Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

Non-oscillatory Solutions of Higher-order Forced Nabla Fractional Difference Equations with Positive and Negative Terms

S. R. Grace[1] | J. M. Jonnalagadda[2] | J. Alzabut[3,4] | S. S. Santra[1,1]

[1]Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Giza - 12221, Egypt.
[2]Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad - 500078, Telangana, India.
[3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh - 11186, Saudi Arabia.
[4]Department of Industrial Engineering, OSTİM Technical University, Ankara - 06374, Turkey.
[5] Department of Mathematics, Applied Science Cluster, University of Petroleum and Energy Studies, Dehradun, Uttarakhand 248007, India.

Correspondence
S. S. Santra, Department of Mathematics, Applied Science Cluster, University of Petroleum and Energy Studies, Dehradun, Uttarakhand 248007, India. Email: shyam01.math@gmail.com or shyamsundar@ddn.upes.ac.in

Summary

In this work, we obtain new sufficient conditions for the non-oscillatory solutions of forced nabla fractional difference equations with positive and negative terms. The results are developed in sense of Caputo nabla fractional difference operator and by the help of Young's inequality as well as an equivalent representation in form of a Volterra-type summation equation. The results improve some existing results in the literature. Further, two examples are presented to support and illustrate the applicability of the deduced results.

KEYWORDS:
Non-oscillatory solutions, Asymptotic behavior, Caputo nabla fractional difference, Nabla fractional difference equations.

1 | INTRODUCTION

Nowadays discrete fractional calculus has gained much attention amongst researchers in the last two decades. Consequently, there has been a burgeoning interest in the theory and applications of fractional difference equations[9]. The fractional difference and summation feature has significantly proved its efficiency and validity due to its nonlocal character and the interpretation of memory. As a result, many papers have appeared that study the qualitative properties of solutions of fractional difference equations. While researchers focused on the oscillation of solutions, the non-oscillation behavior for nonlinear fractional difference equations still needs improvement[1,2,4,3,5].

The study of the oscillation of solutions to nabla fractional difference equations was started by Alzabut et al.[3]. For the following nonlinear nabla fractional difference equations involving the Riemann–Liouville and the Caputo operators of arbitrary order, the authors in[3] defined a number of oscillation criteria.

$$\begin{cases} \nabla^{\mu}_{c+m-2} y(\varrho) + g_1(\varrho, y(\varrho)) = f(\varrho) + g_2(\varrho, y(\varrho)), \quad \varrho \in \mathbb{N}_{c+m-1}, \\ \nabla^{-(1-\mu)}_{c+m-2} y(\varrho)\Big|_{\varrho=c+m-1} = a, \quad a \in \mathbb{R}, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \nabla^{\mu}_{c+m-1*}y(\varrho) + g_1(\varrho, y(\varrho)) = f(\varrho) + g_2(\varrho, y(\varrho)), & \varrho \in \mathbb{N}_{c+m-1}, \\ \nabla^n y(c+m-1) = a_n, \quad a_n \in \mathbb{R}, & n = 0, 1, 2, \cdots, m-1. \end{cases} \tag{1.2}$$

Here $\mu > 0$ and $m \in \mathbb{N}_1$ such that $m-1 < \mu < m$; $\nabla^{\mu}_{c+m-2}y$, $\nabla^{\mu}_{c+m-1*}y$ denote the μ^{th} Riemann–Liouville and Caputo nabla fractional differences of y, respectively; $g_1, g_2 : \mathbb{N}_{c+m-1} \times \mathbb{R} \to \mathbb{R}$; $f : \mathbb{N}_{c+m-1} \to \mathbb{R}$; β, γ are positive real numbers.

Using the fractional Volterra sum equations and Young's inequalities, Abdalla et al.[1] developed new oscillation criteria for (1.1) and (1.2), building on the work in[3]. The authors of[3] noted that the scenarios $\beta > \gamma > 1$ and $\gamma > \beta > 1$ were not taken into account for (1.1). The goal of the study[1] was to fill in this gap and develop additional oscillation criteria that enhance the findings of[3]. Abdalla et al.[2] studied the oscillation of solutions for nabla fractional difference equations with mixed nonlinearities of the following forms:

$$\begin{cases} \nabla^{\mu}_{c+m-2}y(\varrho) - b(\varrho)y(\varrho) + \sum_{j=1}^{k} b_j(\varrho)\,|y(\varrho)|^{\alpha_j - 1} = f(\varrho), & \varrho \in \mathbb{N}_{c+m}, \\ \nabla^{-(m-\mu)}_{c+m-2}y(\varrho)\Big|_{\varrho=c+m-1} = a, \quad a \in \mathbb{R}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \nabla^{\mu}_{c+m-1*}y(\varrho) - b(\varrho)y(\varrho) + \sum_{j=1}^{k} b_j(\varrho)\,|y(\varrho)|^{\alpha_j - 1} = f(\varrho), & \varrho \in \mathbb{N}_{c+m-1}, \\ \nabla^n y(c+m-1) = a_n, \quad a_n \in \mathbb{R}, & n = 0, 1, 2, \cdots, m-1. \end{cases} \tag{1.4}$$

Here $\mu > 0$ and $m \in \mathbb{N}_1$ such that $m-1 < \mu < m$; $\nabla^{\mu}_{c+m-2}y$, $\nabla^{\mu}_{c+m-1*}y$ denote the μ^{th} Riemann–Liouville and Caputo nabla fractional differences of y, respectively; $b, b_j, f : \mathbb{N}_{c+m-1} \to \mathbb{R}$, $j = 1, 2, \cdots k$; α_j $(1 \le j \le k)$ are the ratios of odd positive integers with $\alpha_1 > \cdots > \alpha_i > 1 > \alpha_{i+1} > \cdots > \alpha_k$.

Following the above trend, in[4], Alzabut et al. considered the following forced and damped nabla fractional difference equation:

$$\begin{cases} (1-p(\varrho))\nabla\nabla^{\mu}_{0}y(\varrho) + p(\varrho)\nabla^{\mu}_{0}y(\varrho) + p_2(\varrho)g(y(\varrho)) = p_1(\varrho), & \varrho \in \mathbb{N}_1, \\ \nabla^{-(1-\mu)}_{0}y(\varrho)\Big|_{\varrho=1} = a, \quad a \in \mathbb{R}, \end{cases} \tag{1.5}$$

and established sufficient conditions for the oscillation of the solutions of (1.5). Here $0 < \mu < 1$; ∇y denotes the first nabla difference of y; $\nabla^{\mu}_{0}y$ denotes the μ^{th} Riemann–Liouville nabla fractional difference of y; $g : \mathbb{R} \to \mathbb{R}$; $p, p_1 : \mathbb{N}_1 \to \mathbb{R}$; $p_2 : \mathbb{N}_1 \to \mathbb{R}^+$.

Motivated by the above studies, in this work, we consider the forced nabla fractional difference equation with positive and negative terms of the form

$$\nabla^{x}_{c*}z(\iota) + \phi(\iota, y(\iota)) = \eta(\iota) + \zeta(\iota)y^{\beta}(\iota) + \Phi(\iota, y(\iota)), \quad \iota \in \mathbb{N}_{c+1}, \tag{1.6}$$

where

$$z(\iota) = \nabla^{n-1}\left[d(\iota)\,(\nabla y(\iota))^{\beta}\right], \quad \iota \in \mathbb{N}_c, \quad n \in \mathbb{N}_1, \tag{1.7}$$

$0 < x < 1$, β is the ratio of two odd positive integers, $c \in \mathbb{N}_1$, and $\nabla^{x}_{c*}z$ denotes x^{th} Caputo nabla fractional difference of z. Throughout this work, we nned the following conditions for our work in the sequel.

(i) $\zeta, d : \mathbb{N}_c \to (0, \infty)$, $\eta : \mathbb{N}_c \to \mathbb{R}$ and $\Phi, \phi : \mathbb{N}_c \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(ii) there exist continuous functions $\Theta_1, \Theta_2 : \mathbb{N}_c \to (0, \infty)$ and positive real numbers λ and γ where $\lambda > \gamma$ such that

$$y\phi(\iota, y) \ge \Theta_1(\iota)\,|y|^{\lambda+1}, \quad 0 \le y\Phi(\iota, y) \le \Theta_2(\iota)\,|y|^{\gamma+1}$$

for $y \ne 0$ and $\iota \in \mathbb{N}_c$

A solution of (1.6) that is continuable and nontrivial in any neighborhood of ∞ is considered. Such a solution is said to be *oscillatory* if there exists a sequence $\{\iota_m\} \subseteq \mathbb{N}_{c-n}$ with $\iota_m \to \infty$ as $m \to \infty$ such that $y(\iota_m) = 0$, and it is *non-oscillatory* otherwise.

In this paper, we investigate the asymptotic behavior of the *non-oscillatory* solutions of equation (1.6). Our approach is primarily based on the properties of discrete fractional calculus and some mathematical inequalities. To help in proving the main results, an equivalent representation for equation (1.6) in form of a Volterra-type summation equation is constructed. We will provide numerical examples that will support the validity of theoretical results.

In the sequel, we make use of the following notations, definitions, and known results of nabla fractional calculus[9]. Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ for any $a \in \mathbb{R}$.

Definition 1 (See[9]). For $\iota \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(\iota + r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the generalized rising function is defined by

$$\iota^{\overline{r}} = \frac{\Gamma(\iota + r)}{\Gamma(\iota)}.$$

Further, if $\iota \in \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(\iota + r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $\iota^{\overline{r}} = 0$.

Definition 2 (See[9]). Let $\kappa : \mathbb{N}_a \to \mathbb{R}$. The first backward (nabla) difference of κ is defined by

$$\nabla \kappa(\iota) = \kappa(\iota) - \kappa(\iota - 1), \quad \iota \in \mathbb{N}_{a+1}.$$

Definition 3 (See[9]). Let $\kappa : \mathbb{N}_{a+1} \to \mathbb{R}$ and $x > 0$. The x^{th} nabla fractional sum of κ based at a is given by

$$\nabla_a^{-x} \kappa(\iota) = \frac{1}{\Gamma(x)} \sum_{\iota_1 = a+1}^{\iota} (\iota - \iota_1 + 1)^{\overline{x-1}} \kappa(\iota_1), \quad \iota \in \mathbb{N}_a,$$

where by convention $\nabla_a^{-x} \kappa(a) = 0$.

Definition 4 (See[6]). Let $0 < x < 1$ and $\kappa : \mathbb{N}_a \to \mathbb{R}$. The x^{th} Caputo nabla fractional difference of κ based at a is given by

$$\nabla_{a*}^{x} \kappa(\iota) = \nabla_a^{-(1-x)} \nabla \kappa(\iota), \quad \iota \in \mathbb{N}_{a+1}.$$

2 | PRELIMINARIES

Theorem 2.1 (See[9]). The unique solution to the nabla fractional initial value problem

$$\begin{cases} \nabla_{a*}^{x} \kappa(\iota) = \omega(\iota), & \iota \in \mathbb{N}_{a+1}, \\ \kappa(a) = \kappa_0, \end{cases} \tag{2.1}$$

is given by

$$\kappa(\iota) = \kappa_0 + \frac{1}{\Gamma(x)} \sum_{\iota_1 = a+1}^{\iota} (\iota - \iota_1 + 1)^{\overline{x-1}} \omega(\iota_1), \quad \iota \in \mathbb{N}_a \tag{2.2}$$

where $0 < x < 1$ and $\omega : \mathbb{N}_{a+1} \to \mathbb{R}$.

Lemma 1. Consider the following generalized rising functions are well defined.

1. If $r_3 < \iota \le \iota_1$, then $\iota_1^{\overline{-r_3}} \le \iota^{\overline{-r_3}}$;
2. $\iota^{\overline{r_1}} (\iota + r_1)^{\overline{r_2}} = \iota^{\overline{r_1 + r_2}}$;
3. If $0 < r_3 < 1$ and $\vartheta > 1$, then
$$\left[\iota^{\overline{-r_3}}\right]^{\vartheta} \le \frac{\Gamma(1 + r_3 \vartheta)}{\left[\Gamma(1 + r_3)\right]^{\vartheta}} \iota^{\overline{-r_3 \vartheta}}, \quad \iota > r_3 \vartheta.$$

Lemma 2. Under assumption b, x and p are positive constants with $b > 1$ and $p(x-1) + 1 > 0$ we obtain

$$\sum_{\iota_1 = 1}^{\iota} (\iota - \iota_1 + 1)^{\overline{p(x-1)}} b^{p\iota_1} \le Q b^{p\iota}, \quad \iota \in \mathbb{N}_1,$$

where

$$Q = \left(\frac{b^p}{b^p - 1}\right)^{p(x-1)+1} \Gamma(p(x-1) + 1).$$

Lemma 3. (Young's Inequality) [See[11]] If P and Q are nonnegative, $\frac{1}{\delta} + \frac{1}{\eta} = 1$, and $\delta > 1$ then

$$PQ \le \frac{1}{\delta} P^{\delta} + \frac{1}{\eta} Q^{\eta}, \tag{2.3}$$

where equality holds if and only if $Q = P^{\delta - 1}$.

We denote

$$m(\imath) = \left[\frac{\Theta_2^{\lambda}(\imath)}{\Theta_1^{\gamma}(\imath)}\right]^{\left(\frac{1}{\lambda-\gamma}\right)}, \tag{2.4}$$

and

$$A(\imath, c) = \sum_{\imath_1=c+1}^{\imath} d^{-\frac{1}{\beta}}(\imath_1). \tag{2.5}$$

3 | MAIN RESULTS

In this section, we provide sufficient conditions for which any non-oscillatory solution of (1.6) satisfies

$$|y(\imath)| = O\left(\left[\imath^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\imath}{\beta}} A(\imath, c)\right) \text{ as } \imath \to \infty.$$

Theorem 3.1. Under assumptions $(i) - (ii)$, $0 < x < 1$, $p(x-1)+1 > 0$ for $p > 1$ and

$$\sum_{\imath_1=c+1}^{\infty} \zeta^q(\imath_1)\left[\imath_1^{\overline{n-1}}\right]^q A^{\beta q}(\imath_1, c) < \infty, \quad q = \frac{p}{p-1}, \tag{3.1}$$

$$\lim_{\imath\to\infty}\left[\frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}\left|\eta(\imath_1)\right|\right] < \infty, \tag{3.2}$$

$$\lim_{\imath\to\infty}\left[\frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}m(\imath_1)\right] < \infty, \tag{3.3}$$

every non-oscillatory solution of (1.6) satisfies

$$\limsup_{\imath\to\infty}\frac{|y(\imath)|}{\left[\imath^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\imath}{\beta}} A(\imath, c)} < \infty. \tag{3.4}$$

Proof. Let y be a non-oscillatory solution of (1.6), say $y(\imath) > 0$ for $\imath \in \mathbb{N}_{\imath_1}$ for some $\imath_1 \in \mathbb{N}_{c+1}$. Take $z(c) = c_0$. Letting $F(\imath) = \Phi(\imath, y(\imath)) - \phi(\imath, y(\imath))$, it follows from (1.6) and $(i) - (ii)$, for $\imath \in \mathbb{N}_c$,

$$\begin{aligned}
&\nabla^{n-1}\left[d(\imath)\left(\nabla y(\imath)\right)^{\beta}\right] \\
&= c_0 + \frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}\left[\eta(\imath_1)+\zeta(\imath_1)y^{\beta}(\imath_1)+F(\imath_1)\right] \\
&\le \left|c_0\right| + \frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath_1}(\imath-\imath_1+1)^{\overline{x-1}}\left|F(\imath_1)\right| + \frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}\left|\eta(\imath_1)\right| \\
&\quad + \frac{1}{\Gamma(x)}\sum_{\imath_1=\imath_1+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}\left[\Theta_2(\imath_1)y^{\gamma}(\imath_1)-\Theta_1(\imath_1)y^{\lambda}(\imath_1)\right] \\
&\quad + \frac{1}{\Gamma(x)}\sum_{\imath_1=c+1}^{\imath_1}(\imath-\imath_1+1)^{\overline{x-1}}\zeta(\imath_1)\left|y^{\beta}(\imath_1)\right| \\
&\quad + \frac{1}{\Gamma(x)}\sum_{\imath_1=\imath_1+1}^{\imath}(\imath-\imath_1+1)^{\overline{x-1}}\zeta(\imath_1)y^{\beta}(\imath_1).
\end{aligned} \tag{3.5}$$

Applying Lemma 3 to $\left[\Theta_2(\imath)y^{\gamma}(\imath) - \Theta_1(\imath)y^{\lambda}(\imath)\right]$ with

$$\delta = \frac{\lambda}{\gamma} > 1, \quad X = y^{\gamma}(\imath), \quad Y = \frac{\gamma}{\lambda}\frac{\Theta_2(\imath)}{\Theta_1(\imath)}, \quad \eta = \frac{\lambda}{\lambda-\gamma},$$

we obtain

$$\begin{aligned}
\Theta_2(\iota)y^{\gamma}(\iota)-\Theta_1(\iota)y^{\lambda}(\iota) &= \frac{\lambda}{\gamma}\Theta_1(\iota)\left[y^{\gamma}(\iota)\frac{\gamma}{\lambda}\frac{\Theta_2(\iota)}{\Theta_1(\iota)}-\frac{\gamma}{\lambda}\left(y^{\gamma}(\iota)\right)^{\frac{\lambda}{\gamma}}\right]\\
&= \frac{\lambda}{\gamma}\Theta_1(\iota)\left[XY-\frac{1}{\delta}X^{\delta}\right]\\
&\le \frac{\lambda}{\gamma}\Theta_1(\iota)\left[\frac{1}{\eta}Y^{\eta}\right]\\
&= \left(\frac{\lambda-\gamma}{\gamma}\right)\Theta_1(\iota)\left[\frac{\gamma}{\lambda}\frac{\Theta_2(\iota)}{\Theta_1(\iota)}\right]^{\frac{\lambda}{\lambda-\gamma}}\\
&= (\lambda-\gamma)\left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}}\right]^{\left(\frac{1}{\lambda-\gamma}\right)} m(\iota).
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) and applying Lemma 1 (1), for $\iota \in \mathbb{N}_c$, we obtain

$$\begin{aligned}
&\nabla^{n-1}\left[d(\iota)\left(\nabla y(\iota)\right)^{\beta}\right]\\
&\le \left|c_0\right| + \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota_1}(\iota_1-\iota_1+1)^{\overline{x-1}}\left|F(\iota_1)\right| + \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}\left|\eta(\iota_1)\right|\\
&\quad + \frac{1}{\Gamma(x)}(\lambda-\gamma)\left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}}\right]^{\left(\frac{1}{\lambda-\gamma}\right)}\sum_{\iota_1=\iota_1+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}m(\iota_1)\\
&\quad + \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota_1}(\iota_1-\iota_1+1)^{\overline{x-1}}\zeta(\iota_1)\left|y^{\beta}(\iota_1)\right|\\
&\quad + \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}\zeta(\iota_1)y^{\beta}(\iota_1).
\end{aligned} \tag{3.7}$$

In view of (3.2) and (3.3), we see from (3.7) that, for $\iota \in \mathbb{N}_c$,

$$\nabla^{n-1}\left[d(\iota)\left(\nabla y(\iota)\right)^{\beta}\right] \le C_{n-1} + \frac{1}{\Gamma(x)}\sum_{\iota_1=\iota_1+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}\zeta(\iota_1)y^{\beta}(\iota_1), \tag{3.8}$$

where $C_{n-1} > 0$ is defined by

$$\begin{aligned}
C_{n-1} = \left|c_0\right| &+ \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota_1}(\iota_1-\iota_1+1)^{\overline{x-1}}\left|F(\iota_1)\right| + \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}\left|\eta(\iota_1)\right|\\
&+ \frac{1}{\Gamma(x)}(\lambda-\gamma)\left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}}\right]^{\left(\frac{1}{\lambda-\gamma}\right)}\sum_{\iota_1=\iota_1+1}^{\iota}(\iota-\iota_1+1)^{\overline{x-1}}m(\iota_1)\\
&+ \frac{1}{\Gamma(x)}\sum_{\iota_1=c+1}^{\iota_1}(\iota-\iota_1+1)^{\overline{x-1}}\zeta(\iota_1)\left|y^{\beta}(\iota_1)\right|.
\end{aligned}$$

By the integer order variation of constants formula, it follows from (3.8) that

$$
\begin{aligned}
& d(\iota)\,(\nabla y(\iota))^{\beta} \\
& \le \sum_{k=0}^{n-2} \left(\nabla^{k}\left[d(\iota)\,(\nabla y(\iota))^{\beta}\right]\right)_{\iota=\iota_1-1} \frac{(\iota-\iota_1+1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + \sum_{r=\iota_1}^{\iota} \frac{(\iota-r+1)^{\overline{n-2}}}{\Gamma(n-1)} \left[C_{n-1} + \frac{1}{\Gamma(x)} \sum_{\iota_1=\iota_1+1}^{r} (r-\iota_1+1)^{\overline{x-1}} \zeta(\iota_1) y^{\beta}(\iota_1)\right] \\
& \le \sum_{k=0}^{n-2} \left|\left(\nabla^{k}\left[d(\iota)\,(\nabla y(\iota))^{\beta}\right]\right)_{\iota=\iota_1-1}\right| \frac{(\iota-\iota_1+1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + C_{n-1} \sum_{r=\iota_1}^{\iota} \frac{(\iota-r+1)^{\overline{n-2}}}{\Gamma(n-1)} \\
& \quad + \sum_{r=\iota_1+1}^{\iota} \frac{(\iota-r+1)^{\overline{n-2}}}{\Gamma(n-1)} \left[\frac{1}{\Gamma(x)} \sum_{\iota_1=\iota_1+1}^{r} (r-\iota_1+1)^{\overline{x-1}} \zeta(\iota_1) y^{\beta}(\iota_1)\right] \\
& = \sum_{k=0}^{n-2} \left|\left(\nabla^{k}\left[d(\iota)\,(\nabla y(\iota))^{\beta}\right]\right)_{\iota=\iota_1-1}\right| \frac{(\iota-\iota_1+1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + C_{n-1} \frac{(\iota-\iota_1+1)^{\overline{n-1}}}{\Gamma(n)} \\
& \quad + \sum_{\iota_1=\iota_1+1}^{\iota} \left[\sum_{r=\iota_1}^{\iota} \frac{(\iota-r+1)^{\overline{n-2}}}{\Gamma(n-1)} \frac{(r-\iota_1+1)^{\overline{x-1}}}{\Gamma(x)}\right] \zeta(\iota_1) y^{\beta}(\iota_1) \\
& = \sum_{k=0}^{n-1} C_k \frac{(\iota-\iota_1+1)^{\overline{k}}}{\Gamma(k+1)} + \sum_{\iota_1=\iota_1+1}^{\iota} \frac{(\iota-\iota_1+1)^{\overline{x+n-2}}}{\Gamma(x+n-1)} \zeta(\iota_1) y^{\beta}(\iota_1).
\end{aligned} \tag{3.9}
$$

Here

$$
C_k = \left|\left(\nabla^{k}\left[d(\iota)\,(\nabla y(\iota))^{\beta}\right]\right)_{\iota=\iota_1-1}\right| > 0, \quad k = 0, 1, 2, \cdots, n-2.
$$

Note that (3.9) holds for $n = 1$. Hence, (3.9) holds for all $n \in \mathbb{N}_1$ and for all $\iota \in \mathbb{N}_{\iota_1}$. Next, we proceed to estimate (3.9) as

$$
\begin{aligned}
d(\iota)\,(\nabla y(\iota))^{\beta} & \le \sum_{k=0}^{n-1} C_k \frac{\iota^{\overline{k}}}{\Gamma(k+1)} + \sum_{\iota_1=\iota_1+1}^{\iota} \frac{(\iota-\iota_1)^{\overline{n-1}}(\iota-\iota_1+n)^{\overline{x-1}}}{\Gamma(x+n-1)} \zeta(\iota_1) y^{\beta}(\iota_1) \\
& \le \iota^{\overline{n-1}} \left[\sum_{k=0}^{n-1} \frac{C_k}{k!} + \frac{1}{\Gamma(x+n-1)} \sum_{\iota_1=\iota_1+1}^{\iota} (\iota-\iota_1+1)^{\overline{x-1}} \zeta(\iota_1) y^{\beta}(\iota_1)\right],
\end{aligned}
$$

implying that

$$
d(\iota)\,(\nabla y(\iota))^{\beta} \le \iota^{\overline{n-1}} \left[\Theta_1 + \Theta_2 \sum_{\iota_1=\iota_1+1}^{\iota} (\iota-\iota_1+1)^{\overline{x-1}} \zeta(\iota_1) y^{\beta}(\iota_1)\right], \tag{3.10}
$$

where

$$
\Theta_1 = \sum_{k=0}^{n-1} \frac{C_k}{k!} > 0, \quad \Theta_2 = \frac{1}{\Gamma(x+n-1)} > 0.
$$

Applying Lemmas 1-2 and Holder's inequality to the sum on the far right in (3.10), we get

$$
\begin{aligned}
&\sum_{\iota_1=\iota_1+1}^{\iota} (\iota-\iota_1+1)^{\overline{x-1}}\zeta(\iota_1)y^{\beta}(\iota_1) \\
&= \sum_{\iota_1=\iota_1+1}^{\iota} \left[(\iota-\iota_1+1)^{\overline{x-1}} b^{\iota_1}\right]\left[b^{-\iota_1}\zeta(\iota_1)y^{\beta}(\iota_1)\right] \\
&\le \left(\sum_{\iota_1=\iota_1+1}^{\iota}\left[(\iota-\iota_1+1)^{\overline{x-1}}\right]^{p} b^{p\iota_1}\right)^{1/p}\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^{q}(\iota_1)y^{\beta q}(\iota_1)\right)^{1/q} \\
&\le \left(A\sum_{\iota_1=\iota_1+1}^{\iota}(\iota-\iota_1+1)^{\overline{p(x-1)}} b^{p\iota_1}\right)^{1/p}\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^{q}(\iota_1)y^{\beta q}(\iota_1)\right)^{1/q} \\
&\le (AQb^{p\iota})^{1/p}\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^{q}(\iota_1)y^{\beta q}(\iota_1)\right)^{1/q} \\
&= (AQ)^{1/p} b^{\iota}\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^{q}(\iota_1)y^{\beta q}(\iota_1)\right)^{1/q},
\end{aligned}
\tag{3.11}
$$

where

$$A = \frac{\Gamma(1+(1-x)p)}{[\Gamma(2-x)]^{p}}.$$

Using (3.11) in (3.10), we obtain from (3.10) that

$$d(\iota)\,(\nabla y(\iota))^{\beta} \le \iota^{\overline{n-1}} b^{\iota}\omega(\iota), \tag{3.12}$$

where

$$\omega(\iota) = \Theta_1 + M_3\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^{q}(\iota_1)y^{\beta q}(\iota_1)\right)^{1/q},$$

with

$$M_3 = \Theta_2(AQ)^{1/p} > 0.$$

We rewrite (3.12) as

$$\nabla y(\iota) \le \left(\frac{\iota^{\overline{n-1}} b^{\iota}\omega(\iota)}{d(\iota)}\right)^{\frac{1}{\beta}}, \quad \iota \in \mathbb{N}_{\iota_1}. \tag{3.13}$$

Noting that $\iota^{\overline{n-1}}$, b^{ι}, and $\omega(\iota)$ are all increasing, summing (3.13) from ι_1+1 to ι yields that

$$
\begin{aligned}
y(\iota) &\le y(\iota_1) + \sum_{\iota_1=\iota_1+1}^{\iota}\left[\iota_1^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota_1}{\beta}}\omega^{\frac{1}{\beta}}(\iota_1)d^{-\frac{1}{\beta}}(\iota_1) \\
&\le y(\iota_1) + \left[\iota^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}}\omega^{\frac{1}{\beta}}(\iota)\sum_{\iota_1=\iota_1+1}^{\iota} d^{-\frac{1}{\beta}}(\iota_1) \\
&= y(\iota_1) + \left[\iota^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}}\omega^{\frac{1}{\beta}}(\iota)A(\iota,\iota_1) \\
&= \left(\frac{y(\iota_1)}{\left[\iota^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}}A(\iota,\iota_1)} + \omega^{\frac{1}{\beta}}(\iota)\right)\left[\iota^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}}A(\iota,\iota_1) \\
&\le \left(\frac{y(\iota_1)}{\left[\iota_2^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota_2}{\beta}}A(\iota_2,\iota_1)} + \omega^{\frac{1}{\beta}}(\iota)\right)\left[\iota^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}}A(\iota,\iota_1),
\end{aligned}
$$

holds for $\iota \in \mathbb{N}_{\iota_2}$ with $\iota_2 > \iota_1$. Thus,

$$\frac{y(\iota)}{\left[\overline{\iota^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}} A(\iota, \iota_1)} \leq M_4 + \omega^{\frac{1}{\beta}}(\iota), \quad \iota \in \mathbb{N}_{\iota_2}, \tag{3.14}$$

where

$$M_4 = \frac{y(\iota_1)}{\left[\overline{\iota_2^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota_2}{\beta}} A(\iota_2, \iota_1)}.$$

Applying one of the elementary inequalities

$$(y+z)^q \leq \begin{cases} 2^{q-1}(y^q + z^q), & q \geq 1, \\ y^q + z^q, & 0 < q < 1, \end{cases} \tag{3.15}$$

with y, $z \geq 0$, to (3.14) gives

$$\left(\frac{y(\iota)}{\left[\overline{\iota^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}} A(\iota, \iota_1)} \right)^{\beta} \leq M_5 + M_6 \omega(\iota), \quad \iota \in \mathbb{N}_{\iota_2}, \tag{3.16}$$

where M_5 and $M_6 > 0$ are defined by

$$M_5 = \begin{cases} 2^{\beta-1} M_4^{\beta}, & q \geq 1, \\ M_4^{\beta}, & 0 < q < 1, \end{cases} \tag{3.17}$$

and

$$M_6 = \begin{cases} 2^{\beta-1}, & q \geq 1, \\ 1, & 0 < q < 1. \end{cases} \tag{3.18}$$

Recalling the definition of $\omega(\iota)$, from (3.16), we have that

$$\left(\frac{y(\iota)}{\left[\overline{\iota^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}} A(\iota, \iota_1)} \right)^{\beta} \leq M_7 + M_8 \left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1} \zeta^q(\iota_1) y^{\beta q}(\iota_1) \right)^{1/q}, \tag{3.19}$$

holds for $\iota \in \mathbb{N}_{\iota_2}$, where

$$M_7 = M_5 + \Theta_1 M_6 > 0, \quad M_8 = M_3 M_6 > 0.$$

Applying the inequality (3.15) to (3.19) gives

$$\left(\frac{y(\iota)}{\left[\overline{\iota^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\iota}{\beta}} A(\iota, \iota_1)} \right)^{\beta q} \leq M_9 + M_{10} \sum_{r=\iota_1+1}^{\iota} b^{-qr} \zeta^q(r) y^{\beta q}(r), \tag{3.20}$$

holds for $\iota \in \mathbb{N}_{\iota_2}$, where

$$M_9 = 2^{\beta-1} M_7^q > 0, \quad M_{10} = 2^{\beta-1} M_8^q > 0.$$

Denoting the left-hand side of (3.20) by $w(\iota)$, (3.20) yields

$$w(\iota) \leq M_9 + M_{10} \sum_{\iota_1=\iota_1+1}^{\iota} \left[\overline{\iota_1^{n-1}}\right]^q A^{\beta q}(\iota_1, \iota_1) \zeta^q(\iota_1) w(\iota_1), \tag{3.21}$$

holds for $\iota \in \mathbb{N}_{\iota_2}$, and this can be rewritten as

$$w(\iota) \leq M_{11} + M_{10} \sum_{\iota_1=\iota_2+1}^{\iota} \left[\overline{\iota_1^{n-1}}\right]^q A^{\beta q}(\iota_1, \iota_1) \zeta^q(\iota_1) w(\iota_1), \tag{3.22}$$

holds for $\iota \in \mathbb{N}_{\iota_2}$, where

$$M_{11} = M_9 + M_{10} \sum_{\iota_1=\iota_1+1}^{\iota_2} \left[\overline{\iota_1^{n-1}}\right]^q A^{\beta q}(\iota_1, \iota_1) \zeta^q(\iota_1) w(\iota_1) > 0.$$

Using (3.1) and Gronwall's inequality we have the conclusion of the theorem. The proof for eventually negative solution is similar. So, we omit it here. Thus, the theorem is proved. □

Next, we consider $\beta = 1$ and we provide sufficient conditions for which any non-oscillatory solution of (1.6) is bounded.

Theorem 3.2. Assume that $(i) - (ii)$, $0 < x < 1$, $p(x-1)+1 > 0$ for $p > 1$ and (3.2) - (3.3) hold. Furthermore, assume that there exist real numbers $S > 0$ and $\tau > 1$ such that

$$\left(\frac{\iota^{\overline{n-1}}}{d(\iota)}\right) \le \iota_1 b^{-\tau\iota} \tag{3.23}$$

and

$$\sum_{\iota_1=c+1}^{\infty} b^{-q\iota_1}\zeta^q(\iota_1) < \infty, \quad q = \frac{p}{p-1}, \tag{3.24}$$

hold, then all non-oscillatory solution of (1.6) is bounded.

Proof. Let y be a non-oscillatory solution of (1.6), say $y(\iota) > 0$ for $\iota \in \mathbb{N}_{\iota_1}$ for some $\iota_1 \in \mathbb{N}_{c+1}$. Proceeding as in the proof of Theorem 3.1, we get (3.13) when $\beta = 1$. Since ω is increasing, summing (3.13) from $\iota_1 + 1$ to ι yields

$$\begin{aligned}
y(\iota) &\le y(\iota_1) + \sum_{\iota_1=\iota_1+1}^{\iota} \frac{\iota_1^{\overline{n-1}} b^{\iota_1}\omega(\iota_1)}{d(\iota_1)} \\
&\le y(\iota_1) + \sum_{\iota_1=\iota_1+1}^{\iota} S b^{(1-\tau)\iota_1}\omega(\iota_1) \\
&\le y(\iota_1) + S\omega(t) \sum_{\iota_1=\iota_1+1}^{\iota} b^{(1-\tau)\iota_1} \\
&\le y(\iota_1) + S\omega(t) \sum_{\iota_1=\iota_1+1}^{\iota} \left(\frac{1}{b^{(\tau-1)}}\right)^s \\
&= y(\iota_1) + S\omega(t) \left(\frac{b^{(\tau-1)}}{b^{(\tau-1)}-1}\right)\left[\left(\frac{1}{b^{(\tau-1)}}\right)^{\iota_1+1} - \left(\frac{1}{b^{(\tau-1)}}\right)^{\iota+1}\right] \\
&= y(\iota_1) + S\omega(t) \left(\frac{1}{b^{(\tau-1)}-1}\right)\left[\left(\frac{1}{b^{(\tau-1)}}\right)^{\iota_1} - \left(\frac{1}{b^{(\tau-1)}}\right)^{\iota}\right] \\
&\le y(\iota_1) + S\omega(t) \left(\frac{1}{b^{(\tau-1)}-1}\right)\left(\frac{1}{b^{(\tau-1)}}\right)^{\iota_1}.
\end{aligned}$$

Using the definition of ω, we obtain

$$y(\iota) \le M_{12} + M_{13}\left(\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^q(\iota_1)y^q(\iota_1)\right)^{1/q}, \tag{3.25}$$

for $\iota \in \mathbb{N}_{\iota_2}$, where

$$M_{12} = y(\iota_1) + \Theta_1 S\left(\frac{1}{b^{(\tau-1)}-1}\right)\left(\frac{1}{b^{(\tau-1)}}\right)^{\iota_1} > 0,$$

and

$$M_{13} = M_3 S\left(\frac{1}{b^{(\tau-1)}-1}\right)\left(\frac{1}{b^{(\tau-1)}}\right)^{\iota_1} > 0.$$

Using the inequality (3.15) to (3.25), we have

$$y^q(\iota) \le M_{14} + M_{15}\sum_{\iota_1=\iota_1+1}^{\iota} b^{-q\iota_1}\zeta^q(\iota_1)y^q(\iota_1), \tag{3.26}$$

for $\iota \in \mathbb{N}_{\iota_1}$, where

$$M_{14} = 2^{q-1}M_{12}^q > 0, \quad M_{15} = 2^{q-1}M_{13}^q > 0.$$

Now, using (3.24) and Gronwall's inequality we have the conclusion of the theorem. The proof for eventually negative solution is similar. So, we omit it here. Thus, the theorem is proved. □

4 | EXAMPLES

We conclude this paper with the following examples to illustrate our main results.

Example 1. Consider the equation

$$\nabla_{1*}^{0.75}\left(\nabla^3\left(e^{3\iota}(\nabla y(\iota))^3\right)\right)+\phi(\iota,y(\iota))$$
$$= (\iota-1)^{\overline{-0.9}} + \frac{y(\iota)}{\iota(\iota+1)(\iota+2)e^{\iota/2}} + \Phi(\iota,y(\iota)),\quad \iota\in\mathbb{N}_2. \tag{4.1}$$

Here we have $z(\iota)=\nabla^3\left(e^{3\iota}(\nabla y(\iota))^3\right)$, $n=4$, $x=0.75$, $c=1$, $\beta=3$, $d(\iota)=e^{3\iota}$, $\eta(\iota)=(\iota-1)^{\overline{-0.9}}$, $\zeta(\iota)=\frac{1}{\iota(\iota+1)(\iota+2)e^{\iota/2}}$, and

$$A(\iota,c)=A(\iota,1)=\sum_{\iota_1=2}^{\iota} d^{-\frac{1}{3}}(\iota_1)=\sum_{\iota_1=2}^{\iota} e^{-\iota_1}=\frac{1}{e(e-1)}\left[1-\left(\frac{1}{e}\right)^{\iota-1}\right]\le\frac{1}{e(e-1)}.$$

Clearly, condition (i) holds. Let $b=e$ and $p=2$. Clearly, $p(x-1)+1>0$. Also, we have $q=2$, and

$$\sum_{\iota_1=c+1}^{\infty}\zeta^q(\iota_1)\left[\iota_1^{\overline{n-1}}\right]^q A^{\beta q}(\iota_1,c)\le\frac{1}{e^2(e-1)^2}\sum_{\iota_1=2}^{\infty}e^{-\iota_1}<\infty,$$

implying that (3.1) holds.Considering $\phi(\iota,y(\iota))=\Theta_1(\iota)\,|y(\iota)|^{\lambda-1}\,y(\iota)$ and $\Phi(\iota,y(\iota))=\Theta_2(\iota)\,|y(\iota)|^{\gamma-1}\,y(\iota)$ with $\lambda>\gamma$, $\Theta_1(\iota)=\Theta_2(\iota)=(\iota-1)^{\overline{-0.9}}$, we see (ii) holds. To Check (3.2), we assume

$$\begin{aligned}
\frac{1}{\Gamma(0.75)}\sum_{\iota_1=1+1}^{\iota}(\iota-\iota_1+1)^{\overline{0.75-1}}\left|\eta(\iota_1)\right| &= \frac{1}{\Gamma(0.75)}\sum_{\iota_1=2}^{\iota}(\iota-\iota_1+1)^{\overline{0.75-1}}\left|(\iota_1-1)^{\overline{-0.9}}\right|\\
&=\frac{1}{\Gamma(0.75)}\sum_{\iota_1=2}^{\iota}(\iota-\iota_1+1)^{\overline{0.75-1}}(\iota_1-1)^{\overline{-0.9}}\\
&=\nabla_1^{-0.75}(\iota-1)^{\overline{-0.9}}\\
&=\frac{\Gamma(1-0.9)}{\Gamma(1-0.9+0.75)}(\iota-1)^{\overline{-0.9+0.75}}\\
&=\frac{\Gamma(0.1)}{\Gamma(0.85)}(\iota-1)^{\overline{-0.15}}\\
&\le\frac{\Gamma(0.1)}{\Gamma(0.85)}1^{\overline{-0.15}}\\
&=\Gamma(0.1),
\end{aligned}$$

that is,

$$\lim_{\iota\to\infty}\left[\frac{1}{\Gamma(0.75)}\sum_{\iota_1=1+1}^{\iota}(\iota-\iota_1+1)^{\overline{0.75-1}}\left|e(\iota_1)\right|\right]<\infty.$$

Similarly, it is easy to verify that (3.3) holds. Therefore, all conditions of Theorem 3.1 are satisfied. Thus, every non-oscillatory solution of (1.6) satisfies

$$\limsup_{\iota\to\infty}\frac{|y(\iota)|}{\left[\iota^{\overline{3}}\right]^{\frac{1}{2}}e^{\frac{\iota}{2}}A(\iota,1)}<\infty. \tag{4.2}$$

Example 2. Consider the equation

$$\nabla_{1*}^{0.5}\left(\nabla^2\left(\iota(\iota+1)e^{5\iota}\left(\nabla v(\iota)\right)\right)\right)+\phi(\iota,y(\iota))$$
$$=(\iota-1)^{\overline{-0.75}}+e^{2\iota/3}y(\iota)+\Phi(\iota,y(\iota)),\quad \iota\in\mathbb{N}_2. \tag{4.3}$$

Here we have $z(\iota)=\nabla^2\left(\iota(\iota+1)e^{5\iota}\left(\nabla v(\iota)\right)\right)$, $c=1$, $x=0.5$, $n=3$, $d(\iota)=\iota(\iota+1)e^{5\iota}$, $e(\iota)=(\iota-1)^{\overline{-0.75}}$, and $\zeta(\iota)=e^{2\iota/3}$. Hence, condition (i) holds. Assuming $b=e$, $\iota_1=1$, and $\tau=5$, we find

$$\left(\frac{\iota^{\overline{2}}}{d(\iota)}\right)=e^{-5\iota}.$$

Therefore, (3.23) holds. Now, if we take $p = 3/2$ then we have $q = 3$, and

$$\sum_{\iota_1=c+1}^{\infty} b^{-q\iota_1}\zeta^q(\iota_1) = \sum_{\iota_1=2}^{\infty} e^{-3\iota_1}e^{2\iota_1} = \sum_{\iota_1=2}^{\infty} e^{-\iota_1} = \frac{1}{e(e-1)} < \infty,$$

that is, (3.24) holds. Again if $\phi(\iota, y(\iota)) = \Theta_1(\iota)\,|y(\iota)|^{\lambda-1}\, y(\iota)$ and $\Phi(\iota, y(\iota)) = \Theta_2(\iota)\,|y(\iota)|^{\gamma-1}\, y(\iota)$ with $\lambda > \gamma$, $\Theta_1(\iota) = \Theta_2(\iota) = (\iota-1)^{\overline{-0.75}}$ then it is easy to verify that condition (ii) holds. To check (3.2) holds, we assume

$$\begin{aligned}
\frac{1}{\Gamma(0.5)}\sum_{\iota_1=1+1}^{\iota}(\iota-\iota_1+1)^{\overline{0.5-1}}\left|\eta(\iota_1)\right| &= \frac{1}{\Gamma(0.5)}\sum_{\iota_1=2}^{\iota}(\iota-\iota_1+1)^{\overline{0.5-1}}\left|(\iota_1-1)^{\overline{-0.75}}\right| \\
&= \frac{1}{\Gamma(0.5)}\sum_{\iota_1=2}^{\iota}(\iota-\iota_1+1)^{\overline{0.5-1}}(\iota_1-1)^{\overline{-0.75}} \\
&= \nabla_1^{-0.5}(\iota-1)^{\overline{-0.75}} \\
&= \frac{\Gamma(1-0.75)}{\Gamma(1-0.75+0.5)}(\iota-1)^{\overline{-0.75+0.5}} \\
&= \frac{\Gamma(0.25)}{\Gamma(0.75)}(\iota-1)^{\overline{-0.25}} \\
&\le \frac{\Gamma(0.25)}{\Gamma(0.75)}1^{\overline{-0.25}} \\
&= \Gamma(0.25),
\end{aligned}$$

that is,

$$\lim_{\iota\to\infty}\left[\frac{1}{\Gamma(0.5)}\sum_{\iota_1=1+1}^{\iota}(\iota-\iota_1+1)^{\overline{0.5-1}}\left|e(\iota_1)\right|\right] < \infty.$$

Similarly, it is easy to verify that (3.3) holds. Therefore, all conditions of Theorem 3.2 are satisfied. Thus, every non-oscillatory solution of (4.3) is bounded.

5 | CONCLUSION

In this work, we established some new sufficient conditions for the non-oscillatory solutions of forced nabla fractional difference equations with positive and negative terms. The results are developed in sense of Caputo nabla fractional difference operator and by the help of Young's inequality as well as an equivalent representation in form of a Volterra-type summation equation. The results improved some existing results in the literature. Furthermore, examples are provided to support and illustrate the applicability of the obtained results.

DECLARATION

Availability of data and material
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests
The authors declare that they have no competing interests.

Funding
Not applicable.

Authors' contributions
All authors contributed equally and significantly to this paper. All authors have read and approved the final version of the manuscript.

Acknowledgements
J. Alzabut would like to thank Prince Sultan University and OSTİM Technical University for supporting this research.

References

1. Abdalla B.; Abodayeh K.; Abdeljawad T.; J. Alzabut J., New oscillation criteria for forced nonlinear fractional difference equations. Vietnam Journal of Mathematics, 45(4), 609–618, (2017).

2. Abdalla B.; Alzabut J.; Abdeljawad T., On the oscillation of higher order fractional difference equations with mixed nonlinearities, Hacettepe Journal of Mathematics and Statistics, vol. 47, no. 2, pp. 207–217, 2018.

3. Alzabut, J. O.; Abdeljawad, T. Sufficient conditions for the oscillation of nonlinear fractional difference equations. J. Fract. Calc. Appl. 5 (2014), no. 1, 177–187.

4. Alzabut J.; Abdeljawad T; Alrabaiah H., Oscillation criteria for forced and damped nabla fractional difference equations, J. Comput. Anal. Appl, 24(8), 1387-1394, (2018).

5. Alzabut J.; Muthulakshmi V.; Özbekler A.; Adıgüzel H., On the oscillation of non-linear fractional difference equations with damping, Mathematics, 7(8), 687 (2019).

6. Anastassiou, G. A., Nabla discrete fractional calculus and nabla inequalities. Math. Comput. Modelling 51 (2010), no. 5-6, 562–571.

7. Atıcı, F. M.; Eloe, P. W., Discrete fractional calculus with the nabla operator. Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, no. 3, 12 pp.

8. Chen, F., Fixed points and asymptotic stability of nonlinear fractional difference equations. Electron. J. Qual. Theory Differ. Equ. 2011, No. 39, 18 pp.

9. Goodrich, C.; Peterson, A. C., Discrete fractional calculus. Springer, Cham, 2015.

10. Grace, S. R.; Graef, J. R.; Tunc, E., On the boundedness of non-oscillatory solutions of certain fractional differential equations with positive and negative terms. Appl. Math. Lett. 97 (2019), 114–120.

11. Hardy, G. H.; Littlewood, J. E.; Polya, G., Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

12. Jonnalagadda, J. M., Analysis of a system of nonlinear fractional nabla difference equations. Int. J. Dyn. Syst. Differ. Equ. 5 (2015), no. 2, 149–174.

13. Kelley, Walter G.; Peterson, A. C., Difference equations. An introduction with applications. Second edition. Harcourt/Academic Press, San Diego, CA, 2001.

14. Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J., Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

15. Podlubny, I., Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

How to cite this article: , S. R. Grace, J. M. Jonnalagadda, J. Alzabut, and S. S. Santra (2022), Non-oscillatory Solutions of Higher-order Forced Nabla Fractional Difference Equations with Positive and Negative Terms, *Mathematical Methors in the Applied Sciences*, *2022;00:1–12*.