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Higher order stable schemes for stochastic convection-reaction-diffusion equations driven by additive Wiener noise

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Abstract

In this paper, we investigate the numerical approximation of stochastic convection-reaction-diffusion equations using two explicit exponential integrators. The stochastic partial differential equation (SPDE) is driven by additive Wiener process. The approximation in space is done via a combination of the standard finite element method and the Galerkin projection method. Using the linear functional of the noise, we construct two accelerated numerical methods, which achieve higher convergence orders. In particular, we achieve convergence rates approximately 1 for trace class noise and $\frac{1}{2}$ for space-time white noise. These convergences orders are obtained under less regularities assumptions on the nonlinear drift function than those used in the literature for stochastic reaction-diffusion equations. Numerical experiments to illustrate our theoretical results are provided.

KEYWORDS:
Stochastic convection-reaction-diffusion equations, Strong convergence, Additive noise, Finite element method, Galerkin projection method, Exponential integrators.

1 | INTRODUCTION

This article is devoted to the space-time approximation of the following seimlinear parabolic SPDEs

$$dX(t) = [AX(t) + f(x, X(t))]dt + dW(t), \quad t \in (0, T], \quad x \in \Lambda \tag{1}$$

with initial value $X(0) = X_0$ and Dirichlet boundary conditions or Robin boundary conditions. In (1), Λ is a bounded domain of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with smooth boundary or is a convex polygon. $T > 0$ is a fixed final time. The linear operator A is given by

$$A = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(q_{ij}(x)\frac{\partial}{\partial x_j}\right) - \sum_{j=1}^{d} q_j(x)\frac{\partial}{\partial x_j}, \tag{2}$$

where $q_{ij}, q_j \in L^\infty(\Lambda)$ and $q_{i,j}$ satisfies the following ellipticity condition

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad x \in \overline{\Lambda}, \tag{3}$$

where $c > 0$ is a uniform constant. Precise conditions on the nonlinear f will be given in the next section. Let us introduce the Nemytskij operator $F : L^2(\Lambda) \longrightarrow L^2(\Lambda)$ associated to f in (1), defined by

$$F(u)(x) = f(x, u(x)), \quad x \in \Lambda, \quad u \in L^2(\Lambda). \tag{4}$$

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{F}$ that fulfills the usual conditions, see e.g.,[16, Definition 2.1.11]. The noise term $W(t)$ in (1) is assumed to be a Q-Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, with covariance operator $Q : L^2(\Lambda) \longrightarrow L^2(\Lambda)$, which is assumed to be linear, self-adjoint and positive definite. It is well known (see e.g.,[16]) that the noise $W(t)$ can be represented as follows

$$W(t, x) = \sum_{i=0}^{\infty} \sqrt{q_i} e_i(x) \beta_i(t), \tag{5}$$

where $(q_i, e_i)_{i\in\mathbb{N}}$ are the eigenvalues and eigenfunctions of the covariance operator Q, and $(\beta_i)_{i\in\mathbb{N}}$ are independent and identically distributed standard Brownian motions. It is well known that the linear operator A generates an analytic semi-group $S(t) =: e^{At}$, $t \geq 0$; see e.g.[1,16,15,2]. Under the hypothesis that F is Lipschitz continuous and $X_0 \in L^2(\Omega, L^2(\Lambda))$, the SPDE (1) has up to modifications a unique mild solution $X : [0, T] \times \Omega \longrightarrow \mathcal{H}$, which takes the following form, see e.g.[16,15]

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dW(s), \quad \mathbb{P} - \text{a.s.} \tag{6}$$

SPDEs of type (1) are used to model many real world phenomena such as convection-reaction-diffusion processes. Since explicit solutions of many SPDEs are usually unknown, numerical approaches are good alternatives to provide their realistic approximations. Having a numerical approximation in hand one main question is whether it converges toward the mild solution or not. Another interesting information is to know the rate with which it converges to the true solution. There are mainly two types of convergence: namely strong convergence and weak convergence. Our interest here is on strong convergence. There are numerous numerical methods designed to approximate (1) with linear self-adjoint operator, see e.g.[6,7,8,9]. To classify a numerical method, one also takes in consideration the rate of convergence. In[7], an exponential Euler scheme achieving higher convergence order by exploiting the linear functional of the noise was introduced for semilinear SPDEs driven by space-time white noise. However, assumptions made in[7] to achieve higher convergence order are too restrictive and exclude many nonlinear operators such as $F(v) = \frac{1-v}{1+v^2}$, $v \in \mathcal{H} := L^2(\Lambda)$, see the introduction of[8] for more details. In[8], a modified version of the above mentioned exponential Euler scheme which achieves higher convergence order under more relaxed conditions on F was introduced. In[18], an accelerated exponential integrator was investigated and proved to achieve convergence order 1 for trace class noise under more relaxed assumptions than in[8]. Note that the works in[8,7,18] are only for stochastic diffusion-reaction equations and heavily use the fact that the linear operator A is self-adjoint. Also, such schemes are numerically implementable only if the linear operator is self-adjoint. Here, we are interested in the case of stochastic convection-reaction-diffusion equations, which are more realistic and plays a key role in subsurface processes. For such SPDEs, we are interested on building alternative stable numerical schemes achieving higher convergence order. Recently in[3,4] some exponential integrators and implicit schemes achieving convergence order 1 in time were introduced for stochastic convection-reaction-diffusion equations. The idea behind such schemes, consists on keeping the convection term in the nonlinear part F. As a consequence, these schemes may lose their stability if the velocity field is very high. Moreover, the convergence results in[3,4] exclude the space-time white noise case. Here, we propose novel numerical schemes for stochastic convection-diffusion-reaction equations with general noise (including space-time white noise), which achieve higher convergence order. The idea behind our novel numerical schemes consists on splitting the semi-group appearing in the noise component in two semi-groups, one semi group generated by the self adjoint part and another semi group generated by the advection part (see Section 2 for details). For the convergence proofs of our numerical schemes toward the mild solution, one key argument consists of using an argument based on Miyadera-Voigt perturbation Theorem[1, Chapter III, Corollary 3.16]. In addition, we use relaxed conditions on the nonlinear function F than the ones used in[3,4] and in the literature for stochastic reaction diffusion equations.

The rest of this paper is structured as follows. In Section 2, the well posedness and the numerical schemes are introduced. In Section 3 we prove the strong convergence of our fully discrete schemes toward the mild solution. In Section 4 we provide some numerical experiments to illustrate our results.

2 | MATHEMATICAL SETTINGS AND NUMERICAL SCHEMES

2.1 | Notations and main assumptions

We denote by $\langle .,. \rangle$ the inner product in the Hilbert space $\mathcal{H} = L^2(\Lambda, \mathbb{R}) =: L^2(\Lambda)$. We denote by $\|.\|$ the norm associated to the inner product $\langle .,. \rangle$. For all $p \geq 2$, $L^p(\Omega, \mathcal{H})$ stands for the Banach space of all equivalence classes of p integrable $\mathcal{H}$-valued random variables. Let $\mathcal{L}(\mathcal{H})$ be the space of bounded linear mappings on $\mathcal{H}$ endowed with the usual operator norm $\|.\|_{\mathcal{L}(\mathcal{H})}$. The space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $\mathcal{L}_2(\mathcal{H}) := HS(\mathcal{H})$ and is equipped with the norm $\|l\|^2_{\mathcal{L}_2(\mathcal{H})} := \sum_{i=1}^{\infty} \|le_i\|^2$, $l \in \mathcal{L}_2(\mathcal{H})$, where $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Let $\mathcal{L}_2^0$ be the space of Hilbert Schmidt operator from $Q^{\frac{1}{2}}(\mathcal{H})$ to $\mathcal{H}$. For an $\mathcal{L}_2^0$-valued predictable stochastic process $\phi : [0,T] \times \Omega \longrightarrow \mathcal{L}_2^0$ such that

$$\int_0^t \mathbb{E}\|\phi(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})} ds < \infty, \quad t \in [0,T],$$

the following relation called Itô isometry holds [16,15]

$$\mathbb{E}\|\int_0^t \phi(s)dW(s)\|^2 = \int_0^t \mathbb{E}\|\phi(s)\|^2_{\mathcal{L}_2^0} ds = \int_0^t \mathbb{E}\|\phi(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})} ds, \quad t \in [0,T], \tag{7}$$

For the seek of the convergence analysis of our numerical schemes, we make the following assumptions.

Assumption 2.1. The initial data $X_0 : \Omega \longrightarrow \mathcal{H}$ is assumed to be measurable and $X_0 \in L^2\left(\Omega, \mathcal{D}\left((-A)^{\frac{\beta}{2}}\right)\right)$, $0 \leq \beta \leq 2$.

We assume the covariance operator Q to satisfy the following estimate

$$\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})} < \infty. \tag{8}$$

The nonlinear function F is assumed to be differentiable and there exist $C \geq 0$ and $\eta \in \left(\frac{3}{4}, 1\right)$ such that

$$\|F'(u)v\| \leq C\|v\|, \quad \left\|(-A)^{-\eta}\left(F'(u) - F'(v)\right)\right\|_{\mathcal{L}(\mathcal{H})} \leq C\|u-v\|, \quad u, v \in \mathcal{H}.$$

As a consequence, F satisfies the following Lipschitz condition

$$\|F(u) - F(v)\| \leq C\|u-v\|, \quad \|F(u)\| \leq C(1+\|u\|), \quad u, v \in \mathcal{H}.$$

Remark 1. Let A_1 and A_2 be respectively the self-adjoint and the non self-adjoint parts of A. Using the equivalence of norms (see e.g.[11,2] or[12, (3.3)]), $\|(-A)^{\gamma}v\| \approx \|(-A_1)^{\gamma}v\|$ for all $\gamma \in [-1,1]$ and $v \in \mathcal{D}((-A)^{\gamma})$, it follows that (8) remains true if A is replaced by A_1. The following equivalence of norms holds

$$\|(-A)^{\frac{\gamma}{2}}v\| \approx \|(-A_2)^{\gamma}v\|, \quad \gamma \in [-1,1], \quad v \in \mathcal{D}((-A)^{\frac{\gamma}{2}}).$$

Proposition 1. Under the hypothesis that the function f in (1) is differentiable and there exists a constant $C \geq 0$ such that

$$|f'(z,x) - f'(z,y)| \leq C|x-y|, \quad x, y \in \mathbb{R}, \quad z \in \Lambda, \tag{9}$$

the Nemystkii operator F satisfies the desired properties in Assumption 2.1. Note the the derivation in (9) is respect to the second variable.

Proof. We only prove that $\left\|(-A)^{-\eta}\left(F'(u) - F'(v)\right)\right\|_{\mathcal{L}(\mathcal{H})} \leq C\|u-v\|$, $u, v \in \mathcal{H}$, since proofs of others estimates are similar. The derivative of F is given by

$$(F'(u)(v))(z) = f'(z, u(y)).v(z), \quad z \in \Lambda, \ u, v \in \mathcal{H}.$$

Hence, for $w \in \mathcal{H}$, from the definition of the norm $\|.\|_{L^1(\Lambda,\mathbb{R})}$, using (9), Hölder inequality and the fact that Λ is bounded, it follows that

$$\begin{aligned}
\|\left(F'(u)-F'(v)\right)w\|_{L^1(\Lambda,\mathbb{R})} &= \int_\Lambda |\left(F'(u)-F'(v)\right)w(x)|dx = \int_\Lambda \left|f'(x,u(x))-f'(x,v(x))\right||w(x)|dx \\
&\leq C\int_\Lambda |u(x)-v(x)||w(x)|dx \leq C\left(\int_\Lambda |u(x)-v(x)|^2dx\right)^{\frac{1}{2}}\left(\int_\Lambda |w(x)|^2dx\right)^{\frac{1}{2}} \\
&= C\|u-v\|\|w\|. \qquad (10)
\end{aligned}$$

Using Hölder's inequality, the Sobolev embedding $\mathcal{D}(A^{-\eta}) \hookrightarrow L^\infty(\Lambda,\mathbb{R})$ for $\eta > \frac{d}{4}$, yields

$$\begin{aligned}
\left\|A^{-\eta}\left(F'(u)-F'(v)\right)w\right\| &= \sup_{\|w_1\|\leq 1}\left|\left\langle A^{-\eta}\left(F'(u)-F'(v)\right)w, w_1\right\rangle\right| = \sup_{\|w_1\|\leq 1}\left|\left\langle \left(F'(u)-F'(v)\right)w, (A^*)^{-\eta}w_1\right\rangle\right| \\
&\leq \left\|\left(F'(u)-F'(v)\right)w\right\|_{L^1(\Lambda,\mathbb{R})}\sup_{\|w_1\|\leq 1}\|(A^*)^{-\eta}w_1\|_{L^\infty(\Lambda,\mathbb{R})} \\
&\leq K_1\left\|\left(F'(u)-F'(v)\right)w\right\|_{L^1(\Lambda,\mathbb{R})}\sup_{\|w_1\|\leq 1}\left\|A^\eta(A^*)^{-\eta}w_1\right\|. \qquad (11)
\end{aligned}$$

Using (10) and [17, Lemma 3.1], it follows from (11) that

$$\left\|A^{-\eta}\left(F'(u)-F'(v)\right)w\right\| \leq C\|u-v\|\|w\|\sup_{\|w_1\|\leq 1}\|w_1\| \leq \|u-v\|\|w\|.$$

Therefore.

$$\left\|(-A)^{-\eta}\left(F'(u)-F'(v)\right)\right\|_{\mathcal{L}(\mathcal{H})} \leq C\|u-v\|, \quad u,v\in\mathcal{H}.$$

□

2.2 | Fully discrete schemes and main results

Let $\mathcal{T}_h$ be a set of disjoint intervals of Λ (for $d=1$), a triangulation of Λ (for $d=2$) or a set of tetrahedra (for $d=3$) satisfying the standard regularity assumptions (see[2]). Let $V_h \subset V$ denotes the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_h$. To discretize in space we introduce two projections. Our first projection operator P_h is the $L^2(\Lambda)$ projection onto V_h defined for $u \in L^2(\Lambda)$ by

$$\langle P_h u, \chi\rangle = \langle u, \chi\rangle, \quad \chi \in V_h.$$

Then $A_h : V_h \to V_h$ is the discrete analogue of A defined by

$$\langle A_h\varphi, \chi\rangle = a(\varphi, \chi), \quad \varphi, \chi \in V_h, \qquad (12)$$

where $a(.,.)$ is the corresponding bilinear form associated to the operator A. We denote by S_h the semigroup generated by A_h. The second projection P_N, $N\in\mathbb{N}$ is the projection onto a finite number of spectral modes e_i[1] defined for $u \in L^2(\Lambda)$ by

$$P_N u = \sum_{i\in\mathcal{I}_N}\langle e_i, u\rangle e_i, \quad \text{where} \quad \mathcal{I}_N = \{1,2,...,N\}^d.$$

The semi–discrete version of the problem (1) consists of finding $X_h(t) = X_h(.,t) \in V_h$ such that for $t \in [0,T]$,

$$dX_h = (A_hX_h + P_hF(X_h))dt + P_hP_NdW, \qquad X_h(0) = P_hX_0. \qquad (13)$$

The mild solution of (13) is given by

$$X_h(t) = S_h(t)X_h(0) + \int_0^t S_h(t-s)F(X_h(s))ds + \int_0^t S_h(t-s)P_hP_NdW. \qquad (14)$$

Let $S_1(t) =: e^{A_1t}$ the semi-group generated by A_1 and $S_2(t) =: e^{A_2t}$ the semi-group generated by A_2. Let A_{1h} and A_{2h} be the semi-discrete versions of A_1 and A_2 respectively, defined as in (12). We denote by $S_{1h}(t)$ and $S_{2h}(t)$ the semi-groups generated

[1] Eigenfunctions of the operator A_1 in our case

A_{1h} and A_{2h} respectively. The semi-groups $S_1(t)$, $S_2(t)$, $S_{1h}(t)$ and $S_{2h}(t)$ satisfy the smoothing properties of[13, Proposition 2.2], see also[11,2]. We introduce the following stochastic convolutions

$$O^m_{h,N} := \int_{t_m}^{t_{m+1}} S_h(t_{m+1}-s)P_hP_N dW(s),\ O_m := \int_{t_m}^{t_{m+1}} S(t_{m+1}-s)dW(s),\ O^{h,N}_k = S_{2h}(\Delta t)P_hP_N\int_{t_k}^{t_{k+1}} S_1(t_{k+1}-s)dW(s). \quad (15)$$

To build our numerical schemes, we use the following approximation of the noise: $O^m_{h,N} \approx O^{h,N}_m$. To build our first numerical scheme, we use the approximation $F(X^h(s)) \approx F(X^h(t_m))$, for $s \in [t_m, t_{m+1})$. This yields the following scheme, called accelerated SETD1 (ASETD1): $X^h_0 = P_hX_0$ and recursively by

$$X^h_m = S_h(\Delta t)P_hX_{m-1} + \int_{t_{m-1}}^{t_m} S_h(t_m-s)P_hF(X^h_{m-1})ds + S_{2h}(\Delta t)P_hP_N\int_{t_{m-1}}^{t_m} S_1(t_m-s)dW(s), \quad m \geq 1. \quad (16)$$

Note that the numerical method (16) can be written in the following form, efficient for simulation

$$X^h_m = X^h_{m-1} + \Delta t\varphi_1(\Delta tA_h)\left(A_hX^h_{m-1} + P_hF(X^h_{m-1})\right) + O^{h,N}_{m-1}, \quad m \geq 1, \quad (17)$$

where the linear operator φ_1 is given by (20). To obtain our second numerical scheme, we use the approximation $e^{(t_m-s)A_h}P_hF(X^h(s)) \approx e^{\Delta tA_h}P_hF(X^h(t_{m-1}))$ for $s \in [t_{m-1}, t_m)$. This yields the following scheme, called accelerated SETD0 (ASETD0) $Y^h_0 = P_hX_0$.

$$Y^h_m = S_h(\Delta t)P_hY_{m-1} + \int_{t_{m-1}}^{t_m} S_h(\Delta t)P_hF(Y^h_{m-1})ds + S_{2h}(\Delta t)P_hP_N\int_{t_{m-1}}^{t_m} S_1(t_m-s)dW(s), \quad m \geq 1. \quad (18)$$

The numerical method (18) can be written in the following equivalent form, efficient for simulation

$$Y^h_m = \varphi_0(\Delta tA_h)\left(X^h_{m-1} + \Delta tF(Y^h_{m-1})\right) + O^{h,N}_{m-1}, \quad m \geq 1. \quad (19)$$

The linear operators φ_0 and φ_1 are given respectively by

$$\varphi_0(\Delta tA_h) := e^{A_h\Delta t}, \quad \varphi_1(\Delta tA_h) := \frac{1}{\Delta t}\int_0^{\Delta t} e^{A_h(\Delta t-s)}ds. \quad (20)$$

Note that ASETD1 and ASETD0 are the analogue schemes in[3] that we are improving their stability in this paper.

The following theorem is the main result of this paper.

Theorem 1. [Main results] Let $X(t_m)$ be the mild solution given by (6) and ξ^h_m the numerical approximation (with $\xi^h_m = X^h_m$ for ASETD1 and $\xi^h_m = Y^h_m$ for ASETD0). If Assumption 2.1 is fulfiled, the following strong convergence estimate holds

$$\|X(t_m) - \xi^h_m\|_{L^2(\Omega,\mathcal{H})} \leq C\left[h^{\beta-\epsilon} + \left(\inf_{j\in\mathbb{N}^d\backslash\mathcal{I}_N} \lambda_j\right)^{-\frac{\beta}{2}+\epsilon} + \Delta t^{\min(\beta-\epsilon,1)}\right],$$

where β is given in Assumption 2.1 and $\epsilon \in (0, \beta)$ is any positive number, small enough.

Remark 2. Remember that as in[3] to simulate our accelerated schemes, the eigenfunctions of the linear operator A should be the same as that of the covariance operator Q [2], if not, the projection of the eigenfunctions of Q onto the eigenfunction of A should be done. This is indeed the drawback of the accelerated schemes as in general the projection is costly.

3 | PROOF OF THE MAIN RESULTS

The proofs of the main results need some preliminaries results.

[2]This helps in the computation of $O^{h,N}_{m-1}$

3.1 | Preparatory results

Lemma 1. Let $0 \leq \rho \leq 1$ and $u \in \mathcal{H}$. Then the following sharp integral estimates hold

$$\int_{t_1}^{t_2} \|(-A)^{\frac{\rho}{2}} S(t_2-r)\|^2_{\mathcal{L}(\mathcal{H})} dr \leq C(t_2-t_1)^{1-\rho}, \quad \|(-A)^{\rho} \int_{t_1}^{t_2} S(t_2-r)u dr\| \leq C(t_2-t_1)^{1-\rho}\|u\|, \quad 0 \leq t_1 \leq t_2 \leq T.$$

Proof. The proof of the first estimate can be found in [14, Lemma 2.1]. The proof of the second estimate is similar to [10, Lemma 3.2 (iv)], since this is general and does not use the fact that A is self-adjoint. □

Lemma 2. The following sharp time and space regularities holds

$$\|X(t_2)-X(t_1)\|_{L^2(\Omega,\mathcal{H})} \leq C(t_2-t_1)^{\min\left(\frac{1}{2},\frac{\beta}{2}\right)}, \quad \|(-A)^{\frac{\beta}{2}} X(t_1)\|_{L^2(\Omega,\mathcal{H})} \leq C, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{21}$$

Proof. First of all as in [5, Lemma 2.7], it holds that $\|X(t_2)-X(t_1)\|_{L^2(\Omega,\mathcal{H})} \leq C(t_2-t_1)^{\min\left(\frac{\beta}{2},\frac{1}{\epsilon}-\epsilon\right)}$. Similarly to [5, Theorem 2.6] we have $\|(-A)^{\frac{\beta}{2}} X(t)\|_{L^2(\Omega,\mathcal{H})} < C$ for $\beta \in [0,2)$. For $\beta = 2$ we have

$$AX(t) = S(t)AX_0 + \int_0^t AS(t-s)F(X(t))ds + \int_0^t AS(t-s)\left(F(X(s)) - F(X(t)\right) ds + \int_0^t AS(t-s)dW(s). \tag{22}$$

Taking the norm in both sides of (22) and using Assumption 2.1, Lemma 2 and the stability properties of the semi-group yields

$$\begin{aligned}
\|AX(t)\|_{L^2(\Omega,\mathcal{H})} \leq\; & \|S(t)AX_0\|_{L^2(\Omega,\mathcal{H})} + \|\int_0^t AS(t-s)F(X(t))ds\|_{L^2(\Omega,\mathcal{H})} \\
& + \int_0^t \|AS(t-s)\|_{\mathcal{L}(\mathcal{H})} \|F(X(s)) - F(X(t))\|_{L^2(\Omega,\mathcal{H})} ds + \|\int_0^t AS(t-s)dW(s)\|_{L^2(\Omega,\mathcal{H})} \\
& \leq C\|AX_0\|_{L^2(\Omega,\mathcal{H})} + C\|F(X(t))\|_{L^2(\Omega,\mathcal{H})} + C\int_0^t (t-s)^{-1}\|X(t)-X(s)\|_{L^2(\Omega,\mathcal{H})} ds \\
& + \|\int_0^t AS(t-s)dW(s)\|_{L^2(\Omega,\mathcal{H})}.
\end{aligned} \tag{23}$$

Using the Itô isometry and Assumption 2.1 yields

$$\begin{aligned}
\|\int_0^t AS(t-s)dW(s)\|^2_{L^2(\Omega,\mathcal{H})} &= \int_0^t \|AS(t-s)Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})} ds \leq \int_0^t \|(-A)^{\frac{1}{2}} S(t-s)\|^2_{\mathcal{L}(\mathcal{H})} \|(-A)^{\frac{1}{2}} Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})} ds \\
&\leq C \int_0^t \|(-A)^{\frac{1}{2}} S(t-s)\|^2_{\mathcal{L}(\mathcal{H})} ds \leq C.
\end{aligned} \tag{24}$$

Substituting (24) in (23) yields

$$\|AX(t)\|_{L^2(\Omega,\mathcal{H})} \leq C + C\int_0^t (t-s)^{-1}(t-s)^{\min\left(\frac{\beta}{2},\frac{1}{\epsilon}-\epsilon\right)} ds \leq C.$$

Using the second estimate of (21) one can readily prove that $\|X(t_2)-X(t_1)\|_{L^2(\Omega,\mathcal{H})} \leq C(t_2-t_1)^{\min\left(\frac{1}{2},\frac{\beta}{2}\right)}$. □

3.2 Proof of Theorem 1

We only give the proof for ASETD1, since the case of ASETD0 is similar. Iterating the mild solution (6) yields

$$X(t_m) = S(t_m)X_0 + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)F(X(s))ds + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)dW(s). \tag{25}$$

Iterating the numerical scheme (16) yields

$$X^h_m = S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m-s)P_hF(X^h_k)ds + \sum_{k=0}^{m-1} S_h(t_{m-k-1})O^{h,N}_k. \tag{26}$$

Subtracting (26) from (25) yields

$$\begin{aligned} X(t_m) - X^h_m &= S(t_m)X_0 - S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left[S(t_m-s)F(X(s)) - S_h(t_m-s)P_hF(X^h_k)\right] ds \\ &+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)dW(s) - \sum_{k=0}^{m-1} S_h(t_{m-k-1})O^{h,N}_k \\ &=: II_1 + II_2 + II_3. \end{aligned} \tag{27}$$

Using Lemma 7 in[13] with $r=\alpha=\beta$, it holds that

$$\|II_1\|_{L^2(\Omega,\mathcal{H})} \leq \|\left(S(t_m) - S_h(t_m)P_h\right)X_0\|_{L^2(\Omega,\mathcal{H})} \leq Ch^\beta\|X_0\|_{L^2(\Omega,\dot{\mathcal{H}}^\beta)} \leq Ch^\beta. \tag{28}$$

We can split II_2 in three terms as follows

$$\begin{aligned} II_2 &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left[S(t_m-s)F(X(s)) - S(t_m-s)F(X(t_k))\right] ds + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left[S(t_m-s)F(X(t_k)) - S_h(t_m-s)P_hF(X(t_k))\right] ds \\ &+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left[S_h(t_m-s)P_hF(X(t_k)) - S_h(t_m-s)P_hF(Y^h_k)\right] ds \\ &=: II_{21} + II_{22} + II_{23}. \end{aligned} \tag{29}$$

We start with the estimates of II_{22} and II_{23}, since they are easier than that of II_{21}. Using triangle inequality, Lemma 7 in[13] with $r=\min(\beta, 2-\epsilon)$ and $\alpha=0$, Assumption 2.1, it holds that

$$\begin{aligned} \|II_{22}\| &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|\left(S(t_m-s) - S_h(t_m-s)P_h\right)F(X(t_k))\right\|_{L^2(\Omega,\mathcal{H})} ds \\ &\leq Ch^{\min(\beta,2-\epsilon)}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{\min\left(-\frac{\beta}{2},-1+\frac{\epsilon}{2}\right)}\|F(X(t_k))\|_{L^2(\Omega,\mathcal{H})}ds \\ &\leq Ch^{\min(\beta,2-\epsilon)}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{\min\left(-\frac{\beta}{2},-1+\frac{\epsilon}{2}\right)}ds = Ch^{\min(\beta,2-\epsilon)}\int_0^{t_m}(t_m-s)^{\min\left(-\frac{\beta}{2}-1+\frac{\epsilon}{2}\right)}ds \leq Ch^{\min(\beta,2-\epsilon)}. \end{aligned} \tag{30}$$

Using Assumption 2.1 and the smoothing properties of the semigroup yields

$$\|II_{23}\|_{L^2(\Omega,\mathcal{H})} \leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|S_h(t_m-s)P_h\left(F(X(t_k)) - F(Y^h_k)\right)\right\|_{L^2(\Omega,\mathcal{H})} ds \leq C\Delta t\sum_{k=0}^{m-1}\|X(t_k) - Y^h_k\|_{L^2(\Omega,\mathcal{H})}. \tag{31}$$

Let us now estimate II_{21}. Using Taylor's formula in Banach space yields

$$F(X(s))-F(X(t_k))=\left(\int_0^1 F'\left(X(t_k)+\tau\left(X(s)-X(t_k)\right)\right)d\tau\right)\left(X(s)-X(t_k)\right). \tag{32}$$

From the mild solution we have

$$X(s)-X(t_k)=\left(S(s-t_k)-\mathbf{I}\right)X(t_k)+\int_{t_k}^{s} S(s-r)F(X(r))dr+\int_{t_k}^{s} S(s-r)dW(r). \tag{33}$$

Substituting (33) in (32) yields

$$F(X(s))-F(X(t_k)) = I_{k,s}\left(S(s-t_k)-\mathbf{I}\right)X(t_k)+I_{m,k,s}\int_{t_k}^{s} S(s-r)F(X(r))dr+I_{k,s}\int_{t_k}^{s} S(s-r)dW(r), \tag{34}$$

where $I_{k,s}$ is given by

$$I_{k,s} := \int_0^1 F'\left(X(t_k)+\tau\left(X(s)-X(t_k)\right)\right)d\tau, \quad t_k\leq s\leq t_{k+1}. \tag{35}$$

Note that using Assumption 2.1, one easily check that

$$\|I_{k,s}\|_{\mathcal{L}(\mathcal{H})}\leq C, \quad t_k\leq s\leq t_{k+1}, \quad k\in\{0,1,\cdots,M\}. \tag{36}$$

Substituting (34) in the expression of II_{21} yields

$$\begin{aligned} II_{21} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)I_{k,s}\left(S(s-t_k)-\mathbf{I}\right)X(t_k)ds+\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)I_{k,s}\int_{t_k}^{s} S(s-r)F(X(r))drds \\ &+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)I_{k,s}\int_{t_k}^{s} S(s-r)dW(r)ds \\ &=: II_{21}^{(1)}+II_{21}^{(2)}+II_{21}^{(3)}. \end{aligned} \tag{37}$$

Using Lemma 2, (36), Assumption 2.1 and the stability of the semi-group yields

$$\begin{aligned} \|II_{21}^{(1)}\|_{L^2(\Omega,\mathcal{H})} &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|S(t_m-s)I_{k,s}\left(S(s-t_k)-\mathbf{I}\right)(-A)^{-\frac{\beta}{2}}(-A)^{\frac{\beta}{2}}X(t_k)\|_{L^2(\Omega,\mathcal{H})}ds \\ &\leq C\sum_{k=0}^{m-1}\|(S(s-t_k)-\mathbf{I})(-A)^{-\frac{\beta}{2}}\|_{\mathcal{L}(\mathcal{H})}\|(-A)^{\frac{\beta}{2}}X(t_k)\|_{L^2(\Omega,\mathcal{H})}ds\leq C\Delta t^{\frac{\beta}{2}}. \end{aligned} \tag{38}$$

In view of Assumption 2.1, one can easily get

$$\|II_{21}^{(2)}\|_{L^2(\Omega,\mathcal{H})}\leq C\Delta t. \tag{39}$$

To estimate $II_{21}^{(3)}$ we recast it as follows

$$\begin{aligned} II_{21}^{(3)} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)I_{k,t_k}\int_{t_k}^{s} S(s-r)dW(r)ds+\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)\left(I_{k,s}-I_{k,t_k}\right)\int_{t_k}^{s} S(s-r)dW(r)ds \\ &=: II_{21}^{(31)}+II_{21}^{(32)}. \end{aligned} \tag{40}$$

Along the same lines as [18, (3.13)] we obtain

$$\|II_{21}^{(31)}\|^2_{L^2(\Omega,\mathcal{H})}\leq C\Delta t^{\min(1+\beta,2)}. \tag{41}$$

Using triangle inequality, Hölder inequality and Itô isometry yields

$$
\begin{aligned}
\|II_{21}^{(32)}\|_{L^2(\Omega,\mathcal{H})} &\leq \sum_{k=1}^{m-1}\left\|\int_{t_k}^{t_{k+1}} S(t_m-s)\left(I_{k,s}-I_{k,t_k}\right)\int_{t_k}^{s} S(s-r)dW(r)ds\right\|_{L^2(\Omega,\mathcal{H})} \\
&= \sum_{k=1}^{m-1}\left[\mathbb{E}\left\|\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} S(t_m-s)\left(I_{k,s}-I_{k,t_k}\right)S(s-r)dW(r)\right\|^2 ds\right]^{\frac{1}{2}} \\
&\leq C\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}}\mathbb{E}\left\|\int_{t_k}^{s} S(t_m-s)\left(I_{k,s}-I_{k,t_k}\right)S(s-r)dW(r)\right\|^2 ds\right]^{\frac{1}{2}} \\
&\leq C\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\mathbb{E}\left\|S(t_m-s)P_h\left(I_{k,s}-I_{k,t_k}\right)S(s-r)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} drds\right]^{\frac{1}{2}}. \qquad (42)
\end{aligned}
$$

Using Assumption 2.1 and the stability properties of the semi-group yields

$$
\begin{aligned}
&\mathbb{E}\left\|S(t_m-s)\left(I_{k,s}-I_{k,t_k}\right)S(s-r)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} \\
&\leq \left\|S(t_m-s)(-A)^{\eta}\right\|^2_{\mathcal{L}(\mathcal{H})}\mathbb{E}\left\|(-A)^{-\eta}\left(I_{k,s}-I_{k,t_k}\right)\right\|^2_{L^2(\Omega,\mathcal{H})}\left\|S(s-r)(-A)^{\frac{1-\beta}{2}}(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} \\
&\leq \left\|S(t_{k+1}-s)\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|S(t_m-t_{k+1})(-A)^{\eta}\right\|^2_{\mathcal{L}(\mathcal{H})}\mathbb{E}\left\|(-A)^{-\eta}\left(I_{k,s}-I_{k,t_k}\right)\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|S(s-r)(-A)^{\frac{1-\beta}{2}}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} \\
&\leq Ct_{m-k-1}^{-2\eta}(s-r)^{\min(0,\beta-1)}\mathbb{E}\left\|(-A)^{-\eta}\left(I_{k,s}-I_{k,t_k}\right)\right\|^2_{\mathcal{L}(\mathcal{H})}. \qquad (43)
\end{aligned}
$$

Using the definition of $I^h_{m,k,s}$, Assumption 2.1 and Lemma 2 we arrive at

$$
\begin{aligned}
\|(-A)^{-\eta}\left(I_{k,s}-I_{k,t_k}\right)\|_{\mathcal{L}(\mathcal{H})} &\leq \int_0^1\left\|(-A)^{-\frac{\eta}{2}}\left(F'\left(X(t_{t_k})+\tau\left(X(s)-X(t_k)\right)\right)-F'\left(X(t_k)\right)\right)\right\|_{\mathcal{L}(\mathcal{H})}d\tau \\
&\leq \int_0^1\left\|(-A)^{-\eta}\left(F'\left(X(t_k)+\tau\left(X(s)-X(t_k)\right)\right)-F'\left(X(t_k)\right)\right)\right\|_{\mathcal{L}(\mathcal{H})}d\tau \\
&\leq C\int_0^1\tau\|X(s)-X(t_k)\|d\tau \leq C\|X(s)-X(t_k)\|. \qquad (44)
\end{aligned}
$$

Substituting (44) in (43) and Lemma 2 yields

$$
\begin{aligned}
\mathbb{E}\left\|S(t_m-s)\left(I_{m,k,s}-I_{m,k,t_k}\right)S(s-r)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} &\leq Ct_{m-k-1}^{-2\eta}(s-r)^{\min(0,\beta-1)}\mathbb{E}\|X(s)-X(t_k)\|^2 \\
&\leq Ct_{m-k-1}^{-2\eta}(s-r)^{\min(0,\beta-1)}(s-t_k)^{\min(1,\beta)}. \qquad (45)
\end{aligned}
$$

Substituting (45) in (42) yields

$$\begin{aligned}
\|II_{21}^{(32)}\|_{L^2(\Omega,\mathcal{H})} &\leq C\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} t_{m-k-1}^{-2\eta}(s-r)^{\min(0,\beta-1)}(s-t_k)^{\min(1,\beta)}drds\right]^{\frac{1}{2}}\\
&\leq C\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-2\eta}\int_{t_k}^{s}(s-t_k)^{\min(1,2\beta-1)}drds\right]^{\frac{1}{2}} \leq C\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-2\eta}(s-t_k)^{\min(2,2\beta)}ds\right]^{\frac{1}{2}}\\
&\leq C\Delta t^{\min(1,\beta)}\Delta t^{\frac{1}{2}}\sum_{k=1}^{m-1}\left[\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-2\eta}ds\right]^{\frac{1}{2}} \leq C\Delta t^{\min(1,\beta)}\sum_{k=1}^{m-1}\Delta t\, t_k^{-\eta} \leq C\Delta t^{\min(1,\beta)}.
\end{aligned} \tag{46}$$

Substituting (46) and (41) in (40) yields

$$\|II_{21}^{(3)}\|_{L^2(\Omega,\mathcal{H})} \leq C\Delta t^{\min(1,\beta)}. \tag{47}$$

Substituting (47), (39) and (38) in (37) yields

$$\|II_{21}\|_{L^2(\Omega,\mathcal{H})} \leq C\Delta t^{\min(1,\beta)}. \tag{48}$$

Substituting (48), (31) and (30) in (29) yields

$$\|II_2\|_{L^2(\Omega,\mathcal{H})} \leq C\left(h^{\min(\beta,2-\epsilon)}+\Delta t^{\min(\beta,1)}\right)+C\Delta t\sum_{k=0}^{m-1}\|X(t_k)-X_k^h\|_{L^2(\Omega,\mathcal{H})}. \tag{49}$$

Using triangle inequality, we split II_3 as follows

$$\begin{aligned}
II_3 &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)dW(s)-\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m-t_{k+1})P_hS(t_{k+1}-s)dW(s)\\
&+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m-t_{k+1})P_h\left[S(t_{k+1}-s)-S_1(t_{k+1}-s)\right]dW(s)+\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m-t_{k+1})P_h\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)dW(s)\\
&+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m-t_{k+1})\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)dW(s)\\
&=: II_{31}+II_{32}+II_{33}+II_{34}.
\end{aligned} \tag{50}$$

Note that the term II_{31} can be written as follows

$$\begin{aligned}
II_{31} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left[S(t_m-t_{k+1})-S_h(t_m-t_{k+1})P_h\right]S(t_{k+1}-s)dW(s)\\
&= \int_{t_{m-1}}^{t_m}(\mathbf{I}-P_h)S(t_m-s)dW(s)+\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left[S(t_m-t_{k+1})-S_h(t_m-t_{k+1})P_h\right]S(t_{k+1}-s)dW(s)=II_{31}^{(1)}+II_{31}^{(2)}.
\end{aligned} \tag{51}$$

Let us recall that for any $\gamma\in[0,1]$, the following estimate holds, see e.g. [2]

$$\|(\mathbf{I}-P_h)(-A)^{-\gamma}\|_{\mathcal{L}(\mathcal{H})} \leq Ch^{2\gamma}. \tag{52}$$

Using the Itô isometry, (52), Assumption 2.1 and Lemma 1 yields

$$
\begin{aligned}
\|II_{31}^{(1)}\|^2_{L^2(\Omega,\mathcal{H})} &= \int_{t_{m-1}}^{t_m} \|(\mathbf{I}-P_h)S(t_m-s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds \leq \int_{t_{m-1}}^{t_m} \|(\mathbf{I}-P_h)(-A)^{-\frac{\beta}{2}}S(t_m-s)(-A)^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{H})}\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq \int_{t_{m-1}}^{t_m} \|(\mathbf{I}-P_h)(-A)^{-\frac{\beta}{2}}\|^2_{\mathcal{L}(\mathcal{H})}\|S(t_m-s)(-A)^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{H})}\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq Ch^{2\beta}\int_{t_{m-1}}^{t_m} \|S(t_m-s)(-A)^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{H})}ds \leq Ch^{2\beta}.
\end{aligned} \tag{53}
$$

Using the Itô isometry, Assumption 2.1, the stability properties of the semigroup and [13, Lemma 7] with $r=\beta-\epsilon$ and $\alpha=\max(0,\beta-1)$ yields

$$
\begin{aligned}
\|II_{31}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})} &= \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} \left\|\left[S(t_m-t_{k+1})-S_h(t_m-t_{k+1})P_h\right]S(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq Ch^{2\beta-2\epsilon}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|A^{\frac{\beta-1}{2}}S(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq Ch^{2\beta-2\epsilon}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|S(t_{k+1}-s)\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds \leq Ch^{2\beta-2\epsilon}\sum_{k=0}^{m-2}\Delta t\, t_{m-k-1}^{-1+\epsilon} \leq Ch^{2\beta-2\epsilon}.
\end{aligned} \tag{54}
$$

Substituting (54) and (53) in (51) yields

$$
\|II_{31}\|_{L^2(\Omega,\mathcal{H})} \leq Ch^{\beta-\epsilon}. \tag{55}
$$

Using the Itô isometry and the stability properties of the semi-group, it holds that

$$
\begin{aligned}
II_{32} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|S_h(t_m-t_{k+1})P_h\left[S(t_{k+1}-s)-S_1(t_{k+1}-s)\right]Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}\\
&\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|\left[S(t_{k+1}-s)-S_1(t_{k+1}-s)\right]Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}(\mathcal{H})}ds.
\end{aligned} \tag{56}
$$

Using the Perturbation theorem of Miyadera-Voigt ([1, Theorem 3.14, Chapter III, (3.22)])

$$
S(t_{k+1}-s)-S_1(t_{k+1}-s) = \int_s^{t_{k+1}} S(r)A_2S_1(t_{k+1}-r)dr. \tag{57}
$$

Note that one can easily check that conditions on applying Miyadera-Voigt theorem are fulfilled. This is due to [1, Chapter III, Corollary 3.16].

Inserting $Q^{\frac{1}{2}}$ in (57) yields

$$
\left(S(t_{k+1}-s)-S_1(t_{k+1}-s)\right)Q^{\frac{1}{2}} = \int_s^{t_{k+1}} S(r)A_2S_1(t_{k+1}-r)(-A)^{\frac{1-\beta}{2}}(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}dr. \tag{58}
$$

Taking the norm in both sides of (58), using Assumption 2.1 and the stabilities properties of the semigroup, it follows that

$$
\begin{aligned}
&\left\|\left(S(t_{k+1}-s)-S_1(t_{k+1}-s)\right)Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(\mathcal{H})}\\
&\leq \int_s^{t_{k+1}} \left\|S(r)(-A)^{\frac{1}{2}-\epsilon}\right\|_{\mathcal{L}(\mathcal{H})} \left\|A^{-\frac{1}{2}+\epsilon}A_2^{1-\epsilon}\right\|_{\mathcal{L}(\mathcal{H})} \left\|A_2^{\epsilon}A^{-\epsilon}\right\|_{\mathcal{L}(\mathcal{H})} \left\|A^{\epsilon}S_1(t_{k+1}-r)(-A)^{\frac{1-\beta}{2}}\right\|_{\mathcal{L}(\mathcal{H})} \left\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(\mathcal{H})} dr\\
&\leq C\int_s^{t_{k+1}} r^{-\frac{1}{2}+\epsilon}(t_{k+1}-r)^{\min\left(0,\frac{-1+\beta}{2}-\epsilon\right)}dr \leq Cs^{-\frac{1}{2}+\epsilon}\int_s^{t_{k+1}}(t_{k+1}-r)^{\min\left(0,\frac{-1+\beta}{2}-\epsilon\right)}dr \leq Cs^{-\frac{1}{2}+\epsilon}(t_{k+1}-s)^{\min\left(1,\frac{\beta+1}{2}-\epsilon\right)}. \qquad (59)
\end{aligned}
$$

Substituting (59) in (56) yields

$$
\begin{aligned}
\|II_{32}\|^2_{L^2(\Omega,\mathcal{H})} &\leq C\int_0^{\Delta t} s^{-1+2\epsilon}(\Delta t-s)^{\min(1+\beta-2\epsilon,2)}ds + C\sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}} s^{-1+\epsilon}(t_{k+1}-s)^{\min(2,1+\beta-2\epsilon)}ds\\
&\leq C\Delta t^{\min(1+\beta-\epsilon,2)} + C\sum_{k=1}^{m-1} t_k^{-1+\epsilon}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\min(2,1+\beta-\epsilon)}\\
&\leq C\Delta t^{(2,1+\beta-\epsilon)} + C\Delta t^{\min(2,1+\beta-2\epsilon)}\sum_{k=0}^{m-1}\Delta t\, t_{k+1}^{-1+\epsilon} \leq C\Delta t^{\min(2,2\beta-2\epsilon)}. \qquad (60)
\end{aligned}
$$

Using the Itô isometry and splitting the sum in two parts yields

$$
\begin{aligned}
\|II_{33}\|^2_{\mathcal{L}(\mathcal{H})} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|S_h(t_m-t_{k+1})P_h\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&= \int_{t_{m-1}}^{t_m}\left\|P_h\left(\mathbf{I}-P_N\right)S_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds + \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})P_h\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&= \|II_{33}^{(1)}\|^2_{L^2(\Omega,\mathcal{H})} + \|II_{33}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})}. \qquad (61)
\end{aligned}
$$

We start with the estimate of $\|II_{33}\|^2_{L^2(\Omega,\mathcal{H})}$. Using the stability properties of the semigroup yields

$$
\begin{aligned}
\|II_{33}^{(2)}\|^2_{\mathcal{L}(\mathcal{H})} &= \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})P_h\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})(-A_h)^{\frac{1}{2}-\epsilon}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_h)^{-\frac{1}{2}+\epsilon}P_h\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(H)}ds\\
&\leq C\sum_{k=0}^{m-2}t_{m-k-1}^{-1+\epsilon}\left\|(-A)^{-\frac{1}{2}+\epsilon}\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\sum_{k=0}^{m-2}t_{m-k-1}^{-1+\epsilon}\left\|(-A_1)^{-\frac{1}{2}+\epsilon}\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds.
\end{aligned}
$$

Using the fact that A_1, S_1 and Q are self-adjoint and Assumption 2.1, it follows that

$$
\begin{aligned}
\|II_{33}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})} &= C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{1-\epsilon}\left\|\left(\mathbf{I}-P_N\right)S_1(t_{k+1}-s)Q^{\frac{1}{2}}(-A_1)^{-\frac{1}{2}+\epsilon}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{1-\epsilon}\left\|\left(\mathbf{I}-P_N\right)(-A_1)^{-\frac{\beta}{2}+\epsilon}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta}{2}-\epsilon}S_1(t_{k+1}-s)Q^{\frac{1}{2}}(-A_1)^{-\frac{1}{2}+\epsilon}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\beta+\epsilon}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\|S_1(t_{k+1}-s)\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\beta+\epsilon}\sum_{k=0}^{m-2}\Delta t\, t_{m-k-1}^{-1+\epsilon}\leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\beta+\epsilon}. \qquad (62)
\end{aligned}
$$

Along the same lines as the estimate of $\|II_{33}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})}$, we obtain

$$
\|II_{33}^{(1)}\|^2_{L^2(\Omega,\mathcal{H})}\leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\beta+\epsilon}. \qquad (63)
$$

Substituting (63) and (62) in (61) yields

$$
\|II_{33}\|^2_{L^2(\Omega,\mathcal{H})}\leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\beta+\epsilon}. \qquad (64)
$$

Using the Itô isometry and splitting the sum in two part, it holds that

$$
\begin{aligned}
\|II_{34}\|^2_{L^2(\Omega,\mathcal{H})} &= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&= \int_{t_{m-1}}^{t_m}\left\|(\mathbf{I}-S_{2h}(\Delta t))P_hP_NS_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}\\
&+ \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&=:\ \|II_{34}^{(1)}\|_{L^2(\Omega,\mathcal{H})}+\|II_{34}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})}. \qquad (65)
\end{aligned}
$$

It is well known (see e.g.[11]) that $\|A_{2h}v\|\leq C'\|v\|_1\leq C\|(-A_1)^{\frac{1}{2}}v\|$ for all $v\in V_h$. Hence by interpolation theory, it holds that

$$
\|(-A_{2h})^{\gamma}v\|\leq C'\|(-A_1)^{\frac{\gamma}{2}}v\|,\quad -1\leq\gamma\leq 1,\quad v\in\mathcal{D}((-A)^{\frac{\gamma}{2}})\cap V_h. \qquad (66)
$$

Using the latter estimate together with the smooth properties of the semi-group and [13, Lemma 1] yields

$$
\begin{aligned}
\|II_{34}^{(1)}\|^2_{L^2(\Omega,\mathcal{H})} &= \int_{t_{m-1}}^{t_m} \left\|(\mathbf{I}-S_{2h}(\Delta t))(-A_{2h})^{-\min(\beta,1)}(-A_{2h})^{\min(\beta,1)}P_hP_NS_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq \int_{t_{m-1}}^{t_m} \left\|(\mathbf{I}-S_{2h}(\Delta t))(-A_{2h})^{-\min(\beta,1)}\right\|^2_{L^2(\Omega,\mathcal{H})}\left\|(-A_{2h})^{\min(\beta,1)}P_hP_NS_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta,1)}\int_{t_{m-1}}^{t_m}\left\|(-A_1)^{\frac{\min(\beta,1)}{2}}P_hP_NS_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta,1)}\int_{t_{m-1}}^{t_m}\left\|(-A_1)^{\frac{\min(\beta,1)}{2}}P_NS_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta,1)}\int_{t_{m-1}}^{t_m}\left\|(-A_1)^{\frac{\min(\beta,1)}{2}}S_1(t_m-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds.
\end{aligned}
$$

Using Assumption 2.1 and Lemma 1, it follows that

$$
\begin{aligned}
\|II_{34}^{(1)}\|^2_{L^2(\Omega,\mathcal{H})} &\leq C\Delta t^{2\min(\beta,1)}\int_{t_{m-1}}^{t_m}\left\|(-A_1)^{\frac{\min(\beta,1)}{2}}S_1(t_m-s)(-A_1)^{\frac{1-\beta}{2}}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta,1)}\int_{t_{m-1}}^{t_m}\left\|(-A_1)^{\frac{\min(1,2-\beta)}{2}}S_1(t_m-s)\right\|^2_{\mathcal{L}(\mathcal{H})}ds \leq C\Delta t^{\min(\beta,1)}. \qquad (67)
\end{aligned}
$$

Using the stability properties of the semi-group yields

$$
\begin{aligned}
\|II_{34}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})} &\leq \sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}\left\|S_h(t_m-t_{k+1})(-A_h)^{\frac{1-\epsilon}{2}}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_h)^{\frac{-1+\epsilon}{2}}\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}}t_{m-k-1}^{-1+\epsilon}\left\|(-A_h)^{\frac{-1+\epsilon}{2}}\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds.
\end{aligned}
$$

Similarly to (66), the following estimate holds

$$
\|(-A_h)^{-\frac{\gamma}{2}}v\| \leq C\|(-A_{2h})^{-\gamma}v\|, \quad -1\leq\gamma\leq 1, \quad v\in V_h. \qquad (68)
$$

From the definition of P_N, the following estimate obviously holds

$$
\|(-A_1)^{\frac{\gamma}{2}}P_N(-A_1)^{-\frac{\gamma}{2}}\|_{\mathcal{L}(\mathcal{H})} < C, \quad \gamma\in[-1,1]. \qquad (69)
$$

Inserting and appropriate power of A_{2h}, using (66) and the equivalence of norms [13, Lemma 1] and the fact A_1 is self adjoint, it follows that

$$\begin{aligned}
\|II_{34}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})} &\leq C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|(-A_{2h})^{-1+\epsilon}\left(\mathbf{I}-S_{2h}(\Delta t)\right)P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|(-A_{2h})^{-1+\epsilon}\left(\mathbf{I}-S_{2h}(\Delta t)\right)(-A_{2h})^{1-\beta}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_{2h})^{\beta-1}P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|(-A_{2h})^{-\beta+\epsilon}\left(\mathbf{I}-S_{2h}(\Delta t)\right)\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta-1}{2}}P_hP_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta-\epsilon,1)}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|(-A_1)^{\frac{\beta-1}{2}}P_NS_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds. \qquad (70)
\end{aligned}$$

Using (69), Assumption 2.1 and the smoothing properties of the semigroup, it follows from (70) that

$$\begin{aligned}
\|II_{34}^{(2)}\|^2_{L^2(\Omega,\mathcal{H})} &\leq C\Delta t^{2\min(\beta-\epsilon,1)}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\left\|(-A_1)^{\frac{\beta-1}{2}}P_N(-A_1)^{\frac{1-\beta}{2}}\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta-1}{2}}S_1(t_{k+1}-s)Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(2\beta-\epsilon,1)}\sum_{k=0}^{m-2}\int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\epsilon}\|S_1(t_{k+1}-s)\|^2_{\mathcal{L}(\mathcal{H})}\left\|(-A_1)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})}ds\\
&\leq C\Delta t^{2\min(\beta-\epsilon,1)}\sum_{k=0}^{m-2}\Delta t\, t_{m-k-1}^{-1+\epsilon}ds \leq C\Delta t^{2\min(\beta-\epsilon,1)}. \qquad (71)
\end{aligned}$$

Substituting (71) and (67) in (65) yields

$$\|II_{34}\|_{L^2(\Omega,\mathcal{H})} \leq C\Delta t^{\min(\beta-\epsilon,1)}. \qquad (72)$$

Substituting (72), (64), (60) and (55) in (50) yields

$$\|II_3\|_{L^2(\Omega,\mathcal{H})} \leq C\Delta t^{\min(\beta-\epsilon,1)} + C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\frac{\beta}{2}+\epsilon} + Ch^{\beta-\epsilon}. \qquad (73)$$

Substituting (73), (49) and (28) in (27) yields

$$\|X(t_m)-X_m^h\|_{L^2(\Omega,\mathcal{H})} \leq C\left[Ch^{\beta-\epsilon}+\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-\frac{\beta}{2}+\epsilon}+\Delta t^{\min(\beta-\epsilon,1)}\right] + C\Delta t\sum_{k=0}^{m-1}\|X(t_k)-X_m^h\|_{L^2(\Omega,\mathcal{H})}. \qquad (74)$$

Applying the discrete Gronwall lemma to (74) completes the proof of Theorem 1.

4 | NUMERICAL EXPERIMENTS

As in [3], we consider the following stochastic transport equation

$$dX = \left(D\Delta X - \nabla\cdot(\mathbf{q}X) - \frac{X}{|X|+1}\right)dt + dW \qquad (75)$$

where $D>0$ is the diffusion coefficient, $\mathbf{q}$ is the Darcy's velocity field as in [3] and $D=10^{-2}$. We consider two types of boundary conditions:

- (a) Homogenous Neumann boundary everywhere in the domain

- (b) Mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0,1]\times[0,1]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x, y) \;:\; x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. This is a typical engineering problem indeed.

For boundary condition (a) our function f using in (1) to define our Nemystkii operator F is given by

$$f(z,x) = -\frac{x}{1+|x|}, \quad f'(z,x) = -\frac{1}{(1+|x|)^2}, \quad z\in\Lambda, \quad x\in\mathbb{R}.$$

One can easily check that (9) holds. In fact, simple estimates yields

$$|f'(z,x)-f'(z,y)| = \left|\frac{(|x|-|y|)(2+|x|+|y|)}{(1+|x|)^2(1+|y|)^2}\right| \leq 2|x-y|, \quad z\in\Lambda, \quad x,y\in\mathbb{R}.$$

Therefore from Proposition 1, it follows that the estimates regarding F in Assumption 2.1 is satisfied. For mixed Boundary condition, the Nemystkii operator F also included the trace operator as we can observe in[3]. In this case, Assumption 2.1 is not satisfied as the domain of the trace operator is $H^2(\Lambda)$. Remember that as in[3] to simulate our accelerated schemes, the eigenfunctions of the linear operator A should be the same as that of the covariance operator Q, if not the projection of the eigenfunctions of Q onto the eigenfunction of A should be done. As in[3], our linear operator in all our simulations is the Laplace operator Δ with Neumann boundary everywhere in the domain as the eigenvalues and eigenfunctions are well known in rectangular domain. In the decomposition (5), we have used

$$q_{i,j} = \left(i^2+j^2\right)^{-(\beta+\delta)}, \tag{76}$$

for some small $\delta > 0$ and $\beta > 0$. Since the eigenvalues of the Laplace operator with Neumann boundary are given by $\{\lambda_{i,j}\}_{i,j\geq 0}$ given by $\lambda_{i,j} = (\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2,\ \lambda_i^{(l)} = i\pi$, we obviously have

$$\sum_{(i,j)\in\mathbb{N}^2} \lambda_{i,j}^{\beta-1} q_{i,j} < \pi^2 \sum_{(i,j)\in\mathbb{N}^2} \left(i^2+j^2\right)^{-(1+\delta)} < \infty, \qquad 0<\beta\leq 2,$$

thus Assumption 8 is satisfied. Details on simulation of the accelerated schemes can be found in[3]. In the legends of all of our graphs we use the following notation

- ASETD1 is used for graphs from scheme (17) where the boundary condition (b) is used, while ASETD1=N is used when Neumann boundary condition (a) is used.
- ASETD0 is used for graphs from scheme (18) where the boundary condition (b) is used, while ASETD0=N is used when Neumann boundary condition (a) is used.
- SETD1 is used for graphs from the analogue of ASETD1 scheme in[3] where the boundary condition (b) is used, while ASETD1=N is used when Neumann boundary condition (a) is used.
- ASETD0 is used for graphs from the analogue of ASETD0 scheme in[3] where the boundary condition (b) is used, while ASETD0=N is used when Neumann boundary condition (a) is used.

We study the convergence for the both the small time steps and large time steps in order to show the weak stability of the schemes SETD1 and SETD0 presented in[3] and the good stability properties of our accelerated schemes ASETD1 and ASETD0 that we have proposed in this work. In all our graphs, the exact sample solutions are unknown and the reference sample solutions in each scheme in our errors computations are taken to be the numerical solution samples with that scheme with smallest time step size ($1/15360$ for graphs with small time steps in Figure 1 and $1/240$ for graphs with large time steps in Figure 2). Figure 1 shows the convergence graphs with vey small time steps with $\beta = 1$ and $\beta = 2$. The Peclet number which measures the rate of advection over the diffusion is 24. Although we have used very small time steps, we can observe that the well known SETD1 and SETD0 schemes developed in[3] are unstable at the biggest time step in the graphs.

The convergence rate of all the schemes are close to 1 for $\beta = 1$ (1.04 in Figure 1(a) and 1.01 in Figure 1(b)) and $\beta = 2$ (1.08 in Figure 1(c) and 1.05 in Figure 1(d)). This is in agreement to our theoretical result in Theorem 1. Since for boundary condition (b) Assumption 2.1 is not satisfied for F, we can also conclude that Theorem 1 even holds for large class of nonlinear function F than what we have considered.

In some graphs, we can also observe that the schemes SETD0 and ASETD0 are less accurate comparing to the schemes SETD1 and ASETD1. Indeed this is normal as the deterministic term on F is approximated accurately in SETD1 (ASETD1) than in ASETD0 (ASETD0) scheme. Figure 2 shows the convergence graphs with large time steps with $\beta = 1$ and $\beta = 2$. We

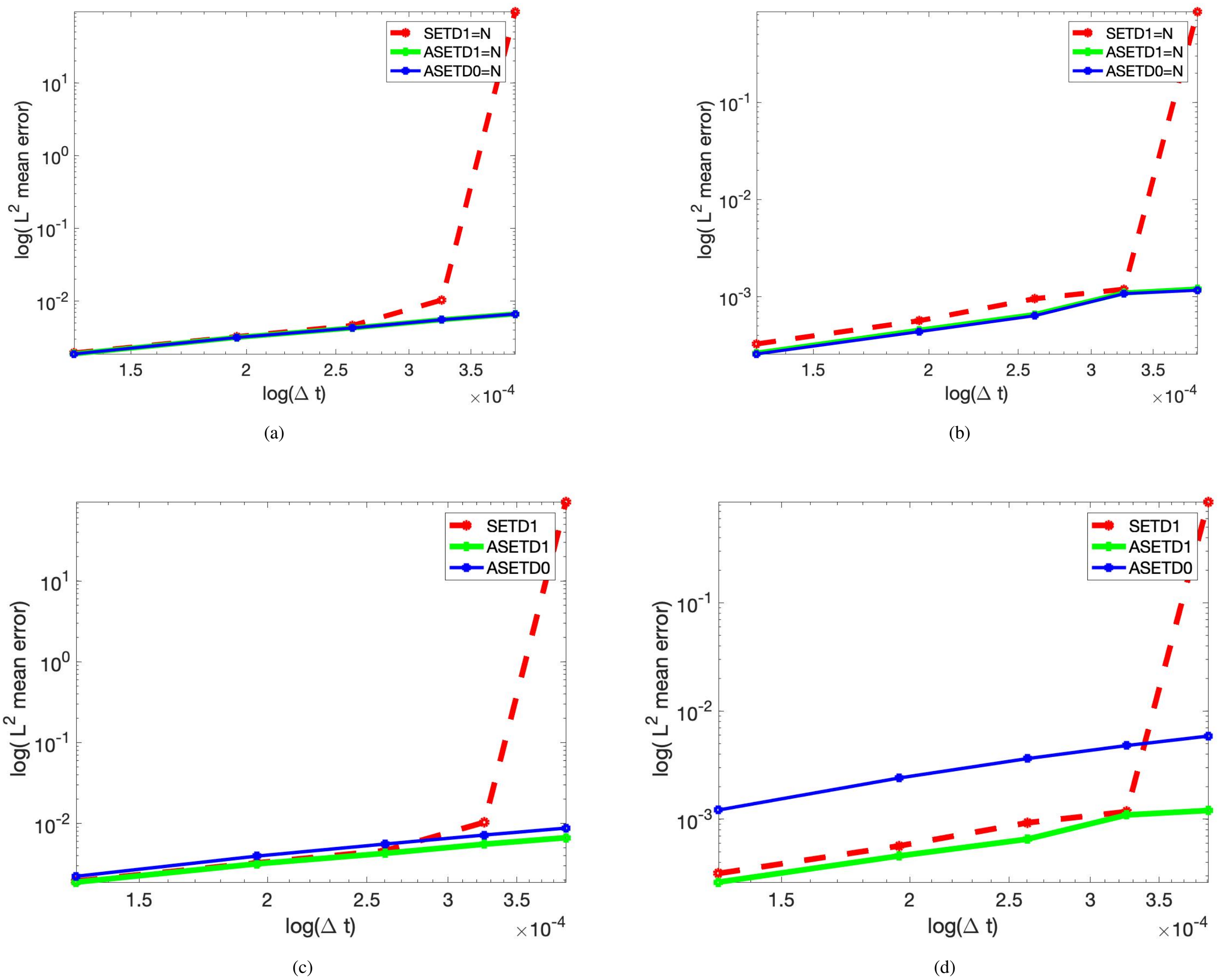

FIGURE 1 (a) Convergence with the root mean square L^2 norm at $T = 1$ as a function of Δt with 50 realizations and $\Delta x = \Delta y = 1/120$, $X_0 = 0$. The noise is white in time and in H^r in space $\beta \in \{1, 2\}$ (with $\delta = 0.05$ in (76)). The temporal order of convergence in time are close to 1 in all the graphs. Graphs in (a) and (b) are for $\beta = 1$, while graphs in (c) and (d) are for $\beta = 2$. The reference solution used in the errors computation of each scheme is the numerical sample solution from that scheme with $1/15360$.

can observe that the schemes ASETD1 and SETD0 are still stable for large time steps and that the convergence rate are still called to the theoretical result in Theorem 1 as we have about 0.95 in Figure2 for $\beta = 1$, and 1.05 for $\beta = 1$.

5 | CONCLUSION

In this paper, we proposed two stable explicit exponential integrators to solve numerically semilinear parabolic partial differential equations driven by additive noise, with a linear operator not necessary self-adjoint. Such equations are also called stochastic convection-recation-diffusion equations. This generalises the known results in the literature for reaction-diffusion equation. Moreover our analysis is done under less regularity assumptions of the nonlinear drift function. Our schemes are accelerated and achieve higher convergence rate. For instance for additive trace class noise converge rate approximately 1 is recovered. We provided numerical experiment to illustrate our findings.

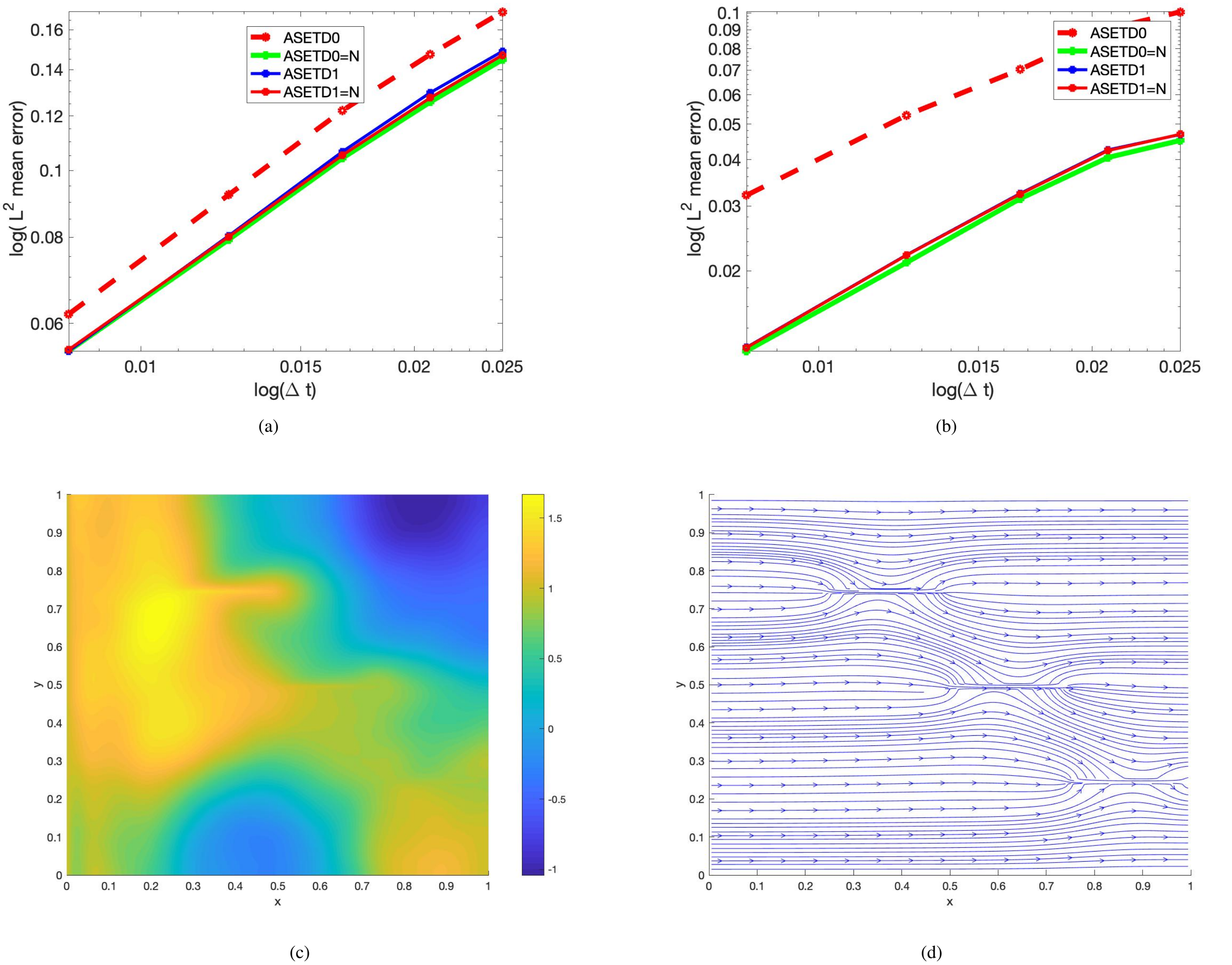

FIGURE 2 (a) Convergence with the root mean square L^2 norm at $T = 1$ as a function of Δt with 50 realizations and $\Delta x = \Delta y = 1/120$, $X_0 = 0$. The noise is white in time and in H^r in space $\beta \in \{1, 2\}$ (with $\delta = 0.05$ in (76)). The temporal order of convergence in time are close to 1 in all the graphs. Graphs in (a) are for $\beta = 1$, while graphs in (b) are for $\beta = 2$. The reference solution used in the errors computation of each scheme is the numerical sample solution from that scheme with $1/240$. A sample of reference solution is in (c) while the streamline of the Darcy's velocity $\mathbf{q}$ is given in (d)

References

1. Engel K J, Nagel R. One-Parameter Semigroup for Linear Evolution Equations. New York: Springer-Verlag; 2000.

2. Fujita H, Suzuki T. Evolutions problems (part1) in: P. G. Ciarlet and J. L. Lions(eds.), Handb. Numer. Anal., vol. II: North-Holland, 1991 pp. 789–928.

3. Gabriel J L, Tambue A. Stochastic exponential integrators for a finite element discretisation of SPDEs with additive noise. Appl. Numer. Math. 2019; 136: 163–182.

4. Gabriel J L, Tambue A. A modified semi-implicit Euler-Maruyama scheme for finite element discretization of SPDEs with additive noise. Appl. Math. Comput. 2018; 332: 105–122.

5. Gabriel J L, Tambue A. Stochastic exponential integrators for the finite element discretization of SPDEs for multiplicative and additive noise. IMA J. Numer. Anal. 2013; 2: 515–543.

6. Kruse R. Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise. IMA J. Numer. 2014; 34: 217–251.

7. Jentzen A, Kloeden P E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. Proc. R. Soc. A, 2009; 456: 649–667.

8. Jentzen A, Kloeden P E, Winkel C. Efficient simulation of nonlinear parabolic SPDEs with additive noise. Ann. Appl. Probab. 2011; 21(3): 908–950.

9. Kloeden P E, Lord G J, Neuenkirch A, Shardlow T. The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds. J. Comput. Appl. Math. 2011; 235: 1245–1260.

10. Kruse R, Larsson S. Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise. Electron. J. Probab. 2012; 17(65): 1–19.

11. Larsson S. Nonsmooth data error estimates with applications to the study of the long-time behavior of finite element solutions of semilinear parabolic problems. Preprint 1992-36, Department of Mathematics, Chalmers University of Technology: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.28.1250

12. Lions J L. Espaces d'interpolation et domaines du puissances fractionnaires d'opérateurs. J. Math. Soc. Jpn. 1962; 14: 233–241.

13. Mukam J D, Tambue A. Strong convergence analysis of the stochastic exponential Rosenbrock scheme for the finite element discretization of semilinear SPDEs driven by multiplicative and additive noise. J. Sci. Comput. 2018; 74(2): 937–978.

14. Mukam J D, Tambue A. A note on exponential Rosenbrock-Euler method for the finite element discretization of a semilinear parabolic partial differential equation. Comput. Math. Appl. 2018; 76(7): 1719–1738.

15. Prato D, Zabczyk G J. Stochastic equations in infinite dimensions. Cambridge, United Kingdom: Cambrige University Press; 2014

16. Prévôt C, and Röckner M. A Concise Course on Stochastic Partial Differential Equations. New York: Springer; 2007.

17. Tambue A, Ngnotchouye J M T. Weak convergence for a stochastic exponential in- tegrator and finite element discretization of stochastic partial differential equation with multiplicative & additive noise. Appl. Num. Math. 2016; 108: 57–86.

18. Wang X, Ruisheng Q. A note on an accelerated exponential Euler method for parabolic SPDEs with additive noise. Appl. Math. Lett. 2015; 46: 31–37.

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