ON AN INITIAL AND NONLOCAL INTEGRAL BOUNDARY CONDITION FOR A MIXED TYPE EQUATION

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Abstract. On an initial and boundary value problem for a mixed type equation is considered. A uniqueness theorem for the solvability of this problem is shown and constructed the solution as the sum of Fourier series. The stability of the solution with respect to initial function is proved.

1. Introduction

We are considered an initial and boundary value problem generated by the mixed hyperbolic-parabolic type equation

$$Lu(x,t) \equiv \begin{cases} u_t - u_{xx} = 0, & t > 0, \\ u_{tt} - u_{xx} = 0, & t < 0, \end{cases} \tag{1.1}$$

on the domain $D = \{x, t | 0 < x < 1, -\alpha < t < \beta\}$, where $\alpha > 0$, $\beta > 0$, with the initial condition

$$u(x, -\alpha) = \psi(x), \quad 0 \leq x \leq 1, \tag{1.2}$$

and nonlocal boundary conditions

$$u_x(0,t) = 0, \quad -\alpha \leq t \leq \beta, \tag{1.3}$$

$$\int_0^1 xu(x,t)dx = 0, \quad -\alpha \leq t \leq \beta. \tag{1.4}$$

It is encountered with parabolic-hyperbolic, elliptic-hyperbolic type equations in electromagnetic events, gas dynamics and similar non-homogeneous processes. In [1], it is shown that the movement of gas in channel with wave equation $u_{tt} = a\Delta u$, out of channel the movement of gas with diffusion equation $u_t = b\Delta u$ where a, b positive physical parameters, Δ is Laplasian are expressed. In [2], it is examined that the distribution of electric waves in semi-infinite line have loss in $0 < x < \ell$ and in the remaining part no loss of distribution. The model of the problem is stated with first order equation systems for induction, electric current, resistance and density. From the equation systems, eliminating

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the electric current the problem is reduced to the boundary problem that is defined initial-boundary and transmission conditions which is hyperbolic in $0 < x < \ell$, parabolic in $\ell < x < \infty$. In [3], electromagnetic movement of liquid is investigated and this process is expressed with boundary problem for parabolic-hyperbolic type equation in multi dimensional space. It is known that there exists applications of hyperbolic-elliptic type equations with Tricomi type in gas dynamics. Boundary problems for mixed type equations attract attention with physcial meanings and investigations are carried out about this subject [4]-[14].

In mathematical physics equations, nonlocal boundary conditions show that physical process not only at the point but also at the whole object. This type boundary conditions are examined in [14]-[18] and many other works for different mathematical physics equations. In [15], it is encountered with this boundary condition for diffusion equation in plasma problem. Some references for mixed type equations with nonlocal boundary conditions and integral boundary conditions are given in [7]-[14]. In this work, parabolic-hyperbolic type equation (1.1) is considered. Firstly, (1.4) boundary condition that is defined with integral is reduced to the nonlocal point boundary condition. In Section 2, this problem is reduced to the spectral problem for ordinary differential equation by the separation of variables method and the spectral problem is examined. In Section 3, for mixed type problem the solution is constructed and the uniqueness of the solution is proved. In Section 4, the existence of the solution and in the last section the the stability of the solution is shown.

Firstly, let us reduce the (1.4) boundary condition into the point boundary condition. For this, in the equation (1.1) fixing the parameter t multiplying by x, then integrating according to the x from ε to $1-\varepsilon$ such as $\varepsilon > 0$ sufficiently small number, we get;

$$\int_{\varepsilon}^{1-\varepsilon} u_t dx - \int_{\varepsilon}^{1-\varepsilon} u_{xx} dx, \quad t > 0,$$

$$\int_{\varepsilon}^{1-\varepsilon} u_{tt} dx - \int_{\varepsilon}^{1-\varepsilon} u_{xx} dx, \quad t < 0.$$

Here when $\varepsilon \to 0$

$$\frac{d}{dt}\left(\int_0^1 xudx\right) = u_x(1,t) - u(1,t) + u(0,t), \quad t \geq 0,$$

$$\frac{d^2}{dt^2}\left(\int_0^1 xudx\right) = u_x(1,t) - u(1,t) + u(0,t), \quad t \leq 0,$$

obtained and as a result (1.4) integral boundary condition is reduced to the

$$u(0,t) - u(1,t) + u_x(1,t) = 0 \tag{1.5}$$

nonlocal condition that contains both ends. Therefore, in this study the solution of equation (1.1) is investigated satisfying

$$u(x,t) \in \Omega = C(\overline{D}) \cap C^1(D) \cap C^1_x(\overline{D}) \cap C^2(D_-) \cap C^2_x(D_+) \tag{1.6}$$

and (1.2) initial condition and (1.3), (1.5) boundary conditions.

2. Spectral Problem

Let us find the non-trivial solution of boundary value problem (1.1)-(1.6) by the separation of variables method with the form $u(x,t) = X(x)T(t)$. Substituting this statement in (1.1) and in the boundary conditions (1.3), (1.5), we encounter a spectral problem for $X(x)$;

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \ell, \tag{2.1}$$

$$X'(0) = 0, \tag{2.2}$$

$$X'(1) + X(0) - X(1) = 0, \tag{2.3}$$

and ordinary differential equation for $T(t)$;

$$T'(t) + \lambda T(t) = 0, \quad 0 < t < \beta, \tag{2.4}$$

$$T''(t) + \lambda T(t) = 0, \quad -\alpha < t < 0, \tag{2.5}$$

where λ is complex parameter. The boundary conditions (2.2), (2.3) are regular and strongly regular ([19] pp 66-67). It shown that the eigenvalues of the boundary value problem (2.1)-(2.3) are the roots of the following equation

$$\mu \sin \mu + \cos \mu - 1 = 0.$$

From the equation we obtain the eigenvalues of the problem (2.1)-(2.3) form two infinite sequences

$$\lambda_{1,k} = (2k\pi)^2, \quad k = 0, 1, 2, ...,$$

$$\lambda_{2,k} = [(2k+1)\pi]^2 \left[1 + O\left(\frac{1}{k}\right)\right], \quad k = N_0, N_0+1, ...,$$

where N_0 is a positive integer and the corresponding eigenfunctions are of the form

$$X_{1,k}(x) = \cos 2k\pi x, \quad k = 0, 1, 2, ...,$$

$$X_{2,k}(x) = \cos(2k+1)\pi x + O\left(\frac{1}{k}\right).$$

According to the results of [20], [21] (see [19]) we have the system of eigenfunctions $X_k(x) = \{X^{(x)}_{k,1}, X^{(x)}_{k,2}\}$ form Riesz basis in $L_2(0,1)$. The system of eigenfunctions of the boundary value problem (2.1)-(2.3) is complete and minimal in $L_2(0,1)$. The minimality of this system follow from the fact that this system has a biorthogonal system consisting of the eigenfunctions of the adjoint boundary value problem (see [19], p.99):

$$Y''(x) + \lambda Y(x) = 0, \quad 0 < x < 1, \tag{2.6}$$

$$Y'(0) - Y(1) = 0, \tag{2.7}$$

$$Y'(1) - Y(1) = 0. \tag{2.8}$$

Boundary conditions (2.7)-(2.8) are regular, also strong regular. Then the eigenfunctions of the adjoint boundary value problem form Riesz basis in $L_2(0,1)$. The eigenvalues of this boundary value problem coincides with the eigenvalues of the spectral problem (2.1)-(2.3). The eigenfunction $y_0(x) = x$ corresponding to the eigenvalue $\lambda_{1,0} = 0$ and to the eigenvalues $\lambda_{1,k}$, $\lambda_{2,k}$ corresponding the eigenfunctions

$$Y_{k,1}(x) = 2k\pi \cos 2k\pi x + \sin 2\pi kx, \quad k = 1, 2, ...,$$

$$Y_{k,2}(x) = 2\cos(2k+1)\pi x + O\left(\frac{1}{k}\right).$$

Let $\{Y_k(x)\} = \{Y_{0,1}, Y_{k,1}, Y_{k,2}\}$ be the eigenfunctions of the adjoint problem (2.6)-(2.8).

3. Uniqueness of the solution

Substituting $\lambda = \mu_k^2$ in (2.4),(2.5) we obtain

$$T_k(t) = \begin{cases} a_k e^{-\mu_k^2 t}, & t > 0, \\ b_k \cos \mu_k t + c_k \sin \mu_k t, & t < 0, \end{cases} \tag{3.1}$$

where a_k, b_k, c_k are arbitrary constants. $u(x,t) \in \Omega$, $u(x,t) = X_k(x)T_k(t)$ also has this property and let us choose the constants a_k, b_k, c_k that below the following conditions holds

$$T_k(0+) = T_k(0-), \quad T_k'(0+) = T_k'(0-). \tag{3.2}$$

When $a_k = c_k$, $b_k = -c_k\mu_k$, the function (3.1) provides the condition (3.2). Therefore, the function (3.1) is found as follow:

$$T_k(t) = \begin{cases} c_k e^{-\mu_k^2 t}, & t > 0, \\ c_k \cos \mu_k t - c_k\mu_k \sin \mu_k t, & t < 0. \end{cases} \tag{3.3}$$

Let $u(x,t)$ be the solution of (1.1)-(1.5). Let us take account the following function:

$$u_k(t) = \int_0^1 u(x,t)Y_k(x)dx, \tag{3.4}$$

and find the differential equation satisfying the $u_k(t)$. First we form the following function:

$$u_{k,\varepsilon}(t) = \int_\varepsilon^{1-\varepsilon} u(x,t)Y_k(x)dx, \tag{3.5}$$

here $\varepsilon > 0$ sufficiently small number. Derivating the equation (3.5) according to the t once when $t > 0$, twice when $t < 0$, we obtain

$$u'_{k,\varepsilon}(t) = \int_{\varepsilon}^{1-\varepsilon} u_t(x,t)Y_k(x)dx = \int_{\varepsilon}^{1-\varepsilon} u_{xx}(x,t)Y_k(x)dx, \tag{3.6}$$

$$u''_{k,\varepsilon}(t) = \int_{\varepsilon}^{1-\varepsilon} u_{tt}(x,t)Y_k(x)dx = \int_{\varepsilon}^{1-\varepsilon} u_{xx}(x,t)Y_k(x)dx. \tag{3.7}$$

Applying two times partial integration by the right side of (3.6) and (3.7) and taking to the limit when $\varepsilon \to 0$, according to the conditions (2.2), (2.3), we find the following equations:

$$u'_k(t) + \mu_k^2 u_k(t) = 0, \quad t > 0, \tag{3.8}$$

$$u''_k(t) + \mu_k^2 u_k(t) = 0, \quad t < 0. \tag{3.9}$$

For $\lambda = \mu_k^2$, (3.8) and (3.9) differential equations coincide with the equations (2.4) and (2.5) and $u_k(t) \equiv T_k(t)$ for $-\alpha \le t \le \beta$. Hence the functions $u_k(t)$ identified with (3.3). To find the coefficients c_k, let us supply the initial value (2.2):

$$u_k(-\alpha) = \int_0^1 u(x,-\alpha)Y_k(x)dx = \int_0^1 \psi(x)Y_k(x)dx = \psi_k. \tag{3.10}$$

Then, from (3.3) and (3.10) we obtain

$$c_k[\cos\mu_k\alpha + \mu_k \sin\mu_k\alpha] = \psi_k. \tag{3.11}$$

From here when

$$d(k) = \cos\mu_k\alpha + \mu_k \sin\mu_k\alpha \neq 0, \tag{3.12}$$

$$c_k = \frac{\psi_k}{\cos\mu_k\alpha + \mu_k \sin\mu_k\alpha} = \frac{\psi_k}{d(k)}, \tag{3.13}$$

is found. Putting (3.13) into the (3.3)

$$u_k(t) = \begin{cases} \frac{\psi_k}{d(k)} e^{-\mu_k^2 t}, & t > 0, \\ \frac{\cos\mu_k t - \mu_k \sin\mu_k t}{d(k)} \psi_k, & t < 0, \end{cases} \tag{3.14}$$

is obtained.

Now we assume that $\psi(x) \equiv 0$ and (3.12) provided for all $k = 1, 2, \ldots$. Then $\psi_k \equiv 0$ and according to the formulas (3.14), (3.4) for every $t \in [-\alpha, \beta]$

$$\int_0^1 u(x,t)Y_k(x)dx = 0, \quad (k = 1, 2, \ldots).$$

Since $\{Y_k(x)\}$ forms basis in $L_2\,(0,1)$, it is also complete sequence. Using this property from the last relation for $\forall\ t \in [-\alpha, \beta]$, $u(x,t) \equiv 0$ in almost everywhere. Since $u(x,t)$ is continuous in the closed $\overline{D}$ region, in this region $u(x,t) \equiv 0$. Below the following theorem is proved.

Theorem 3.1. *If the solution of the boundary value problem (1.1)-(1.6) exists, this solution is unique if and only if the condition (3.12) holds.*

Assume that for a α and $k = p$ numbers, (3.12) doesn't hold, then $\delta(p) = \cos\mu_p\alpha + \mu_p \sin\mu_p\alpha = 0$. In that case, it is found non-zero solution of homogeneous problem (1.1)-(1.6) $(\psi(x) \equiv 0)$ as follows:

$$u_p(x,t) = \begin{cases} c_p e^{-\mu_p^2 t} X_p(x), & t > 0, \\ c_p(\cos\mu_p t - \mu_p \sin\mu_p t) X_p(x), & t < 0, \end{cases} \tag{3.15}$$

where $c_p \neq 0$ arbitrary constants.

4. Existence of the Solution

We assume that $d(k) \neq 0$ and there exists such a C_0 that $|d(k)| \geq C_0 > 0$ holds. The solution of (1.1)-(1.6) can be shown as

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x). \tag{4.1}$$

It is obvious that $v_k(x,t) = u_k(t)X_k(x)$ holds the equation (1.1). To show that (3.1) is the solution of the boundary problem (1.1)-(1.6), we must prove that (3.1) series are uniform convergence in the closed $\overline{D}$ region and derivative once according to the t, twice according to the x when $t < 0$ and when $t > 0$ two times according to the t and x term to term.

Lemma 4.1. *For every $k \in \mathbb{N}^+$ the following assessments hold:*

$$|u_k(t)| \leq A_1 k |\psi_k|, \quad |u_k'(t)| \leq A_2 k^2 |\psi_k|, \tag{4.2}$$

for $t \in [-\alpha, \beta]$ and

$$|u_k''(t)| \leq A_3 k^3 |\psi_k|, \tag{4.3}$$

for $t \in [-\alpha, 0]$.

Proof. From (3.14) for $t \in [0, \beta]$

$$|u_k(t)| = \left| \frac{e^{-\mu_k^2 t}}{d(k)} \psi_k \right| \leq \frac{|\psi_k|}{C_0} \leq \widetilde{A}_1 k |\psi_k|,$$

$$|u_k'(t)| = \left| \frac{-\mu_k^2 e^{-\mu_k^2 t}}{d(k)} \psi_k \right| \leq \frac{\mu_k^2}{C_0} |\psi_k| \leq \widetilde{A}_2 k^2 |\psi_k|$$

is found. Similarly, for $t \in [-\alpha, 0]$

$$|u_k(t)| = \left| \frac{\psi_k}{d(k)} (\cos\mu_k t - \mu_k \sin\mu_k t) \right| \leq \frac{|\psi_k|}{C_0} \sqrt{1 + \mu_k^2} \leq \widetilde{A}_3 k |\psi_k|,$$

$$|u_k'(t)| = \left| \frac{\psi_k \ \mu_k}{d(k)} (\sin\mu_k t + \mu_k \cos\mu_k t) \right| \leq \frac{\mu_k |\psi_k|}{C_0} \sqrt{1 + \mu_k^2} \leq \widetilde{A}_4 k^2 |\psi_k|$$

is obtained. From (3.9)

$$\left|u_k''(t)\right| = \mu_k^2 \left|u_k(t)\right| \leq \widetilde{A}_3 k^3 \left|\psi_k\right|,$$

here, $\widetilde{A}_j$ $(j = 1, 2, 3, 4)$ are positive constants. When $t \in [0, \beta]$ and $t \in [-\alpha, 0]$, for the functions $u_k(t), u_k'(t), u_k''(t)$ (4.2), (4.3) inequalities are found from the obtained assessments. The proof is completed.

□

Applying Lemma 3.1, the series (4.1) and the first order derivatives of (4.1) in the closed region $\overline{D}$, the second order derivatives of (4.1) appropriately in $\overline{D}_+$ and $\overline{D}_-$ with the assessments according to the (4.2), (4.3) can be limited from above below the following series:

$$A_4 \sum_{k=1}^{\infty} k^3 \left|\psi_k\right|. \tag{4.4}$$

Now let us obtain assessments for ψ_k.

Lemma 4.2. $\psi_k \in C^4[0, \ell]$ *and let boundary conditions are satisfied. Then,*

$$\psi_k = \frac{\psi_k^{(4)}}{\left|\mu_k\right|^4}, \quad k \in \mathbb{Z}^+, \tag{4.5}$$

$$\sum_{k=0}^{\infty} \left|\psi_k^{(1)}\right|^2 \leq \left\|\psi^{(4)}\right\|_{L_2(0,1)}. \tag{4.6}$$

Proof. Let take account the integral (3.14):

$$u_k(-\alpha) = \int_0^1 u(x, -\alpha) Y_k(x) dx = \int_0^1 \psi(x) Y_k(x) dx = \psi_k.$$

According to the (3.1)

$$\psi_k = \int_0^1 \psi(x) Y_k(x) dx = -\frac{1}{\left|\mu_k\right|^2} \int_0^1 \psi(x) Y_k''(x) dx.$$

By integrating twice successively and taking consider the conditions of lemma, we find

$$\psi_k = -\frac{1}{\mu_k^2} \int_0^1 \psi''(x) Y_k(x) dx = -\frac{1}{\mu_k^2} \psi_k^{(2)}. \tag{4.7}$$

As a result of the following similar process we obtain

$$\psi_k^{(2)} = -\frac{1}{\mu_k^2} \int_0^1 \psi_k^{(4)} Y_k(x) dx = -\frac{1}{\mu_k^2} \psi_k^{(4)}. \tag{4.8}$$

Putting (4.8) into (4.7) we obtain the formula (4.5).

According to the condition of lemma $\psi_k^{(4)} \in C\,[0,1]\,,$ then according to the Fourier series theorem $\sum_{k=1}^{\infty} |\psi_k|^2$ is convergent and (4.6) Bessel inequality is hold. The proof is completed. □

Under the conditions Lemma 4.2, (4.4) is bounded from above below the following numeric series:

$$A_5 \sum_{k=1}^{\infty} \frac{1}{k} \left|\psi_k^{(4)}\right|. \tag{4.9}$$

Therefore, since (4.9) is convergent and according to the Weierstrass criterion, (4.1) series

$$u_t(x,t) = \sum_{k=1}^{\infty} u_k'(t) X_k(x), \quad t > 0, \tag{4.10}$$

$$u_{tt}(x,t) = \sum_{k=1}^{\infty} u_k^{''}(t) X_k(x), \ t < 0, \tag{4.11}$$

$$u_{xx}(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k''(x), \ t < 0, \tag{4.12}$$

series are absolute and uniform convergent appropriately in closed $\overline{D}_+$ and $\overline{D}_-$ regions. Hence, the sum of (4.1) series $u(x,t) \in \Omega,$ then the condition (1.6) holds. Substituting (4.1), (4.10), (4.11), (4.12) into (1.1);

$$u_t - u_{xx} = \sum_{k=1}^{\infty} \left[u_k'(t) X_k(x) - u_k(t) X_k''(x)\right] \equiv 0, \ t > 0,$$

$$u_{tt} - u_{xx} = \sum_{k=1}^{\infty} \left[u_k''(t) X_k(x) - u_k(t) X_k''(x)\right] \equiv 0, \ t < 0,$$

are obtained. Then (3.1) satisfied the equation (1.1). As a result, below the following theorem is proved.

Theorem 4.1. *There exists only one solution of (1.1)-(1.6) boundary problem and defined with the series (3.1).*

We note that, let α is a rational number, for a $k = p = k_1, k_2, ..., k_m$ if $\delta(p) = 0$, the necessary and sufficient condition to be solvable of the problem (1.1)-(1.6) is

$$\psi_k = \int_0^1 \psi(x) Y_k(x) dx = 0, \tag{4.13}$$

where $1 \leq k_1 < k_2 < ... < k_m \leq k_0$, k_i, $i = 1, 2, ..., m$, $m \in \mathbb{N}^+$ given numbers. In that case the solution is defined with the form as:

$$u(x,t) = \left(\sum_{k=1}^{k_1-1} +... \sum_{k=k_{m-1}+1}^{k_m-1} + \sum_{k=k_m+1}^{\infty}{}^{\infty} \right) u_k(t) X_k(x) + \sum_p B_p u_p(x,t). \tag{4.14}$$

In the last term the number p takes the values $k_1, k_2, ..., k_m$, B_p is arbitrary constant, $u_p(x,t)$ function is expressed as formula (3.15) such as if at the left side of the (4.14) upper bounds is less than lower bounds, this term is equal to zero.

5. The Stability of the Solution

As the stability of the solution of the boundary value problem (1.1)-(1.6), according to the function that is given initial, let us obtain assessments. Before this, we define below the following norms:

$$\|u(x,t)\|_{L_2[0,1]} = \left(\int_0^1 |u(x,t)|^2 \, dx \right)^{\frac{1}{2}}, \|u(x,t)\|_{C(\overline{D})} = \max_D |u(x,t)|.$$

Theorem 5.1. *For the $u(x,t) \in \Omega$ solution of the boundary value problem (1.1)-(1.6) below the following inequality holds:*

$$\|u(x,t)\|_{C(\overline{D})} \leq M \|\psi''(x)\|_{C[0,1]},$$

here $M > 0$ and doesn't depend on $\psi(x)$.

Proof. Let $(x,t) \in \overline{D}$ be any point. From the statement of $X_k(x)$ it is obtained that $|X_k(x)| \leq M_1 > 0$, M_1 is constant. According to the Cauchy-Schwartz inequality

$$\begin{aligned} |u(x,t)| &\leq \sum_{k=1}^{\infty} |u_k(t)| \ \ |X_k(x)| \leq M_1 \sum_{k=1}^{\infty} k \ \ |\psi_k| \leq M_2 \sum_{k=1}^{\infty} \frac{1}{k} \ \left| \psi_k^{(2)} \right| \leq \\ &\leq M_2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left| \psi_k^{(2)} \right|^2 \right)^{\frac{1}{2}} \leq M_3 \|\psi''(x)\|_{L_2(0,1)}, \end{aligned} \tag{5.1}$$

here M_i, $i = 1, 2, 3$ positive constants. The proof is completed.

□

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