Local Stability Theory for Caputo Fractional Planar System and Application to Predator-Prey Model with Group Defense

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Abstract

In this manuscript we show a new approach into analyzing the local stability of equilibrium points of non-linear Caputo fractional planar systems. It is shown that the equilibrium points of such systems can exhibit an unstable focus or stable focus under suitable conditions. Further, it is shown that for α close to 1, global stability can be concluded, under suitable conditions, and without the use of a Lyapunov function. Lastly, our results are applied to a redator prey model with group defense, in which we show that it had equilibrium points that undergo an unstable focus and a stable focus.

1 Introduction

Fractional Differential Equations (FDE) have been growing in popularity in the field of applied mathematics, in particular in the field of mathematical modeling, see [2, 5, 6, 9–12, 17]. The primary modeling approach in the references is via Dynamical Systems with $\alpha \in (0,1)$, where α is the order of the derivative. Such models are particularly popular in modeling disease spread of Predator-Prey interactions (ecosystems), see [3, 7, 20, 21]. Traditionally, the authors in those papers are interested in determining the qualitative behavior of the system near its equilibria points, by employing the classical theory of local stability analysis or bifurcation theory. Similarly, for the fractional case, authors attempt to do the same.

However, due to the complexity of the fractional derivative, the results obtained are not always as strong as the classical. non-fractional, case. That is to say, for the classical case the local stability theory is well developed and it is easy to justify the qualitative behavior of a system near its equilibria point, as well as provide a complete characterization of the solutions. However, for the fractional case this is not the case.

In this paper we show that the stability and qualitative behaviour of equilibria points of a Caputo Fractional Planar System (CFPS) undergo the same qualitative behavior,under suitable conditions, as the classical case for α close to 1. This extends, the well known result, to the Caputo Fractional Planar Systems, where [19] has a similar result. Our method of proof is different, by using an asymptotic expansion of the Mittag-Leffler function, as provided in [4]. Also, our result differs from that in [19], as illustrated in Theorem 3.2 and example 3.1.

We show these new results by making use of some results obtained in [4,16]. In particular, first we show that the there exists some neighborhood of the equilibrium point of (CFPS) such that the stability, and characterization of the equilibrium point behave the same as the classical case for α close to 1, see Theorem 3.1. This result is new. Secondly, we show that there exists a range for $\alpha \in (0,1)$, whre α need not be arbitrarily close to 1, such that, under suitable conditions, a equilibrium point can undergo an unstable or stable focus, Lemma 3.3 and Theorem 3.2, respectively. The conclusion is then summarized in Theorem 3.3. Moreover, the foregoing results allow us to provide a way of concluding global stability for CFPS where α is close to 1, without the need to use a Lyapunov function, see Lemma 3.5. The approach that is taken in this paper is also different then that taken in [19] and it provides insight into how to approach more complex problems in this area.

We use Theorem 3.3 to improve the result obtained in [17], Theorem 5.5 and Theorem 5.6. Specifically, we show that the equilibrium point (u_{1*}, u_{2*}) obtained in [17] undergo an unstable and stable spiral under suitable conditions. Previously, the local stability of (u_{1*}, u_{2*}) is classified only as asymptotically stable or unstable.

Numerical results are provided using the Matlab function fde12 solver, see [18].

2 Preliminaries

Definition 2.1. [1, p. 13] Let $\alpha \geq 0$. The operator J_a^α, defined on $L^1[a,b]$ by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx \tag{2.1}$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α. Here and in what follows $\Gamma(\cdot)$ is the Gamma function.

Remark 2.1. For $\alpha = 0$, we set $J_a^0 := I$, the identity operator.

Definition 2.2. Let $0 < \alpha < 1$. Then, we define the Caputo fractional differential operator cD_a^α as

$$cD_a^\alpha f(t) := J_a^{1-\alpha} f'(t) \tag{2.2}$$

whenever $f, f' \in L^1[a,b]$.

Definition 2.3. Let $\alpha > 0$, $\beta > 0$. The function E_α, defined by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \tag{2.3}$$

whenever the series converges is called the two parameter Mittag-Leffler function with parameters α and β.

Lemma 2.1. *[4, p. 32] If $0 < \alpha < 1$, $\beta \in \mathbb{C}$ and $\mu \in \mathbb{R}$ such that*

$$\alpha\frac{\pi}{2} < \mu < \alpha\pi,$$

then for a arbitrary integer $p \geq 1$ the following expansion holds:

$$E_\alpha(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\Big(|z|^{-1-p}\Big) \quad |z| \to \infty,\ |arg(z)| \leq \mu \tag{2.4}$$

or

$$E_\alpha(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\Big(|z|^{-1-p}\Big) \quad |z| \to \infty.\ \mu \leq |arg(z)| \leq \pi \tag{2.5}$$

Remark 2.2. Note, that the terms

$$\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\Big(|z|^{-1-p}\Big)$$

become arbitrary small as $|z| \to \infty$. Fix $\beta = 1$, then

$$E_\alpha(z) = \frac{1}{\alpha} e^{z^{1/\alpha}} \quad |z| \to \infty,\ |arg(z)| \leq \mu$$

3 Local Stability Theory of Planar Fractional System

In this section we provide the stability theory that we will use in the paper for Planar Fractional Systems. Specifically, we build on the the classical theory and extend the results to the fractional case.

$$\begin{cases} cD_0^\alpha x(t) = f(x,y), \\ cD_0^\alpha y(t) = g(x,y) \end{cases} \tag{3.1}$$

subject to the initial condition:

$$(x(0), y(0)) = (x_0, y_0)$$

where $f, g \in C^1(\mathbb{R}^2)$. Recall that (x, y) is said to be a solution of (3.1) if $x, y \in AC(\mathbb{R}_+)$ and satisfy the above initial value problem. A solution (x, y) is said to be positive if $x, y \in P$ where

$$P = \{x \in C^1(R_+) : x(t) \geq 0 \quad \text{for } t \in R_+\}.$$

Since, $f, g \in C^1(\mathbb{R}^2)$, it is well known that for any $(x_0, y_0) \in \mathbb{R}^2$ the initial value problem (3.1) has a unique solution.

We denote by $A(x, y)$ the Jacobian matrix of f and g at (x, y), that is,

$$A(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{3.2}$$

and by $|A(x, y)|$ and $\mathrm{tr}(A(x, y))$ the determinant and trace of $A(x, y)$, respectively.

Definition 3.1. A point $(x^*, y^*) \in \mathbb{R}^2$ is called an equilibrium point of (3.1) if $f(x^*, y^*) = 0$, and $g(x^*, y^*)$.

Below we define the linearized system of (3.1) about the equilibrium point (x^*, y^*).

Definition 3.2. Let A be the matrix defined in (3.2) is evaluated at the equilibrium point (x^*, y^*). Then,

$$cD_0^\alpha X = A^* X, \tag{3.3}$$

where $X = (x, y)^T$, is the linearization of system (3.1) at the equilibrium point (x^*, y^*).

The following Lemma is a special case $(n = 2)$ of Lemma 3.2 in [17].

Lemma 3.1. *[17] Let (x^*, y^*) be an equilibrium point of (3.1) and A be defined as in (3.2). Let λ_1 and λ_2 be the eigenvalues of A. Then, the following assertions hold.*

(1) *The equilibrium point (x^*, y^*) is locally asymptotically stable if and only if $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$.*

(2) *The equilibrium point (x^*, y^*) is stable if and only if $|arg(\lambda_{1,2})| \geq \frac{\alpha\pi}{2}$ and the eigenvalues with $|arg(\lambda_{1,2})| = \frac{\alpha\pi}{2}$ have the same geometric multiplicity and algebraic multiplicity.*

(3) *The equilibrium point (x^*, y^*) is unstable if and only if $|arg(\lambda_1)| < \frac{\alpha\pi}{2}$.*

Lemma 3.2. *[16][Theorem 3] If the origin O is a hyperbolic equilibrium point of (3.1), then vector field $(f(x,y), g(x,y))$ is topologically equivalent with its linearization vector field given by the linear system $cD_0^\alpha X = AX$ in the neighborhood of the origin O.*

Theorem 3.1. *Consider the Caputo planar system*

$$\begin{cases} cD_a^\alpha x(t) = f(x(t)), \\ x(t_0) = x_0, \end{cases} \tag{3.4}$$

where $\alpha \in (0,1)$. Let $f \in C^1(E)$, where $E \subset \mathbb{R}$. Let x_e be an equilibrium point of (3.4). Define by

$$N_{\delta_0}(x_e) := \{x : |x - x_e| < \delta_0\},$$

and

$$N_{\delta_1}(x_e) := \{x : |x - x_e| < \delta_1\},$$

for some $\delta_0 > 0$ and $\delta_1 > 0$. Then, N_{δ_0} and N_{δ_1} are two neighborhoods about the equilibrium point x_e. Further, let x_α and x_1 represents the solutions to (3.4) with $\alpha \in (0,1)$ and $\alpha = 1$, respectively. Then, for

$$x_\alpha \in N_{\delta_0}(x_e) \quad \textit{and} \quad x_1 \in N_{\delta_1}(x_e),$$

$$\lim_{\alpha \to 1} x_\alpha(t) = x_1(t) \quad \textit{for } t \in [t_0, \infty) \textit{ for every } t_0 \geq 0.$$

Proof. By Lemma 3.2 we have that the local behavior of the nonlinear system (3.4) near a hyperbolic equilibrium point x_e is qualitatively determined by the behavior of the linear system

$$cD_0^\alpha x(t) = Ax,$$

where A is a 2×2 matrix such that $A = Df(x_e)$, near the equilibrium point x_e. The linear function $Ax = Df(x_e)x$ is called the linear part of f at x_e.

Since, we are considering the linearization near an equilibrium point x_e, then we assume that our system has attained the equilibrium state. In particular, we will consider system (3.4) for $t \geq t_0$ for any $t_0 \geq 0$.

Let λ_1 and λ_2 be the two eigenvalue of A. Furthermore, suppose that $\lambda_1 \neq \lambda_2$. Then, the solution can be represented by the form

$$x_\alpha(t) = c_1 u^{(1)} E_{\alpha,1}(\lambda_1 t^\alpha) + c_2 u^{(2)} E_{\alpha,1}(\lambda_2 t^\alpha),$$

where $c_1, c_2 \in \mathbb{R}$ and $u^{(1)}$ and $u^{(2)}$ are the eigenvectors of λ_1 and λ_2, respectively.

Consider,

$$\begin{aligned}
\lim_{\alpha\to 1} x_\alpha(t) &= \lim_{\alpha\to 1}\left(c_1 u^{(1)} E_{\alpha,1}(\lambda_1 t^\alpha) + c_2 u^{(2)} E_{\alpha,1}(\lambda_2 t^\alpha)\right)\\
&= c_1 u^{(1)} \lim_{\alpha\to 1} E_{\alpha,1}(\lambda_1 t^\alpha) + c_2 u^{(2)} \lim_{\alpha\to 1} E_{\alpha,1}(\lambda_2 t^\alpha)\\
&= c_1 u^{(1)} e^{\lambda_1 t} + c_2 u^{(2)} e^{\lambda_2 t} = x_1(t)
\end{aligned}$$

The case for $\lambda_1 = \lambda_2$ follows in a similar manner. □

Remark 3.1. Lemma (3.1) has an important implication that has not received much attention from previous authors in the past. In particular, it shows that the stability and just as important the characterization of the stability of the solutions near the equilibrium point, for $\alpha \in (1-\epsilon, 1)$ for some $\epsilon > 0$, behave the same as the classical case. Therefore, the classical theory can be applied to study the stability of the Caputo system for $\alpha \in (1-\epsilon, 1)$.

Lemma 3.3. *Let $\alpha \in (0,1)$. Suppose that the matrix A defined in (3.2) has the following eigenvalues: $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ with $a > 0$ and $b \neq 0$. Then, the general solution X of*

$$cD_0^\alpha X(t) = AX, \tag{3.5}$$

where $X = (x, y)^T$, has the following form

$$\begin{aligned}
X(t) = c_1 u_1 [\frac{1}{\alpha} e^{t^\alpha a/\alpha} (cos(bt^\alpha) + isin(bt^\alpha))^{1/\alpha}]\\
+ c_2 u_2 [\frac{1}{\alpha} e^{t^\alpha a/\alpha} (cos(bt^\alpha) - isin(bt^\alpha))^{1/\alpha}]
\end{aligned} \tag{3.6}$$

as $t \to \infty$ and $|arg(\lambda_{1,2})| \leq \frac{\alpha\pi}{2}$.

Where, c_1 and c_2 are arbitrary constants and u_1 and u_2 are eigenvectors corresponding to λ_1 and λ_2, respectively.

Proof. (1) Let $\lambda_1 = a+ib$ for $a, b \in \mathbb{R}$, since $a > 0$, then if $|arg(\lambda_{1,2})| \leq \frac{\alpha\pi}{2}$, together with (2.5), we have $E_\alpha(\lambda_1 t^\alpha) = (1/\alpha)e^{(\lambda_1 t^\alpha)^{1/\alpha}}$ as $t \to \infty$.

Also, we have

$$\begin{aligned}(1/\alpha)e^{(\lambda_1 t^\alpha)^{1/\alpha}} &= (1/\alpha)e^{((a+ib)t^\alpha)^{1/\alpha}} = (1/\alpha)e^{(at^\alpha+ibt^\alpha)^{1/\alpha}} \\ &= (1/\alpha)\Big[e^{at^\alpha}\Big(\cos bt^\alpha + i\sin bt^\alpha\Big)\Big]^{(1/\alpha)} \\ &= (1/\alpha)e^{a^{\frac{1}{\alpha}}t}\Big(\cos bt^\alpha + i\sin bt^\alpha\Big)^{(1/\alpha)}.\end{aligned}$$

Thus,

$$E_\alpha(\lambda_1 t^\alpha) = (1/\alpha)e^{(\lambda_1 t^\alpha)^{1/\alpha}} = (1/\alpha)e^{a^{\frac{1}{\alpha}}t}\Big(\cos bt^\alpha + i\sin bt^\alpha\Big)^{(1/\alpha)} \text{as } t \to \infty.$$

Similarly, it can be show that for $\lambda = \lambda_2 = a - ib$, that

$$E_\alpha(\lambda_1 t^\alpha) = (1/\alpha)e^{(\lambda_1 t^\alpha)^{1/\alpha}} = (1/\alpha)e^{a^{\frac{1}{\alpha}}t}\Big(\cos bt^\alpha - i\sin bt^\alpha\Big)^{(1/\alpha)} \text{as } t \to \infty.$$

Since, λ_1 and λ_2 are distinct eigenvalues, the general solution takes the form

$$y(t) = c_1 u_1 E_\alpha(\lambda_1 t^\alpha) + c_2 u_2 E_\alpha(\lambda_2 t^\alpha) \tag{3.7}$$

and,

$$\begin{aligned}X(t) = c_1 u_1 [&\frac{1}{\alpha}e^{t^\alpha a/\alpha}(cos(bt^\alpha) + isin(bt^\alpha))^{1/\alpha}] \\ &+ c_2 u_2[\frac{1}{\alpha}e^{t^\alpha a/\alpha}(cos(bt^\alpha) - isin(bt^\alpha))^{1/\alpha}].\end{aligned}$$

It is shown in [2] that expression (3.7) can be expressed as a real valued solution, thus the foregoing solution can be expressed as a real valued solution.

as $t \to \infty$ and $|arg(\lambda_{1,2})| \leq \frac{\alpha\pi}{2}$. □

Remark 3.2. Lemma 3.3 is new. It shows that the solution of a Caputo Fractional linear system behaves as a spiral (focus) under suitable conditions. However, the solutions need not maintain their spiral behaviour if $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$. We illustrate later that this eventually is the case.

Theorem 3.2. *Let $\alpha \in (0,1)$. Let λ_1 and λ_2 be eigenvalues of the matrix A defined in (3.2) and suppose that $\lambda_1 = a+ib$ and $\lambda_2 = a-ib$, with $a > 0$ and $b \neq 0$. If $|Arg(\lambda_1)| < \frac{\alpha\pi}{2}$, then the following holds*

1. *If $\alpha \in (\alpha^*, 1)$, then the origin is an unstable focus.*
2. *There exists an ϵ_0, such that for some $\epsilon_0 > 0$, the origin is a stable focus whenever $\alpha \in (\alpha^* - \epsilon_0, \alpha^*)$ and Asymptotically stable whenever $\alpha \in (0, \alpha^* - \epsilon_0]$*

Proof. Note, that $\alpha^* = \frac{2}{\pi}|Arg(\lambda_{1,2})| \in (0,1)$ if and only if $|arg(\lambda_{1,2})| < \frac{\pi}{2}$ and since, $a > 0$ then this condition is satisfied.

(1) By condition (1) we have that $|A| > 0$, $a = tr(A) > 0$ and $b = tr^2(A) - 4|A| \neq 0$. Then, it is well known that the origin is an unstable focus of (3.5) with $\alpha = 1$. Consider, $\alpha \in (0,1)$. Since, $\alpha \in (\alpha^*, 1)$, then $|arg(\lambda_{1,2})| < \frac{\alpha\pi}{2}$.

By application of Lemma 3.3 we have that the solution X of (3.5) can be expressed as

$$X(t) = c_1 u_1[\frac{1}{\alpha} e^{t^\alpha a/\alpha}(cos(bt^\alpha) + isin(bt^\alpha))^{1/\alpha}] + c_2 u_2[\frac{1}{\alpha} e^{t^\alpha a/\alpha}(cos(bt^\alpha) - isin(bt^\alpha))^{1/\alpha}] \tag{3.8}$$

as $t \to \infty$.

Thus the characterization of the solution is a spiral, and by applying Lemma 3.1 we obtain the behavior of the stability. The result follows.

(2) Since, $0 < \alpha < \alpha^*$, then $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$. Thus, the solution X need not maintain the spiral behavior, as given by (3.6). However,taking the following limit: $\alpha \to \alpha^*$ of the above yields,

$$\lim_{\alpha \to \alpha^*} X_\alpha(t) = X_{\alpha^*}(t).$$

Hence, there exists $\epsilon_0 > 0$ such that for $\alpha \in (\alpha^* - \epsilon_0, \alpha^*)$, X_α can be made arbitrarily close to $X_{\alpha^{*-}}$. Thus, the characteristic of the solution will be preserved. Moreover, since $\alpha \in (\alpha^* - \epsilon_0, \alpha^*)$ by Lemma 3.1 we have that the stability is interchanged. That is to say, it is now stable. For $\alpha \in (0, \alpha^* - \epsilon_0]$, the solutions need not preserve the characteristic of a stable focus, and thus we can only conclude, by Lemma 3.1, that it is asymptotically stable, and result (2) follows.

□

The following Theorem follows from Lemma 3.1 and Theorem 3.2, where the conditions are expressed in terms of $\mathrm{tr}(A(x^*, y^*))$ and $|A(x^*, y^*)|$.

Theorem 3.3. *Let $\alpha \in (0, 1)$. If (x^*, y^*) is a equilibrium point of (3.1), then the following assertions hold.*

(i) *If $|A(x^*, y^*)| < 0$, then (x^*, y^*) is unstable. (3.1).*

(ii) *Let $\delta > 0$. If $|A(x^*, y^*)| > 0$, $\mathrm{tr}(A(x^*, y^*)) > 0$ and $(\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$, then (x^*, y^*) is unstable.*

(iii) *If the conditions of Theorem 3.2 hold, and $|A(x^*, y^*)| > 0$, $\mathrm{tr}(A(x^*, y^*)) > 0$ and $(\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$, then (x^*, y^*) is an unstable focus of (3.1) for $\alpha \in (\alpha^*, 1)$; stable focus of (3.1) for $\alpha \in (\alpha^* - \delta, \alpha^*]$; Locally Asymptotically stable if $\alpha \in (0, \alpha - \delta)$ where $\alpha^* = (2/\pi)|arg(\lambda_1)|$.*

(iv) *If $|A(x^*, y^*)| > 0$, $\mathrm{tr}(A(x^*, y^*)) < 0$, then (x^*, y^*) is Locally Asymptotically stable.*

Remark 3.3. We note that result (iii) of Theorem 3.3 is not the same as that provided in [19] Theorem 4 (f) and (g). In fact, in [19], the matrix A is not the same as the one defined in (3.2). Furthermore, Theorem 3.3 (iii) guarantees that the equilibrium point (x^*, y^*) is only a stable focus for α in some left neighborhood of α^*, provided that the eigenvalues are complex conjugates with a strictly positive real part. As opposed to Theorem 4 (g) in [19], which implies that an equilibrium point is a stable focus for $\alpha \in (0, \alpha^*)$, provided that the eigenvalues are complex conjugates.

Our method of proof is also different, in this paper we use asymptotic expansion of the Mittag-Leffler function, as shown in 2.1, to prove our result. Below we provide an example illustrating the result obtained in Theorem 3.3 and show the importance that $\alpha \in (0, 1)$ has in determining the qualitative behavior of an equilibrium point.

Example 3.1. Consider the following Caputo Fractional linear system

$$cD_0^\alpha X(t) = AX, \tag{3.9}$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}.$$

Then, clearly A has two eigenvalues that are complex conjugates with the real part begin strictly positive, specifically

$$\lambda_1 = 0.5000 + 1.9365i \quad \lambda_2 = 0.5000 - 1.9365i.$$

Furthemore, it can be shown that $\alpha^* = 0.8391$.

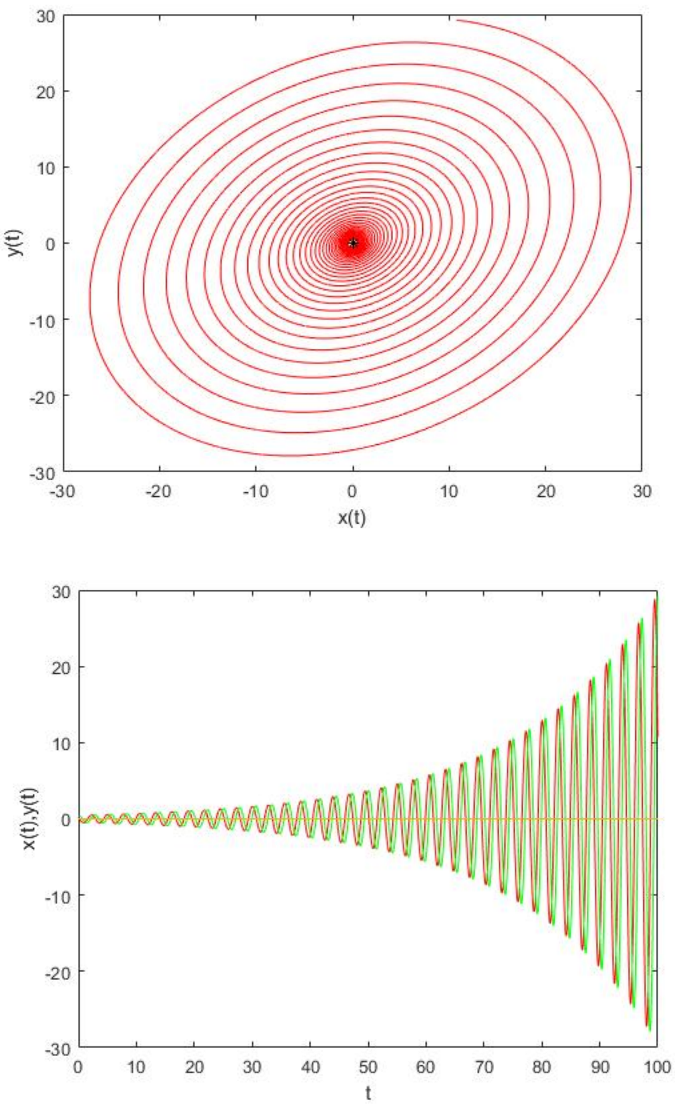

Figure 1: The origin is an unstable focus for of (3.9) with initial conditions $(x_0, y_0) = (0.15, 0.4)$ and $\alpha = 0.849$.

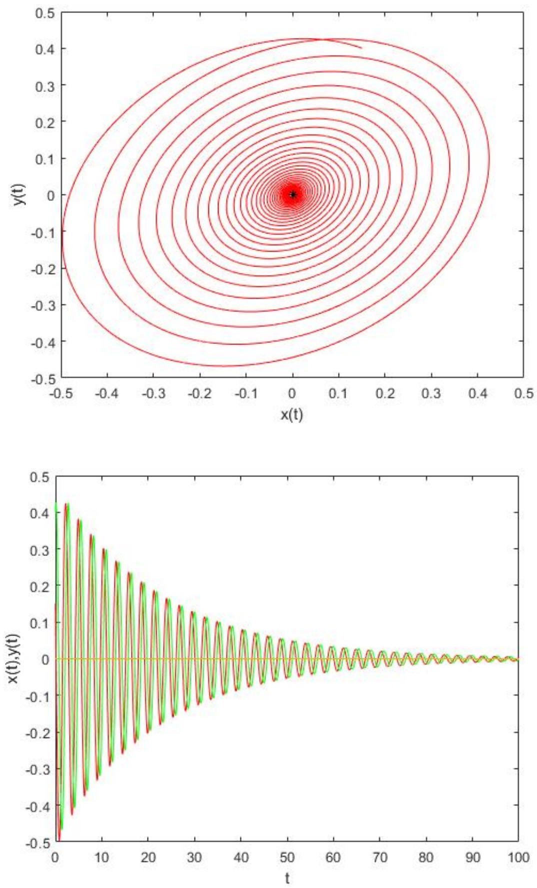

Figure 2: The origin is a stable focus for of (3.9) with initial conditions $(x_0, y_0) = (0.15, 0.4)$ and $\alpha = 0.829$.

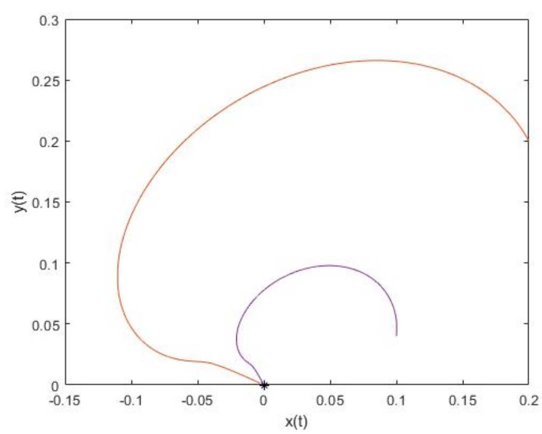

Figure 3: The origin is locally asymptotically stable for (3.9) with initial conditions $(x_0, y_0) = (0.1, 0.04)$, $(x_0, y_0) = (0.2, 0.2)$, and $\alpha = 0.5$. It can be seen that the spiral behavior is no longer present

Below we state a result that is used in the classical theory.

Lemma 3.4. *Let $\alpha = 1$. Assume that each positive solution of (3.1) with $(x_0, y_0) \in \mathbb{R}^2_+$ is contained in a bounded closed subset B of $\mathbb{R}^2$. Assume that B contains only one equilibrium (x^*, y^*) of (3.1) and (x^*, y^*) belongs to the boundary of B. Then each positive solution of (3.1) converges to (x^*, y^*).*

As an consequence of Theorem 3.1 we obtain the following result, an extension of the above Lemma 3.4 to the fractional case.

Lemma 3.5. *Assume that each positive solution of (3.1) with $(x_0, y_0) \in \mathbb{R}^2_+$ is contained in a bounded closed subset B of $\mathbb{R}^2$. Assume that B contains only one equilibrium (x^*, y^*) of (3.1) and (x^*, y^*) belongs to the boundary of B. Then, there exists a $\delta_0 > 0$ such that for $\alpha \in (\alpha - \delta_0, 1)$ we have that each positive solution of (3.1) converges to (x^*, y^*).*

Remark 3.4. Lemma 3.5 is new. It shows that Lemma 3.4 can be extended to the fractional case for Caputo fractional planar systems for some range of $\alpha \in (0, 1)$. This allows one to conclude global stability without the use of Luyapunov function, which is the most used approach for concluding global stability in such systems.

4 Application and Numerical Simulation

In this section we illustrate, via numerical simulation, the results obtained in section 2 by applying Theorem 3.3 (*iii*) to a fractional predator-prey model with group defense, see [17]. Further, we improve the result stated in Theorem 5.5 and Theorem 5.6 in [17], pertaining to the model below.

$$\begin{cases} cD_0^{\theta}u_1(t) = \gamma_1\left[1 - \frac{u_1(t)}{k}\right]u_1(t) - \gamma_2 u_1(t)^{\sigma}u_2(t) - \frac{\rho_1\rho_2 u_1(t)}{\rho_2\rho_3+\rho_4 u_1(t)} := f(u_1, u_2), \\ cD_0^{\theta}u_2(t) = -\gamma_3 u_2(t) + \gamma_2\gamma_4 u_1(t)^{\sigma}u_2(t) := g(u_1, u_2), \end{cases} \tag{4.1}$$

where $u_1(t)$, and $u_2(t)$ represent the prey and predator densities at time t, respectively. γ_1 represents the logistic growth rate, γ_2 is the search efficiency of predator for prey, γ_3 is the mortality rate of predator, γ_4 is the biomass conversion coefficient, k represents the carrying capacity of the environment, and σ denotes the aggregation efficiency, ρ_1, ρ_2 denote the catch-ability parameter, the effort applied to harvest the prey species, respectively, ρ_3 and ρ_4 denote the appropriate real constants. For biological purposes, all of the parameters are assumed to be positive.

Let

$$u_{1*} := \left[\frac{\gamma_3}{\gamma_2\gamma_3 k^\sigma}\right]^{\frac{1}{\sigma}} \text{ and } u_{2*} := u_{1*}^{1-\sigma}\left[1 - u_{1*} - \frac{\rho_1\rho_2}{\gamma_1(\rho_2\rho_3 + u_{1*}\rho_4 k)}\right].$$

It is shown in Theorem (5.1) of [17] that the equilibrium point (u_{1*}, u_{2*}) is positive if $(1 - u_{1*})\gamma_1(\rho_2\rho_3 + u_{1*}\rho_4 k) > \rho_1\rho_2$.

The notation and Lemmas below were obtained in [17], as Theorem 5.5 and Theorem 5.6, respectively.

Notation.

$$\begin{cases} a_{11} := \gamma_1 - \frac{2\gamma_1 u_{1*}}{k} - \gamma_2\sigma u_{1*}^{\sigma-1}u_{2*} - \frac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_{1*})^2} - \gamma_2 u_{1*}^\sigma, \\ a_{12} := \gamma_2 u_{1*}^\sigma, \\ a_{21} := \sigma\gamma_2\gamma_4 u_{1*}^{\sigma-1}u_{2*}, \\ a_{22} := -\gamma_3 + \gamma_2\gamma_4 u_{1*}, \\ \alpha := a_{11} + a_{22}, \\ \beta := a_{11}a_{22} - a_{12}a_{21}. \end{cases}$$

Lemma 4.1. *If one of the following inequalities is satisfied: (a) $\alpha \leq 0$ or (b) $\alpha > 0$, $\alpha^2 - 4\beta < 0$ and $\frac{\sqrt{\alpha^2-4\beta}}{\alpha} > \tan\frac{\theta\pi}{2}$. Then, the equilibrium point (u_{1*}, u_{2*}) is locally asymptotically stable.*

Lemma 4.2. *If one of the following inequalities is satisfied: (a) $\alpha > 0$, $\alpha^2 - 4\beta \geq 0$ or (b) $\alpha > 0$, $\alpha^2 - 4\beta < 0$ and $\frac{\sqrt{\alpha^2-4\beta}}{\alpha} < \tan\frac{\theta\pi}{2}$. Then, the equilibrium point (u_{1*}, u_{2*}) is unstable.*

By applying Theorem 3.3 (*iii*) we improve the result given in Lemma 4.1 and Lemma 4.2. Below we state the result.

Theorem 4.1. *Suppose that $\alpha > 0$ and $\alpha^2 - 4\beta < 0$, then the following assertions hold*

1. *If $\theta \in (\theta^*, 1)$, then the equilibrium point (u_{1*}, u_{2*}) is an unstable focus.*

2. *There exists a $\delta_0 > 0$ such that (u_{1*}, u_{2*}) is a stable focus whenever $\theta \in (\theta^* - \delta_0, \theta^*]$, and locally asymptotically stable whenever $\theta \in (0, \theta^* - \delta_0)$.*

Proof. Suppose that $\alpha > 0$ and $\alpha^2 - 4\beta < 0$, then, as shown in [17], we have that $\mathrm{tr}(A(u_{1*}, u_{2*})) > 0$ and $\mathrm{tr}(A(u_{1*}, u_{2*}))^2 - 4\det(A(u_{1*}, u_{2^*})) < 0$. By application of Theorem 3.3, the result follows. □

Remark 4.1. Theorem 4.1 improves the result stated in Theorem 5.5 and Theorem 5.6 in [17]. specifically, it shows that the equilibrium point (u_{1*}, u_{2*}) can be an unstable or stable focus under suitable conditions. This provides a more complete understanding of the dynamical behavior of (4.1).

Below we provide numerical simulation for Theorem 4.1. This is done by considering the linearized part of (4.1) at the equilibrium point (u_{1*}, u_{2*}), where the linearized part is defined in (3.3), where A^* is evaluated at the equilibrium point (u_{1*}, u_{2*}).

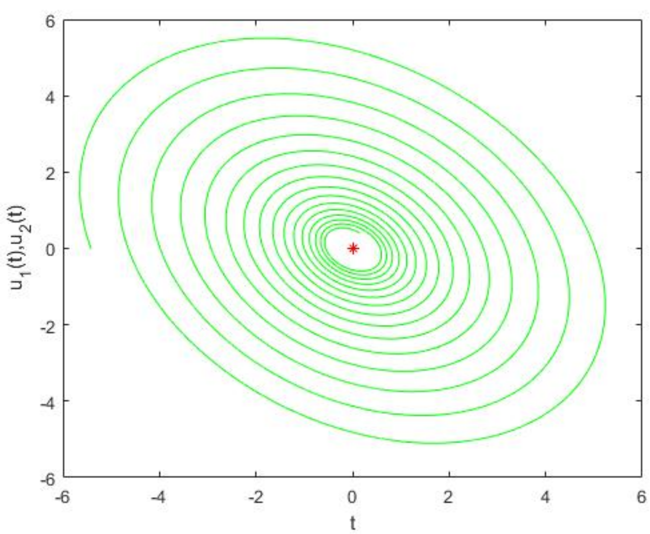

Figure 4: The trajectories of the linear part of system (4.1) with $\theta = 0.93$, $\gamma = 2$, $\gamma_2 = 3$, $\gamma_3 = 0.2$, $\gamma_4 = 6/9$, $\rho_1 = 0.29$, $\rho_2 = 0$, $\rho_3 = 1.2$, $\rho_4 = 0.5$, $s_1 = 0.255$ and $k = 500/499$. The equilibrium point $(0, 0)$ is an unstable focus. The initial conditions used here are $(u_{10}, u_{20}) = (0.15, 0.4)$.

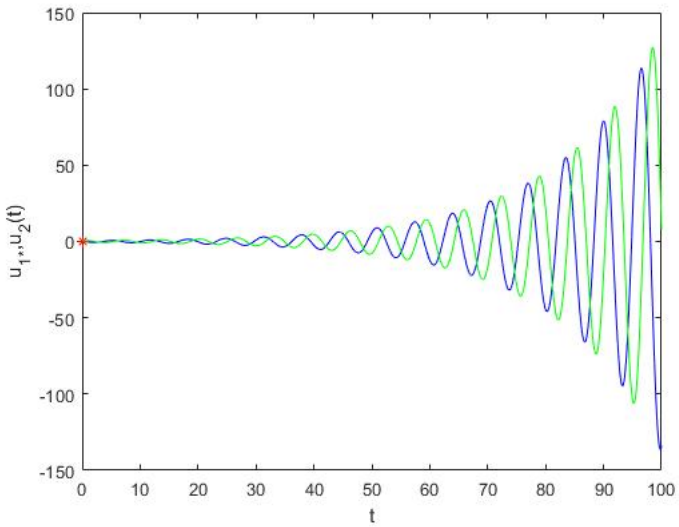

Figure 5: The trajectories of the linear part of system (4.1) vs time, with $\theta = 0.93$, $\gamma = 2$, $\gamma_2 = 3$, $\gamma_3 = 0.2$, $\gamma_4 = 6/9$, $\rho_1 = 0.29$, $\rho_2 = 0$, $\rho_3 = 1.2$, $\rho_4 = 0.5$, $s_1 = 0.255$ and $k = 500/499$. The equilibrium point $(0, 0)$ is an unstable focus. The initial conditions used here are $(u_{10}, u_{20}) = (0.15, 0.4)$.

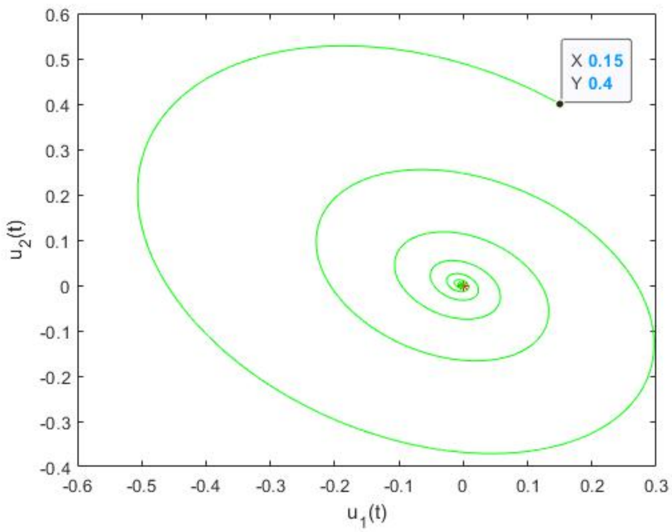

Figure 6: The trajectories of the linear part of system (4.1) with $\theta = 0.83$, $\gamma = 2$, $\gamma_2 = 3$, $\gamma_3 = 0.2$, $\gamma_4 = 6/9$, $\rho_1 = 0.29$, $\rho_2 = 0$, $\rho_3 = 1.2$, $\rho_4 = 0.5$, $s_1 = 0.255$ and $k = 500/499$. The equilibrium point $(0, 0)$ is a stable focus. The initial conditions used here are $(u_{10}, u_{20}) = (0.15, 0.4)$.

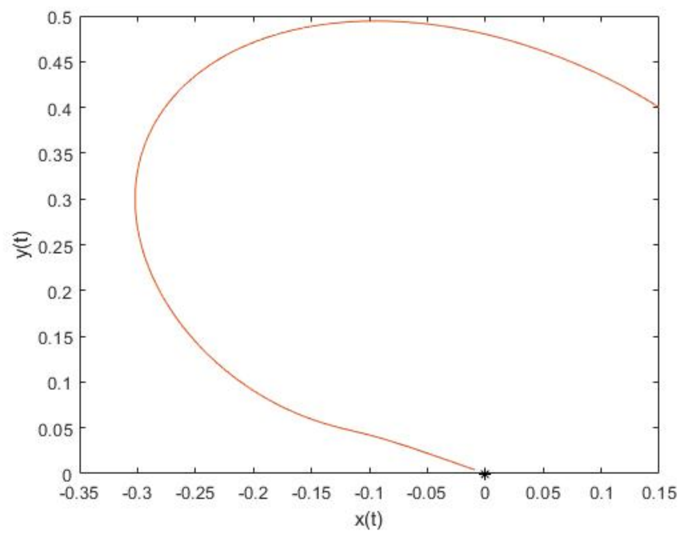

Figure 7: The trajectories of the linear part of system (4.1) vs time, with $\theta = 0.65$, $\gamma = 2$, $\gamma_2 = 3$, $\gamma_3 = 0.2$, $\gamma_4 = 6/9$, $\rho_1 = 0.29$, $\rho_2 = 0$, $\rho_3 = 1.2$, $\rho_4 = 0.5$, $s_1 = 0.255$ and $k = 500/499$. The equilibrium point $(0,0)$ is Locally Asymptotically Stable. The initial conditions used here are $(u_{10}, u_{20}) = (0.15, 0.4)$.

Remark 4.2. It should be noted that similar figures, such as Figure 6 can be found in [17]. However, the equilibrium point is classified only as locally asymptotically stable. Theorem 4.1 shows that it can be classified as a stable focus, and figure 6 supports this. Furthermore, the above figures clearly illustrate Theorem 3.3 as it is applied to system (4.1) via Theorem 4.1.

Conclusion

This manuscript concerns itself with the local stability theory and qualitative behavior of the equilibria points of Caputo Fractional Planar Systems. In particular, it shows how the well established result of the classical planar systems (regular ODE) can be extended to study the stability of Caputo Fractional Planar Systems (CFPS) near its equilibria points, see Theorem 3.1, which shows that the local stability theory of the classical planar system can be employed to determine the local stability of equilibria points of CFPS for $\alpha \in (\alpha - \epsilon, 1)$ for some $\epsilon > 0$. Moreover, Theorem 3.2 shows that under suitable conditions, the equilibrium point of CFPS can be an unstable focus for a specific range of α. Consequently, improving Lemma 3.1. The method of proof here is different then the one given in [19] and our result shows that if α is small enough, then the spiral behavior need not be preserved, and we do not have stable focus. This is not mentioned in [19]. Lastly, by employing Theorem 3.3, we provide a more complete picture on the local stability of the equilibrium point (u_{1*}, u_{2*}) found [17]. Essentially, improving on Theorem 5.5 and Theorem 5.6 in [17]. Also, using LEmma 3.5 we can show that, under suitable conditions, an equilibrium point of (3.1) is globally asymptotically stable without the need to use a Lyapunov function, which in some cases could be difficult to determine. Thus, we have shown that Lemma 3.4 could be applied to the CFPS for some α close to 1.

We remark, that these results and our method of proof only apply to hyperbolic equilibrium points. The case for non-hyperbolic equilibrium points is much more difficult and is part of the future work.

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