GLOBAL STRONG SOLUTION TO A THERMODYNAMIC COMPRESSIBLE DIFFUSE INTERFACE MODEL WITH TEMPERATURE DEPENDENT HEAT-CONDUCTIVITY IN 1-D

YAZHOU CHEN, QIAOLIN HE, BIN HUANG, AND XIAODING SHI

Abstract. In this paper, we investigate the wellposedness of the non-isentropic compressible Navier-Stokes/Allen-Cahn system with the heat-conductivity proportional to a positive power of the temperature. This system describes the flow of a two-phase immiscible heat-conducting viscous compressible mixture. The phases are allowed to shrink or grow due to changes of density in the fluid and incorporates their transport with the current. We established the global existence and uniqueness of strong solutions for this system in 1-D, which means no phase separation, vacuum, shock wave, mass or heat or phase concentration will be developed in finite time, although the motion of the two-phase immiscible flow has large oscillations and the interaction between the hydrodynamic and phase-field effects is complex. Our result can be regarded as a natural generalization of the Kazhikhov-Shelukhin's result ([Kazhikhov-Shelukhin. J. Appl. Math. Mech. 41 (1977)]) for the compressible single-phase flow with constant heat conductivity to the non-isentropic compressible immiscible two-phase flow with degenerate and nonlinear heat conductivity.

1. Introduction

The immiscible two-phase flows appear widely in many fields, such as thermal power engineering, nuclear energy engineering, cryogenic engineering and aerospace, etc. An important feature of immiscible two-phase flow is the coexistence of two fluids with different phase states or components and the existence of the interface. The theoretical analysis of the immiscible two-phase flow is much more difficult than that of single-phase flow. A common treatment method is the so-called separated flow model, which holds that the concept and method of single-phase flow can be applied to each phase of the two-phase flow system, and the interface properties and motion between the two phases are considered.

Understanding the geometry and distribution of the interface is very important for determining the flow of immiscible two-phase flow. The treatment of such two-phase flow interface is derived from the idea of physicist J.D. Van der Waals [34], who regarded the interface of immiscible two-phase flow as a region with a certain thickness. Mathematical models based on this idea are often called diffusion interface models, such as the famous Navier-Stokes/Allen-Cahn system, which can be used to study the immiscible two-phase flow, such as phase transformation, chemical reactions, etc. see [1]– [15], [28], [30], and the references therein. In these literatures, by introducing diffusion interface instead of sharp interface, the authors can avoid the difficulty of processing interface boundary conditions.

Date: July 23, 2020.

2010 *Mathematics Subject Classification.* 35Q30, 76T30, 35C20.

Key words and phrases. compressible Navier-Stokes equations, Allen-Cahn equation, diffuse interface, immiscible two-phase flow, global strong solution.

Here is a brief review of the model. For compressible immiscible two-phase flow, taking any one of the volume particles in the flow, we assume M_i the mass of the components in the representative material volume V, $\phi_i = \frac{\rho_i}{\rho}$ the mass concentration, $\rho_i = \frac{M_i}{V}$ the apparent mass density of the fluid i $(i = 1, 2)$. The total density is given by $\rho = \rho_1 + \rho_2$, and $\phi = \phi_1 - \phi_2$. We call ϕ the difference of the two components for the fluid mixture. Obviously, ϕ describes the distribution of the interface. Therefore, on this basis, Blesgen [6], Heida-Málek-Rajagopal [15] coupled the Navier-Stokes system describing the flow of the single fluid and the Allen-Cahn equation describing the change of the phase field (Allen-Cahn [1]), the Navier-Stokes/Allen-Cahn system is proposed (hereinafter called as NSAC equations):

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}\big(\rho\mathbf{u}\otimes\mathbf{u}\big) = \operatorname{div}\mathbb{T}, \\ (\rho\phi)_t + \operatorname{div}\big(\rho\phi\mathbf{u}\big) = -\mu, \\ \rho\mu = \rho\dfrac{\partial f}{\partial\phi} - \operatorname{div}\big(\rho\dfrac{\partial f}{\partial\nabla\phi}\big), \\ (\rho E)_t + \operatorname{div}(\rho E\mathbf{u}) = \operatorname{div}\big(\mathbb{T}\mathbf{u} + \kappa(\theta)\nabla\theta - \mu\dfrac{\partial f}{\partial\nabla\phi}\big), \end{cases} \tag{1.1}$$

where $\tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^N$, N is the spatial dimension, $t > 0$ is the time. The unknown functions $\rho(\tilde{\mathbf{x}}, t)$, $\mathbf{u}(\tilde{\mathbf{x}}, t)$, $\phi(\tilde{\mathbf{x}}, t)$, $\theta(\tilde{\mathbf{x}}, t)$ denote the total density, the velocity, the difference of the two components for the fluid mixture, and the absolute temperature respectively. $\mu(\tilde{\mathbf{x}}, t)$ is called the chemical potential of the fluid. $\epsilon > 0$ is the thickness of the diffuse interface. div and ∇ are the divergence operator and gradient operator respectively. The Cauchy stress-tensor is represented by

$$\mathbb{T} = 2\nu\mathbb{D}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\mathbb{I} - p\mathbb{I} - \rho\nabla\phi\otimes\frac{\partial f}{\partial\nabla\phi}, \tag{1.2}$$

where f is the fluid-fluid interfacial free energy density, and it has the following form (Lowengrub-Truskinovsky [30], Heida-Málek-Rajagopal [15]):

$$f(\rho, \phi, \nabla\phi) \overset{\text{def}}{=} \frac{1}{4\epsilon}(1-\phi^2)^2 + \frac{\epsilon}{2\rho}|\nabla\phi|^2, \tag{1.3}$$

$\mathbb{I}$ is the unit matrix, $\mathbb{D}\mathbf{u}$ is the so-called deformation tensor

$$\mathbb{D}\mathbf{u} = \frac{1}{2}\big(\nabla\mathbf{u} + \nabla^\top\mathbf{u}\big), \tag{1.4}$$

here and hereafter, superscript $\top$ denotes the transpose and all vectors are column ones. $\nu > 0, \lambda > 0$ are viscosity coefficients, satisfying

$$\nu > 0, \quad \lambda + \frac{2}{N}\nu \geq 0. \tag{1.5}$$

$\kappa(\theta)$ is the heat conductivity satisfying

$$\kappa(\theta) = \tilde{\kappa}\theta^\beta, \tag{1.6}$$

with constants $\tilde{\kappa} > 0$ and $\beta > 0$. The total energy density ρE is given by

$$\rho E = \rho e + \rho f + \frac{1}{2}\rho\mathbf{u}^2, \tag{1.7}$$

where ρe is the internal energy, $\frac{\rho \mathbf{u}^2}{2}$ is the kinetic energy, $p = p(\rho, \theta)$, $e = e(\rho, \theta)$ and $f = f(\rho, \phi, \nabla\phi)$ obey the second law of thermodynamics (Lions [29]):

$$ds = \frac{1}{\theta}\big(d(e+f) + pd(\frac{1}{\rho})\big), \tag{1.8}$$

where s is the entropy. Then we deduce from (1.8):

$$\frac{\partial s}{\partial \theta} = \frac{1}{\theta}\frac{\partial(e+f)}{\partial \theta}, \quad \frac{\partial s}{\partial \rho} = \frac{1}{\theta}\big(\frac{\partial(e+f)}{\partial \rho} - \frac{p}{\rho^2}\big), \tag{1.9}$$

which implies the following compatibility equation

$$p = \rho^2\frac{\partial(e+f)}{\partial \rho} + \theta\frac{\partial p}{\partial \theta} = \rho^2\frac{\partial e(\rho,\theta)}{\partial \rho} - \frac{\epsilon}{2}|\nabla\phi|^2 + \theta\frac{\partial p}{\partial \theta}. \tag{1.10}$$

Throughout the paper, we concentrate on ideal polytropic gas, that is, p satisfies

$$p(\rho,\theta) = R\rho\theta - \frac{\epsilon}{2}|\nabla\phi|^2, \tag{1.11}$$

and e satisfies

$$e = c_v\theta + \text{constant}, \tag{1.12}$$

where c_v is the specific heat capacity. Substituting (1.2), (1.7), (1.3), (1.10) and (1.12) into (1.1), then the NSAC system is simplified to

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \rho\mathbf{u}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - 2\tilde{\nu}\operatorname{div}\mathbb{D}\mathbf{u} - \tilde{\lambda}\nabla\operatorname{div}\mathbf{u} = -\operatorname{div}\big(\epsilon\nabla\phi\otimes\nabla\phi - \frac{\epsilon}{2}|\nabla\phi|^2 + \theta\frac{\partial p}{\partial\theta}\big), \\ \rho\phi_t + \rho\mathbf{u}\cdot\nabla\phi = -\mu, \\ \rho\mu = \frac{\rho}{\epsilon}(\phi^3 - \phi) - \epsilon\Delta\phi, \\ c_v\big(\rho\theta_t + \rho\mathbf{u}\cdot\nabla\theta\big) + \theta p_\theta\operatorname{div}\mathbf{u} - \operatorname{div}(\kappa(\theta)\nabla\theta) = 2\tilde{\nu}|\mathbb{D}\mathbf{u}|^2 + \tilde{\lambda}(\operatorname{div}\mathbf{u})^2 + \mu^2. \end{cases} \tag{1.13}$$

In this paper, we consider the one-dimensional problem of the system (1.13):

$$\begin{cases} \rho_t + (\rho u)_{\tilde{x}} = 0, \\ \rho u_t + \rho u u_{\tilde{x}} + (R\rho\theta)_{\tilde{x}} = \nu u_{\tilde{x}\tilde{x}} - \frac{\epsilon}{2}(\phi_{\tilde{x}}^2)_{\tilde{x}}, \\ \rho\phi_t + \rho u\phi_{\tilde{x}} = -\mu, \\ \rho\mu = \frac{\rho}{\epsilon}(\phi^3 - \phi) - \epsilon\phi_{\tilde{x}\tilde{x}}, \\ c_v(\rho\partial_t\theta + \rho u\theta_{\tilde{x}}) + R\rho\theta u_{\tilde{x}} - (\kappa(\theta)\theta_{\tilde{x}})_{\tilde{x}} = \nu u_{\tilde{x}}^2 + \mu^2, \end{cases} \tag{1.14}$$

for $(\tilde{x}, t) \in [0,1]\times[0,+\infty)$, where $\nu = 2\tilde{\nu} + \tilde{\lambda} > 0$. The initial boundary conditions are following

$$(u, \phi_{\tilde{x}}, \theta_{\tilde{x}})(0,t) = (u, \phi_{\tilde{x}}, \theta_{\tilde{x}})(1,t) = 0, \tag{1.15}$$

$$(\rho, u, \theta, \phi)\big|_{t=0} = (\rho_0, u_0, \theta_0, \phi_0), \quad \tilde{x} \in (0,1). \tag{1.16}$$

Without loss of generality, we assume that

$$\int_0^1 \rho_0(\tilde{x})d\tilde{x} = 1, \tag{1.17}$$

and

$$\nu = R = c_v = \tilde{\kappa} = 1. \tag{1.18}$$

In Lagrange coordinates

$$x=\int_0^{\tilde{x}}\rho(\xi,t)d\xi, \tag{1.19}$$

the system (1.14)–(1.16) can be rewritten as

$$\begin{cases} v_t-u_x=0,\\ u_t+(\dfrac{\theta}{v})_x=(\dfrac{u_x}{v})_x-\dfrac{\epsilon}{2}(\dfrac{\phi_x^2}{v^2})_x,\\ \phi_t=-v\mu,\\ \mu=\dfrac{1}{\epsilon}(\phi^3-\phi)-\epsilon(\dfrac{\phi_x}{v})_x,\\ \theta_t+\dfrac{\theta}{v}u_x-(\dfrac{\theta^\beta\theta_x}{v})_x=\nu\dfrac{u_x^2}{v}+v\mu^2, \end{cases} \tag{1.20}$$

with the boundary condition

$$(u,\phi_x,\theta_x)(0,t)=(u,\phi_x,\theta_x)(1,t)=0,\quad t\geq 0. \tag{1.21}$$

and the initial value condition

$$(v,u,\theta,\phi)(x,0)=(v_0,u_0,\theta_0,\phi_0)(x),\qquad x\in(0,1), \tag{1.22}$$

where

$$v=\frac{1}{\rho}.$$

There are a lot of works on the research of the global existence and large time behavior of solutions to the compressible heat-conducting Navier-Stokes system. Kazhikhov-Shelukhin [26] first proposed the global existence of solutions with large initial data. In the year since, significant progress has been made, see [2]– [5], [23], [24], [27] and the references therein. It should be noted that these works are given under the assumption that the heat conductivity is a positive constant. If κ depends on temperature, Kawohl [20], Jiang [21,22] and Wang [35] established the global existence of smooth solutions for compressible heat-conducting Navier-Stokes system with boundary condition of $(u,\theta_x)(0,t)=(u,\theta_x)(1,t)=0$. and the methods used there relies heavily on the non-degeneracy of the heat conductivity κ which cannot be applied directly to the degenerate and nonlinear case ($\beta>0$). Pan-Zhang [32] generalize the above results to the degenerate case (1.6) where $\beta\in(0,\infty)$, and more improvement results associated with this degenerate case, please refer to Duan-Guo-Zhu [12], Huang-Shi-Sun [16], Huang-Shi [17] and the references therein.

In terms of compressible two-phase flows, so far as we know, most of the work focuses on the isentropic case. Feireisl-Petzeltová-Rocca-Schimperna [13], Chen-Wen-Zhu [9] proved the global existence of the weak solution one after another, the method the used is based on the renormalization weak solution framework for the compressible Navier-Stokes system introduced by Lions [29]. Kotschote [28] showed the existence and uniqueness of local strong solutions for arbitrary initial data. Ding-Li-Luo [11] established the existence and uniqueness of global strong solution in 1D for initial density without vacuum states. Chen-Guo [8] generalized the result of Ding-Li-Luo [11] to the case that the initial vacuum is allowed.

In this paper, we focus on the non-isentropic compressible flow for two-phase immiscible mixture. Our purpose is to study the existence and uniqueness of global strong solution for the non-isentropic NSAC systems (1.20) even with large initial data. More specifically, for general initial conditions without vacuum state, we study the global existence of the solution for the system (1.20)–(1.22). Now we give our main result as following

Theorem 1.1. *Assume that*

$$(v_0, \theta_0) \in H^1(0,1), \quad \phi_0 \in H^2(0,1), \quad u_0 \in H_0^1(0,1), \tag{1.23}$$

and

$$\inf_{x\in(0,1)} v_0(x) > 0, \quad \inf_{x\in(0,1)} \theta_0(x) > 0, \quad \phi_0(x) \in [-1,1]. \tag{1.24}$$

Then, the initial boundary value problem (1.20)-(1.22) *has a unique strong solution* (v, u, θ, ϕ) *such that for fixed* $T > 0$, *satisfying*

$$\begin{cases} v, \theta \in L^\infty(0,T; H^1(0,1)), \quad u, \phi_x \in L^\infty(0,T; H_0^1(0,1)), \\ v_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)), \\ \phi_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)), \\ u_t, \theta_t, \phi_{xt},, u_{xx}, \theta_{xx}, \phi_{xxx} \in L^2((0,1)\times(0,T)). \end{cases} \tag{1.25}$$

Moreover, there exist a positive constant C *depending on the initial data and* T, *satisfying*

$$C^{-1} \le v(x,t) \le C, \quad C^{-1} \le \theta(x,t) \le C, \quad \phi(x,t) \in [-1,1]. \tag{1.26}$$

Remark 1.1. *Our result can be regarded as a natural generalization of the Kazhikhov-Shelukhin's result ([26]) for the compressible single-phase flow with constant heat conductivity to the non-isentropic compressible immiscible two-phase flow with degenerate and nonlinear heat conductivity.*

Remark 1.2. *The global existence and uniqueness of strong solutions for NSAC system* (1.20) *means no vacuum, phase separation, shock wave, mass or heat or phase concentration will be developed in finite time, although the motion of the two-phase immiscible flow has large oscillations and the interaction between the hydrodynamic and phase-field effects is complex.*

Now we give some notes on the proof of the main theorem. The key to the proof is to get the upper and lower bounds of v, θ and ϕ, see (2.16), (2.23), (2.75). Inspired by the idea of Kazhikhov [25], we first obtain a key expression of v, see (2.8), combining with the energy inequality (2.3), Jensen's inequality, we obtain the positive lower bound of v and θ. Then after getting the upper and lower bounds on ϕ, observing that $\max_{x\in[0,1]} \big(\frac{\phi_x}{v}\big)^2(x,t)$ can be bounded by $C\big(\max_{x\in[0,1]} \theta(x,t) + 1 + V(t)\big)$ (see (2.33)), we obtain the upper bound of v by the expression of v. Further, after observing that the inequalities (2.36) and (2.53), we get the estimates on the $L^\infty(0,T;L^2)$-norm of θ_x, and the upper bound of temperature θ is achieved. The details of the proof will be shown in the next section.

2. The Proof of Theorem

The following Lemma is the existence and uniqueness of local strong solutions which can be obtained by the fixed point method. From here to the end of this paper, $C > 0$ denotes the generic positive constant depending only on $\|(v_0, u_0, \theta_0\|_{H^1(0,1)}$, $\|\phi_0\|_{H^2(0,1)}$, $\inf\limits_{x\in[0,1]} v_0(x)$, and $\inf\limits_{x\in[0,1]} \theta_0(x)$. Moreover, by using conservation of energy, combining with

(1.17), without loss of generality, the following assumption is given

$$\int_0^1 v_0dx=1,\quad \int_0^1 (\frac{u_0^2}{2}+\theta_0+\frac{1}{4\epsilon}(\phi_0^2-1)^2+\frac{\epsilon}{2}\frac{\phi_{0x}^2}{v_0})dx=1. \tag{2.1}$$

Lemma 2.1. *Let (1.23) and (1.24) hold. Then there exists some $T_*>0$ such that the initial boundary value problem (1.20)-(1.22) has a unique strong solution (v,u,θ,ϕ) satisfying*

$$\begin{cases} v,\theta\in L^\infty(0,T_*;H^1(0,1)),\quad u,\phi_x\in L^\infty(0,T_*;H_0^1(0,1)),\\ v_t\in L^\infty(0,T_*;L^2(0,1))\cap L^2(0,T_*;H^1(0,1)),\\ \phi_t\in L^\infty(0,T_*;L^2(0,1))\cap L^2(0,T_*;H^1(0,1)),\\ u_t,\theta_t,\phi_{xt},u_{xx},\theta_{xx},\phi_{xxx}\in L^2((0,1)\times(0,T_*)). \end{cases} \tag{2.2}$$

Theorem 1.1 can be achieved by extending the local solutions globally in time based on the following series of prior estimates.

Lemma 2.2. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds*

$$\sup_{0\le t\le T}\int_0^1 (\frac{u^2}{2}+\frac{1}{4\epsilon}(\phi^2-1)^2+\frac{\epsilon}{2}\frac{\phi_x^2}{v}+(v-\ln v)+(\theta-\ln\theta)\big)dx+\int_0^T V(s)ds\le E_0, \tag{2.3}$$

where

$$V(t)\overset{def}{=}\int_0^1 (\frac{\theta^\beta\theta_x^2}{v\theta^2}+\frac{u_x^2}{v\theta}+\frac{v\mu^2}{\theta})dx, \tag{2.4}$$

and

$$E_0\overset{def}{=}\int_0^1 (\frac{u_0^2}{2}+\frac{1}{4\epsilon}(\phi_0^2-1)^2+\frac{\epsilon}{2}\frac{\phi_{0x}^2}{v}+(v_0-\ln v_0)+(\theta_0-\ln\theta_0)\big)dx. \tag{2.5}$$

Proof. From (1.20) and (1.21), combining with (2.1), we have

$$\int_0^1 vdx=1,\quad \int_0^1 (\frac{u^2}{2}+\theta+\frac{1}{4\epsilon}(\phi^2-1)^2+\frac{\epsilon}{2}\frac{\phi_x^2}{v})dx=1. \tag{2.6}$$

Multiplying $(1.20)_1$ by $1-\frac{1}{v}$, $(1.20)_2$ by u, $(1.20)_3$ by μ, $(1.20)_5$ by $1-\frac{1}{\theta}$, and adding them together, we get

$$\begin{aligned}&(\frac{u^2}{2}+\frac{1}{4\epsilon}(\phi^2-1)^2+\frac{\epsilon}{2}\frac{\phi_x^2}{v}+(v-\ln v)+(\theta-\ln\theta)\big)_t+(\frac{\theta^\beta\theta_x^2}{v\theta^2}+\frac{u_x^2}{v\theta}+\frac{v\mu^2}{\theta})\\ &=u_x+(\frac{uu_x}{v}-\frac{u\theta}{v})_x+\big((1-\theta^{-1})\frac{\theta^\beta\theta_x}{v}\big)_x+\epsilon(\frac{\phi_x\phi_t}{v})_x-\frac{\epsilon}{2}(\frac{\phi_x^2u}{v^2})_x,\end{aligned} \tag{2.7}$$

integrating (2.7) over $[0,1]\times[0,T]$ by parts, together with the boundary condition (1.21), we obtain (2.3), and the proof of Lemma 2.2 is finished. □

Lemma 2.3. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it has the following expression of v*

$$v(x,t)=D(x,t)Y(t)+\int_0^t\frac{D(x,t)Y(t)\big(\theta(x,\tau)+\frac{\epsilon}{2}\frac{\phi_x^2(x,\tau)}{v(x,\tau)}\big)}{D(x,\tau)Y(\tau)}d\tau, \tag{2.8}$$

where

$$D(x,t)=v_0(x)e^{\int_0^x\big(u(y,t)-u_0(y)\big)dy}e^{\big(-\int_0^1 v\int_0^x udydx+\int_0^1 v_0\int_0^x u_0dydx\big)}, \tag{2.9}$$

and

$$Y(t) = e^{-\int_0^t \int_0^1 \left(u^2+\theta+\frac{\epsilon}{2}\frac{\phi_x^2}{v}\right)dxds}. \tag{2.10}$$

Proof. We introduce the following σ

$$\sigma \overset{\text{def}}{=} \frac{u_x}{v} - \frac{\theta}{v} - \frac{\epsilon}{2}\frac{\phi_x^2}{v^2}. \tag{2.11}$$

Firstly, integrating $(1.20)_2$ over $[0, x]$, we have

$$\left(\int_0^x udy\right)_t = \sigma - \sigma(0,t), \tag{2.12}$$

multiply both sides of (2.12) by v, we get

$$v\sigma(0,t) = v\sigma - v\left(\int_0^x udy\right)_t,$$

Integrating the above equation over $[0, 1]$, combining with (2.1) and (1.21), we obtain

$$\begin{aligned}\sigma(0,t) &= \int_0^1 v\sigma dx - \int_0^1 v\left(\int_0^x udy\right)_t dx \\ &= \int_0^1 (u_x - \theta - \frac{\epsilon}{2}\frac{\phi_x^2}{v})dx - (\int_0^1 v\int_0^x udydx)_t + \int_0^1 u_x\int_0^x udydx \\ &= -\left(\int_0^1 v\int_0^x udydx\right)_t - \int_0^1\left(u^2+\theta+\frac{\epsilon}{2}\frac{\phi_x^2}{v}\right)dx. \end{aligned} \tag{2.13}$$

Secondly, from $(1.20)_1$, we have

$$\sigma = (\ln v)_t - \frac{\theta}{v} - \frac{\epsilon}{2}\frac{\phi_x^2}{v^2},$$

which combining with (2.12) and (2.13), we get

$$\left(\int_0^x udy\right)_t = (\ln v)_t - \frac{\theta}{v} - \frac{\epsilon}{2}\frac{\phi_x^2}{v^2} + \left(\int_0^1 v\int_0^x udydx\right)_t + \int_0^1\left(\theta+u^2+\frac{\epsilon}{2}\frac{\phi_x^2}{v}\right)dx,$$

integrating both sides of the above equation with respect to t, we have

$$v(x,t) = D(x,t)Y(t)e^{\int_0^t\left(\frac{\theta}{v}+\frac{\epsilon}{2}\frac{\phi_x^2}{v^2}\right)ds}, \tag{2.14}$$

with $D(x,t)$ and $Y(t)$ as defined in (2.9) and (2.10) respectively.

Finally, introducing the following g

$$g = \int_0^t (\frac{\theta}{v} + \frac{\epsilon}{2}\frac{\phi_x^2}{v^2})ds, \tag{2.15}$$

by using (2.14), we get the following ordinary differential equation for g

$$g_t = \frac{\theta(x,t) + \frac{\epsilon}{2}\frac{\phi_x^2(x,t)}{v(x,t)}}{v(x,t)} = \frac{\theta(x,t) + \frac{\epsilon}{2}\frac{\phi_x^2(x,t)}{v(x,t)}}{D(x,t)Y(t)e^g},$$

and this gives us an expression for e^g

$$e^g = 1 + \int_0^t \frac{\theta(x,\tau) + \frac{\epsilon}{2}\frac{\phi_x^2(x,\tau)}{v(x,\tau)}}{D(x,\tau)Y(\tau)}d\tau,$$

substituting it into the expression (2.14), and thus the proof of Lemma 2.3 is finished. □

Lemma 2.4. *Let (v, u, θ, ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall (x,t)\in[0,1]\times[0,T]$*

$$v(x,t)\geq C^{-1}, \qquad \theta(x,t)\geq C^{-1}. \tag{2.16}$$

Proof. Firstly, since the function $x-\ln x$ is convex, by using Jensen's inequality, we have

$$\int_0^1 \theta dx - \ln\int_0^1 \theta dx \leq \int_0^1 (\theta-\ln\theta)dx, \tag{2.17}$$

combining with (2.3) and (2.6), we get

$$\bar\theta(t) \overset{\text{def}}{=} \int_0^1 \theta(x,t)dx \in [\alpha_1, 1], \tag{2.18}$$

where $0<\alpha_1<\alpha_2$ are the two roots of the following algebraic equation

$$x-\ln x = E_0. \tag{2.19}$$

Secondly, from (2.6), by using Cauchy's inequality, we have

$$\Big|\int_0^1 v\int_0^x u dy dx\Big| \leq \int_0^1 v\Big|\int_0^x u dy\Big| dx \leq \int_0^1 v\Big(\int_0^1 u^2 dy\Big)^{1/2} dx \leq C,$$

and then by the definition (2.9) of D, we get

$$C^{-1}\leq D(x,t)\leq C. \tag{2.20}$$

Moreover, from (2.6), combining with (2.18), we also have

$$\Big|\int_0^1 \ln v dx\Big| + \int_0^1 \Big(u^2+\theta+\frac{\epsilon}{2}\frac{\phi_x^2}{v}\Big)dx \leq C,$$

which follows that, for $\forall \tau\in[0,t)$,

$$e^{-2t}\leq Y(t)\leq 1, \ \ \text{and} \ \ e^{-2(t-\tau)}\leq \frac{Y(t)}{Y(\tau)}\leq e^{-\alpha_1(t-\tau)}. \tag{2.21}$$

By using (2.14), (2.20) and (2.21), we obtain, there exist some positive constant C, such that

$$v(x,t)\geq C^{-1}, \qquad \forall (x,t)\in[0,1]\times[0,T]. \tag{2.22}$$

Finally, $\forall p>2$, multiplying $(1.20)_5$ by θ^{-p}, integrate over $[0,1]$ with respect to x, by using (2.22), we have

$$\begin{aligned}\frac{1}{p-1}\frac{d}{dt}\int_0^1 \left(\theta^{-1}\right)^{p-1}dx + \int_0^1 \frac{u_x^2}{v\theta^p}dx &\leq \int_0^1 \frac{u_x}{v\theta^{p-1}}dx \\ &\leq \frac12\int_0^1 \frac{u_x^2}{v\theta^p}dx + \frac12\int_0^1\frac{1}{v\theta^{p-2}}dx \\ &\leq \frac12\int_0^1 \frac{u_x^2}{v\theta^p}dx + C\left\|\theta^{-1}\right\|_{L^{p-1}}^{p-2}.\end{aligned}$$

Applying Gronwall's inequality to the above result, we obtain

$$\sup_{0\leq t\leq T}\left\|\theta^{-1}(\cdot,t)\right\|_{L^{p-1}}\leq C, \quad \forall p>2,$$

moreover, letting p tends to infinity, we do eventually get the lower bound of θ. The proof of Lemma 2.4 is completed. □

Lemma 2.5. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall(x,t)\in[0,1]\times[0,T]$*

$$|\phi(x,t)|\le C,\qquad v(x,t)\le C. \tag{2.23}$$

Proof. Firstly, from (2.3), we have

$$\frac{\epsilon}{4}\int_0^1(\phi^2-1)^2\le E_0,$$

which implies that

$$\int_0^1\phi^4\le C. \tag{2.24}$$

Moreover, for $\forall(x,t)\in[0,1]\times[0,T]$,

$$\begin{aligned}
|\phi(x,t)| &\le \left|\int_0^1\big(\phi(x,t)-\phi(y,t)\big)dy\right|+\left|\int_0^1\phi(y,t)dy\right|\\
&\le \left|\int_0^1\Big(\int_y^x\phi_\xi(\xi,t)d\xi\Big)dy\right|+CE_0\\
&\le \Big(\int_0^1\frac{\phi_x^2}{v}dx\Big)^{\frac12}+CE_0\\
&\le C.
\end{aligned}\tag{2.25}$$

Next, for $0<\alpha<1$ and $0<\varepsilon<1$, integrating $(1.20)_5$ multiplied by $\theta^{-\alpha}$ over $(0,1)\times(0,T)$ yields

$$\begin{aligned}
&\int_0^T\int_0^1\frac{\alpha\theta^\beta\theta_x^2}{v\theta^{\alpha+1}}dxdt+\int_0^T\int_0^1\frac{u_x^2+(v\mu)^2}{v\theta^\alpha}dxdt\\
&=\frac{1}{1-\alpha}\int_0^1\left(\theta^{1-\alpha}-\theta_0^{1-\alpha}\right)dx+\int_0^T\int_0^1\frac{\theta^{1-\alpha}u_x}{v}dxdt\\
&\le C(\alpha)+\frac12\int_0^T\int_0^1\frac{u_x^2}{v\theta^\alpha}dxdt+C\int_0^T\int_0^1\theta^{2-\alpha}dxdt\\
&\le C(\alpha)+\frac12\int_0^T\int_0^1\frac{u_x^2}{v\theta^\alpha}dxdt+C\int_0^T\max_{x\in[0,1]}\theta^{1-\alpha}\int_0^1\theta dxdt\\
&\le C(\alpha,\varepsilon)+\frac12\int_0^T\int_0^1\frac{u_x^2}{v\theta^\alpha}dxdt+\varepsilon\int_0^T\max_{x\in[0,1]}\theta dt,
\end{aligned}\tag{2.26}$$

where in the first inequality we have used (2.3) and (2.16). Then, for $\alpha=\min\{1,\beta\}/2$, using (2.16), we get

$$\begin{aligned}
\int_0^T\max_{x\in[0,1]}\theta dt &\le C+C\int_0^T\int_0^1|\theta_x|dxdt\\
&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{1+\alpha}}dxdt+C\int_0^T\int_0^1\frac{v\theta^{1+\alpha}}{\theta^\beta}dxdt
\end{aligned}$$

$$\leq C + C\int_0^T\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^{1+\alpha}}dxdt + \frac{1}{2}\int_0^T \max_{x\in[0,1]}\theta dt, \tag{2.27}$$

which together with (2.26) yields that

$$\int_0^T \max_{x\in[0,1]}\theta dt \leq C, \tag{2.28}$$

and then that for $0<\alpha<1$,

$$\int_0^T\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^{\alpha+1}}dxdt \leq C(\alpha). \tag{2.29}$$

Finally, we can give the upper bounds of v. In fact, combining the expression of v (2.8) with the upper and lower bound estimates (2.20)–(2.22), we have the following inequality

$$\begin{aligned} v(x,t) &= D(x,t)Y(t) + \int_0^t \frac{D(x,t)Y(t)\big(\theta(x,\tau)+\frac{\epsilon}{2}\frac{\phi_x^2(x,\tau)}{v(x,\tau)}\big)}{D(x,\tau)Y(\tau)}d\tau \\ &\leq C + C\int_0^t e^{-\alpha_1(t-\tau)}\Big(\max_{x\in[0,1]}\theta(x,\tau) + \max_{x\in[0,1]}\big(\frac{\phi_x(x,\tau)}{v(x,\tau)}\big)^2\max_{x\in[0,1]}v(x,\tau)\Big)d\tau. \end{aligned} \tag{2.30}$$

In view of $(1.20)_4$, we derive that

$$\epsilon\Big(\frac{\phi_x}{v}\Big)_x = -\mu + \frac{1}{\epsilon}(\phi^3-\phi), \tag{2.31}$$

and then combining with (2.3), (2.6), (2.16), (2.25), we obtain

$$\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2\frac{v}{\theta}dx \leq C\big(1+V(t)\big). \tag{2.32}$$

Using (2.3), (2.6), (2.22), (2.25), (2.32), we get

$$\begin{aligned} \max_{x\in[0,1]}\big(\frac{\phi_x}{v}\big)^2(x,t) &\leq C\int_0^1 \frac{\phi_x}{v}\Big(\frac{\phi_x}{v}\Big)_x dx \\ &\leq C\int_0^1 \frac{\theta}{v^2}\frac{\phi_x^2}{v}dx + \int_0^1\Big(\frac{\phi_x}{v}\Big)_x^2\frac{v}{\theta}dx \\ &\leq C\big(\max_{x\in[0,1]}\theta(x,t) + 1 + V(t)\big). \end{aligned} \tag{2.33}$$

Substituting (2.33) into (2.30), we obtain

$$v(x,t) \leq C + C\int_0^t \big(\max_{x\in[0,1]}\theta(x,t) + 1 + V(t)\big)\max_{x\in[0,1]}v(x,\tau)d\tau, \tag{2.34}$$

by using Gronwall's inequality, we get the upper bound of v

$$v(x,t) \leq C, \qquad \forall (x,t)\in[0,1]\times[0,T]. \tag{2.35}$$

The proof of Lemma 2.5 is completed. □

Lemma 2.6. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall (x,t)\in[0,1]\times[0,T]$*

$$\sup_{0\leq t\leq T}\int_0^1 v_x^2dx \leq C, \qquad \int_0^T\int_0^1 \phi_{xx}^2dx \leq C, \qquad \int_0^T\int_0^1 \phi_t^2dx \leq C. \tag{2.36}$$

Proof. Firstly, we rewrite the momentum equation in Lagrange coordinate system $(1.20)_2$ as follows

$$\Big(u-\frac{v_x}{v}\Big)_t=-\Big(\frac{\theta}{v}+\frac{\epsilon}{2}(\frac{\phi_x}{v})^2\Big)_x \tag{2.37}$$

Multiplying (2.37) by $u-\frac{v_x}{v}$, integrating by parts over $[0,1]$ with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1\Big(u-\frac{v_x}{v}\Big)^2(x,t)dx-\frac{1}{2}\int_0^1\Big(u-\frac{v_x}{v}(x,0)\Big)^2dx\\
&=\int_0^t\int_0^1\Big(\frac{\theta v_x}{v^2}-\frac{\theta_x}{v}-\epsilon\frac{\phi_x}{v}(\frac{\phi_x}{v})_x\Big)\Big(u-\frac{v_x}{v}\Big)dxdt\\
&=-\int_0^t\int_0^1\frac{\theta v_x^2}{v^3}dxdt+\int_0^t\int_0^1\frac{\theta u v_x}{v^2}dxdt\\
&\quad-\int_0^t\int_0^1\frac{\theta_x}{v}\Big(u-\frac{v_x}{v}\Big)dxdt-\int_0^t\int_0^1\epsilon\frac{\phi_x}{v}(\frac{\phi_x}{v})_x\Big(u-\frac{v_x}{v}\Big)dxdt.
\end{aligned}\tag{2.38}$$

Now we give the last three terms on the right side of (2.38). First, by using (2.3), (2.22), (2.28) and (2.35), we have

$$\begin{aligned}
\Big|\int_0^t\int_0^1\frac{\theta u v_x}{v^2}dxdt\Big|&\le\frac{1}{2}\int_0^t\int_0^1\frac{\theta v_x^2}{v^3}dxdt+\frac{1}{2}\int_0^t\int_0^1\frac{u^2\theta}{v}dxdt\\
&\le\frac{1}{2}\int_0^t\int_0^1\frac{\theta v_x^2}{v^3}dxdt+C\int_0^t\max_{x\in[0,1]}\theta dt\\
&\le\frac{1}{2}\int_0^t\int_0^1\frac{\theta v_x^2}{v^3}dxdt+C.
\end{aligned}\tag{2.39}$$

Next, for the third term on the righthand side of (2.38), by using (2.3) and (2.16) we obtain

$$\begin{aligned}
\Big|\int_0^t\int_0^1\frac{\theta_x}{v}\Big(u-\frac{v_x}{v}\Big)dxdt\Big|&\le\int_0^t\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^2}dxdt+\frac{1}{2}\int_0^t\int_0^1\frac{\theta^2}{v\theta^\beta}\Big(u-\frac{v_x}{v}\Big)^2dxdt\\
&\le C+C\int_0^t\max_{x\in[0,1]}\theta^2\int_0^1\Big(u-\frac{v_x}{v}\Big)^2dxdt.
\end{aligned}\tag{2.40}$$

Moreover, it follows from (2.3), (2.16), (2.23), (2.28) and (2.29) that, for any $\varepsilon>0$,

$$\begin{aligned}
\int_0^T\max_{x\in[0,1]}\theta^2dt&\le C\int_0^T\max_{x\in[0,1]}\Big|\theta^2-\int_0^1\theta^2dx\Big|dt+C\int_0^T\max_{x\in[0,1]}\theta dt\\
&\le C+\int_0^T\int_0^1\theta|\theta_x|dxdt\\
&\le C+C\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^{\alpha+1}}dxdt+\varepsilon\int_0^T\int_0^1 v\theta^{3+\alpha-\beta}dxdt\\
&\le C+C\varepsilon\int_0^1\max_{x\in[0,1]}\theta^2dxdt,
\end{aligned}\tag{2.41}$$

which follows that

$$\int_0^T\max_{x\in[0,1]}\theta^2dt\le C. \tag{2.42}$$

Moreover, integrating $(1.20)_5$ over $[0,1]\times[0,T]$, combining with (2.22), (2.35), (2.42), we have

$$\begin{aligned}\int_0^T\int_0^1 \frac{u_x^2+(v\mu)^2}{v}dxdt &= \int_0^1 \theta dx-\int_0^1 \theta_0 dx+\int_0^T\int_0^1 \frac{\theta}{v}u_x dx\\ &\le C+\frac{1}{2}\int_0^T\int_0^1 \frac{u_x^2}{v}dxdt, \end{aligned}\tag{2.43}$$

which implies that

$$\int_0^T\int_0^1 \big(u_x^2+(v\mu)^2\big)dxdt\le C,\tag{2.44}$$

Finally, for the fourth term on the righthand side of (2.38), by using (2.16), (2.32), (2.35), we get

$$\begin{aligned}&\Big|\int_0^t\int_0^1 \epsilon\frac{\phi_x}{v}(\frac{\phi_x}{v})_x\Big(u-\frac{v_x}{v}\Big)dxdt\Big|\\ &\le C\int_0^t\int_0^1 \Big(|(\frac{\phi_x}{v})_x|^2+|\frac{\phi_x}{v}|^2\big(u-\frac{v_x}{v}\big)^2\Big)dxdt\\ &\le C+C\int_0^t \max_{x\in[0,1]}|\frac{\phi_x}{v}|^2\int_0^1 \big(u-\frac{v_x}{v}\big)^2dxdt \end{aligned}\tag{2.45}$$

Substituting (2.39), (2.40), (2.45) into (2.38), combining with (2.33), (2.42) and Gronwall's inequality, we have

$$\sup_{0\le t\le T}\int_0^1 \Big(u-\frac{v_x}{v}\Big)^2dx+\int_0^T\int_0^1 \frac{\theta v_x^2}{v^3}dxdt\le C,\tag{2.46}$$

together with (2.3), (2.16) and (2.23), we derive that

$$\sup_{0\le t\le T}\int_0^1 v_x^2dx\le C.$$

Secondly, we rewrite $(1.20)_{3,4}$ as follows

$$\phi_t-\epsilon\phi_{xx}=-\epsilon\frac{\phi_x v_x}{v}-\frac{v}{\epsilon}(\phi^3-\phi),\tag{2.47}$$

multiplying (2.47) by ϕ_{xx} and integrating the resultant over $[0,1]$, by using (2.16), (2.23), (2.33), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \phi_x^2dx+\epsilon\int_0^1 \phi_{xx}^2dx\\ &=\epsilon\int_0^1 \frac{\phi_x v_x}{v}\phi_{xx}dx+\frac{1}{\epsilon}\int_0^1 v(\phi^3-\phi)\phi_{xx}dx\\ &\le C\Big(\int_0^1 \phi_x^2v_x^2dx+\int_0^1 (\phi^3-\phi)^2dx\Big)+\frac{\epsilon}{2}\int_0^1 \phi_{xx}^2dx\\ &\le C\Big(\max_{x\in[0,1]}\phi_x^2(x,t)\int_0^1 v_x^2dx+\int_0^1 \phi^2dx\Big)+\frac{\epsilon}{2}\int_0^1 \phi_{xx}^2dx\\ &\le C\big(\max_{x\in[0,1]}\theta(x,t)+1+V(t)\big)+\frac{\epsilon}{2}\int_0^1 \phi_{xx}^2dx, \end{aligned}\tag{2.48}$$

combining with (2.3) and (2.28), we have

$$\int_0^T\int_0^1 \phi_{xx}^2 dx \leq C. \tag{2.49}$$

Finally, for ϕ_t, from (2.47), we get

$$\phi_t = \epsilon\phi_{xx} - \epsilon\frac{\phi_x v_x}{v} - \frac{v}{\epsilon}(\phi^3-\phi), \tag{2.50}$$

integrating (2.50) over $[0,1]$, we arrive

$$\begin{aligned}\int_0^1 \phi_t^2 dx &\leq C\Big(\int_0^1 \phi_{xx}^2 dx + \int_0^1 \phi_x^2 v_x^2 dx + \int_0^1 (\phi^3-\phi)^2 dx\Big)\\ &\leq C\Big(\int_0^1 \phi_{xx}^2 dx + \int_0^1 v_x^2 dx \int_0^1 \phi_{xx}^2 dx + 1\Big)\\ &\leq C\Big(\int_0^1 \phi_{xx}^2 dx + 1\Big),\end{aligned} \tag{2.51}$$

then, by using (2.49), we arrive

$$\int_0^T\int_0^1 \phi_t^2 dx \leq C. \tag{2.52}$$

The proof of Lemma 2.6 is finished. □

In order to get the a priori estimates on u, we present some prior estimates for higher derivatives of ϕ.

Lemma 2.7. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall(x,t)\in[0,1]\times[0,T]$*

$$\sup_{0\leq t\leq T}\int_0^1 \phi_{xx}^2 dx + \int_0^T\int_0^1 \Big(\phi_{xt}^2 + \big(\frac{\phi_x}{v}\big)_{xx}^2\Big) dx \leq C. \tag{2.53}$$

Proof. We rewrite (2.47) as

$$\frac{\phi_t}{v} - \epsilon\Big(\frac{\phi_x}{v}\Big)_x = -\frac{1}{\epsilon}(\phi^3-\phi), \tag{2.54}$$

then, differentiating (2.54) with respect to x, we have

$$\Big(\frac{\phi_x}{v}\Big)_t - \epsilon\Big(\frac{\phi_x}{v}\Big)_{xx} = -\frac{1}{\epsilon}(\phi^3-\phi)_x + \frac{\phi_t v_x}{v^2} - \frac{\phi_x u_x}{v^2}, \tag{2.55}$$

multiplying (2.55) by $\big(\frac{\phi_x}{v}\big)_t$ and integrating the resultant over $[0,1]$, by using (2.3), (2.16), (2.23), (2.36), (2.44) and (2.51), we obtain

$$\begin{aligned}&\int_0^1 \Big(\frac{\phi_x}{v}\Big)_t^2 dx + \frac{\epsilon}{2}\frac{d}{dt}\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2 dx\\ &= -\frac{1}{\epsilon}\int_0^1 (\phi^3-\phi)_x\big(\frac{\phi_x}{v}\big)_t dx + \int_0^1 \frac{\phi_t v_x}{v^2}\big(\frac{\phi_x}{v}\big)_t dx - \int_0^1 \frac{\phi_x u_x}{v^2}\big(\frac{\phi_x}{v}\big)_t dx\\ &\leq C\Big(\int_0^1 (3\phi^2-1)^2\phi_x^2 dx + \int_0^1 \phi_t^2 v_x^2 dx + \int_0^1 \phi_x^2 u_x^2 dx\Big) + \frac{1}{3}\int_0^1 \big(\frac{\phi_x}{v}\big)_t^2 dx\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(1+\|\phi_t\|_{L^\infty}^2\int_0^1 v_x^2dx+\|\frac{\phi_x}{v}\|_{L^\infty}^2\int_0^1 u_x^2dx\Big)+\frac{1}{3}\int_0^1 (\frac{\phi_x}{v})_t^2dx\\
&\le C\Big(1+\int_0^1 (\phi_t^2+2|\phi_t\phi_{xt}|)dx+\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2dx\int_0^1 u_x^2dx\Big)+\frac{1}{3}\int_0^1 (\frac{\phi_x}{v})_t^2dx\\
&\le C\Big(1+\int_0^1 \phi_{xx}^2dx+\int_0^1 \phi_{xt}^2dx+\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2dx\int_0^1 u_x^2dx\Big)+\frac{1}{3}\int_0^1 (\frac{\phi_x}{v})_t^2dx\\
&\le C\Big(1+\int_0^1 \phi_{xx}^2dx+\int_0^1 u_x^2dx\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2dx\Big)+\frac{1}{2}\int_0^1 (\frac{\phi_x}{v})_t^2dx, \qquad (2.56)
\end{aligned}
$$

where in the last inequality we have used $\phi_{xt}=\big((\frac{\phi_x}{v})_t+\frac{\phi_x u_x}{v^2}\big)v$. Thus, by Gronwall's inequality, we have

$$
\sup_{t\in[0,T]}\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2dx+\int_0^T\int_0^1 \Big(\frac{\phi_x}{v}\Big)_t^2dx\le C, \qquad (2.57)
$$

combining with (2.3), (2.36), we get

$$
\sup_{0\le t\le T}\int_0^1 \phi_{xx}^2dx+\int_0^T\int_0^1 \phi_{xt}^2dx\le C. \qquad (2.58)
$$

Moreover, by using the Sobolev embedding theorem for one dimension, (2.57) also implies

$$
\max_{(x,t)\in[0,1]\times[0,T]}\Big|\frac{\phi_x}{v}\Big|^2\le C+C\sup_{t\in[0,T]}\int_0^1 \Big(\frac{\phi_x}{v}\Big)_x^2dx\le C. \qquad (2.59)
$$

Further, by using (2.55) and the estimates obtained above, we achieve

$$
\int_0^T\int_0^1 (\frac{\phi_x}{v})_{xx}^2dx\le C, \qquad (2.60)
$$

the proof of Lemma 2.7 is completed. □

After the above preparation work, now we can give a prior estimate of the velocity u.

Lemma 2.8. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall(x,t)\in[0,1]\times[0,T]$*

$$
\sup_{0\le t\le T}\int_0^1 u_x^2dx+\int_0^T\int_0^1 (u_t^2+u_{xx}^2)dx\le C. \qquad (2.61)
$$

Proof. Multiplying $(1.20)_2$ by u_{xx} and integrating the resultant over $(0,1)\times(0,T)$, by using (2.16), (2.23), (2.36), (2.42), (2.57), (2.59), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 u_x^2dx+\int_0^T\int_0^1 \frac{u_{xx}^2}{v}dxdt\\
&\le C+\frac{1}{2}\int_0^T\int_0^1 \frac{u_{xx}^2}{v}dxdt+C\int_0^T\int_0^1 \Big(\theta_x^2+\theta^2v_x^2+\Big|\frac{\phi_x}{v}\Big|^2\Big|\Big(\frac{\phi_x}{v}\Big)_x\Big|^2+u_x^2v_x^2\Big)dxdt\\
&\le C+\frac{1}{2}\int_0^T\int_0^1 \frac{u_{xx}^2}{v}dxdt+C\int_0^T\int_0^1 \theta_x^2dx+C\int_0^T\max_{x\in[0,1]}\theta^2\int_0^1 v_x^2dxdt\\
&\quad+C\max_{(x,t)\in[0,1]\times[0,T]}\Big|\frac{\phi_x}{v}\Big|^2\int_0^T\int_0^1 \Big|\Big(\frac{\phi_x}{v}\Big)_x\Big|^2dxdt+C\int_0^T\max_{x\in[0,1]}u_x^2\int_0^1 v_x^2dxdt
\end{aligned}
$$

$$\leq C+\frac{3}{4}\int_0^T\int_0^1\frac{u_{xx}^2}{v}dxdt+C_1\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v}dxdt, \tag{2.62}$$

where we have used the following and the Sobolev embedding inequality in one dimension

$$\begin{aligned}\int_0^T\max_{x\in[0,1]}u_x^2dt &\leq C(\delta)\int_0^T\int_0^1 u_x^2dxdt+\delta\int_0^T\int_0^1\frac{u_{xx}^2}{v}dxdt\\ &\leq C(\delta)+\delta\int_0^T\int_0^1\frac{u_{xx}^2}{v}dxdt.\end{aligned} \tag{2.63}$$

Multiplying $(1.20)_5$ by θ, integrating the resultant over $(0,1)\times(0,T)$, by using (2.42), (2.44) and (2.63), we have

$$\begin{aligned}&\frac{1}{2}\int_0^1\theta^2dx+\int_0^T\int_0^1\frac{\theta^\beta\theta_x^2}{v}dxdt\\ &\leq C+C\int_0^T\int_0^1\theta^2|u_x|dx+C\int_0^T\int_0^1(u_x^2+\mu^2)\theta dxdt\\ &\leq C+C\int_0^T\int_0^1\theta u_x^2dxdt++C\int_0^T\int_0^1\theta^3dxdt+\int_0^T\max_{x\in[0,1]}\theta dt\\ &\leq C+C\int_0^T\max_{x\in[0,1]}u_x^2dt+C\int_0^T\max_{x\in[0,1]}\theta^2dt\\ &\leq C(\delta)+C\delta\int_0^T\int_0^1\frac{u_{xx}^2}{v}dxdt.\end{aligned} \tag{2.64}$$

From (2.62) and (2.64), choosing δ small enough, we obtain

$$\sup_{t\in[0,T]}\int_0^1(\theta^2+u_x^2)dx+\int_0^T\int_0^1\theta^\beta\theta_xdxdt+\int_0^T\int_0^1u_{xx}^2dxdt\leq C. \tag{2.65}$$

Then, rewriting $(1.20)_2$ as

$$u_t=-\big(\frac{\theta}{v}\big)_x+\frac{u_{xx}}{v}-\frac{u_xv_x}{v^2}-\epsilon\frac{\phi_x}{v}\big(\frac{\phi_x}{v}\big)_x, \tag{2.66}$$

combining with (2.36), (2.42), (2.59), (2.63), (2.53) and (2.65), we achieve

$$\begin{aligned}\int_0^T\int_0^1u_t^2dxdt &\leq C\int_0^T\int_0^1\Big(u_{xx}^2+u_x^2v_x^2+\theta_x^2+\theta^2v_x^2+\big|\frac{\phi_x}{v}\big|^2\big|\big(\frac{\phi_x}{v}\big)_x\big|^2\Big)dxdt\\ &\leq C,\end{aligned} \tag{2.67}$$

together with (2.65), the energy inequality (2.61) is obtained. The proof of Lemma 2.8 is completed. □

Lemma 2.9. *Let (v,u,θ,ϕ) be a smooth solution of (1.20)-(1.22) on $[0,1]\times[0,T]$. Then it holds that for $\forall(x,t)\in[0,1]\times[0,T]$*

$$\sup_{0\leq t\leq T}\int_0^1\theta_x^2dx+\int_0^T\int_0^1(\theta_t^2+\theta_{xx}^2)dx\leq C. \tag{2.68}$$

Proof. Multiplying $(1.20)_5$ by $\theta^\beta\theta_t$ and integrating the resultant over $(0,1)$, by using (2.16), (2.23), (2.65), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 \frac{(\theta^\beta\theta_x)^2}{v}dx\Big) + \int_0^1 \theta^\beta\theta_t^2 dx \\
&= -\frac{1}{2}\int_0^1 \frac{(\theta^\beta\theta_x)^2 u_x}{v^2}dx + \int_0^1 \frac{\theta^\beta\theta_t\big(-\theta u_x + u_x^2 + v^2\mu^2\big)}{v}dx \\
&\le C\max_{x\in[0,1]}|u_x|\theta^{\frac{\beta}{2}}\int_0^1 \theta^{\frac{3\beta}{2}}\theta_x^2 dx + \frac{1}{2}\int_0^1 \theta^\beta\theta_t^2 dx + C\int_0^1 \theta^{\beta+2}u_x^2 dx + C\int_0^1 \theta^\beta\big(u_x^4+\mu^4\big)dx \\
&\le C\int_0^1 \theta^\beta\theta_x^2 dx\int_0^1 (\theta^\beta\theta_x)^2 dx + \frac{1}{2}\int_0^1 \theta^\beta\theta_t^2 dx + C\max_{x\in[0,1]}\big(\theta^{2\beta+2}+u_x^4+\mu^4\big) + C. \qquad (2.69)
\end{aligned}$$

Now we deal with the term $\max_{x\in[0,1]}\big(\theta^{2\beta+2}+u_x^4+\mu^4\big)$ in the last inequality of (2.69). Combining Lemma 2.8, direct computation shows that

$$\begin{aligned}
&\int_0^T \max_{x\in[0,1]} u_x^4 dt \\
\le\ & C\int_0^T\int_0^1 u_x^4 dxdt + C\int_0^T\int_0^1 |u_x^3 u_{xx}|dxdt \\
\le\ & C\int_0^T \max_{x\in[0,1]} u_x^2\int_0^1 u_x^2 dxdt + C\int_0^T \max_{x\in[0,1]} u_x^2\Big(\int_0^1 u_x^2 dx\Big)^{\frac{1}{2}}\Big(\int_0^1 u_{xx}^2 dx\Big)^{\frac{1}{2}}dt \\
\le\ & \frac{1}{2}\int_0^T \max_{x\in[0,1]} u_x^4 dt + C\int_0^T\int_0^1 (u_x^2+u_{xx}^2)dxdt \\
\le\ & \frac{1}{2}\int_0^T \max_{x\in[0,1]} u_x^4 dt + C, \qquad (2.70)
\end{aligned}$$

so we get immediately that

$$\int_0^T \max_{x\in[0,1]} u_x^4 dt \le C. \qquad (2.71)$$

By the same way above, combining with (2.53), we have

$$\int_0^T \max_{x\in[0,1]} \mu^4 dt \le C. \qquad (2.72)$$

Moreover, by using Sobolev embedding theorem, we get

$$\max_{x\in[0,1]}\theta^{2\beta+2} \le C + C\int_0^1 (\theta^\beta\theta_x)^2 dx, \qquad (2.73)$$

Substituting (2.71), (2.72), (2.73) into (2.69), by using Gronwall's inequality and (2.5), we obtain

$$\sup_{0\le t\le T}\int_0^1 (\theta^\beta\theta_x)^2 dx + \int_0^T\int_0^1 \theta^\beta\theta_t^2 dxdt \le C. \qquad (2.74)$$

Therefore, in view of (2.73), we achieve

$$\max_{(x,t)\in[0,1]\times[0,T]}\theta \le C. \qquad (2.75)$$

Thus, both (2.73) and (2.74) lead to

$$\sup_{0\le t\le T}\int_0^1 \theta_x^2 dx + \int_0^T\int_0^1 \theta_t^2 dxdt \le C. \tag{2.76}$$

Let's go back to (1.20) again, we have

$$\frac{\theta^\beta\theta_{xx}}{v} = \theta_t - \frac{\beta\theta^{\beta-1}\theta_x^2}{v} + \frac{\theta^\beta\theta_x v_x}{v^2} + \frac{\theta u_x}{v} - \frac{u_x^2+(v\mu)^2}{v}, \tag{2.77}$$

which yields that

$$\begin{aligned}\int_0^T\int_0^1 \theta_{xx}^2 dxdt &\le C\int_0^T\int_0^1 \left(\theta_x^4+\theta_x^2v_x^2+u_x^4+\mu^4+u_x^2+\theta_t^2\right)dxdt\\ &\le C + C\int_0^T \max_{x\in[0,1]}\theta_x^2 dt\\ &\le C + C\int_0^T\int_0^1 \theta_{xx}^2 dxdt. \end{aligned}\tag{2.78}$$

Meanwhile, by using maximum principle, $-1\le\phi\le 1$. The proof of Lemma 2.9 is completed. □

Proof of Theorem 1.1. By Lemma 2.1, there exists a $T_*>0$, such that the system (1.20)-(1.22) has a unique strong solution (v,u,θ,ϕ) on $(0,T_*]$ satisfying (2.2). Suppose that T_0 is the maximum existence time of the unique strong solution (v,u,θ,ϕ) to (1.20)-(1.22). Therefore, $T_0\ge T_*>0$. We claim that

$$T_0=+\infty. \tag{2.79}$$

If not , $T_0<+\infty$, then by using Lemma 2.2–2.9, the global a priori estimates of the solutions ensure $(v,u,\theta,\phi)\in C([0,T_0];H^1)$ and that

$$\|(v,u,\theta,\phi)(t)\|_{H^1}\le C(T_0)<+\infty,\quad \forall t\in[0,T_0], \tag{2.80}$$

where $C(T_0)$ is a positive constant depending only on T_0, $\inf\limits_{x\in[0,1]} v_0(x)$, $\inf\limits_{x\in[0,1]}\theta_0(x)$, and $\|(v_0,u_0,\theta_0,\phi_0)\|_{H^1(0,1)}$. Thus, $(v,u,\theta,\phi)(x,T_0)$ is finite and well-defined. Thus, follows from Lemma 2.1, there exists a positive constant $T_1>0$ such that (1.20)-(1.22) has a unique strong solution on $[T_0,T_0+T_1]$, which contradicts the definition of T_0. Thereby, the claim (2.79) is true. The proof of Theorem 1.1 is completed.

Acknowledgement

This research was partially supported by the National Natural Science Foundation of China, No. 11901025 (the first author), No. 11971020(the second author) and No. 11671027(the fourth author).

References

[1] S.M. Allen, J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta Metallurgica*, **27**, 1085-1095, (1979).

[2] A.A. Amosov, A.A. Zlotnik, Global generalized solutions of the equations of the one-dimensional motion of a viscous heat-conducting gas, *Soviet Math. Dokl.*, **38**, 1-5, (1989).

[3] A.A. Amosov, A.A. Zlotnik, Solvability "in the large" of a system of equations of the one-dimensional motion of an inhomogeneous viscous heat-conducting gas. *Math. Notes*, **52**, 753-763, (1992).

[4] A.A. Amosov, A.A. Zlotnik, On the stability of generalized solutions of equations of one-dimensional motion of a viscous heat-conducting gas, *Sib. Math. J.*, **38**, 663-684,(1997).
[5] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids. Amsterdam, New York: North-Holland, 1990.
[6] T. Blesgen, A generalization of the Navier-Stokes equations to two-phase flows, *J. Physics D*, **32**, 1119-1123, (1999).
[7] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.*, **28**, 258-267, (1958).
[8] M. Chen and X. Guo, Global large solutions for a coupled compressible Navier-Stokes/Allen-Cahn system with initial vacuum, *Nonl. Analysis: Real World Appl.*, **37**, 350-373, (2017).
[9] S. Chen, H. Wen, C. Zhu, Golbal existence of weak solution to compressible Navier-Stokes/Allen-Cahn system in three dimensions. *J.Math. Anal. Appl.* **477**, 1265-1295,(2019).
[10] Y. Chen, Q. He, M. Mei, and X. Shi, Asymptotic stability of solutions for 1-D compressible Navier-Stokes-Cahn-Hilliard system, *J. Math. Anal. Appl.*, **467**, 185-206, (2018)
[11] S. Ding, Y. Li, and W. Luo, Global solutions for a coupled compressible Navier-Stokes/Allen-Cahn system in 1D, *J. Math. Fluid Mech.*, **15**, 335-360, (2013).
[12] R. Duan, A. Guo, C. Zhu, Global strong solution to compressible Navier-Stokes equations with density dependent viscosity and temperature dependent heat conductivity. *J. Differential Equations*, **262**, 4314-4335, (2017).
[13] E. Feireisl, H. Petzeltová, E. Rocca, G. Schimperna, Analysis of a phase-field model for two-phase compressible fluids, *Mathematical Models and Methods in Applied Sciences*, **20**(7), 1129-1160, (2010).
[14] Q. He, C. Liu, and X. Shi, Numerical study of phase transition in van der Waals fluid, *Discret. Contin. Dyn. Syst.(B)*, **23**, 4519-4540, (2018).
[15] M. Heida, J. Malek, and K.R. Rajagopal,On the development and generalizations of Allen-Cahn and Stefan equations within a thermodynamic framework, *Z. Angew. Math. Phys.*, **63**, 759-776, (2012).
[16] B. Huang, X. Shi, Y. Sun, Global strong solutions to compressible Navier-Stokes system with degenerate heat conductivity and density-depending viscosity, Commun. Math. Sci. preprint, (2020).
[17] B. Huang, X. Shi, Nonlinearly exponential stability of compressible Navier-Stokes system with degenerate heat-conductivity, *J. Differential Equations*, (2019).
[18] F. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, *Arch. Ration. Mech. Anal.*, **197**, 89-116, (2010).
[19] X. Huang, J. Li, and Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Comm. Pure Appl. Math.*, **65**, 549-585, (2012).
[20] B. Kawohl, Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting,one-dimensional real gas, *J. Differential Equations*, **58**,76-103, (1985).
[21] S. Jiang, On initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas, *J. Differential Equations*, **110**, 157-181, (1994), .
[22] S. Jiang, On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas, *Math. Z.*, **216**, 317-336, (1994).
[23] S. Jiang, Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain. *Comm. Math. Phys.*, **178**, 339-374, (1996).
[24] S. Jiang, Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, *Annli Mat. Pura Appl.*, **175**, 253-275, (1998).
[25] A.V. Kazhikhov, To a theory of boundary value problems for equations of one-dimensional nonstationary motion of viscous heat-conduction gases, in: Boundary Value Problems for Hydrodynamical Equations, No. 50, Institute of Hydrodynamics, Siberian Branch Acad. USSR, 1981, pp. 37-62, in Russian.
[26] A.V. Kazhikhov, V.V. Shelukhin, Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.*, **41**, 273-82, (1977).
[27] S. Kawashima, Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications, *Proc. R. Soc. Edinb. A*, **106**, 169-194, (1987).

[28] M. Kotschote, Strong solutions of the Navier-Stokes Equations for a Compressible Fluid of Allen-Cahn Type, *Arch. Rational Mech. Anal.*, **206**, 489-514, (2012).
[29] P.L. Lions, Mathematical topics in fluid mechanics, Vol. 2. Compressible models, Oxford University Press, New York, 1998.
[30] J. Lowengrub, and L. Truskinovsky, Quasi-incompressible Cahn-Hilliard fluids and topological transitions, *Proc. Royal Soc. A: Math. Phys. Eng. Sci.*, **454**, 2617-2654, (1998).
[31] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, **20**, 67-104, (1980).
[32] R. Pan, W. Zhang, Compressible Navier-Stokes equations with temperature dependent heat conductivities, *Commun. Math. Sci.*, **13**, 401-425, (2015).
[33] P. Poláčik, Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^n$: I. Asymptotic symmetry for the Cauchy problem,*Commun. P.D.E.*, **30**, 1567-1593, (2005).
[34] J.D. Van der Waals, Therinodynamische theorie der kapillaritat unter voraussetzung stetiger dichteanderung, *Z. Phys. Chem.*, **13**, 657-725, (1894). (English transl. in *J. Stat. Phys.*, **20**, 197-200, (1979).)
[35] D.H. Wang, Global solutions of the Navier-Stokes equations for viscous compressible flows, *Nonlinear Anal.* , **52**, 1867-1890, (2003).

College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China
E-mail address: chenyz@mail.buct.edu.cn

School of Mathematics, Sichuan University, Sichuan 610065, China
E-mail address: qlhejenny@scu.edu.cn

College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China
E-mail address: abinhuang@gmail.com

College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing 100029, China
E-mail address: shixd@mail.buct.edu.cn