frac{1}{2}\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}(\phi + C), \tag{67}$$

in which

$$\phi = kx + \frac{\lambda^2 (2d+1) k}{\mu\alpha \left(d^3 + 5d^2 + 8d + 4\right)} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{68}$$

Set V:

$$L = 0,\quad M = 0,\quad N = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k},\quad k = k,\quad l = \frac{\lambda^2 (2d+1) k}{\mu \left(d^3 + 5d^2 + 8d + 4\right)},\quad a_0 = -\frac{1}{2} \frac{q_2 \lambda (2d+1) r_1}{\mu p_1 (d+2)}, \tag{69}$$

$$a_1 = \frac{1}{2} \frac{q_2 \lambda (2d+1)}{\mu p_1 (d+2)},\quad p_1 = p_1,\quad p_2 = 0,\quad q_1 = -p_1,\quad q_2 = q_2,\quad r_1 = r_1,\quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_5(\phi) = \left(\frac{1}{2} \frac{(2d+1)\lambda \left(\sinh(\chi(\phi)) - \cosh(\chi(\phi))\right)}{\mu (d+2)\cosh(\chi(\phi))}\right)^{\frac{1}{d}},\quad \chi(\phi) = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k}(\phi + C), \tag{70}$$

in which

$$\phi = kx + \frac{\lambda^2 (2d+1) k}{\mu\alpha \left(d^3 + 5d^2 + 8d + 4\right)} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{71}$$

Set VI:

$$L = 0,\quad M = 0,\quad N = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k},\quad k = k,\quad l = \frac{\lambda^2 (2d+1) k}{\mu \left(d^3 + 5d^2 + 8d + 4\right)},\quad a_0 = \frac{1}{2} \frac{q_2 \lambda (2d+1) r_1}{\mu p_1 (d+2)}, \tag{72}$$

$$a_1 = -\frac{1}{2} \frac{q_2 \lambda (2d+1)}{\mu p_1 (d+2)},\quad p_1 = p_1,\quad p_2 = 0,\quad q_1 = p_1,\quad q_2 = q_2,\quad r_1 = r_1,\quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_6(\phi) = \left(-\frac{1}{2} \frac{(2d+1)\lambda \left(\sinh(\chi(\phi)) + \cosh(\chi(\phi))\right)}{\mu (d+2)\cosh(\chi(\phi))}\right)^{\frac{1}{d}},\quad \chi(\phi) = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k}(\phi + C), \tag{73}$$

in which

$$\phi = kx + \frac{\lambda^2 (2d+1) k}{\mu\alpha \left(d^3 + 5d^2 + 8d + 4\right)} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{74}$$

Set VII:

$$L = 0,\quad M = 0,\quad N = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k},\quad k = k,\quad l = \frac{\lambda^2 (2d+1) k}{\mu \left(d^3 + 5d^2 + 8d + 4\right)},\quad a_0 = \frac{1}{2} \frac{q_2 \lambda (2d+1) r_1}{\mu p_1}, \tag{75}$$

$$a_1 = -\frac{1}{2} \frac{q_2 \lambda (2d+1)}{\mu p_1},\quad p_1 = p_1,\quad p_2 = (d+1) q_2,\quad q_1 = p_1,\quad q_2 = q_2,\quad r_1 = r_1,\quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_7(\phi) = \left(-\frac{1}{2} \frac{(2d+1)\lambda \left(\sinh(\chi(\phi)) + \cosh(\chi(\phi))\right)}{\mu \left(\sinh(\chi(\phi)) d + \sinh(\chi(\phi)) + \cosh(\chi(\phi))\right)}\right)^{\frac{1}{d}},\quad \chi(\phi) = \frac{1}{2} \frac{\sqrt{-\mu (d+1)(2d+1)}\lambda d}{\mu (d+1)(d+2) k}(\phi + C), \tag{76}$$

in which

$$\phi = kx + \frac{\lambda^2 (2d+1) k}{\mu\alpha \left(d^3 + 5d^2 + 8d + 4\right)} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{77}$$

Set VIII:

$$L=0,\quad M=0,\quad N=\frac{1}{2}\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k},\quad k=k,\quad l=\frac{\lambda^2\,(2\,d+1)\,k}{\mu\,(d^3+5\,d^2+8\,d+4)},\quad a_0=-\frac{1}{2}\frac{q_2\lambda\,(2\,d+1)\,r_1}{\mu\,p_1}, \tag{78}$$

$$a_1=\frac{1}{2}\frac{q_2\lambda\,(2\,d+1)}{\mu\,p_1},\quad p_1=p_1,\quad p_2=-(d+1)q_2,\quad q_1=-p_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=0.$$

We, therefore, gained the following generalized solitary solution

$$u_8(\phi)=\left(-\frac{1}{2}\frac{(2\,d+1)\,\lambda\,(\sinh(\chi(\phi))-\cosh(\chi(\phi)))}{\mu\,(\sinh(\chi(\phi))\,d+\sinh(\chi(\phi))-\cosh(\chi(\phi)))}\right)^{\frac{1}{d}},\quad \chi(\phi)=\frac{1}{2}\frac{\sqrt{-\mu\,(d+1)\,(2\,d+1)}\lambda\,d}{\mu\,(d+1)\,(d+2)\,k}(\phi+C), \tag{79}$$

in which

$$\phi=kx+\frac{\lambda^2\,(2\,d+1)\,k}{\mu\alpha\,(d^3+5\,d^2+8\,d+4)}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{80}$$

V. GRAPHICAL REPRESENTATION

The graphical description of derived soliton and other solutions have been expressed in the mentioned figures by allotting the different values of the parameters. The 3D plot for some solutions of the considered equation for four fractional order cases $\alpha=0.25, 0.5, 0.85, 0.99$ have been shown. When the all obtained exact solutions for the fractional generalized Korteweg-de Vries equation are examined, the exact solution (31) is similar to the solution of Sahadevan and Bakkyaraj [51], the solution of Li, Guo, and Zhao [52] (47) and the solution of Akbulut and Taşcan [53] (73) in the literature. The other obtained exact solutions that are not included in the literature, and it can be said that they are new exact solutions obtained by the new version trial equation method. Also, two and three dimensional graphics of the obtained solution functions are illustrated in Fig. 1-6 which demonstrate with suitable parametric choices.

VI. CONCLUSION

We used the direct truncation method to find the explicit solution of fractional generalized Korteweg-de Vries equation, comparing with the known results in the literatures, we get some novel solitary wave solutions, including bright, dark, kink and periodic solitons. Numerical simulation have been performed by using the Maple software. The solution of every PDEs are always utilized for understanding the system and various phenomena described by it. The new analytical expansion method is helpful to obtained the the solutions in the form of hyperbolic and ergometric forms which are exact and helpful in understanding the fractional forms of it. Finally, a transformation is used to draw a soliton solutions of Eq. (1) by the use of Maple software. So, this gives the efficient applications of new analytical expansion for the fractional PDEs.

References

[1] K. Fellnerand, C. Schmeiser, Burgers-Poisson: a nonlinear dispersive wave equation, SIAM J. Appl. Math., 64 (2004) 1509-1525.

[2] M. Gaur and K. Singh, On group invariant solutions of fractional order Burgers-Poisson equation, App. Math and Comp., 244 (2014) 870-877.

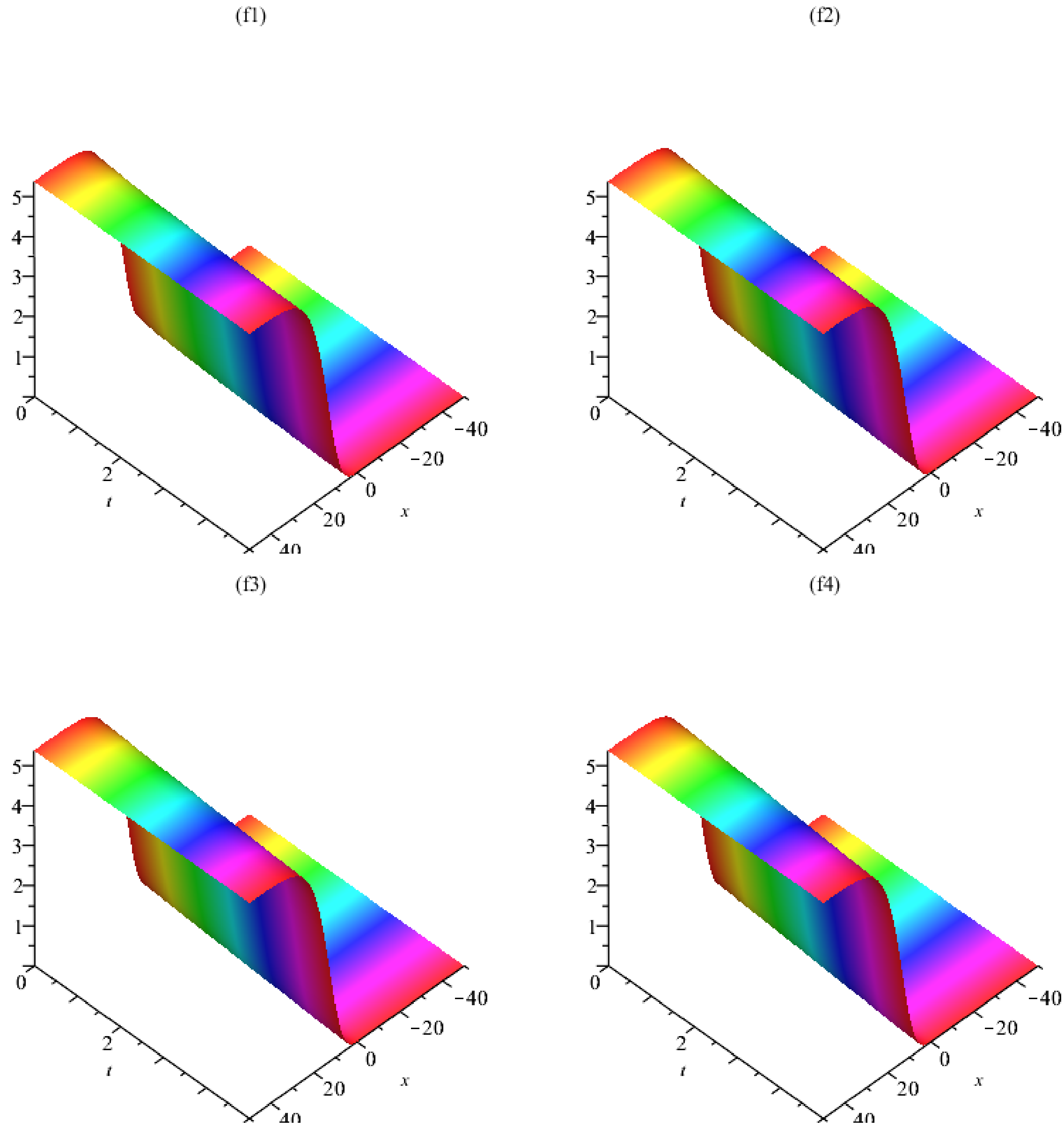

FIG. 6: The 3D plot of (79) at $d = 0.2, \mu = -1, r_2 = 1.5, q_2 = 2, \lambda = 2.2, M = 3, N = 2, k = 3$ when (f1) $\alpha = 0.25$, (f2) $\alpha = 0.5$, (f3) $\alpha = 0.85$, and (f4) $\alpha = 0.99$.

[3] M. Kato, Large time behavior of solutions to the Burgers-Poisson equations, Osaka J. Math., 4 (2007) 923-943.

[4] S. Momani, An explicit and numerical solutions of the fractional KdV equation, Math. Comput. Simul., 70 (2005) 110-118.

[5] M. Antonova, A. Biswas, Adiabatic parameter dynamics of perturbed solitary waves, Commun. Nonlinear Sci. Num. Simul., 14 (2009) 734-748.

[6] J. He, Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering, Dalian, China, (1998) 288-291.

[7] J. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol, 15 (1999) 86-90.

[8] J. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods. Appl. Mech. Eng, 167 (1998) 57-68.

[9] I. Grigorenko, E. Grigorenko, Chaotic dynamics of the fractional lorenz system, Phys. Rev. Lett, 91 (2003) 034101.

[10] S. Momani, N. T. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, Appl.Math. Comput. 182 (2006) 1083-1092.

[11] S. Momani, M. A. Noor, Numerical methods for fourth-order fractional integrodifferential equations, Appl. Math. Comput, 182 (2006) 754-760.

[12] V. D. Gejji, H. Jafari, Solving a multi-order fractional differential equation, Appl. Math. Comput, 189 (2007) 541-548.

[13] O. Guner, A. Bekir, H. Bilgil,, A note on exp-function method combined with complex transform method applied to fractional differential equations, Adv. Nonlinear Anal., 4 (3) (2015) 201-208.

[14] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A, 365 (2007) 345-350.

[15] N. H. Sweilam, M. M. Khader, R. F. Al-Bar, Numerical studies for a multi-order fractional differential equation, Phys. Lett. A, 371 (2007) 26-33.

[16] B. Zheng, C. Wen, Exact solutions for fractional partial differential equations by a new fractional sub-equation method, Advances in Difference Equations, 2013 (2013) 199.

[17] B. Lu, The first integral method for some time fractional differential equations, J. Math. Anal. Appl. 395 (2012) 684-693.

[18] H. Jafari, S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, Commu. Nonlinear Sci. Num. Simu, **14** (2009) 1962-1969.

[19] A., Bekir, O., Guner, A.H., Bhrawy, A., Biswas, Solving nonlinear fractional differential equations using exp-function and (G'/G)-expansion methods, Rom. J. Phys., 60 3-4 (2015) 360-378.

[20] M. A. Herzallah and K. A. Gepreel, Approximate solution to time-space fractional cubic nonlinear Schrodinger equation, Applied Mathematical Modeling, 36 (11) (2012) 5678-5685.

[21] M. S. Mohamed, Analytical treatment of Abel integral equations by optimal homotopy analysis transform method, Journal of Information and Computing Science; 10 (1) (2015) 19-28.

[22] K. A. Gepreel and M. S. Mohamed, An optimal homotopy analysis method nonlinear fractional differential equation, Journal of Advanced Research in Dynamical and Control Systems, 6(1) (2014) 1-10.

[23] S. Kemple, H. Beyer, Global and causal solutions of fractional differential equations, Transform methods and special functions: Varna 96, Proceedings of 2nd international workshop (SCTP), Singapore, **96** (1997) 210-216.

[24] Arshed S. Two reliable techniques for the soliton solutions of perturbed Gerdjikov-Ivanov equation. Optik. 2018;164:93-99.

[25] Biswas A, Yildirim Y, Yasar E, et al. Optical soliton perturbation with Gerdjikov-Ivanov equation by modified simple equation method. Optik. 2018;157:1235-1240.

[26] Biswas A, Alqahtani RT. Chirp-free bright optical solitons for perturbed Gerdjikov-Ivanov equation by semi-inverse variational principle. Optik-International journal for light and electron optics. 2017;147:72-76.

[27] Biswas A, Yildirim Y, Yasar E, et al. Solitons for perturbed Gerdjikov-Ivanov equation in optical fibers and PCF by extended Kudryashov's method. Optical and quantum electronics. 2018;50:1.

[28] Yasar E, Yildirim Y. New optical solitons of space-time conformable fractional perturbed Gerdjikov-Ivanov equation by sine-Gordon equation method. Results in physics. 2018;9:1666-1672.

[29] Biswas A, Ekici M, Sonmezoglu A, et al. Optical soliton perturbation for Gerdjikov-Ivanov equation by extended trial equation method. Optik. 2018;158:747-752.

[30] I. Podlubny, Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, New York: Academic Press, (1999).

[31] S.G. Samko, AA. Kilbas, OI. Marichev, Fractional integrals and derivatives: theory and applications, Yverdon, Gordon and Breach, (1993).

[32] L. Debanth, Recents applications of fractional calculus to science and engineering, Int. J. Math. Math. Sci, **54** (2003) 3413-3442.

[33] V.E. Zakharov, E.A. Kuznetsov, On three–dimensional solitons, Sov. Phys, **39** (1974) 285-288.

[34] J.H. He, X.H. Wu, Exp–function method for nonlinear wave equations, Chaos, Solitons Fractals, **30** (2006) 700-708.

[35] M.A. Abdou, Generalized solitonary and periodic solutions for nonlinear partial differential equations by the Exp–function method, Nonlinear Dyn, **52** (2008) 1-9.

[36] A. Boz, A. Bekir, Application of Exp–function method for (3+1)-dimensional nonlinear evolution equations, Comput. Math. Appl, **56** (2008) 1451-1456.

[37] R. Y. Molliq, B. Batiha, Approximate analytic solutions of fractional Zakharov-Kuznetsov equations by fractional complex transform, Int. J. Eng. Tech., **1** (2012) 1-13.

[38] X. Zhao, H. Zhou, Y. Tang, H. Jia, Travelling wave solutions for modified Zakharov–Kuznetsov equation, Appl. Math. Comput, **181** (2006) 634-648.

[39] Z. B. Li, J. H. He, Application of the fractional complex transform to fractional differential equations, Nonlinear Sci. Lett. A, **2** (2011) 121-126.

[40] J. H. He, S. K. Elagan, Z. B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Phys. Lett. A, **376** (2012) 257-259.

[41] X. J. Yang, General Fractional Derivatives: Theory, Methods and Applications, CRC Press, New York, USA, 2019.

[42] X. J. Yang, A new fractal nonlinear Burgers' equation arising in the acoustic signals propagation, Math Meth Appl Sci., 42 (2019) 7539-7544.

[43] X. J. Yang, Y. Y. Feng, C. Cattani, M. Inc, Fundamental solutions of anomalous equation with the decay exponential kernel, Math Meth Appl Sci., 42 (2019) 4054-4060.

[44] X. J. Yang, A new integral transform operator for solving the heat-diffusion problem, Appl. Math. Lett., 64 (2017) 193-197.

[45] J. G. Liu, X. J. Yang, Y. Y. Feng, M. Iqbal, Group analysis to the time fractional nonlinear wave equation, International Journal of Mathematics, 31(4) (2020) 20500299.

[46] J. G. Liu, X. J. Yang, Y. Y. Feng, H. Y. Zhang, Analysis of the time fractional nonlinear diffusion equation from diffusion process, Journal of Applied Analysis and Computation, 10(3) (2020) 1060-1072.

[47] J. G. Liu, X. J. Yang, Y. Y. Feng, On integrability of the time fractional nonlinear heat conduction equation, Journal of Geometry and Physics, 144 (2019) 190-198.

[48] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel, Theory and application to heat transfer model. Therm Sci. (2016) 20:763-9. doi: 10.2298/TSCI16011 1018A

[49] A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivative with fractional order, Chaos Solitons Fractals, 89 (2016) 447-454.

[50] K.M. Saad, A. Atangana, D. Baleanu, New fractional derivatives with non-singular kernel applied to the burgers equation, Chaos, 28 (6) (2018) 063-109.

[51] R. Sahadevan, T. Bakkyaraj, Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations, Journal of Mathematical Analysis and Applications, 393(215) (2012) 341-347.

[52] R. Sahadevan, T. Bakkyaraj, New solitary wave solutions of (2+1)-dimensional space-time fractional Burgers equation and Korteweg-de Vries equation, Computers & Mathematics with Applications, 77 (2019) 2255-2262.

[53] A. Akbulut, F. Taşcan, Lie symmetries, symmetry reductions and conservation laws of time fractional modified Korteweg-de Vries (mkdv) equation, Chaos, Solitons & Fractals, 100 (2017) 1-6.