The initial value problem for a tissue growth mathematical model

Zenggui Wang

School of Mathematical Sciences, Liaocheng University, Liaocheng 252059,PR China

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Abstract: This paper considers the initial value problem for a normal hyperbolic curvature flow derived by the cell-based mathematical models of tissue growth to account for the mechanistic influence of curvature on cell evolution. The equations satisfied by support functions under this flow is a hyperbolic Monge-Ampère equation. The equation for both perimeter and area of closed curves under the flow are also obtained. Based on this, we show that for a closed curve, if the initial velocity $v_0 < 0$, the solution of this flow converges to a point in finite time; if $v_0 > 0$, the solution of this flow exists for all $t \in [0, \infty)$.

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1. Introduction

In this paper we consider a family of closed curves $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ satisfying the following evolution equation

$$\begin{cases} \gamma_{tt} = -v^2 k\mathbf{n} - vv_s\mathbf{t}, \\ \gamma|_{t=0} = \gamma_0, \gamma_t|_{t=0} = v_0\mathbf{n}_0, \end{cases} \tag{1.1}$$

where subscripts denote partial derivatives, v is the outer normal velocity of the interface γ, $\mathbf{n}$ is the outer unit normal of the curve γ, v_t corresponds to the normal acceleration of the interface, i.e., $v_t = \gamma_{tt} \cdot \mathbf{n}$, s is the arc length such that $ds = gdu = |\gamma_u|du$, γ_0 is a convex closed curve, v_0 is the initial velocity of the curve γ_0, $\mathbf{n_0}$ is the outer unit normal of the γ_0. In fact, the evolution equation (1.1) can be derived by the normal curve flow

$$\begin{cases} \gamma_t = v\mathbf{n}, \\ v_t = -v^2 k, \\ \gamma|_{t=0} = \gamma_0, \gamma_t|_{t=0} = v_0\mathbf{n}_0, \end{cases} \tag{1.2}$$

The cell-based mathematical models of tissue growth to account for the mechanistic influence of curvature on cell evolution is proposed by Alias [1], which can be reduces to a specific type of hyperbolic curvature flow

$$\begin{cases} \gamma_t = v\mathbf{n}, \\ v_t = -v^2 k + Dv_{ss} - Av, \\ \gamma|_{t=0} = \gamma_0, \gamma_t|_{t=0} = v_0\mathbf{n}_0, \end{cases} \tag{1.3}$$

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[1]Corresponding author: Zenggui Wang

[2]E-mail addresses: wangzenggui@lcu.edu.cn(Z.Wang)

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in which Dv_{ss} represents cell diffusion parallel to the interface with diffusivity D, and the term $-Av$ denotes the depletion of active cells at rate A. The hyperbolic character of this mathematical models gives rise to a rich set of interface movement patterns.

In the hyperbolic versions of the mean curvature flow one studies the motion of a hypersurface (or curve) whose normal acceleration is equal to a function of mean curvature and normal velocity. Many results have been obtained over the years. Gurtin and Podio-Guidugli [2] proposed a mechanical theory for the evolution of a two-dimensional interface possessing an effective inertia. A crystalline algorithm to study the motion of closed polygonal curves under hyperbolic mean curvature flow was developed by Rostein, Brandon and Novick-Cohen [3]. Yau [4] proposed the motion of a hypersurface whose normal acceleration equals to its mean curvature. The local solvability of this problem is established in [5] and the relation between the equations for hyperbolic mean curvature flow and the equations for extremal surfaces in the Minkowski space time is also discussed. Lefloch and Smoczyk [6] studied a hyperbolic geometry equation which stems from a geometrically natural action containing kinetic and internal energy terms, The lifespan of classical solution to the Cauchy problem for this flow is established by [7]. Kong, Liu and Wang [8] studied the hyperbolic curve shortening problem and proved the limit behavior of plane curves. In Kong and Wang [9] they further considered the formation of singularities in the motion of closed plane curves under hyperbolic mean curvature flow. Subsequently, Wang et al. [10]-[15] studied the hyperbolic mean curvature flow with different forced term, some new results are obtained. Dong et al. [16] investigated exact solutions of a hyperbolic Monge-Ampere equation by using Lie symmetry analysis method, similarly, Gao and Zhang [17] obtained invariant solutions of the normal hyperbolic mean curvature flow with dissipation. He et al [18] considered self-similar solutions to the hyperbolic mean curvature flow. Ginder [19]-[20] investigated hyperbolic mean curvature flow using an approximation method.

Aside from the hyperbolic mean curvature flow, there are other hyperbolic curvature flows for convex hypersurfaces. In Chou and Wo [21] they proposed hyperbolic Gauss curvature flows and a new class of fully nonlinear Euclidean invariant hyperbolic equations are given. Furthermore, Wo et al. [22] introduced a new hyperbolic version of affine invariant curve flow. The local solvability, finite time blow-up of the flow and global existence are established. Recently, a dissipative hyperbolic affine curve flow is studied by Wang [23]. Notz [24] introduced and studied a new geometric flow equation, which describes the motion of closed hypersurfaces in Riemannian manifolds. Yan [25] studied the motion of closed hypersurfaces in the central force fields which generalized the results in [24]. Mao et al. [26]-[28] introduced different geometry flows such as forced hyperbolic mean curvature flow, hyperbolic inverse mean curvature flow and generalized hyperbolic mean curvature flow.

The aim of this paper is to study the hyperbolic curvature flow (1.1). By the support functions of the curve, which is reduced to a single fully nonlinear hyperbolic equation

$$h_{tt} = \frac{h_{\theta t}^2 - h_t^2}{h_{\theta\theta} + h}, \tag{1.4}$$

where $h(\theta, t)$ is the support function of γ. In this paper, we will study the local solvability, finite time blow-up and global existence of Equation (1.4).

The paper is organized as follows. In Section 2, a hyperbolic Monge-Ampere equation (1.4) will be derived and the evolution equation for other geometric quantities of the curve associated to the flow are obtained. In Section 3, we prove the local existence of the flow and show that the evolving curve governed by this flow remains convex as long as it exists. Finite time blow-up and global existence for closed curves driven by this flow are studied.

2.Hyperbolic evolution equations

To study the motion of convex curves, it is convenient to express the flow in terms of the support function $h(\theta,t)$. Let $\theta \in [0,2\pi)$ be the normal angle of the curves, then

$$\mathbf{n} = (\cos\theta, \sin\theta), \qquad \mathbf{t} = (-\sin\theta, \cos\theta),$$

and the support function of γ is given by

$$h(\theta,t) = \langle \tilde{\gamma}(\theta,t), \mathbf{n} \rangle,$$

where $\tilde{\gamma}(\theta,t) = \gamma(u(\theta,t),t) = (x(\theta,t), y(\theta,t))$ is the point of the curve whose normal angle θ. By a direct computation, we have

$$\tilde{\gamma} = h\mathbf{n} + h_\theta \mathbf{t},$$

$$\tilde{\gamma}_t = \gamma_u \cdot u_t + \gamma_t,$$

and

$$h_{\theta\theta} + h = \langle \tilde{\gamma}_\theta(\theta,t), \mathbf{t} \rangle = \langle \tilde{\gamma}_s \cdot s_\theta, \mathbf{t} \rangle = s_\theta = \frac{1}{k}.$$

So we have

$$h_{\theta t} = \langle \tilde{\gamma}_t, \mathbf{t} \rangle = \langle \gamma_u \cdot u_t, \mathbf{t} \rangle = |\gamma_u| \cdot u_t,$$

then

$$h_t = \langle \gamma_t, \mathbf{n} \rangle = \tilde{v}(\theta,t).$$

and

$$\begin{aligned} h_{tt} =& \tilde{v}_t = v_u \cdot u_t + v_t \\ =& \tilde{v}_\theta \cdot \theta_s \cdot s_u \cdot u_t + v_t \\ =& h_{\theta t} \cdot k \cdot |\gamma_u| \cdot u_t + v_t \\ =& h_{\theta t}^2 k - \tilde{v}^2 k \\ =& h_{\theta t}^2 k - h_t^2 k \\ =& \frac{h_{\theta t}^2 - h_t^2}{h_{\theta\theta} + h}. \end{aligned}$$

i.e.,

$$h h_{tt} + h_{tt} h_{\theta\theta} - h_{\theta t}^2 + h_t^2 = 0. \tag{2.1}$$

Therefore, the evolution equation of the curves (1.1) is rewritten as the following

$$\begin{cases} h h_{tt} + h_{tt} h_{\theta\theta} - h_{\theta t}^2 + h_t^2 = 0, \\ h(\theta,0) = f(\theta) = \langle \gamma_0, \mathbf{n} \rangle, \\ h_t(\theta,0) = \tilde{v}_0(\theta), \end{cases} \tag{2.2}$$

where $f(\theta)$ is the support function of γ_0.

The following propositions will be useful for further discussion.

Lemma 2.1 *Similar to [29], for the normal flow (1.2), we have*

$$\begin{aligned}
&\frac{\partial g}{\partial t} = vkg,\\
&\frac{\partial s}{\partial t} = vks\\
&\frac{\partial}{\partial t}\frac{\partial}{\partial s} = -kv\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}\\
&\frac{\partial}{\partial t}\frac{\partial^2}{\partial s^2} = -2kv\frac{\partial^2}{\partial s^2} - k_s v\frac{\partial}{\partial s} - kv_s\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}\\
&\frac{\partial \mathbf{t}}{\partial t} = \frac{\partial v}{\partial s}\mathbf{n},\\
&\frac{\partial \mathbf{n}}{\partial t} = -\frac{\partial v}{\partial s}\mathbf{t},\\
&\frac{\partial k}{\partial t} = -vk^2 - \frac{\partial^2 v}{\partial s^2}.
\end{aligned}$$

Furthermore, we can get

$$\frac{\partial (vg)}{\partial t} = \frac{\partial v}{\partial t}g + v\frac{\partial g}{\partial t} = -v^2kg + v^2kg = 0,$$

which means that

$$v = \frac{v_0 g_0}{g}.$$

Proposition 2.2 *The perimeter $L(t)$ for any closed curves $\gamma(\cdot, t)$ of the flow* (1.1) *satisfies*

$$\frac{dL(t)}{dt} = \int_0^{2\pi} h_t d\theta = \int_0^{2\pi} \tilde{v}(\theta) d\theta$$

and

$$\frac{d^2L(t)}{dt^2} = \int_0^{2\pi} (h_{\theta t}^2 - h_t^2)k d\theta.$$

Proof. By the definition of the perimeter, we have

$$L(t) = \int_0^{2\pi} |\tilde{\gamma}_\theta| d\theta = \int_0^{2\pi} k^{-1} d\theta.$$

By taking derivative we have

$$\frac{dL(t)}{dt} = -\int_0^{2\pi} k^{-2}k_t d\theta = \int_0^{2\pi} (h_{\theta\theta t} + h_t) d\theta = \int_0^{2\pi} h_t d\theta = \int_0^{2\pi} \tilde{v}(\theta) d\theta$$

and

$$\frac{d^2L(t)}{dt^2} = \int_0^{2\pi} h_{tt} d\theta = \int_0^{2\pi} (h_{\theta t}^2 - h_t^2)k d\theta.$$

Proposition 2.3 *The area $A(t)$ enclosed by the closed curves $\gamma(\cdot, t)$ satisfies*

$$\frac{dA(t)}{dt} = \int_0^{2\pi} \frac{h_t}{k} d\theta$$

and

$$\frac{d^2A(t)}{dt^2} = 0.$$

Proof. By the definition of the perimeter, we have

$$A(t) = -\frac{1}{2}\int_{\gamma(\cdot,t)} < \tilde{\gamma}, \mathbf{n} > ds = \frac{1}{2}\int_0^{2\pi} \frac{h}{k} d\theta.$$

By taking derivative, we have

$$\begin{aligned}\frac{dA(t)}{dt} =& \frac{1}{2}\int_0^{2\pi} (\frac{h_t}{k} - \frac{hk_t}{k^2}) d\theta = \frac{1}{2}\int_0^{2\pi} [\frac{h_t}{k} + h(S_{\theta\theta t} + h_t)] d\theta \\ =& \frac{1}{2}\int_0^{2\pi} [\frac{h_t}{k} + h_t(h_{\theta\theta} + h)] d\theta = \int_0^{2\pi} \frac{h_t}{k} d\theta.\end{aligned}$$

and

$$\begin{aligned}\frac{d^2A(t)}{dt^2} =& \int_0^{2\pi} (\frac{h_{tt}}{k} - h_t k^{-2} k_t) d\theta = \int_0^{2\pi} \left[h_{\theta t}^2 - h_t^2 + h_t(h_{\theta\theta t} + h_t)\right] d\theta \\ =& \int_0^{2\pi} \left[h_{\theta t}^2 - h_t^2 + h_t^2 - h_{\theta t}^2\right] d\theta = 0,\end{aligned}$$

i.e.,

$$\frac{dA(t)}{dt} \equiv \frac{dA(t)}{dt}\Big|_{t=0} = \int_0^{2\pi} \frac{\tilde{v}_0(\theta)}{k_0(\theta)} d\theta = -c_0,$$

which means

$$A(t) = A(0) - c_0 t.$$

Proposition 2.4 *Under the Proposition 2.2, the following inequality holds*

$$\left(\frac{\partial \widetilde{v}}{\partial \theta}\right)^2 - \widetilde{v}^2 > 0 \quad \text{for all} \quad t \in [0, T).$$

Proof. Since

$$\frac{\partial v}{\partial t} = -v^2 k < 0 \quad \text{for} \quad \text{all} \quad t \in [0, T),$$

then

$$v(u, t_2) < v(u, t_1) \quad \text{for all} \;\; t_2 \geq t_1, t_1, t_2 \in (0, T),$$

i.e.,

$$\widetilde{v}(\theta, t_2) = v(u, t_2) < v(u, t_1) = \widetilde{v}(\theta, t_1) \quad \text{for} \quad \text{all} \quad t_2 \geq t_1, t_1, t_2 \in (0, T),$$

hence,

$$\frac{\partial \widetilde{v}}{\partial t} < 0 \quad \text{for all} \quad t \in [0, T).$$

On the other hand, by the chain rule,

$$\begin{aligned}\frac{\partial v(u,t)}{\partial t} =& \frac{\partial \widetilde{v}(\theta,t)}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \widetilde{v}(\theta,t)}{\partial t} \\ =& \frac{\partial \widetilde{v}(\theta,t)}{\partial \theta}\frac{\partial v(u,t)}{\partial s} + \frac{\partial \widetilde{v}(\theta,t)}{\partial t} \\ =& -\frac{\partial \widetilde{v}(\theta,t)}{\partial \theta}\frac{\partial \widetilde{v}(\theta,t)}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial \widetilde{v}(\theta,t)}{\partial t},\end{aligned}$$

hence,

$$\frac{\partial \widetilde{v}}{\partial t} = \left[\left(\frac{\partial \widetilde{v}}{\partial \theta}\right)^2 - \widetilde{v}^2\right] k < 0,$$

therefore,

$$\left(\frac{\partial \widetilde{v}}{\partial \theta}\right)^2 - \widetilde{v}^2 < 0 \quad \text{for} \quad \text{all} \quad t \in [0, T).$$

Thus, the proof is completed. ■

3. Finite time blow up for closed curves under the flow(1.1)

Theorem 3.1 *(Local existence) Suppose γ_0 is a strictly convex closed curve and the initial normal velocity v_0 is larger than 0. Then there exists a family convex closed curves $\gamma(\cdot, t)$ with $t \in [0, T)(T > 0)$ such that $\gamma(\cdot, t)$ satisfies (1.1), provided that $v_0(u)$ is a smooth function on S^1.*

Proof Similar to [28], we obtain the local solvability of hyperbolic flow of plane curves by the linearization method [30]. Firstly, in terms of γ_0 is a smooth strictly convex closed curve, the curvature k_0 of the curve γ_0 is larger than 0. Then the linear wave equation

$$\begin{cases} h_{tt} = h_{\theta\theta} + \tilde{v}_0^2 \tilde{k}_0, \\ h(\theta, 0) = f(\theta), \\ h_t(\theta, 0) = \tilde{v}_0. \end{cases} \tag{3.1}$$

has a unique solution $\bar{h} \in C^\infty(S^1 \times [0, T_0)$ with some $T_0 > 0$.

Secondly, we consider the linearization of (2.2) around $\bar{h}$. For equation (2.2), denote

$$\Phi(\theta, h_{\theta\theta}, h_{\theta t}, h_\theta, h_t, h) = \frac{h_{\theta t}^2 - h_t^2}{h_{\theta\theta} + h}.$$

Assume $h_\varepsilon := \bar{h} + \varepsilon h$, we can obtain the linearized operator $L_{\bar{h}}$ of $\frac{\partial^2}{\partial t^2} - \Phi$ around $\bar{h}$ as follows:

$$\begin{aligned}
L_{\bar{h}} h :=& \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\frac{\partial^2 h_\varepsilon}{\partial t^2} - \Phi(\theta, (h_\varepsilon)_{\theta\theta}, (h_\varepsilon)_{\theta t}, (h_\varepsilon)_\theta, (h_\varepsilon)_t, h_\varepsilon)\right) \\
=& h_{tt} - \left[\frac{\partial \Phi}{\partial (h_\varepsilon)_{\theta\theta}} \frac{d(h_\varepsilon)_{\theta\theta}}{d\varepsilon} + \frac{\partial \Phi}{\partial (h_\varepsilon)_{\theta t}} \frac{d(h_\varepsilon)_{\theta t}}{d\varepsilon} + \frac{\partial \Phi}{\partial (h_\varepsilon)_\theta} \frac{d(h_\varepsilon)_\theta}{d\varepsilon} + \frac{\partial \Phi}{\partial (h_\varepsilon)_t} \frac{d(h_\varepsilon)_t}{d\varepsilon} + \frac{\partial \Phi}{\partial (h_\varepsilon)} \frac{d(h_\varepsilon)}{d\varepsilon}\right]\Big|_{\varepsilon=0} \\
=& h_{tt} - \left[\frac{\partial \Phi}{\partial (\bar{h})_{\theta\theta}} h_{\theta\theta} + \frac{\partial \Phi}{\partial (\bar{h})_{\theta t}} h_{\theta t} + \frac{\partial \Phi}{\partial (\bar{h})_\theta} h_\theta + \frac{\partial \Phi}{\partial (\bar{h})_t} h_t + \frac{\partial \Phi}{\partial (\bar{h})} h\right] \\
=& h_{tt} - \left[\frac{\bar{h}_t^2 - \bar{h}_{\theta t}^2}{(\bar{h}_{\theta\theta} + \bar{h})^2} h_{\theta\theta} + 2\frac{\bar{h}_{\theta t}}{\bar{h}_{\theta\theta} + \bar{h}} h_{\theta t} - 2\frac{\bar{h}_t}{\bar{h}_{\theta\theta} + \bar{h}} h_t + \frac{\bar{h}_t^2 - \bar{h}_{\theta t}^2}{(\bar{h}_{\theta\theta} + \bar{h})^2} h\right],
\end{aligned}$$

consider the principal matrix

$$\begin{pmatrix} -1 & \frac{\bar{h}_{\theta t}}{\bar{h}_{\theta\theta} + \bar{h}} \\ \frac{\bar{h}_{\theta t}}{\bar{h}_{\theta\theta} + \bar{h}} & \frac{\bar{h}_t^2 - \bar{h}_{\theta t}^2}{(\bar{h}_{\theta\theta} + \bar{h})^2} \end{pmatrix}$$

which can be transformed into

$$\begin{pmatrix} -1 & 0 \\ 0 & \bar{h}_t^2 \bar{k}^2 \end{pmatrix}$$

At $t=0$, $\tilde{v}_0^2\bar{k}_0^2>0$, thus $L_{\bar{h}}$ is uniformly hyperbolic in some time interval $[0,T)$. Therefore, the linerization of (2.2) around $\bar{h}$ given by

$$\begin{cases} L_{\bar{h}}h := h_{tt} - \left[\dfrac{\bar{h}_t^2-\bar{h}_{\theta t}^2}{(\bar{h}_{\theta\theta}+\bar{h})^2}h_{\theta\theta} + 2\dfrac{\bar{h}_{\theta t}}{\bar{h}_{\theta\theta}+\bar{h}}h_{\theta t} - 2\dfrac{\bar{h}_t}{\bar{h}_{\theta\theta}+\bar{h}}h_t + \dfrac{\bar{h}_t^2-\bar{h}_{\theta t}^2}{(\bar{h}_{\theta\theta}+\bar{h})^2}h\right], \\ h(\theta,0)=f(\theta), \\ h_t(\theta,0)=\tilde{v}_0. \end{cases} \tag{3.2}$$

has a unique solution $h\in C^\infty(S^1\times[0,T))$.

Next, we translate the solvability of (2.2) to the invertibility of the operator A given by

$$h\to Ah := h_{tt}-\Phi(\theta,h_{\theta\theta},h_{\theta t},h_\theta,h_t,h)=\frac{h_{\theta t}^2-h_t^2}{h_{\theta\theta}+h}.$$

By the inverse function theorem, if $DA(\bar{h})$ is a linear homeomorphism from h to Ah, then there exists a neighborhood $O_{\bar{h}}$ such that

$$A: O_{\bar{h}}\to A(O_{\bar{h}})$$

is a homeomorphism. Let $\bar{h}$ is the solution of (2.2), then $DA(\bar{h}): h\to DA(\bar{h})(h)=L_{\bar{h}}h$, we know that there exists a unique solution h to (3.1) and $DA(\bar{h})$ is invertible. Furthermore, A is invertible in a neighborhood $O_{\bar{h}}$ of $\bar{h}$, then the conclusion of Theorem 3.1 is obtained.

Example 3.1 *Consider $\gamma(\cdot,t)$ to be a family of round circles with the radius $r(t)$ centered at the origin. The support function and the curvature are given by $h(\theta,t)=r(t)$ and $k(\theta,t)=\frac{1}{r(t)}$, respectively. Then the evolution equation (2.2) is rewritten as*

$$\begin{cases} r''=-\dfrac{r'^2}{r}, \\ r(0)=r_0>0, \quad r'(0)=r_1. \end{cases} \tag{3.3}$$

Eq. (3.3) is rewritten as

$$\begin{cases} rr''=-r'^2\ , \\ r(0)=r_0>0, \quad r'(0)=r_1. \end{cases} \tag{3.4}$$

By integrating this equation once we obtain

$$rr'=r_0r_1,$$

integrating once again,

$$r(t)=\sqrt{r_0^2+2r_0r_1t},$$

In other words $\gamma(0,t)=\{(x,y)\to\mathbb{R}^2|x^2+y^2=r_0^2+2r_0r_1t\}$ is a solution of the normal hyperbolic curvature flow with $\gamma(0,t)=\{(x,y)\to\mathbb{R}^2|x^2+y^2=r_0^2\}$ as initial curve and r_1 as initial velocity. We observe that if $r_1=0$, $r(t)=r_0$ for all time; if $r_1>0$, $r(t)$ expands to ∞ as $t\to\infty$; if $r_1<0$, $r(t)$ converges to a point as $t\to t_c=-\dfrac{r_0}{2r_1}$.

Proposition 3.1(Containment principle) *Let γ_1 and $\gamma_2: S^1\times[0,T)\to\mathbb{R}^2$ be two convex solutions of* (1.1) *(or* (1.2)*). Suppose that $\gamma_2(\cdot,0)$ lies in the domain enclosed by $\gamma_1(\cdot,0)$, and $f_2(p)\geq f_1(p)$. Then $\gamma_2(\cdot,t)$ is contained in the domain enclosed by $\gamma_1(\cdot,t)$ for all $t\in[0,T)$.*

Proof. Let $S_1(\theta,t)$ and $S_2(\theta,t)$ be the support functions of $\gamma_1(\cdot,t)$ and $\gamma_2(\cdot,t)$ respectively. Then h_1 and h_2 satisfies the same equation (1.2) with $h_2(\theta,0)\le h_1(\theta,0)$ and $h_{2t}(\theta,0)\le h_{1t}(\theta,0)$ for $\theta\in[0,2\pi]$.

Let

$$w(\theta,t)=h_2(\theta,t)-h_1(\theta,t),$$

then w satisfies the following equation

$$\begin{cases} w_{tt}=(h_{1t}h_{2t}-h_{1\theta t}h_{2\theta t})\,k_1k_2(w_{\theta\theta}+w)+(k_1h_{1\theta t}+k_2h_{2\theta t})w_{\theta t}+(k_1h_{1t}+k_2h_{2t})w_t, \\ w_t(\theta,0)=f_1(\theta)-f_2(\theta)=w_1(\theta), \\ w(\theta,0)=h_2(\theta)-h_1(\theta)=w_0(\theta). \end{cases} \tag{3.5}$$

We can define the operator L by

$$L[w]=aw_{\theta\theta}+2bw_{\theta t}+cw_{tt}+dw_\theta+ew_t+gw \tag{3.6}$$

In view of (3.5), we know that

$$\begin{aligned} a=&g=(h_{1t}h_{2t}-h_{1\theta t}h_{2\theta t})\,k_1k_2,\quad b=\frac{(k_1h_{1\theta t}+k_2h_{2\theta t})}{2}, \\ c=&-1,\quad d=0\quad\text{and}\quad e=(k_1h_{1t}+k_2h_{2t}) \end{aligned}$$

are continuously differentiable functions of θ and t. By a direct computation, we get

$$\begin{aligned} b^2-ac=&\frac{(k_1h_{1\theta t}+k_2h_{2\theta t})^2}{4}-(h_{1t}h_{2t}-h_{1\theta t}h_{2\theta t})\,k_1k_2\cdot(-1) \\ =&\frac{(k_1h_{1\theta t}-k_2h_{2\theta t})^2}{4}+h_{1t}h_{2t}>0. \end{aligned}$$

Hence the operator L is defined by (3.5) is hyperbolic in $S^1\times[0,T)$ and it is *uniformly hyperbolic* in $S^1\times[0,T)$.

By the maximum principle in hyperbolic equation (to see [31]), we deduce that

$$h_2(\theta,t)\le h_1(\theta,t)$$

for all $t\in[0,T)$. Thus, the proof is completed. ■

Proposition 3.2 (Preserving convexity) *Let k_0 be the mean curvature of initial curve γ_0 and set*

$$\delta=\min_{\theta\in[0,2\pi]}\{k_0(\theta)\}.$$

*Assume that $\delta>0$ and the initial velocity satisfies $k_t(0)\ge 0$. Then for any C^2-solution k of (**??**), one has*

$$k(\theta,t)\ge\delta$$

for $t\in[0,T)$, where $[0,T)$ is the maximal time interval for the solution $\gamma(\cdot,t)$ of (1.1).

Proof. Since the initial curve is strictly convex, by Theorem 2.1 we know that the solution of (2.5) remains strictly convex on some short time interval $[0,T)$ with some $T\le T$ and its support function satisfies

$$h_{tt}=(h_{\theta t}^2-h_t^2)k$$

for all $(\theta, t) \in [0, 2\pi] \times [0, T)$. Taking derivative with respect to t, we have

$$k_t = \left(\frac{1}{h + h_{\theta\theta}}\right)_t = -\frac{1}{(h + h_{\theta\theta})^2}(h_t + h_{\theta\theta t}) = -k^2(h_t + h_{\theta\theta t}).$$

$$h_t + h_{\theta\theta t} = -(h + h_{\theta\theta})^2 k_t = -\frac{1}{k^2}k_t,$$

$$h_{tt} + h_{\theta\theta tt} = (-\frac{1}{k^2}k_t)_t = \frac{2}{k^3}k_t^2 - \frac{1}{k^2}k_{tt},$$

$$h_{\theta t} + h_{\theta\theta\theta t} = (-\frac{1}{k^2}k_t)_\theta = \frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t},$$

and

$$\begin{aligned} h_{tt} + h_{tt\theta\theta} =& (h_{\theta t}^2 - h_t^2)k + [(h_{\theta t}^2 - h_t^2)k]_{\theta\theta} \\ =& [(h_{\theta t}^2 - h_t^2)_\theta k + (h_{\theta t}^2 - h_t^2)k_\theta]_\theta + (h_{\theta t}^2 - h_t^2)k \\ =& (h_{\theta t}^2 - h_t^2)(k + k_{\theta\theta}) + 2(h_{\theta t}^2 - h_t^2)_\theta k_\theta + (h_{\theta t}^2 - h_t^2)_{\theta\theta} k \\ =& (h_{\theta t}^2 - h_t^2)(k + k_{\theta\theta}) - \frac{2}{k}h_{\theta t}k_{\theta t} - 8h_{\theta t}h_t k_\theta + \frac{2}{k^3}k_t^2 + \frac{6}{k}h_t k_t, \end{aligned}$$

hence, the mean curvature k satisfies

$$\begin{aligned} k_{tt} =& \frac{2}{k}k_t^2 - (h_{tt} + h_{\theta\theta tt})k^2 \\ =& (h_t^2 - h_{\theta t}^2)k^2 k_{\theta\theta} + 2kh_{\theta t}k_{\theta t} + 8h_{\theta t}h_t k^2 k_\theta - 6h_t k k_t - 3(h_t^2 - h_{\theta t}^2)k^2. \end{aligned}$$

Define the operator L as

$$L[k] := (h_t^2 - h_{\theta t}^2)k^2 k_{\theta\theta} + 2kh_{\theta t}k_{\theta t} - k_{tt} - 8k^2 h_{\theta t}h_t k_\theta + 6kh_t k_t.$$

We know that

$$a = (h_t^2 - h_{\theta t}^2)k^2, b = kh_{\theta t}, c = -1$$

are twice continuously differentiable functions of θ and t. So we have

$$b^2 - ac = (kh_{\theta t})^2 - (h_t^2 - h_{\theta t}^2)k^2 \cdot (-1) = h_t^2 k^2 > 0,$$

which implies that the operator L is hyperbolic in $S^1 \times [0, T)$. Define $k(\theta, t)$ is given by the following equation

$$\begin{cases} (L + g)[k] := (h_t^2 - h_{\theta t}^2)k^2 k_{\theta\theta} + 2kh_{\theta t}k_{\theta t} + 8h_{\theta t}h_t k^2 k_\theta - 6h_t k k_t - 3(h_t^2 - h_{\theta t}^2)k^2, \\ k(\theta, 0) = k_0(\theta) \\ \dfrac{\partial k}{\partial \nu} := -bk_\theta - ck_t = \rho(\theta) \geq 0. \qquad on \ \ \Gamma_0. \end{cases} \tag{3.7}$$

we find that k satisfies

$$\begin{cases} (L + g)[k - \delta] = 3(h_t^2 - h_{\theta t}^2)\delta \geq 0, \quad in \ \ S^1 \times [0, T_0), \\ k_0(\theta) - \delta \geq 0, \\ \dfrac{\partial(k - \delta)}{\partial \nu} = \rho(\theta) \geq 0. \quad on \ \ \Gamma_0. \end{cases} \tag{3.8}$$

Thus, we apply maximum principle for hyperbolic equation (to see [31]), we deduce that

$$k(\theta, t) \geq \delta$$

for all $(\theta, t) \in S^1 \times [0, T)$. Thus, the proof is completed. ■

Theorem 3.2 *Let the initial curve γ_0 be a smooth, strictly convex closed curve. If $v_0 < 0$, the solution of the hyperbolic curvature flow* (1.1) *exists only a finite-time interval. Moreover precisely, the solution of the flow converges to a point, i.e, the curvature of the curve blows up at finite time. If $v_0 > 0$, the solution of the hyperbolic curvature flow* (1.1) *exists for all time.*

Proof. If $v_0 < 0$, suppose the initial γ_0 is enclosed by a large circle Γ_0, evolve Γ_0 by the flow (1.1) to get a solution $\Gamma(\cdot, t)$. By Example (3.1), we know that the solution $\Gamma(\cdot, t)$ exists only at a finite time interval $[0, T_*)$, and $\Gamma(\cdot, t)$ converges to a point as $t \to T_*$. In terms of Proposition 3.1 and [32], $\gamma(\cdot, t)$ is always enclosed by $\Gamma(\cdot, t)$ for all $t \in [0, T_*)$. Thus we conclude that the solution $\gamma(\cdot, t)$ must become singular at some time $T \leq T_*$.

By
$$v_t = -v^2 k \leq 0,$$
we have
$$v \leq v_0 < 0, \quad i.e., \quad \tilde{v} \leq \tilde{v}_0 < 0,$$
then
$$\frac{dL(t)}{dt} = \int_0^{2\pi} h_t d\theta = \int_0^{2\pi} \tilde{v} d\theta < 0,$$
$$\frac{d^2 L(t)}{dt^2} = \int_0^{2\pi} (h_{\theta t}^2 - h_t^2) k d\theta < 0.$$
We deduce $L(t)$ becomes zero in finite time.

On the other hand,
$$\frac{dA(t)}{dt} = \frac{dA(t)}{dt}\Big|_{t=0} = \int_0^{2\pi} \frac{\tilde{v}_0}{k_0} d\theta = -c_0 < 0,$$
$$A(t) = A(0) - c_0 t,$$
we deduce that the area $A(t)$ enclosed by $\gamma(\cdot, T_*)$ decreases to zero as $t \to t_c = \dfrac{A(0)}{c_0}$, i.e., $\gamma(\cdot, T_*)$ is either a point or a line segment.

In case the flow converges to a line segment as the enclosed area tends to zero, it is clear that
$$\min_{\theta \in S^1} \{k(\theta, t)\} \to 0.$$
However in the proof of Proposition 3.2 we have shown that the curvature of $\gamma(\cdot, t)$ has a uniform positive lower bound. Hence the flow must converge to a point.

If $v_0 > 0$, suppose a small circle Γ_1 is enclosed by the initial γ_0, evolve Γ_1 by the flow (1.1) to get a solution $\Gamma_1(\cdot, t)$. By Example (3.1), we know that the solution $\Gamma_1(\cdot, t)$ exists for all time, and $\Gamma_1(\cdot, t)$ expands to infinity as $t \to \infty$. In terms of Proposition 3.1 and [32], $\Gamma_1(\cdot, t)$ is always enclosed by $\gamma(\cdot, t)$ for all $t \in [0, \infty)$. Thus we conclude that the solution $\gamma(\cdot, t)$ expands to infinity as $t \to \infty$. Therefore we have completed the proof of the theorem. ■

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