On the Stefan Problem With Nonlinear Thermal Conductivity

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Abstract

The solution is obtained and validated by an existence and uniqueness theorem for the following nonlinear boundary value problem

$$\frac{d}{dx}(1+\delta y+\gamma y^2)^n \frac{dy}{dx}] + 2x\frac{dy}{dx} = 0,\ \ x > 0,\ y(0) = 0,\ \ y(\infty) = 1,$$

which was proposed in 1974 by [1] to represent a Stefan problem with a nonlinear temperature-dependent thermal conductivity on the semi-infinite line $(0, \infty)$. The modified error function of two parameters $\varphi_{\delta,\gamma}$ is introduced to represent the solution of the problem above, and some properties of the function are established. This generalizes the results obtained in [3, 4].

Keywords: Boundary value problem, existence and uniqueness theorem, Modified error function, Stefan problem.

1 Introduction

The subject of moving boundary problems is an active area of research that attracted many researchers since the nineteenth century, and has become a rapidly growing field with a great deal of interest in this direction. This type of problems model phase-change processes that occur naturally and industrially, such as melting of ice, freezing of liquids, diffusion of Oxygen, and so many other processes. As the name suggests, the boundary moves when the phase change occur, and in order to solve these problems, a "Stefan condition" is usually imposed and used to help in calculating the phase front, and free boundary problems with a Stefan condition are called: "Stefan problems". Generally, Stefan problems are nonlinear in nature and restricted to heat transfer, which justifies their importance in describing many physical and industrial processes. The old classical Stefan problem deals with constant thermal quantities, such as thermal conductivity and specific heat for example. However, recent developments and research suggest more realistic models in which thermal parameters are temperature-dependent.

In 1978, Cho and Sunderland [1] assumed a linear thermal conductivity and investigated the nonlinear problem

$$\left((1+\delta y)\frac{dy}{dx}\right)' + 2x\frac{dy}{dx} = 0,\ 0 < x < \infty \tag{1.1}$$

$$y(0) = y(\infty) = 1, \tag{1.2}$$

The solution of the problem was obtained numerically and defined as the modified error function φ_δ, where δ is the thermal coefficient of the thermal conductivity $1 + \delta y$, and y represents the temperature distribution.

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Noting that when $\delta = 0$, Problem (1.1)-(1.2) becomes linear and the solution y reduces to the classical error function

$$y = \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi. \tag{1.3}$$

Then they proposed the following problem as a generalization of (1.1)-(1.2)

$$\begin{cases} \frac{d}{dx}\left[(1+\delta y+\gamma y^2)^n \frac{dy}{dx}\right] + 2x\frac{dy}{dx} &= 0,\ 0 < x < \infty, \\ y(0) = 0,\ y(\infty) &= 1. \end{cases} \tag{1.4}$$

Oliver and Sunderland [2] investigated a more general model of (1.1) in which thermal condutivity and specific heat are both linear in time, and a numerical solution were obtained. No existence and uniqueness theorems have been established in [1, 2]. This has motivated many researchers to develop existence and uniqueness theorems for the solutions of this problem which has been ever since the subject of many research papers, see for example [3]- [14]. The authors in [3] proved the existence and uniqueness of the modified error function for small values of $\delta > 0$, and the general case $\delta > -1$ was proved in [4].
The purpose of this paper is to provide an existence and uniqueness theorem for the solution of (1.4) that was proposed in [1], which represents a Stefan problem with a nonlinear thermal conductivity of the form $(1 + \delta y + \gamma y^2)^n$, where $\delta > -1$ and $\gamma > -1$. To the best of our knowledge, the question of existence and uniqueness of the solution of Pr. (1.4) has not been answered since proposing the problem in 1974.
The paper is organized as follows: In section 2, we prove some useful lemmas. In section 3, an exact solution for the problem is obtained together with upper and lower bounds. In section 4, we establish existence and uniqueness theorems for the solution obtained. In section 5 we provide an analysis for the results. The modified error function of two parameters $\varphi_{\delta,\gamma}$ is defined as the solution to the Pr. (1.4). The analysis shows that $\varphi_{\delta,\gamma}$ generalizes the modified error function defined in [1] and [3] and shares some properties with the classical error function.

2 Fundamental lemmas

Writing the nonlinear term of Pr.(1.4): $1 + \delta y + \gamma y^2$ in the form

$$1+\delta y+\gamma y^2 = \gamma\left(y^2 + \frac{\delta}{\gamma}y + \frac{1}{\gamma}\right) = \gamma\left(y + \frac{\delta}{2\gamma}\right)^2 + 1 - \frac{\delta^2}{4\gamma},\ \gamma > -1. \tag{2.1}$$

Thus, the nonlinear differential equation of Pr.(1.4) becomes

$$\frac{d}{dx}\left[\left(\gamma(y+\frac{\delta}{2\gamma})^2 + 1 - \frac{\delta^2}{4\gamma}\right)^n \frac{dy}{dx}\right] + 2x\frac{dy}{dx} = 0,\ 0 < x < \infty, \tag{2.2}$$

or

$$\frac{d}{dx}\left[\left(\gamma(y+\frac{\delta}{2\gamma})^2 + 1 - \frac{\delta^2}{4\gamma}\right)^n \frac{d}{dx}(y+\frac{\delta}{2\gamma})\right] + 2x\frac{d}{dx}(y+\frac{\delta}{2\gamma}) = 0,\ 0 < x < \infty. \tag{2.3}$$

By the change of variable

$$u = y + \frac{\delta}{2\gamma}. \tag{2.4}$$

Eq. (2.3) becomes

$$\frac{d}{dx}\left[\left(\gamma u^2 + \kappa\right)^n \frac{du}{dx}\right] + 2x\frac{du}{dx} = 0,\ 0 < x < \infty, \tag{2.5}$$

where $\kappa = 1 - \frac{\delta^2}{4\gamma}$. Hence

$$\left(\gamma u^2 + \kappa\right)^n \frac{d^2u}{dx^2} + n\left(\gamma u^2 + \kappa\right)^{n-1}\left(2\gamma u\left(\frac{du}{dx}\right)^2\right) + 2x\frac{du}{dx} = 0,\ 0 < x < \infty. \tag{2.6}$$

Consequently,

$$\frac{d^2u}{dx^2} + 2\gamma n \frac{u\left(\frac{du}{dx}\right)^2}{\gamma u^2+\kappa} + 2x\frac{\frac{du}{dx}}{\left(\gamma u^2+\kappa\right)^n} = 0,\ 0<x<\infty. \tag{2.7}$$

Thus

Lemma 1 *Pr.(1.4) can be converted to the nonlinear boundary value problem*

$$\left\{\begin{array}{rcl} u''+f(x,u,u') &=& 0,\ 0<x<\infty, \\ u(0)=\frac{\delta}{2\gamma},\ u(\infty) &=& 1+\frac{\delta}{2\gamma}, \end{array}\right. \tag{2.8}$$

where

$$f(x,u,u') = 2\gamma n\frac{u(u')^2}{\gamma u^2+\kappa} + 2x\frac{u'}{(\gamma u^2+\kappa)^n},\ u=y+\frac{\delta}{2\gamma}. \tag{2.9}$$

Lemma 2

$$\frac{\delta}{2\gamma} \le u(x) \le 1+\frac{\delta}{2\gamma},\ 0\le x<\infty. \tag{2.10}$$

Proof. Since $0\le y\le 1$ [1,2]. Thus u belongs to the interval $I=[\frac{\delta}{2\gamma}, 1+\frac{\delta}{2\gamma}]$ for $\gamma>-1$ and $\delta>-1$. ■

3 Solutions

First of all, we give a generalization of Theorem 2.1 [3].

Theorem 3 *The solution y of Pr.(1.1) can be expressed by*

$$y = C\int_0^x \frac{1}{\Psi(\eta)}\exp\left(-2\int_0^\eta \frac{\xi}{\Psi(\xi)}d\xi\right)d\eta,\ 0\le x<\infty, \tag{3.1}$$

where $\Psi(x)=(1+\delta y+\gamma y^2)^n$ and

$$C = \left[\int_0^\infty \frac{1}{\Psi(x)}\exp\left(-2\int_0^x \frac{\xi}{\Psi(\xi)}d\xi\right)dx\right]^{-1}. \tag{3.2}$$

Proof. Rewrite the first equation of Pr.(2.8) as

$$\frac{u''}{u'} + n\frac{(\gamma u^2+\kappa)'}{\gamma u^2+\kappa} + \frac{2x}{(\gamma u^2+\kappa)^n} = 0. \tag{3.3}$$

So that ,

$$\frac{u''}{u'} = -n\frac{(\gamma u^2+\kappa)'}{\gamma u^2+\kappa} - \frac{2x}{(\gamma u^2+\kappa)^n}. \tag{3.4}$$

Case 1: $\gamma\ge 0$.
Since $0\le y\le 1$. Thus for $\gamma\ge 0$, we have $0\le\gamma y^2\le\gamma$. Hence

- If $\delta\ge 0$, then $0\le\delta y\le\delta$. So that $1\le 1+\delta y+\gamma y^2\le 1+\delta+\gamma$.
- If $-1<\delta<0$, then $\delta\le\delta y\le 0$. So that $\delta+1\le 1+\delta y+\gamma y^2\le 1+\gamma$.

Case 2: $-1<\gamma<0$.
For $-1<\gamma<0$, we have $\gamma\le\gamma y^2\le 0$. Hence

- If $\delta\ge 0$, then $0\le\delta y\le\delta$. So that $1+\gamma\le 1+\delta y+\gamma y^2\le 1+\delta$.

In view of these, we have $1+\delta y+\gamma y^2>0$, that is $\gamma u^2(x)+\kappa>0$ for $\gamma>-1$ and $\delta>-1$.
Integrating now from 0 to x, we obtain

$$u' = \frac{C}{(\gamma u^2(x)+\kappa)^n}\exp\left(-2\int_0^x \frac{\eta}{(\gamma u^2(\eta)+\kappa)^n}d\eta\right). \tag{3.5}$$

where C is an unknown constant.
Hence

$$u=\frac{\delta}{2\gamma}+C\int_0^x \frac{1}{(\gamma u^2(\eta)+\kappa)^n}\exp\left(-2\int_0^\eta \frac{\xi}{(\gamma u^2(\xi)+\kappa)^n}d\xi\right)d\eta,\ 0\le x<\infty. \tag{3.6}$$

The constant C can be determined by using the second boundary condition $u(\infty)=1+\frac{\delta}{2\gamma}$. Thus

$$C=\frac{1}{\int_0^\infty \frac{1}{(\gamma u^2(x)+\kappa)^n}\exp\left(-2\int_0^x \frac{\xi}{(\gamma u^2(\xi)+\kappa)^n}d\xi\right)dx}. \tag{3.7}$$

Substituting $u=y+\frac{\delta}{2\gamma}$ and $\Psi(x)=(1+\delta y+\gamma y^2)^n$ into (3.6)-(3.7), we obtain the desired result. ■

Remark 4 *When $\gamma=0$ and $n=1$, this reduces to Theorem 2.1 [3].*

Now we prove the double inequalities for the lower and upper bounds of the solution $u(x)$ and $u'(x)$ for different values of δ and γ that guarantee the existence of the solution of Pr.(2.8).

Theorem 5 *There are upper and lower bounds of the solution $u(x)$ of Pr.(2.8) such that we have*

1.

$$u_1(x)\le u(x)\le u_2(x)\ \textit{for}\ \delta\ge 0,\ 0\le x<+\infty, \tag{3.8}$$

2.

$$u_1^*(x)\le u(x)\le u_2^*(x)\ \textit{for}\ -1<\delta<0,\ 0\le x<+\infty, \tag{3.9}$$

where

$$u_1=\frac{\delta}{2\gamma}+C_1\frac{\sqrt{\pi}}{2(\gamma+\delta+1)^n}\,\mathrm{erf}\,(x), \tag{3.10}$$

$$u_2=\frac{\delta}{2\gamma}+C_1\frac{\sqrt{(\gamma+\delta+1)^n\pi}}{2}\,\mathrm{erf}\Big(\frac{x}{\sqrt{(\gamma+\delta+1)^n}}\Big), \tag{3.11}$$

$$u_1^*=\frac{\delta}{2\gamma}+C_2\frac{\sqrt{(\delta+1)^n\pi}}{2(\gamma+1)^n}\,\mathrm{erf}\left(\frac{x}{\sqrt{(\delta+1)^n}}\right), \tag{3.12}$$

$$u_2^*=\frac{\delta}{2\gamma}+C_2\frac{\sqrt{(\gamma+1)^n\pi}}{2(\delta+1)^n}\,\mathrm{erf}\Big(\frac{x}{\sqrt{(\gamma+1)^n}}\Big) \tag{3.13}$$

and the constants $C_i,\ i=1,2$ satisfy

$$\frac{2}{\sqrt{(\gamma+\delta+1)^n\pi}}\le C_1\le \frac{2(\gamma+\delta+1)^n}{\sqrt{\pi}} \tag{3.14}$$

and

$$\frac{2(\delta+1)^n}{\sqrt{(\gamma+1)^n\pi}}\le C_2\le \frac{2(\gamma+1)^n}{\sqrt{(\delta+1)^n\pi}}. \tag{3.15}$$

Our Proof of this theorem makes use of Lemma 2.
Proof. Using the above inequalities, we have

$$\frac{1}{(\gamma+\delta+1)^n} \leq \frac{1}{(\gamma u^2+\kappa)^n} \leq 1 \text{ for } \delta \geq 0 \tag{3.16}$$

and

$$\frac{1}{(\gamma+1)^n} \leq \frac{1}{(\gamma u^2+\kappa)^n} \leq \frac{1}{(\delta+1)^n} \text{ for } -1<\delta<0. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.6) and (3.7), we obtain the desired inequalities.
This completes the proof. ■
Similarly, we have

Theorem 6 *There are upper and lower bounds of $u'(x)$ such that we have*

1.

$$u_1^+(x) \leq u'(x) \leq u_2^+(x) \textit{ for } \delta \geq 0,\ 0 \leq x < +\infty, \tag{3.18}$$

2.

$$u_1^-(x) \leq u'(x) \leq u_2^-(x) \textit{ for } -1<\delta<0,\ 0 \leq x < +\infty, \tag{3.19}$$

where

$$u_1^+ = \frac{C_1}{(1+\delta+\gamma)^n} \exp\left(-x^2\right), \tag{3.20}$$

$$u_2^+ = C_1 \exp\left(-\frac{x^2}{(1+\delta+\gamma)^n}\right), \tag{3.21}$$

$$u_1^- = \frac{C_2}{(1+\gamma)^n} \exp\left(-\frac{x^2}{(1+\delta)^n}\right) \tag{3.22}$$

and

$$u_2^- = \frac{C_2}{(1+\delta)^n} \exp\left(-\frac{x^2}{(1+\gamma)^n}\right). \tag{3.23}$$

Proof. *Substituting the above inequalities (3.16) and (3.17) into (3.5), we obtain the desired inequalities. This completes the proof.* ■

4 Existence and uniqueness theorem

4.1 Existence of the solution

In this section we make use of the following result (Theorem 3.1, see [15])

Theorem 7 *For the given boundary value problem*

$$\left\{ \begin{array}{rcl} v'' &=& h(x,v,v'),\ 0<x<\infty, \\ -\alpha v(0) &=& \beta v'(0) = r,\ v(\infty)=0, \end{array} \right. \tag{4.1}$$

where $\alpha>0, \beta \geq 0$ and r is a given constant. If $h(x,v,p)$ is continuous and satisfies:

- *There is a constant $M \geq 0$ such that $vh(x,v,0) \geq 0$ for $\mid v \mid > M$.*

- *There are functions* $A(x,v)>0$ *and* $B(x,v)>0$ *which are bounded when* v *varies in a bounded set and if* $\mid h(x,v,p) \mid \leq A(x,v)p^2 + B(x,v)$.
- *There is a continuous function* φ *such that* $\varphi(x) \to 0$ *as* $x \to \infty$ *and* $\mid v(x) \mid \leq \varphi(x)$ *for* $0 \leq x < \infty$.

Then this boundary value problem has at least one solution in $\mathbb{C}^2[0,\infty)$.

We also have the following lemmas

Lemma 8 *The given boundary problem Pr.(2.8) can be converted to*

$$\left\{ \begin{array}{rcl} v'' & = & g(x,v,v'),\ 0<x<\infty, \\ v(0) & = & -1,\ v(\infty)=0, \end{array} \right. \tag{4.2}$$

where $g(x,v,v')$ *is continuous and defined on* $[0,\infty)\times[-1,0]\times\mathbb{R}$ *by*

$$g(x,v,v') = -2\gamma n \frac{(v+1+\frac{\delta}{2\gamma})(v')^2}{\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa} - 2x\frac{v'}{(\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa)^n}. \tag{4.3}$$

Proof. Setting $u = v+1+\frac{\delta}{2\gamma}$ into Pr.(2.8). ■

Lemma 9 *For the given BVP (4.2), there are functions* $A(x,v), B(x,v)>0$ *which are bounded when* v *varies in a bounded set* $[-1,0]$ *such that*

$$\mid g(x,v,p) \mid \leq A(x,v)p^2 + B(x,v). \tag{4.4}$$

Proof. From the upper bounds of $u'(x)$, we have

$$u'(x) \leq C_1 \exp\left(-\frac{x^2}{(1+\delta+\gamma)^n}\right) \quad \text{for } \delta \geq 0 \tag{4.5}$$

and

$$u'(x) \leq \frac{C_2}{(1+\delta)^n} \exp\left(-\frac{x^2}{(1+\gamma)^n}\right) \quad \text{for } -1<\delta<0, \tag{4.6}$$

that is

$$v'(x) \leq C_1 \exp\left(-\frac{x^2}{(1+\delta+\gamma)^n}\right) \quad \text{for } \delta \geq 0 \tag{4.7}$$

and

$$v'(x) \leq \frac{C_2}{(1+\delta)^n} \exp\left(-\frac{x^2}{(1+\gamma)^n}\right) \quad \text{for } -1<\delta<0. \tag{4.8}$$

Thus

$$\mid g(x,v,v') \mid \leq 2\gamma n \frac{(v+1+\frac{\delta}{2\gamma})}{\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa}(v')^2 + \frac{2C_1 x \exp\left(-\frac{x^2}{(1+\delta+\gamma)^n}\right)}{(\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa)^n},\ \delta \geq 0 \tag{4.9}$$

and

$$\mid g(x,v,v') \mid \leq 2\gamma n \frac{(v+1+\frac{\delta}{2\gamma})}{\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa}(v')^2 + \frac{2C_1 x \exp\left(-\frac{x^2}{(1+\gamma)^n}\right)}{(1+\delta)^n(\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa)^n},\quad -1<\delta<0. \tag{4.10}$$

Hence

$$A(x,v) = 2\gamma n \frac{(v+1+\frac{\delta}{2\gamma})}{\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa},\ B(x,v) = \frac{2C_1 x \exp\left(-\frac{x^2}{(1+\delta+\gamma)^n}\right)}{(\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa)^n},\ \delta \geq 0 \tag{4.11}$$

and

$$A(x,v) = 2\gamma n \frac{(v+1+\frac{\delta}{2\gamma})}{\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa},\ B(x,v) = \frac{2C_1 x \exp\left(-\frac{x^2}{(1+\gamma)^n}\right)}{(1+\delta)^n(\gamma(v+1+\frac{\delta}{2\gamma})^2+\kappa)^n},\quad -1<\delta<0. \tag{4.12}$$

When v varies in a bounded set $[-1,0]$, we have

$$\mid A(x,v) \mid \leq\ 2\gamma n + n\delta,\ \mid B(x,v) \mid \leq 2C_1,\quad \delta \geq 0 \tag{4.13}$$

and

$$\mid A(x,v) \mid \leq\ 2\gamma n + n\delta,\ \mid B(x,v) \mid \leq \frac{2C_1}{(1+\delta)^n},\quad -1<\delta<0. \tag{4.14}$$

■

Lemma 10 *For the given BVP (4.2), there is a continuous function $\varphi(x)$ such that $\varphi(x) \to 0$ as $x \to \infty$ and $\mid v(x) \mid \leq \varphi(x)$ for $0 \leq x < \infty$.*

Proof. From the upper bounds of $u(x)$

$$u(x) \leq \frac{\delta}{2\gamma} + C_1 \frac{\sqrt{(\gamma+\delta+1)^n\pi}}{2} \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+\delta+1)^n}}\right),\ \delta \geq 0 \tag{4.15}$$

and

$$u(x) \leq \frac{\delta}{2\gamma} + C_2 \frac{\sqrt{(\gamma+1)^n\pi}}{2(\delta+1)^n} \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+1)^n}}\right),\ -1<\delta<0. \tag{4.16}$$

If we choose, for example $C_1 = \frac{2}{\sqrt{(\gamma+\delta+1)^n\pi}}$ and $C_2 = \frac{2(\delta+1)^n}{\sqrt{(\gamma+1)^n\pi}}$, then

$$u(x) \leq \frac{\delta}{2\gamma} + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+\delta+1)^n}}\right),\ \delta \geq 0 \tag{4.17}$$

and

$$u(x) \leq \frac{\delta}{2\gamma} + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+1)^n}}\right),\ -1<\delta<0, \tag{4.18}$$

and in view of $v = u - 1 - \frac{\delta}{2\gamma}$, we obtain

$$v(x) \leq -1 + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+\delta+1)^n}}\right), 0 \leq x < \infty \text{ for } \delta \geq 0, \tag{4.19}$$

and

$$v(x) \leq -1 + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+1)^n}}\right), 0 \leq x < \infty \text{ for } -1<\delta<0. \tag{4.20}$$

This means that there exists a continuous function $\varphi(x)$ such that $v(x) \leq \varphi(x)$, where

$$\varphi(x) = -1 + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+\delta+1)^n}}\right) \to 0 \text{ as } x \to \infty \text{ for } \delta \geq 0 \tag{4.21}$$

and

$$\varphi(x) = -1 + \operatorname{erf}\left(\frac{x}{\sqrt{(\gamma+1)^n}}\right) \to 0 \text{ as } x \to \infty \text{ for } -1<\delta<0. \tag{4.22}$$

■

We are now ready to prove the existence of the solution.

Theorem 11 *The given boundary problem Pr.(4.2) has at least one solution v in $\mathbb{C}^2[0,\infty)$.*

Proof. Since the function $g(x,v,p)$ satisfies Lemmas 7-9. Furthermore $g(x,v,0)=0$. Thus, we can conclude from Theorem 3.1 (see [15]) that Pr.(4.2) has at least one solution $v(x)$. ■

4.2 Uniqueness of the solution

Theorem 12 *If $g(x, v, p)$ is monotone increasing in v for each fixed $x \in [0, \infty)$ and $p \in \mathbb{R}$. Then, the boundary problem Pr.(4.2) has at most one solution v in $\mathbb{C}^2[0, \infty)$.*

Proof. Let v and w be two solutions of Pr. (4.2) such that $v(x) \neq w(x)$. It is well known that if $g(x, v, v')$ is monotone increasing in v, then the boundary value boundary on a finite interval [16]:

$$\begin{cases} v'' &= g(x, v, v'),\ 0 < x < b, \\ v(0) &= \sigma_1,\ v(b) = \sigma_2 \end{cases} \tag{4.23}$$

has at most one solution.
Thus $v(x) = w(x)$ on $0 \leq x \leq b$ and $v(x) > w(x)$ on $x > b$ or $v(x) < w(x)$ on $x > b$. It follows that

$$(v(x) - w(x))'' = g(x, v, p) - g(x, w, p) \geq 0 \text{ for } x \geq b. \tag{4.24}$$

Thus $(v(x) - w(x))'$ is increasing $x > b$, that is $(v(x) - w(x))' \geq z'(b)$, where $z'(b) = v'(b) - w'(b)$. This implies $v(x) - w(x) \to \infty$ as $x \to \infty$, which is impossible since $v(x) \to 0$ and $w(x) \to 0$ as $x \to \infty$. This shows that $v(x) = w(x)$. ■

5 Properties of the Modified Error Function of two parameters $\varphi_{\delta,\gamma}$

The analysis of the result obtained in the preceding sections reveals two important observations regarding the nature and behavior of the solution y to Pr. (1.4).

1. The solution (3.1) of Pr. (1.4) can be viewed as a generalization of the modified error function established in [4] for $\delta > -1$, and in [3] when $\delta > 0$. Pr. (1.4) represents a Stefan's problem with a nonlinear thermal conductivity, and when $n = 1$ the thermal conductivity becomes quadratic of the form $K(y) = \gamma y^2 + \delta y + 1$. If the quadratic coefficient parameter $\gamma = 0$ then $K = \delta y + 1$, which is linear. The solution (3.1) in this case reduces to the solution of Pr (1.1) (Theorem 2.1) [3], which was defined as the modified error function of the parameter $\delta > 0$. Further, if $\delta = 0$ then substituting in (3.1) and (3.7) reduces y to the classical error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$. Therefore, the solution (3.1) of Pr. (1.4) can be viewed as the generalization of the modified error function defined in [3]. Hence we define the modified error function of two parameters
$$\varphi_{\delta,\gamma} = C \int_0^x \frac{1}{\Psi(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi(\xi)} d\xi\right) d\eta \tag{5.1}$$
as the solution to the nonlinear BVP (1.4).

2. Reducing the case to a linear thermal conductivity by substituting $n = 1$ and $\gamma = 0$ we substitute the upper and lower bounds of C_1 into (3.8) and the upper and lower bounds of C_2 into (3.9), we obtain
$$\frac{1}{(\delta+1)^{3/2}} \operatorname{erf}(x) \leq y \leq (\delta+1)^{3/2} \operatorname{erf}\left(\frac{x}{\sqrt{\delta+1}}\right), \quad \text{for } \delta \geq 0, \tag{5.2}$$
and
$$(\delta+1)^{3/2} \operatorname{erf}\left(\frac{x}{\sqrt{\delta+1}}\right) \leq y \leq \frac{1}{(\delta+1)^{3/2}} \operatorname{erf}(x), \quad \text{for } -1 < \delta < 0. \tag{5.3}$$
Taking $\delta \to 0^+$ in (5.2) and $\delta \to 0^-$ in (5.3) we obtain $y = \operatorname{erf}(x)$. This shows that the upper and lower bounds in Theorem 5 are finite and they approach to $\operatorname{erf}(x)$ as $\delta \to 0^{\pm}$ in full agreement with the discussion above.

The modified error function $\varphi_{\delta,\gamma}$ shares some basic properties with the classical error function. This was established in [3] for the modified error φ_δ where $\delta > 0$. It is a direct consequence from theorem 6 that $y' > 0$ for all $\gamma \in \mathbb{R}$ and $\delta > -1$, and $y' \to 0$ as $x \to \infty$ which is in agreement with corollary 3.2 [10]. This also implies that we either have $y'' < 0$ for all $x > 0$ or $y'' < 0$ for some $x > x_0$. That is; either y is concave down on its domain or eventually concave down. Direct computations of y'' from (1.4) yields

$$y'' = -\frac{y'}{(1 + \delta y + \gamma y^2)^n} \left[2x + n(1 + \delta y + \gamma y^2)^{n-1}(\delta + 2\gamma y)y'\right]. \tag{5.4}$$

In view of the values of the parameters γ and δ that were considered above, we have two cases:

1. $\gamma > -1$ and $\delta > 0$. Then it is easy to see that $y'' < 0$ for $x > 0$. That is, the error function $\varphi_{\delta,\gamma}$ is increasing and concave down for all $x > 0$ and $n \geq 1$.

2. $\gamma > -1$ and $-1 < \delta < 0$, or $-1 < \gamma < 0$ and $\delta > 0$ If $y > \dfrac{-\delta + \sqrt{\delta^2 - 4\gamma}}{2\gamma}$, then $y'' < 0$. This implies that $\varphi_{\delta,\gamma}$ eventually concave down.

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The author declare that I have no competing interests.

Authors contributions

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