Decay mild solutions of fractional differential hemivariational inequalities *

Xiuwen Li[1] Zhenhai Liu[1,2†]

[1] Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis,
Guangxi University for Nationalities,
Nanning 530006, Guangxi Province, P.R. China

[2] Guangxi Colleges and Universities Key Laboratory of Complex System
Optimization and Big Data Processing,
Yulin Normal University, Yulin 537000, P.R. China.

Abstract

In this paper, we consider fractional differential hemivariational inequalities (FDHVIs, for short) in the framework of Banach spaces. The aim of this paper is three folds. The first one is to investigate the existence of mild solutions for FDHVIs and by means of a fixed point technique we are able to avoid the hypothesis of compactness on the semigroup. The second aim is to study the existence of decay mild solutions for FDHVI via giving asymptotic behavior of Mittag-Leffler function. Finally, a mathematical model is provided to illustrate our abstract results.

Key words: *Decay mild solutions; fractional differential hemivariational inequalities; fixed point theorem; Measure of noncompactness; Mittag-Leffler function.*

1 Introduction and problem formulation

Differential variational inequalities (DVIs, for short) were firstly systematically discussed by Pang–Stewart [32] in Euclidean spaces. Since DVIs can describe various models in mechanical impact problems in engineering, operations research, and physical sciences such as electrical circuits with ideal diodes, coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks and so forth, more and more researchers have paid their attentions to the study of DVIs and a considerable effort has been made in their analysis and numerical approximation. We refer the reader to [7, 11, 19, 25, 26, 32] for some recent results on solvability, stability, and bifurcation to finite dimensional DVIs. The study of DVIs in infinite dimensional spaces is more recent. The reason is that, in contrast with the case of finite dimensional spaces, the

*Project supported by NNSF of China Grants Nos. 11671101, 11661001, NSF of Guangxi Grant No. 2018GXNSFDA138002, and the funding from the European Unions Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie (823731 CONMECH). Tel.: +86-771-3265663/3260370.

†Corresponding author. 641542785@qq.com (X. Li), zhhliu@hotmail.com (Z. Liu)

study of DVIs in infinite dimensional spaces requires regularity results of the solution of the associated variational inequalities such as measurability, continuity, or condensivity, which enables us to convert them to differential inclusions. For more details, we refer to [16, 17, 20, 21, 31] and the references therein.

It is noteworthy that fractional differential equations arise in a natural manner as mathematical models of dynamic systems that exhibit such properties as long-term memory and self-similarity. For more details on this topics the reader is welcome to consult [1, 11, 13, 14, 18, 23, 24, 34, 37, 38] and the references therein. In contrast to the more conventional DVIs, FDHVIs represent an important extension of DVIs, which couple fractional differential or partial differential equations with a hemivariational inequality or a variational-hemivariational inequality. Very recently, Ke, Loi, and Obukhovskii [11] studied the decay solutions for a class of FDVIs in finite Banach spaces. However, until now, FDVIs in infinite dimensional spaces have not been investigated. To fill this gap, in this paper, we study a FDHVI in Banach spaces. In the present paper, by combining the topological methods and the fractional calculus we consider the existence of decay mild solutions to FDHVI. To introduce the problem we need some notations as follows.

Everywhere in this work, let E be a Banach space and U be a reflexive Banach space, endowed with the norms $\|\cdot\|_E$ and $\|\cdot\|_U$, respectively. We denote by U^* the strong topological dual of U and by $\langle\cdot,\cdot\rangle$ the duality paring mapping between U^* and U. Let 0_U represent the zero element of U. Moreover, the notation $\mathcal{L}(U,E)$ denotes the space of linear bounded operators from E to U endowed with the usual norm $\|\cdot\|_{\mathcal{L}(U,E)}$ and we abbreviate this notation to $\mathcal{L}(E)$ when $U=E$. Let $\mathbb{R}_+=[0,+\infty)$. Below, I denotes either a bounded interval of the form $[0,b]$ with $b>0$, or the unbounded interval $\mathbb{R}_+$. By $C(I;E)$ we denote the space of continuous functions on I with values in E. In the rest of the manuscript we shall use the standard notation for the Lebesque and Sobolev spaces.

Let $A:D(A)\subseteq E\to E$ be the infinitesimal generator of a C_0-semigroup $T(t)$ $(t\geq 0)$ on E and let $B:I\times E\to\mathcal{L}(U,E)$, $F:I\times E\to\mathcal{P}(E)$ (where $\mathcal{P}(E)$ is defined in Section 2). Consider also the set $K\subset U$, the functions $g:I\times E\to U^*$, $G:U\to U^*$, $\varphi:U\times U\to\mathbb{R}$ and $J:U\to\mathbb{R}$. We suppose that J is a locally Lipschitz function and J^0 denotes its generalized (Clarke) directional derivative. With these data, we consider the system consisting of a fractional differential hemivariational inequality (FDHVI, for short) as follows:

$$^{C}D_t^{\alpha}x(t)\in Ax(t)+B(t,x(t))u(t)+F(t,x(t)),\qquad a.e.\ t\in I,\tag{1.1}$$

$$u(t)\in SOL(K,g(t,x(t))+G(\cdot),\varphi,J),\qquad a.e.\ t\in I,\tag{1.2}$$

$$x(0)=x_0\in E.\tag{1.3}$$

Here $SOL(K,g(t,x(t))+G(\cdot),\varphi,J)$ stands for the solution set of the hemivariational inequality (HVI, for short): find $u=u(t)\in K$ such that

$$\begin{aligned}&\langle g(t,x(t))+G(u(t)),v-u(t)\rangle+\varphi(u(t),v)-\varphi(u(t),u(t))\\&\qquad+J^0(u(t);v-u(t))\geq 0\quad\text{for all }\ v\in K\end{aligned}\tag{1.4}$$

and $^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha\in(0,1)$ with the lower limit zero, i.e.,

$$^{C}D_t^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}[x(s)-x(0)]ds,\qquad t>0,\ 0<\alpha<1.$$

Therefore, inclusion (1.2) stands for u satisfying inequality (1.4) a.e. $t \in I$. With this remark, we note that FDHVI represents a system which couples the fractional differential equation (1.1) with the hemivariational inequalities (1.4), associated to the initial condition (1.3). Therefore, following the terminology in [16, 17, 20–22], we refer to (1.1)–(1.3) as a fractional differential hemivariational inequality. The solution of FDHVI is understood in the following sense.

Definition 1.1. *A pair of functions (x, u), with $x \in C(I; E)$ and $u : I \to K(\subset U)$ measurable, is said to be a mild solution of FDHVI (1.1)–(1.3) if there exists $f \in L^p(I; E)$ $(p > \frac{1}{\alpha})$ such that $f(t) \in F(t, x(t))$ for a.e. $t \in I$ and*

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[B(s, x(s))u(s) + f(s)]ds, \quad a.e.\ t \in I, \tag{1.5}$$

where $u(t) \in SOL(K, g(t, x(t)) + G(\cdot), \varphi, J)$ for a.e. $t \in I$ and

$$S_\alpha(t) = \int_0^\infty M_\alpha(\theta)T(t^\alpha\theta)d\theta, \qquad T_\alpha(t) = \alpha\int_0^\infty \theta M_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$M_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(1-\alpha(1+n))} = \frac{1}{\pi}\sum_{n=0}^\infty \frac{(-z)^n}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha).$$

If (x, u) is a mild solution of FDHVI (1.1)–(1.3), then $x(t)$ is called the mild trajectory and $u(t)$ is called the variational control trajectory.

The rest of the manuscript is structured as follows. In Section 2 we recall some basic definitions and results needed throughout this paper. In Section 3, the existence of the mild solution for problem FDHVI (1.1)–(1.3) is presented. In Section 4, the existence of the decay mild solution associated to FDHVI (1.1)–(1.3) is obtained. Finally, a mathematical model is provided to illustrate our abstract results.

2 Background material

In this section we review some prerequisites that are necessary in the rest of the manuscript.

A function $\varphi : U \to \mathbb{R}$ is proper if it is not identically equal to $+\infty$, i.e., the effective domain $dom\varphi = \{x \in U : \varphi(x) < +\infty\} \neq \emptyset$. It is lower semicontinuous (l.s.c., for short) if $x_n \to x$ in U, as $n \to +\infty$ implies $\varphi(x) \leq \liminf_{n\to+\infty} \varphi(x_n)$.

Following Clarke [8], we present the generalization of the gradient operator for functionals which are no longer convex, but are locally Lipschitz.

Definition 2.1. *Let $J : U \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of J at $x \in U$ in the direction $v \in U$ is defined by*

$$J^0(x; v) := \limsup_{\lambda\to 0^+,\, \xi\to x} \frac{J(\xi + \lambda v) - J(\xi)}{\lambda}.$$

The generalized gradient of $J : U \to \mathbb{R}$ at $x \in U$ is the subset of U^* given by

$$\partial J(x) := \{\xi \in U^* : \ J^0(x; v) \geq \langle \xi, v\rangle, \ \forall v \in U\}.$$

The statement below collects some basic properties.

Lemma 2.2. *([8, Proposition 2.1.2]) If $J : U \to \mathbb{R}$ is a locally Lipschitz function, then there hold:*

(i) *for all $x, v \in U$, one has $J^0(x; v) = \max\{\langle \xi, v\rangle : \xi \in \partial J(x)\}$;*

(ii) *for every $x \in U$, $\partial J(x)$ is a nonempty, convex, weak*-compact subset of U^* and $\|\xi\|_{U^*} \leq K$ for any $\xi \in \partial J(x)$, where $K > 0$ is the Lipschitz constant of J near x.*

(iii) *For every $x \in U$, the function $U \ni v \mapsto J^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $J^0(x; \lambda v) = \lambda J^0(x; v)$ for all $\lambda \geq 0$, $v \in U$ and $J^0(x; v_1 + v_2) \leq J^0(x; v_1) + J^0(x; v_2)$ for all v_1, $v_2 \in U$, respectively.*

In the sequel, we proceed with the definition of some classes of operators.

Definition 2.3. *[29, Definition 9] An operator $G : U \to U^*$ is said to be:*

(a) *monotone, if for all u, $v \in U$, we have $\langle G(u) - G(v), u - v\rangle \geq 0$;*

(b) *bounded, if G maps bounded sets of U into bounded sets of U^*;*

(c) *pseudomonotone, if it is bounded and $u_n \to u$ weakly in U with*

$$\limsup\langle G(u_n), u_n - u\rangle \leq 0$$

imply $\liminf\langle G(u_n), u_n - v\rangle \geq \langle G(u), u - v\rangle$ *for all $v \in U$.*

Next, by $\mathcal{P}(E)$ [$\mathcal{P}_{cl}(E)$, $\mathcal{P}_b(E)$, $\mathcal{P}_{cv}(E)$, $\mathcal{P}_{(w)cp}(E)$], we denote the collections of all nonempty [respectively, nonempty closed, nonempty bounded, nonempty convex, nonempty (weakly) compact] subsets of the Banach space E. Now, we list the following definition.

Definition 2.4. *[24, Definition 2.5] A multimap $F : E \to \mathcal{P}(U)$ is said to be:*

(i) *upper semicontinuous (u.s.c., for short), if for every open subset $O \subset U$ the set*

$$F^{+1} = \{x \in E : F(x) \subset O\}$$

is open in E;

(ii) *closed if its graph*

$$\{(x, y) : x \in E,\ y \in F(x)\}$$

is a closed subset of $E \times U$;

(iii) *compact, if its range $F(E)$ is relatively compact in U, i.e. $\overline{F(E)}$ is compact in U;*

(iv) *quasicompact, if its restriction to any compact subset $K \subset E$ is compact.*

Moreover in the sequel, we will use the following results.

Definition 2.5. *[12, Definition 1.1.5]. Let E, U be two Banach spaces. A multimap $F : E \to \mathcal{P}(U)$ is said to be locally weakly compact, if for any $x \subset E$, there exists a $\delta > 0$ such that the set $\bigcup_{z \in B_E(x,\delta)} F(z)$ (where $B_E(x, r)$ denotes the open ball of center $x \in E$ and radius $r > 0$) is relatively compact with respect to the weak topology w.*

Proposition 2.6. *[6, Proposition 4]. Let $F : E \to \mathcal{P}(U)$ be a strongly-weakly closed graph (i.e., if $x_n \to x$ in E and $f_n \to f$ weakly in U with $f_n \in F(x_n)$, then $f \in F(x)$) locally weakly compact multimap. Then F is u.s.c. from E to U_w (the symbol U_w stands for Y equipped with the weak topology).*

We now briefly focus on a few facts about the measure of noncompactness (cf. [5, 12]).

Definition 2.7. *Let E be a Banach space. A map $\beta : \mathcal{P}_b(E) \to \mathbb{R}_+$ is called a measure of noncompactness (MNC, for short) in E if $\beta(\overline{co}\Omega) = \beta(\Omega)$ for every $\Omega \in \mathcal{P}_b(E)$.*

In particular, a MNC β is called:

(i) *monotone, if $\Omega_1,\ \Omega_2 \in \mathcal{P}_b(E),\ \Omega_1 \subseteq \Omega_2$ implies $\beta(\Omega_1) \leq \beta(\Omega_2)$;*

(ii) *nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E,\ \Omega \in \mathcal{P}_b(E)$;*

(iii) *invariant with respect to flection through the origin, if $\beta(-\Omega) = \beta(\Omega)$ for every $\Omega \in \mathcal{P}_b(E)$;*

(iv) *algebraically semiadditive, if $\beta(\Omega_1+\Omega_2) \leq \beta(\Omega_1)+\beta(\Omega_2)$ for every $\Omega_1,\ \Omega_2 \in \mathcal{P}_b(E)$;*

(v) *regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.*

An important example of the MNC possessing all of above properties is the Hausdorff MNCχ which can be defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \ \varepsilon\text{-net}\},$$

for all $\Omega \in \mathcal{P}_b(E)$.

For $I = [0, b]$ with $b > 0$, it is known that the Hausdorff MNC on the space $C(I; \mathbb{R}^n)$ is given by

$$\chi_b(\Omega) = \frac{1}{2} \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{t_1,t_2 \in I, |t_1-t_2| \leq \delta} \|x(t_1) - x(t_2)\|_{\mathbb{R}^n}.$$

The last measure can be seen as the modulus of equicontinuity of a subset in the space $C(I; \mathbb{R}^n)$. However, on the space $C(I; E)$ with E being of infinite dimension, there is no such formulation as above. In fact, if $\Omega \subset C(I; E)$ is an equicontinuous set, then

$$\chi_b(\Omega) = \sup_{t \in I} e^{-Lt} \chi(\Omega(t)),$$

where χ is the Hausdorff MNC in E.

Now, consider the space $BC(\mathbb{R}_+; E)$ of bounded continuous functions on $\mathbb{R}_+$ taking values on E. Denote by π_b the restriction operator on this space, i.e., $\pi_b(x)$ is the restriction of x to $[0, b]$. Then

$$\chi_\infty(\Omega) = \sup_{b>0} \chi_b(\pi_b(\Omega)), \quad \Omega \subset BC(\mathbb{R}_+; E),$$

is a MNC. We give some measures of noncompactness as follows

$$d_b(\Omega) = \sup_{u \in \Omega} \sup_{t \geq b} \|u(t)\|,$$

$$d_\infty(\Omega) = \lim_{b \to \infty} d_b(\Omega),$$

$$\chi^*(\Omega) = \chi_\infty(\Omega) + d_\infty(\Omega).$$

The proof for the regularity of MNC χ^* is similar to that of [11, Lemma 2.1].

In the sequel, we would like to employ a relation between the so-called k-condensing and k-Lipschitz properties of a nonlinear map. Let $\widetilde{E}$ be another Banach space and $\widetilde{\chi}$ the Hausdorff MNC on $\widetilde{E}$. A mapping $G : E \to \widetilde{E}$ is said to be condensing with respect to a constant k (k-condensing) if

$$\widetilde{\chi}(G(\Omega)) \leq k\chi(\Omega), \quad \forall\, \Omega \in \mathcal{P}(E).$$

It is well known from [2] that if G is a Lipschitz map with a constant k (k-Lipschitz), i.e.,

$$\|G(x) - G(y)\|_{\widetilde{E}} \leq k\|x - y\|_E, \quad \forall\, x,\ y \in E,$$

then G is k-condensing.

Then the following property is evident.

Lemma 2.8. *Let χ be the Hausdorff MNC on a Banach space E, $\Omega \in \mathcal{P}_b(E)$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset \Omega$ such that*

$$\chi(\Omega) \leq 2\chi(\{x_n\}_{n=1}^\infty) + \varepsilon, \quad \forall \varepsilon > 0.$$

Let $\chi : \mathcal{P}_b(E) \to \mathbb{R}_+$ be a MNC in E. The next definition is necessary.

Definition 2.9. *([12, Definition 2.2.6]). A multimap $\mathcal{F} : E \to \mathcal{P}_{cp}(E)$ is called χ-condensing if for every $\Omega \subseteq E$ that is not relatively compact we have $\chi(\mathcal{F}(\Omega)) \ngeq \chi(\Omega)$.*

In the sequel, we briefly focus on the following notion. A multimap $F : I \to \mathcal{P}(E)$ is called p-time integrably bounded, if there exists a function $\delta \in L^p(I; \mathbb{R}_+)$ such that

$$\|\Phi(t)\|_E := \sup\{\|\phi(t)\|_E : \phi(t) \in \Phi(t)\} \leq \delta(t), \quad \text{for a.e.} \quad t \in I.$$

We will also use the following definition in this paper.

Definition 2.10. *[24, Definition 2.12] The sequence $\{f_n\}_{n=1}^\infty \subset L^p(I; E)$ is said to be p-time semicompact if it is p-time integrably bounded and the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact in E for a.e. $t \in I$.*

The key tool to get our main results is the following fixed point theorem.

Theorem 2.11. *([12, Corollary 3.3.1]). If Ω is a bounded convex closed subset of E, and $\mathcal{F} : \Omega \to \mathcal{P}_{cv,cp}(\Omega)$ is an u.s.c. χ-condensing multimap, where χ is a nonsingular MNC defined on subsets of Ω, then* $\operatorname{Fix} \mathcal{F} := \{x \in \Omega : x \in \mathcal{F}(x)\} \neq \emptyset$.

3 Existence of mild solutions

In this section, we consider the existence of mild solutions for the FDHVI on a bounded interval $I = [0, b]$ with $b > 0$. Before stating and proving the main results of this section, we consider the following hypotheses.

H$(A)_1$ The operator A is the infinitesimal generator of a uniformly bounded C_0-semigroup $T(t)$ $(t \geq 0)$ (i.e., $\sup\limits_{t\in[0,+\infty)} \|T(t)\| \leq M$) on Banach space E.

H(B) $B : I \times E \to \mathcal{L}(U, E)$ is such that

(1) $B(\cdot, x) : I \to \mathcal{L}(U, E)$ is continuous for all $x \in E$;

(2) there is a constant $L_B > 0$ such that

$$\|B(t, x) - B(t, y)\|_{\mathcal{L}(U,E)} \leq L_B \|x - y\|_E, \quad \text{for a.e.} \;\; t \in I, \quad \text{all} \;\; x,\; y \in E;$$

(3) there exists a constant $d > 0$ such that

$$\|B(t, 0)\|_{\mathcal{L}(U,E)} \leq d, \quad \text{for all} \;\; t \in I.$$

H(F) $F : I \times E \to \mathcal{P}_{cv,cp}(E)$ is such that

(1) for all $x \in E$, $t \to F(t, x)$ is measurable;

(2) for a.e. $t \in I$, $F(t, \cdot)$ has a strongly-weakly closed graph;

(3) there are a function $a \in L^p(I; \mathbb{R}_+)(p > \frac{1}{\alpha})$ and a constant $c > 0$ such that

$$\|F(t, x)\| := \sup\{\|f\|_E : \; f \in F(t, x)\} \leq a(t) + c\|x\|_E,$$

for a.e. $t \in I$, all $x \in E$;

(4) for every bounded subset $\Omega \subset E$, there is a constant $M_F > 0$ such that

$$\chi(F(t, \Omega)) \leq M_F \chi(\Omega), \quad \text{for a.e. } t \in I,$$

where χ stands for the Hausdorff MNC in the space E.

H(K) K is a closed convex subset of U such that $0_U \in K$.

H(G) $G : U \to U^*$ is such that

(1) it is pseudomonotone;

(2) there exists a constant $\alpha_G > 0$, β, $\gamma \in \mathbb{R}$ and $u_0 \in K$ such that

$$\langle G(v), v - u_0\rangle \geq \alpha_G \|v\|_U^2 - \beta\|v\|_U - \gamma, \quad \text{for all} \;\; v \in U;$$

(3) strongly monotone, i.e., there exists $m_G > 0$ such that

$$\langle G(v_1) - G(v_2), v_1 - v_2\rangle \geq m_G \|v_1 - v_2\|_U^2, \quad \text{for all} \;\; v_1,\; v_2 \in U.$$

H(φ) : The functional $\varphi : U \times U \to \mathbb{R}$ is such that

(1) $\varphi(\eta, \cdot) : U \to \mathbb{R}$ is convex and lower semicontinuous for all $\eta \in U$;

(2) $\varphi(u, \lambda v) = \lambda\varphi(u, v)$, $\forall\; u,\; v \in U,\; \lambda > 0$;

(3) $\varphi(v, v) \geq 0$, $\forall\; v \in U$;

(4) there exists $\alpha_\varphi > 0$ such that

$$\varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \leq \alpha_\varphi \|\eta_1 - \eta_2\|_U \|v_1 - v_2\|_U,$$

for all $\eta_1,\; \eta_2,\; v_1,\; v_2 \in U$.

H(J) : The locally Lipschitz functional $J : U \to \mathbb{R}$ is such that

(1) there exists $\kappa_0,\ \kappa_1 \geq 0$ such that

$$\|\partial J(v)\|_{U^*} \leq \kappa_0 + \kappa_1 \|v\|_U \quad \text{for all} \quad v \in U;$$

(2) there exists $\alpha_J > 0$ such that

$$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \leq \alpha_J \|v_1 - v_2\|_U^2, \qquad \text{for all} \quad v_1,\ v_2 \in U.$$

H(g) $g : I \times E \to U^*$ is such that

(1) $g(\cdot, x) : I \to U^*$ is continuous for all $x \in E$;

(2) there is a constant $L_g > 0$ such that

$$\|g(t,x) - g(t,y)\|_{U^*} \leq L_g \|x - y\|_E, \quad \text{for a.e.} \quad t \in I, \quad \text{all} \quad x,\ y \in E;$$

(3) there exists a constant $\ell > 0$ such that

$$\|g(t,x)\| \leq \ell, \quad \text{for a.e. } t \in I, \quad \text{all} \quad x \in E.$$

Now, we are first concerned with the following HVI. Throughout this paper, we denote

$$Q(z) = \{u \in K : \langle G(u), v - u\rangle + \varphi(u,v) - \varphi(u,u) + J^0(u; v - u) \geq \langle z, v - u\rangle,\ \forall\ v \in K\}.$$

The following result is due to the properties of the subdifferential, the surjectivity result for pseudomonotone operators and the Banach fixed point theorem, whose proof can be found in [29, Theorem 18].

Lemma 3.1. *Let* H(K), H(G), H(φ)(1), (4) *and* H(J) *be satisfied. Then, for each* $z \in U^*$*, the solution set* $Q(z)$ *is a singleton provided that*

$$\alpha_\varphi + \alpha_J < m_G,\ \alpha_J < \alpha_G. \tag{3.1}$$

In the sequel, for a fixed $x \in E$, consider the original form of (1.4).

$$\langle g(t,x) + G(u), v - u\rangle + \varphi(u,v) - \varphi(u,u) + J^0(u; v-u) \geq 0,\ \forall\ v \in K, \tag{3.2}$$

which can be denoted by the following from the definition of Q:

$$\begin{aligned} Q(-g(t,x)) = \{u \in K : \langle G(u), v - u\rangle &+ \varphi(u,v) - \varphi(u,u) \\ &+ J^0(u; v-u) \geq \langle -g(t,x), v - u\rangle,\ \forall\ v \in K\}. \end{aligned}$$

We now proceed with the following result.

Lemma 3.2. *Let* H(K), H(G), H(J), H(φ)(1), (4), H(g) *and (3.1) be satisfied. Then, for each* $x \in E$*, there exists a unique solution* $u \in U$ *of HVI (3.2). Moreover, the solution set* Q *is Lipschitzian, i.e.,*

$$\|Q(-g(t,x)) - Q(-g(t,y))\|_U \leq \frac{L_g}{m_B - \alpha_\varphi - \alpha_J}\|x - y\|_E \qquad \forall\, t \in I.$$

In addition, if H(φ)(2), (3) *are also satisfied, then the solution of HVI (3.2) satisfies the bound*

$$\|u\|_U \leq \frac{1}{m_G - \alpha_J}(\|G(0_U)\|_{U^*} + \ell + \kappa_0) := \rho.$$

Proof. From Lemma 3.1, it is easy to show that HVI (3.2) has a unique fixed point for each given $x \in E$.

Now, let us show that the set Q is Lipschitzian. It is clear from (3.2) that

$$\langle g(t,x) + G(u_1), v - u_1\rangle + \varphi(u_1, v) - \varphi(u_1, u_1) + J^0(u_1, v - u_1) \geq 0, \quad \forall\, v \in K, \quad (3.3)$$

$$\langle g(t,y) + G(u_2), v - u_2\rangle + \varphi(u_2, v) - \varphi(u_2, u_2) + J^0(u_2, v - u_2) \geq 0, \quad \forall\, v \in K. \quad (3.4)$$

We now take $v = u_2$ in (3.3) and $v = u_1$ in (3.4), then adding the resulting inequalities yield that

$$\begin{aligned}
&\langle G(u_1) - G(u_2), u_1 - u_2\rangle \\
&\quad \leq \varphi(u_1, u_2) - \varphi(u_1, u_1) + \varphi(u_2, u_1) - \varphi(u_2, u_2) \\
&\qquad + J^0(u_1; u_2 - u_1) + J^0(u_2; u_1 - u_2) \\
&\qquad\quad + \langle g(t,y) - g(t,x), u_1 - u_2\rangle.
\end{aligned}$$

From the above equality and using assumptions H$(G)(3)$, H$(\varphi)(4)$, H$(J)(2)$ and H(g), we can obtain that

$$(m_G - \alpha_\varphi - \alpha_J)\,\|u_1 - u_2\|_U \leq L_g\,\|x - y\|_E,$$

which implies that Q is Lipschitzian.

Next, we check that the solution mapping satisfies the bound

$$\|u\|_U \leq \frac{1}{m_G - \alpha_J}(\|G(0_U)\|_{U^*} + \ell + \kappa_0).$$

To this end, take $v = 0_U \in K$ in (3.2), then we use assumptions H$(\varphi)(1)$, (2) to obtain

$$\langle G(u), u\rangle \leq J^0(u; -u) - \langle g(t,x), u\rangle.$$

We now write $G(u) = G(u) - G(0_U) + G(0_U)$ and use the condition H$(G)(3)$ of the operator G and hypothesis H(g) to see that

$$m_G\|u\|_U^2 \leq (\|G(0_U)\|_{U^*} + \ell)\|u\|_U + J^0(u; -u). \quad (3.5)$$

On the other hand, taking $v_1 = u$ and $v_2 = 0_U$ in H$(J)(2)$ we find that

$$J^0(u; -u) \leq \alpha_J\|u\|_U^2 - J^0(0_U; u). \quad (3.6)$$

Moreover, using Lemma 2.1(iii) we have

$$-J^0(0_U; u) \leq |J^0(0_U; u)| \leq |\max_{\xi \in \partial J(0_U)} \langle \xi, u\rangle| \leq \max_{\xi \in \partial J(0_U)} |\langle \xi, u\rangle| \leq \max_{\xi \in \partial J(0_U)} \|\xi\|_{U^*}\|u\|_U.$$

and, using condition H$(J)(2)$ with $v = 0_U$ yields

$$-J^0(0_U; u) \leq \kappa_0\|u\|_U. \quad (3.7)$$

We now combine inequalities (3.6) and (3.7) to see that

$$J^0(u; -u) \leq \alpha_J\|u\|_U^2 + \kappa_0\|u\|_U.$$

then we use this inequality in (3.5) to deduce that

$$(m_G - \alpha_J)\|u\|_U \leq \|G(0_U)\|_{U^*} + \ell + \kappa_0. \quad (3.8)$$

Inequality (3.8) is now a direct consequence of the smallness assumption the Lemma. The proof is complete.

□

To solve FDHVI (1.1)–(1.3), we convert it to a differential inclusion. To this end, we define $V : I \times E \to \mathcal{P}(E)$ as follows:

$$V(t,x) := \{B(t,x)u + F(t,x) : u \in Q(-g(t,x))\}.$$

It is easy to see that V has convex and compact values. And from Proposition 2.6, V is weakly u.s.c. Moreover, by H(B)(2), (3), H(F)(3) and Lemma 3.2, we have

$$\begin{aligned}
\|V(t,x)\|_E &:= \sup\{\|v\|_E : v \in V(t,x)\} \\
&\leq \|B(t,x)\|\|u\|_U + a(t) + b\|x\|_E \\
&\leq (\|B(t,x) - B(t,0)\| + \|B(t,0)\|)\|u\|_U + a(t) + c\|x\|_E \\
&\leq d\rho + a(t) + (L_B\rho + c)\|x\|_E. \qquad (3.9)
\end{aligned}$$

It follows from H(B)(2), (3) and Lemma 3.2 that

$$\|B(t,x)Q(-g(t,x)) - B(t,y)Q(-g(t,y))\| \leq \left(L_B\rho + \frac{dL_g}{m_G - \alpha_\varphi - \alpha_J}\right)\|x-y\|.$$

Then, we have

$$\begin{aligned}
\chi(V(t,\Omega)) &:= \chi(B(t,\Omega)Q(-g(t,\Omega)) + F(t,\Omega)) \\
&\leq \left(L_B\rho + \frac{dL_g}{m_G - \alpha_\varphi - \alpha_J}\right)\chi(\Omega) + M_F\chi(\Omega) := \varpi\chi(\Omega). \qquad (3.10)
\end{aligned}$$

By the aforementioned setting, the problem (1.1)–(1.3) is converted to

$$\begin{cases} {}^C D_t^\alpha(t) \in Ax(t) + V(t,x(t)), & a.e.\ t \in I, \\ x(0) = x_0 \in E. \end{cases} \qquad (3.11)$$

We define

$$\begin{aligned}
\mathcal{N}_V^p &: C(I;E) \to \mathcal{P}(L^p(I;E)) \ (p > \frac{1}{\alpha}), \\
\mathcal{N}_V^p(x) &= \{v \in L^p(I;E) : v(t) \in V(t,x(t)) \text{ a.e. } t \in I\}. \qquad (3.12)
\end{aligned}$$

To obtain our main results, we list the following proposition from [24, Lemma 3.5].

Proposition 3.3. *Suppose that $S : L^p(I;E) \to C(I;E)$ be an operator satisfying the following conditions:*

(S1) *there exists $D \geq 0$ such that*

$$\|Sf - Sg\|_{C(I;E)} \leq D\|f-g\|_{L^p(I;E)}, \qquad \textit{for every } f,\ g \in L^p(I;E);$$

(S2) *for any compact $K \subset E$ and sequence $\{f_n\} \subset L^p(I;E)$ such that $\{f_n(t)\} \subset K$ for a.e. $t \in I$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $Sf_n \to Sf_0$.*

Then for every p-time semicompact sequence $\{f_n\} \subset L^p(I;E)$, the sequence $\{Sf_n\}$ is relatively compact in $C(I;E)$, and moreover, if $f_n \rightharpoonup f_0$, then $Sf_n \to Sf_0$.

Now, we define the operator $\mathcal{S} : L^p(I;E) \to C(I;E)$ as

$$(\mathcal{S}v)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)v(s)ds, \quad v \in L^p(I;E). \tag{3.13}$$

Borrowing some ideas from [24], we can establish the following assertion.

Lemma 3.4. *Under the condition* $\mathrm{H}(A)_1$*, the operator* $\mathcal{S}$ *defined above satisfies the properties* $(S1)$ *and* $(S2)$*. Moreover, if the assumptions* $\mathrm{H}(B)$ *and* $\mathrm{H}(F)$ *are also satisfied, then the composition*

$$\mathcal{S} \circ \mathcal{N}_V^p : L^p(I;E) \to C(I;E)$$

is a closed multivalued operator with compact convex values and is u.s.c.

Proof. It follows from [38, Lemma 3.2] that

$$\begin{aligned} \|(\mathcal{S}f)(t) - (\mathcal{S}g)(t)\|_E &\le \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|f(s)-g(s)\|ds \\ &\le \frac{M}{\Gamma(\alpha)}\Big(\frac{p-1}{p\alpha-1}\Big)^{1-\frac{1}{p}} b^{\alpha-\frac{1}{p}}\bigg(\int_0^t \|f(s)-g(s)\|^p ds\bigg)^{\frac{1}{p}} \\ &:= D\bigg(\int_0^t \|f(s)-g(s)\|^p ds\bigg)^{\frac{1}{p}}, \end{aligned} \tag{3.14}$$

which implies that property $(S1)$ follows.

To check property $(S2)$, we note that for every compact $K \subset E$ the set $\Xi \subset E$,

$$\Xi = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)Kds, \quad t \in I, \tag{3.15}$$

is relatively compact. Then for every sequence $\{v_n\}_{n=1}^\infty \subset L^p(I;E)$ such that $\{v_n(s)\}_{n=1}^\infty \subseteq K$ for a.e. $s \in I$, we can obtain $\{(\mathcal{S}v_n)(t)\}_{n=1}^\infty \subset \Xi$, so the sequence $\{(\mathcal{S}v)(t)\}_{n=1}^\infty \subset E$ is relatively compact for every $t \in I$.

By assumption $\mathrm{H}(A)_1$ and similar to the proof of [18, Step 2 of Theorem 3.3], we know that the sequence of functions $\{\mathcal{S}v_n\}_{n=1}^\infty \subset C(I;E)$ is equicontinuous. We can also easily check that $\{\mathcal{S}v_n\}_{n=1}^\infty$ is uniformly bounded. Hence, it follows from the well known Arzela-Ascoli criterion that the sequence of $\{\mathcal{S}v_n\}_{n=1}^\infty$ is relatively compact.

It is easy to see that condition $(S1)$ implies that $\mathcal{S}$ is a bounded linear operator from the space $L^p(I;E)$ into $C(I;E)$. Therefore it is continuous if these spaces are endowed with the topology of weak sequential convergence and $v_n \rightharpoonup v$ implies $\mathcal{S}v_n \to \mathcal{S}v$. And the relative compactness of the sequence $\{\mathcal{S}v_n\}_{n=1}^\infty$ implies that $\mathcal{S}v_n \to \mathcal{S}v$ is in the norm of the space $C(I;E)$, so we can conclude that $(S2)$ also holds.

Now, let $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \subset C(I;E)$, $x_n \to x_0$, $z_n \in \mathcal{S} \circ \mathcal{N}_V^p(x_n)$ and $z_n \to z_0$. Take an arbitrary sequence $\{v_n\}_{n=1}^\infty \subset L^p(I;E)$ such that $v_n \in \mathcal{N}_V^p(x_n)$, $z_n = \mathcal{S}v_n$ $(n \ge 1)$. By (3.10), we have

$$\chi(\{v_n\}_{n=1}^\infty) \le \varpi\chi(\{x_n\}_{n=1}^\infty) = 0,$$

for a.e. $t \in I$, i.e., the set $\{v_n\}_{n=1}^\infty$ is relatively compact for a.e. $t \in I$. From (3.9) it follows that, without loss of generality, there exists $v_0 \in L^p(I;E)$ such that the sequence $v_n \rightharpoonup v_0$ as $n \to \infty$. From the previous proof, we know that $z_n = \mathcal{S}v_n \to \mathcal{S}v_0 = z_0$. On the other hand, it follows from [27, Lemma 11] that $v_0 \in \mathcal{N}_V^p(x_0)$. Therefore $z_0 \in \mathcal{S} \circ \mathcal{N}_V^p(x_0)$, i.e., the multioperator $\mathcal{S} \circ \mathcal{N}_V^p$ is closed.

For a given $x \in C(I;E)$, $v_n^* \in \mathcal{N}_V^p(x)(n \geq 1)$, the same argument as above implies that the sequence $\{\mathcal{S}v_n^*\}_{n=1}^{\infty} \subset C(I;E)$ is relatively compact. Since $\mathcal{S} \circ \mathcal{N}_V^p$ is closed we deduce that the set $\mathcal{S} \circ \mathcal{N}_V^p$ is compact. Furthermore, from the Definition 2.4(iv), the multioperator $\mathcal{S} \circ \mathcal{N}_V^p$ is quasicompact. By [12, Theorem 1.1.12], we can deduce that $\mathcal{S} \circ \mathcal{N}_V^p$ is u.s.c. Clearly, for any $x \in C(I;E)$, $\mathcal{S} \circ \mathcal{N}_V^p(x)$ is convex by the convexity of $\mathcal{N}_V^p(x)$. The proof is complete. □

We also need the following important result which can be found in [24, Lemma 3.9].

Lemma 3.5. *Let the sequence of functions $\{v_n\}_{n=1}^{\infty} \subset L^p(I;E)$ satisfy the following conditions:*

$$\|v_n\|_E \leq \sigma(t),$$

and

$$\chi(\{v_n(t)\}_{n=1}^{\infty}) \leq \varsigma(t),$$

for all $n = 1,\ 2, \cdots,\ t \in I$, where $\sigma,\ \varsigma \in L^p(I;\mathbb{R})$. Then

$$\chi(\{\mathcal{S}v_n(t)\}_{n=1}^{\infty}) \leq 2^{1+\frac{1}{p}} D\bigg(\int_0^t (\varsigma(s))^p ds\bigg)^{\frac{1}{p}},$$

for all $t \in I$, where $D \geq 0$ is the constant in condition $(S1)$.

In the sequel, we turn to considering the multivalued operator $\mathcal{F} : C(I;E) \to 2^{C(I;E)}$:

$$\mathcal{F}(x) = \Big\{y \in C(I;E) : y(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)v(s)ds : v \in \mathcal{N}_V^p(x)\Big\}.$$

As an immediate consequence of Lemma 3.4, we know that the multivalued operator $\mathcal{F}$ is u.s.c. with compact values. Next, our goal is to check that $\mathcal{F}$ is χ_b-condensing.

Theorem 3.6. *If the hypotheses* $\mathrm{H}(A)_1$, $\mathrm{H}(B)$, $\mathrm{H}(F)$, $\mathrm{H}(K)$, $\mathrm{H}(G)$, $\mathrm{H}(\varphi)$, $\mathrm{H}(J)$, $\mathrm{H}(g)$ *and (3.1) are satisfied, then the multivalued operator $\mathcal{F}$ is χ_b-condensing.*

Proof. For $\Omega \in \mathcal{P}_b(C(I;E))$ being a not relatively compact subset of $C(I;E)$, let us assume that

$$\chi_b(\Omega) \leq \chi_b(\mathcal{F}(\Omega)), \tag{3.16}$$

In fact, for $\forall\ t \in I$, it is easy to see that

$$\mathcal{F}(\Omega)(t) \subset S_\alpha(t)x_0 + \mathcal{S} \circ \mathcal{N}_V^p(\Omega)(t). \tag{3.17}$$

According to Lemma 2.8, there exists $\{x_n\}_{n=1}^{\infty} \subset \Omega$ such that for $\forall \epsilon > 0$, we have

$$\chi_b(\mathcal{F}(\Omega)) \leq 2\chi_b(\{\mathcal{F}(x_n)\}_{n=1}^{\infty}) + \epsilon. \tag{3.18}$$

In fact, for $\{x_n\}_{n=1}^{\infty}$, there exists $v_n \in \mathcal{N}_V^p(x_n)$, such that

$$\mathcal{F}(x_n)(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)v_n(s)ds,\ t \in I,\ n \geq 1.$$

For $s \in I$, it follows from (3.10) that

$$\begin{aligned}\chi(\{v_n(s)\}_{n=1}^{\infty}) &\leq \varpi\chi(\{x_n(s)\}_{n=1}^{\infty}) = \varpi e^{Ls} \cdot e^{-Ls}\chi(\{x_n(s)\}_{n=1}^{\infty}) \\ &\leq \varpi e^{Ls} \cdot \sup_{\tau \in I} e^{-L\tau}\chi(\{x_n(\tau)\}_{n=1}^{\infty}) = \varpi e^{Ls} \cdot \chi_b(\{x_n\}_{n=1}^{\infty}).\end{aligned}$$

where the positive constant L is chosen such that

$$L > \frac{2^{p+1} D^p \varpi^p}{p}, \tag{3.19}$$

here $D \geq 0$ is the constant in (3.14) and $\varpi > 0$ is defined in (3.10).

It follows from Lemma 3.5 that

$$\begin{aligned}
\chi_b(\{\mathcal{S}v_n\}_{n=1}^{\infty}) &= \sup_{t\in I} e^{-Lt}\chi(\{(\mathcal{S}v_n)(t)\}_{n=1}^{\infty}) \\
&\leq \sup_{t\in I} e^{-Lt} 2^{1+\frac{1}{p}} D \bigg(\int_0^t (\varpi e^{Ls})^p ds \bigg)^{\frac{1}{p}} \chi_b(\{x_n\}_{n=1}^{\infty}) \\
&\leq 2^{1+\frac{1}{p}} D\varpi \sup_{t\in I} e^{-Lt} \bigg(\frac{e^{Lpt}-1}{Lp} \bigg)^{\frac{1}{p}} \chi_b(\{x_n\}_{n=1}^{\infty}) \\
&\leq \frac{2^{1+\frac{1}{p}} D\varpi}{(Lp)^{\frac{1}{p}}} \chi_b(\{x_n\}_{n=1}^{\infty}).
\end{aligned}$$

From (3.16) and (3.17), we have

$$\chi_b(\{x_n\}_{n=1}^{\infty}) \leq \chi_b(\mathcal{F}(\{x_n\}_{n=1}^{\infty})) \leq \chi_b(\mathcal{S}\circ \mathcal{N}_V^p(\{x_n\}_{n=1}^{\infty})) \leq \frac{2^{1+\frac{1}{p}} D\varpi}{(Lp)^{\frac{1}{p}}} \chi_b(\{x_n\}_{n=1}^{\infty}).$$

It follows from (3.19) that

$$\chi_b(\{x_n\}_{n=1}^{\infty}) = 0, \tag{3.20}$$

hence,

$$\chi_b(\mathcal{F}(\{x_n\}_{n=1}^{\infty})) = 0. \tag{3.21}$$

From (3.18) and the arbitrariness of ε, we know that

$$\chi_b(\Omega) \leq \chi_b(\mathcal{F}(\Omega)) \leq 2\chi_b(\{\mathcal{F}(x_n)\}_{n=1}^{\infty}) = 0.$$

This implies that Ω is relatively compact by regularity of χ_b (see item (v) in Definition 2.7), which comes to the conclusion that $\mathcal{F}$ is χ_b-condensing. The proof is complete. □

Now, we are in the position to present our first main result in this section.

Theorem 3.7. *If the hypotheses* $\mathrm{H}(A)_1$, $\mathrm{H}(B)$, $\mathrm{H}(F)$, $\mathrm{H}(K)$, $\mathrm{H}(G)$, $\mathrm{H}(\varphi)$, $\mathrm{H}(J)$, $\mathrm{H}(g)$ *and (3.1) are satisfied, then the problem (3.11) has at least one mild solution on* $C(I;E)$.

Proof. It is clear that a solution of the problem (3.11) is a fixed point of the multioperator $\mathcal{F}$. From Lemma 3.4 and Lemma 3.6, the multioperator $\mathcal{F} : C(I;E) \to \mathcal{P}_{cv,cp}(C(I;E))$ is u.s.c. and ν-condensing on every bounded subset of $\Omega \subset E$. So according to Theorem 2.11, we only need to check that $\mathcal{F}$ maps the bounded set into itself.

To prove this, we introduce the ball $B_R = \{x \in C(I;E) : \|x\|^*_{C(I;E)} \leq R\}$ in the space $C(I;E)$, where $R > 0$ is chosen such that

$$R > \frac{M\|x_0\|_E + \frac{Md\rho b^\alpha}{\Gamma(\alpha+1)} + \frac{Mb^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{1-\frac{1}{p}}\|a\|_{L^p}}{1 - \frac{M(L_B\rho+c)}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha-1}\right)^{1-\frac{1}{p}} b^{\alpha-\frac{1}{p}}\left(\frac{1}{Lp}\right)^{\frac{1}{p}}},$$

and $\|x\|^*_{C(I;E)} = \sup_{t\in I} e^{-Lt}\|x(t)\|_E$, the positive constant L is chosen so that the inequality (3.14) be satisfied.

Now we show that the multioperator $\mathcal{F}$ maps the ball B_R into itself. In fact, for $x \in B_R$ and $y \in \mathcal{F}(x)$, it follows from Lemma 3.2 of [38] that

$$\begin{aligned}
e^{-Lt}\|y(t)\|_E &\le e^{-Lt}\|S_\alpha(t)x_0\|_E + e^{-Lt}\int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|v(s)\|_E ds \\
&\le M\|x_0\|_E + \frac{Me^{-Lt}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\big(d\rho + a(s) + (L_B\rho + c)\|x(s)\|_E\big)ds\Big) \\
&\le M\|x_0\|_E + \frac{Md\rho b^\alpha}{\Gamma(\alpha+1)} + \frac{Mb^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\big(\frac{p-1}{p\alpha-1}\big)^{1-\frac{1}{p}}\|a\|_{L^p} \\
&\quad + \frac{M(L_B\rho + c)e^{-Lt}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}e^{Ls}e^{-Ls}\|x(s)\|_E ds\Big) \\
&\le M\|x_0\|_E + \frac{Md\rho b^\alpha}{\Gamma(\alpha+1)} + \frac{Mb^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\big(\frac{p-1}{p\alpha-1}\big)^{1-\frac{1}{p}}\|a\|_{L^p} \\
&\quad + \frac{M(L_B\rho + c)R}{\Gamma(\alpha)}\big(\frac{p-1}{p\alpha-1}\big)^{1-\frac{1}{p}} b^{\alpha-\frac{1}{p}}\big(\frac{1}{Lp}\big)^{\frac{1}{p}} < R.
\end{aligned}$$

which reads as $\|y\|^*_{C(I;E)} \le R$. At this point, we know that the multivalued map $\mathcal{F}$ fulfils all the requirements of Theorem 2.11. Hence, we can conclude that the operator $\mathcal{F}$ has a fixed point. The proof is complete.

□

4 Existence of Decay Mild Solutions

The subject of this section is to consider the existence of decay mild solutions of the FDHVI (1.1)–(1.3) on the space $BC(\mathbb{R}_+; E)$. For positive number $\alpha,\ \gamma,\ R$, let

$$B^\alpha_R(\gamma) = \big\{x \in B_R : \sup_{t\in\mathbb{R}_+} t^\alpha\|x(t)\|_E \le \gamma\big\},$$

where B_R is the ball in $BC(\mathbb{R}_+; E)$ centered at the origin with radius $R > 0$. It is clear that $B^\alpha_R(\gamma)$ is closed, bounded and convex in $BC(\mathbb{R}_+; E)$.

Before stating and proving the main results of this section, we impose

$\underline{H(A)_2}$ The strongly continuous semigroup $T(t)$ $(t \ge 0)$ generated by A is exponentially stable on E (i.e., there are positive numbers $w,\ M$ such that $\|T(t)\| \le Me^{-wt}$) with $T(t)$ compact linear operator for every $t > 0$.

To provide our main result in this section, we first recall the following useful lemma.

Lemma 4.1. *(Theorem 1.1 of [13]) If $\varphi \in L^1(I),\ g \in L^p(I), 1 \le p \le \infty$. Then for almost every $t \in I$, the function $s \mapsto \varphi(t-s)g(s)$ is integrable on I. Moreover, the convolution $\varphi * g$, given by*

$$(\varphi * g)(t) = \int_0^t \varphi(t-s)g(s)ds,$$

belongs to $L^p(I)$ and

$$\|\varphi * g\|_{L^p(I)} \le \|\varphi\|_{L^1(I)}\|g\|_{L^p(I)}.$$

In what follows we recall with the Mittag–Leffer function $E_\alpha(z)$ with $\alpha > 0$ which is defined by the following series representation, valid in the whole plane:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha > 0, \ z \in \mathbb{C}.$$

The two-parameter Mittag–Leffer function $E_{\alpha,\beta}(z)$ with $\alpha, \ \beta > 0$ is defined by the following series representation:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \ \beta > 0, \ z \in \mathbb{C}.$$

To obtain our main results of this section, we first give some results for asymptotic behavior of MittagCLeffler functions based on [34, Theorem 1.1.3] and [34, Theorem 1.1.4] or [37, Lemma 1.1].

Lemma 4.2. *Let $\alpha \in (0, 2)$ and $\beta \in \mathbb{R}$ be arbitrary. Then for $q = [\frac{\beta}{\alpha}]$, the following asymptotic expansions hold:*

(1) $E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) - \sum\limits_{k=1}^{q} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O(z^{-1-q}), \quad$ *as* $z \to \infty$.

(2) $E_{\alpha,\beta}(z) = -\sum\limits_{k=1}^{q} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O(|z|^{-1-q}), \quad$ *as* $z \to -\infty$.

Remark 4.3. *From Lemma 4.2, we know that $E_{\alpha,\beta}(-wt^\alpha)$ $(w > 0)$ is bounded for any $t > 0$. Therefore, there are constants $M_1, \ M_2, \ M_3 > 0$ such that*

$$E_\alpha(-wt^\alpha) \le M_1, \quad E_{\alpha,\alpha}(-wt^\alpha) \le M_2, \quad E_{\alpha,\alpha+1}(-wt^\alpha) \le M_3. \tag{4.1}$$

Moreover, we also have that $t^\alpha E_{\alpha,\beta}(-wt^\alpha)$ $(w > 0)$ is bounded for any $t > 0$, i.e., there are constants $\widetilde{M}_1, \ \widetilde{M}_2, \ \widetilde{M}_3 > 0$ such that

$$t^\alpha E_\alpha(-wt^\alpha) \le \widetilde{M}_1, \quad t^\alpha E_{\alpha,\alpha}(-wt^\alpha) \le \widetilde{M}_2, \quad t^\alpha E_{\alpha,\alpha+1}(-wt^\alpha) \le \widetilde{M}_3. \tag{4.2}$$

We now proceed with the following result which is brought from [11, Proposition 2.1].

Proposition 4.4. *If the strongly continuous semigroup $T(t)$ $(t \ge 0)$ generated by A is exponentially stable, i.e., there are positive numbers $w, \ M$ such that*

$$\|T(t)\| \le Me^{-wt},$$

then $\|S_\alpha(t)\| \le ME_\alpha(-wt^\alpha), \ \|T_\alpha(t)\| \le ME_{\alpha,\alpha}(-wt^\alpha)$ for all $t \ge 0$.

For each $x \in B_R^\alpha(\gamma)$, we first have the following lemma.

Lemma 4.5. *If the hypotheses* $\mathrm{H}(A)_2, \ \mathrm{H}(B), \ \mathrm{H}(F), \ \mathrm{H}(K), \ \mathrm{H}(G), \ \mathrm{H}(\varphi), \ \mathrm{H}(J), \ \mathrm{H}(g)$ *and* (3.1) *are satisfied, the constant d in condition* $\mathrm{H}(B)$ *is equal to zero, the function $a(\cdot)$ in assumption* $\mathrm{H}(F)$ *belongs to $L^\infty(\mathbb{R}_+)$ with $\|a\|_\mu := \sup\limits_{t\ge 0} \frac{a(t)}{\mu(t)} < \infty$ and μ satisfies*

$$\sup_{t\ge 0} t^\alpha \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \mu(s) ds := M_\mu < \infty.$$

Moreover,

$$M\widetilde{M}_1(L_B\rho + c)\Gamma(1-\alpha) < 1,$$

where $\widetilde{M}_1$ *is given in Remark 4.3,* ρ *is defined in Lemma 3.2,* $M,\ L_B,\ c$ *are appeared in hypotheses* $\mathrm{H}(A)_2,\ \mathrm{H}(B)$ *and* $\mathrm{H}(F)$, *respectively. Then* $\mathcal{F}(B_R^\alpha(\gamma)) \subset (B_R^\alpha(\gamma))$.

Proof. To show that $\mathcal{F}(B_R^\alpha(\gamma)) \subset (B_R^\alpha(\gamma))$, we suppose that the contrary that for each $n \in \mathbb{N}$, there exists $x_n \in B_R^\alpha(\gamma)$, but

$$\sup_{t\geq 0} t^\alpha \|\mathcal{F}(x_n)(t)\|_E > n.$$

It follows from the formulation of $\mathcal{F}$ and proposition 4.4, Remark 4.3 that

$$\begin{aligned}
&\|\mathcal{F}(x_n)(t)\|_E \\
\leq\ & \|S_\alpha(t)x_0\|_E + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|v_n(s)\|_E ds \\
\leq\ & ME_\alpha(-wt^\alpha)\|x_0\|_E + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|a(s)ds \\
& + \int_0^t (t-s)^{\alpha-1} ME_{\alpha,\alpha}(-w(t-s)^\alpha)(L_B\rho + c)\|x_n(s)\|_E ds \\
\leq\ & ME_\alpha(-wt^\alpha)\|x_0\|_E + \|a\|_\mu \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\mu(s)ds \\
& + M(L_B\rho + c)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-w(t-s)^\alpha)s^{-\alpha}s^\alpha\|x_n(s)\|_E ds \\
\leq\ & ME_\alpha(-wt^\alpha)\|x_0\|_E + \|a\|_\mu \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\mu(s)ds \\
& + M\widetilde{M}_3(L_B\rho + c)\Gamma(1-\alpha)E_\alpha(-wt^\alpha)n.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\sup_{t\geq 0} t^\alpha\|\mathcal{F}(x_n)(t)\|_E \ \leq\ & \sup_{t\geq 0}\Big[Mt^\alpha E_\alpha(-wt^\alpha)\|x_0\|_E + \|a\|_\mu t^\alpha \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\mu(s)ds \\
& + M(L_B\rho + c)\Gamma(1-\alpha)t^\alpha E_\alpha(-wt^\alpha)n\Big] \\
\leq\ & M\widetilde{M}_1\|x_0\|_E + \|a\|_\mu M_\mu + M\widetilde{M}_1(L_B\rho + c)\Gamma(1-\alpha)n.
\end{aligned}$$

It follows from the above inequality that

$$1 < \frac{\sup_{t\geq 0} t^\alpha\|\mathcal{F}(x_n)(t)\|_E}{n} \leq \frac{M\widetilde{M}_1\|x_0\|_E + \|a\|_\mu M_\mu}{n} + M\widetilde{M}_1(L_B\rho + c)\Gamma(1-\alpha).$$

Passing to the limit in the inequality above, we get a contradiction. The proof is complete. □

In view of the Lemma 4.5, from now on we can consider

$$\mathcal{F} : B_R^\alpha(\gamma) \to \mathcal{P}(B_R^\alpha(\gamma)).$$

We now are concerned with the condensivity property of $\mathcal{F}$.

Lemma 4.6. *Let the hypotheses of Lemma 4.5 hold. Then $\mathcal{F}$ is χ^*-condensing.*

Proof. Let $\Omega \subset B_R^\alpha(\gamma)$ be a bounded set. Recall that $\chi^*(\Omega) = \chi_\infty(\Omega) + d_\infty(\Omega)$. Now we first show that $d_\infty(\Omega) = 0$. In fact, since $\|x\| \le R$ for all $x \in \Omega$, by Lemma 4.5, we obtain

$$t^\alpha \|\mathcal{F}(x)(t)\|_E \le \gamma, \quad \forall t \ge 0,$$

which implies

$$\|\mathcal{F}(x)(t)\|_E \le \gamma t^{-\alpha}, \quad \forall x \in \Omega, \quad \forall t \ge 0.$$

Equivalently, for large $b > 0$, one has $d_b(\mathcal{F}(\Omega)) \le \gamma b^{-\alpha}$. Then

$$d_\infty(\mathcal{F}(\Omega)) = \lim_{b\to\infty} d_b(\mathcal{F}(\Omega)) = 0.$$

In the rest of the proof, we will show that $\chi_\infty(\mathcal{F}(\Omega)) = 0$. Obviously, for $\forall b > 0$, it follows from Lemma 2.8 that, there exists $\{x_n\}_{n=1}^\infty \subset \Omega$ such that for $\forall \epsilon > 0$, we have

$$\chi_b(\pi_b(\mathcal{F}(\Omega))) \le 2\chi_b(\{\mathcal{F}(x_n)\}_{n=1}^\infty) + \epsilon.$$

In fact, for $\{x_n\}_{n=1}^\infty$, there exists $v_n \in \mathcal{N}_V^p(x_n)$, such that

$$\mathcal{F}(x_n)(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) v_n(s) ds, \ t \in I, \ n \ge 1.$$

By (3.10), Lemma 4.1 and Lemma 3.5, we obtain

$$\begin{aligned}
&\chi(\mathcal{F}(\{x_n\}(t))_{n=1}^\infty) \\
\le\ & 4\int_0^t \chi\Big[(t-s)^{\alpha-1}T_\alpha(t-s)\{v_n(s)\}_{n=1}^\infty\Big] ds \\
\le\ & 4\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-w(t-s)^\alpha)(L_B\rho + M_F)\chi(\{x_n(s)\}_{n=1}^\infty) ds \\
\le\ & 4(L_B\rho + M_F)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-w(t-s)^\alpha)e^{Ls}e^{-Ls}\chi(\{x_n(s)\}_{n=1}^\infty) ds \\
\le\ & 4(L_B\rho + M_F)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-w(t-s)^\alpha)e^{Ls} ds \cdot \chi_b(\{x_n\}_{n=1}^\infty) \\
\le\ & 4(L_B\rho + M_F)\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-w(t-s)^\alpha) ds \int_0^t e^{Ls} ds \cdot \chi_b(\{x_n\}_{n=1}^\infty) \\
\le\ & 4(L_B\rho + M_F)t^\alpha E_{\alpha,\alpha+1}(-wt^\alpha)\frac{e^{Lt}-1}{L}\chi_b(\{x_n\}_{n=1}^\infty) \\
<\ & \frac{4(L_B\rho + M_F)\widetilde{M}_3 e^{Lt}}{L}\chi_b(\{x_n\}_{n=1}^\infty)
\end{aligned}$$

It is inferred that

$$\chi_b(\{\mathcal{F}(\{x_n\}_{n=1}^{\infty})\} \leq \frac{4(L_B\rho + M_F)\widetilde{M}_3}{L}\chi_b(\{x_n\}_{n=1}^{\infty})$$

Therefore, we get

$$\chi_b(\pi_b(\mathcal{F}(\Omega))) \leq \frac{4(L_B\rho + M_F)\widetilde{M}_3}{L}\chi_b(\pi_b(\Omega))$$

Hence, we know that

$$\chi_\infty(\mathcal{F}(\Omega)) = \sup_{b>0}\chi_b(\pi_b(\mathcal{F}(\Omega))) \leq \frac{4(L_B\rho + M_F)\widetilde{M}_3}{L}\chi_\infty(\Omega)$$

Now if $\chi_\infty(\Omega) \leq \chi_\infty(\mathcal{F}(\Omega))$, then

$$\chi_\infty(\Omega) \leq \frac{4(L_B\rho + M_F)\widetilde{M}_3}{L}\chi_\infty(\Omega).$$

This implies that $\chi_\infty(\Omega) = 0$ choice by $L > 4(L_B\rho + M_F)\widetilde{M}_3$. Hence, Ω is relatively compact. The proof is complete.

□

Lemma 4.7. *Let the hypotheses of Lemma 4.5 hold. Then* $\Gamma : B_R^\alpha(\gamma) \to \mathcal{P}(B_R^\alpha(\gamma))$ *is u.s.c.*

Proof. According to Proposition 2.6, we first show the closeness of $\mathcal{F}$. Let $x_n \in B_R^\alpha(\gamma)$, $x_n \to x_*$, $\varphi_n \in \mathcal{F}(x_n)$ and $\varphi_n \to \varphi_*$ in $B_R^\alpha(\gamma)$. We will check that $\varphi_* \in \mathcal{F}(x_*)$, i.e.,

$$\varphi_*(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)v_*(s)ds, \quad v_* \in \mathcal{N}_V^p(x_*), \quad \forall t \in \mathbb{R}_+.$$

To this end, take an arbitrary sequence $\{v_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R}_+; E)$ such that $v_n \in \mathcal{N}_V^p(x_n)$, $\varphi_n(t) = S_\alpha(t)x_0 + \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)v_n(s)ds$ $(n \geq 1)$. By (3.10), we have

$$\chi_b(\{v_n\}_{n=1}^{\infty}) \leq (L_B\rho + M_F)\chi_b(\{x_n\}_{n=1}^{\infty}) = 0,$$

for a.e. $t \in \mathbb{R}_+$, i.e., the set $\{v_n\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in \mathbb{R}_+$. Without loss of generality, there exists $v_* \in L^p(\mathbb{R}_+; E)$ such that the sequence $v_n \rightharpoonup v_*$ as $n \to \infty$. On the other hand, it follows from [27, Lemma 11] that $v_* \in \mathcal{N}_V^p(x_*)$. Therefore $\varphi_*(t) = S_\alpha(t)x_0 + \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)v_*(s)ds$, i.e., $\varphi_* \in \mathcal{F}(x_*)$. So the multioperator $\mathcal{F}$ is closed.

For a given $x \in C(\mathbb{R}_+; E)$, $v_n \in \mathcal{N}_V^p(x)(n \geq 1)$, the same argument as above implies that the sequence $\{v_n\}_{n=1}^{\infty} \subset C(\mathbb{R}_+; E)$ is relatively compact. Since $\mathcal{F}$ is closed we deduce that the set $\mathcal{F}$ is compact. Furthermore, from the Definition 2.4(iv), the multioperator $\mathcal{F}$ is quasicompact. By [12, Theorem 1.1.12], we can deduce that $\mathcal{F}$ is u.s.c. The proof is complete. □

The following theorem is our main result.

Theorem 4.8. *Let the hypotheses of Lemma 4.5 hold, then the problem* (1.1)-(1.3) *has a compact set of solutions on* $\mathbb{R}_+$ *satisfying*

$$t^\alpha \|x(t)\|_E = O(1) \quad as \quad t \to \infty.$$

Proof. $\mathcal{F}$ is χ^*-condensing due to Lemma 4.6. By Lemma 4.7, $\mathcal{F}$ is u.s.c. Employing Theorem 2.11 again, we get the conclusion. □

5 An application

In this section we introduce and study a mathematical model for which the results of Section 3 and Section 4 can be applied. The classical formulation of the problem is the following.

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}x(t,z) = \delta x_{zz}(t,z) - cx(t,z) + B(t,x(t,z))u(t,z) + f(t,z), && (5.1)\\
&\qquad\qquad \text{for all } t\in\mathbb{R}_+,\ \text{a.e. } z\in(0,1),\ 0<\alpha<1, &&\\
&f(t,z) = \mu f_1(t,z,x(t,z)) + (1-\mu)f_2(t,z,x(t,z)),\quad \mu\in[0,1], && (5.2)\\
&u_{zz}(t,z)) - g^*(t,z,x(t,z)) \in \partial h(z,u(t,z)),\quad \text{for all } t\in\mathbb{R}_+,\ \text{a.e. } z\in(0,1), && (5.3)\\
&u_z(t,0) = u(t,0) = 0,\quad -u_z(t,1)\le\lambda,\qquad \text{for all } t\in\mathbb{R}_+, && (5.4)\\
&x_z(t,0) = x(t,1) = 0,\qquad \text{for all } t\in\mathbb{R}_+, && (5.5)\\
&x(0,z) = x_0(z)\quad \text{a.e. } z\in(0,1). && (5.6)
\end{aligned}
$$

Here, (5.1) represents the state equation, (5.3) expresses the control equation. In our problem, $x(t,z)$ is the displacement function, $\delta>0$ is the diffusion coefficient, the parameter c is used to regulate the convergence speed, f is an arbitrary external forcing functions, $u(t,z)$ is the control function.

Next, take $E=L^2(0,1)$ and define the operator $A: D(A)(\subset E)\to E$ as follows

$$
\begin{cases}
[Ax](z) = \delta x''(z) - cx(z),\\
D(A) = \{x\in H^2(0,1)\mid x'(0)=0, x'(1)=0\}.
\end{cases}
$$

It is well know from [33] that A satisfies assumption $\mathrm{H}(A)_1$ on the space $E=L^2(0,1)$. Moreover, the semigroup $T(t)=e^{tA}$ generated by A is exponential stable, that is,

$$
\|T(t)\|_{\mathcal{L}(L^2(0,1))} \le e^{-(c+\frac{\delta\pi^2}{4})t}\qquad \text{for all } t\in\mathbb{R}_+. \tag{5.7}
$$

This shows that the assumption $\mathrm{H}(A)_2$ holds, too. Moreover, we know from [37, Remark 2.2] that

$$
E_\alpha\left(-\left[c+\frac{\delta\pi^2}{4}\right]t^\alpha\right) \le \frac{1}{1+(c+\frac{\delta\pi^2}{4})t^\alpha}. \tag{5.8}
$$

We now define the multi-valued function

$$
F:\mathbb{R}_+\times E\to\mathcal{P}(E)
$$

$$
F(t,x)(z) = \{\mu f_1(t,z,x(z)) + (1-\mu)f_2(t,z,x(z)),\ \mu\in[0,1]\}.
$$

Then (5.1), (5.2) and (5.5) can be reformulated as

$$
{}^{C}D_t^{\alpha}x(t) \in Ax(t) + B(t,x(t))u(t) + F(t,x(t)),\quad t\in\mathbb{R}_+. \tag{5.9}
$$

We suppose, in addition, that $f_1,\ f_2$ are Lipschitz continuous, i.e., there exist nonnegative functions $\vartheta_1,\ \vartheta_2\in L^2(0,1)$ such that

$$
|f_1(t,z,x) - f_1(t,z,y)| \le \vartheta_1(z)|x-y|,\ \forall\, t\in\mathbb{R}_+,\ z\in(0,1),\ x,\ y\in\mathbb{R},
$$

$$|f_2(t,z,x) - f_2(t,z,y)| \le \vartheta_2(z)|x-y|, \ \forall\, t \in \mathbb{R}_+, \ z \in (0,1), \ x, \ y \in \mathbb{R},$$

and $f_1(t,z,0) \equiv 0$, $f_2(t,z,0) \equiv 0$. Similar to the work [31], we can check that the multi-valued function F is fulfilled the hypotheses H(F).

As far as the functions g^* and h are concerned, we suppose that

$\underline{\mathrm{H}(g^*)}$ $g^* : \mathbb{R}_+ \times (0,1) \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $g^*(\cdot,\cdot,\cdot) : \mathbb{R}_+ \times (0,1) \times \mathbb{R} \to \mathbb{R}$ is continuous for all $(t,z,\vartheta) \in \mathbb{R}_+ \times (0,1) \times \mathbb{R}$;

(2) there exists $L_{g^*} > 0$ such that

$$|g^*(t,z,\vartheta_1) - g^*(t,z,\vartheta_2)| \le L_{g^*}|\vartheta_1 - \vartheta_2| \ \text{ for all } t \in \mathbb{R}_+, \ z \in (0,1), \ \vartheta_1, \ \vartheta_2 \in \mathbb{R};$$

(3) there exists a constant $\ell > 0$ such that

$$|g(t,z,\vartheta)| \le \ell \ \text{ for all } t \in \mathbb{R}_+, \ z \in (0,1), \ \vartheta \in \mathbb{R}.$$

$\underline{\mathrm{H}(h)}$ The functional $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $h(\cdot,\vartheta)$ is measurable on $(0,1)$ for all $\vartheta \in \mathbb{R}$ and there exists $\zeta \in L^2(0,1)$ such that $h(\cdot,\zeta(\cdot)) \in L^2(0,1)$;

(2) $h(z,\cdot)$ is locally Lipschitz continuous on $\mathbb{R}$, a.e. $z \in (0,1)$;

(3) there exists $\overline{\kappa}_0, \ \overline{\kappa}_1 \ge 0$ such that

$$|\partial h(z,\vartheta)| \le \overline{\kappa}_0 + \overline{\kappa}_1|\vartheta| \quad \text{for all } \ \vartheta \in \mathbb{R}, \quad \text{a.e. } x \in (0,1);$$

(4) there exists $\alpha_h > 0$ such that

$$h^0(z,\vartheta_1;\vartheta_2 - \vartheta_1) + h^0(z,\vartheta_2;\vartheta_1 - \vartheta_2) \le \alpha_h|\vartheta_1 - \vartheta_2|^2,$$

for all $\vartheta_1, \ \vartheta_2 \in \mathbb{R}$, a.e. $x \in (0,1)$.

We now perform an integration by parts on $u_{zz}(t,z) - g^*(t,z,x(t,z))$ with the condition $u_z(t,0) = u(t,0) = 0$ to obtain

$$\begin{aligned} &\int_0^1 u_{zz}(t,z)(v(t,z) - u(t,z))dz - \int_0^1 g^*(t,z,x(t,z))(v(t,z) - u(t,z))dz \\ &\quad = u_z(t,1)\big(v(t,1) - u(t,1)\big) - \int_0^1 u_z(t,z)(v_z(t,z) - u_z(t,z))dz \\ &\qquad - \int_0^1 g^*(t,z,x(t,z))(v(t,z) - u(t,z))dz. \end{aligned} \tag{5.10}$$

It follows from (5.3) and Definition 2.1 that

$$(u_{zz}(t,z) - g^*(t,z,x(t,z)))\vartheta \le h^0(z,u;\vartheta), \ \text{ in } (0,1), \ \text{for all } \vartheta \in \mathbb{R}.$$

Using this inequality and $-u_z(t,1) \leq \kappa$ in (5.10), we obtain the following variational-hemivariational inequality.

$$\int_0^1 u_z(t,z)(v_z(t,z) - u_z(t,z))dz + \int_0^1 g^*(t,z,x(t,z))(v(t,z) - u(t,z))dz$$
$$+ \lambda v(t,1)^+ - \lambda u(t,1)^+ + \int_0^1 h^0(t,z,x(t,z); v(t,z) - u(t,z))dz \geq 0. \quad (5.11)$$

where $v^+ = \max\{0, v\}$. In the sequel, we take $E = U = L^2(0,1)$. We denote in what follows $x(t) \in E,\ u(t) \in U$ such that $x(t)(z) = x(t,z),\ u(t)(z) = u(t,z)$. And define the set K by

$$K = \{\, v \in H^1(0,1) \mid v'(0) = v(0) = 0 \,\} \subset U, \quad (5.12)$$

which is a closed convex subset of U. Also, we define the operator $G : U \to U^*$ and the functions $g : I \times U \to U^*$, $\varphi : U \times U \to \mathbb{R}$, $J : U \to \mathbb{R}$, by

$$\langle G(u), v\rangle = \langle -u_{zz}, v\rangle := \int_0^1 u_z v_z dz \quad \text{for all} \ \ u,\ v \in K, \quad (5.13)$$

$$\langle g(t,x), v\rangle = \int_0^1 g^*(t, x(z))\, v(z)\, dz, \quad \text{for all} \ \ t \in \mathbb{R}_+,\ x \in E,\ v \in U. \quad (5.14)$$

$$\varphi(u,v) = \lambda v^+, \quad \text{for all} \ \ u,\ v \in U, \quad (5.15)$$

$$J(v) = \int_0^1 h(v)dz, \quad \text{for all} \quad v \in U. \quad (5.16)$$

With this data, the following variational formulation of the problem could be derived: *Find a control $u(t) \in K$ such that*

$$\langle G(u(t)) + g(t,x(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) + J^0(u(t); v - u(t)) \ \geq 0\rangle \quad \forall\, v \in K,$$

for all $t \in \mathbb{R}_+$, which is the equivalence form of (HVI)(1.2).

It follows from Gagliardo–Nirenberg–Sobolev inequality ([9, Theorem 5.6.1]) that $\langle G(u), u\rangle = \|u\|_{H_0^1(0,1)} \geq C\|u\|_U^2$, so the operators $G = -u_{zz}$ given by (5.3) is fulfilled the condition H(G) with $\alpha_G = C$. It is easy to see that the function φ defined by (5.4) satisfies condition H(φ). Next, using the standard arguments on subdifferential calculus, the hypothesis H(J) is a consequence of H(h), which holds with $\alpha_J = \alpha_h,\ \kappa_0 = \overline{\kappa}_0$ and $\kappa_1 = \overline{\kappa}_1$ In addition, we can also check that the function g defined by (5.6) satisfies the hypothesis H(g) from the assumption H($\widetilde{g}$).

So, we can apply Theorems 4.8 and obtain the following result concerning the existence of decay mild solution of the system (5.1)–(5.6).

Theorem 5.1. *Suppose that* H(g^*) *and* H(j) *hold. Moreover, assume that in addition, $m_{\mathcal{G}} > \alpha_j$ and*

$$\left(\frac{L_B(\ell + \kappa_0)}{m_{\mathcal{G}} - \alpha_j} + \max\{\|\vartheta_1\|_E,\ \|\vartheta_2\|_E\}\right)\Gamma(1-\alpha) < c + \frac{\delta\pi^2}{4} \quad (5.17)$$

hold. Then problem (5.1)–(5.6) has a mild solution on $\mathbb{R}_+$ *satisfying* $t^\alpha \|x(t)\|_E = O(1)$ *as* $t \to \infty$*..*

Proof. Note that (5.17) guarantees that the inequality of lemma 4.5 holds. Therefore, Theorem 5.1 is a direct consequence of Theorem 4.8. □

References

[1] R. Agarwal, S. Hristova, D. O'Regan, A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations, *Fract. Calc. Appl. Anal.,* **19(2)** (2016), 290–318.

[2] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser: Boston-Basel-Berlin, 1992.

[3] J.P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, New York, 1984.

[4] D. Baleanu, A.K. Golmankhaneh, On electromagnetic field in fractional space, *Nonlinear Anal.:RWA.,* **11** (2010), 288–292.

[5] J. Banas, K. Goebal, *Measure of Noncompactness in Banach Spaces*, Marcel Dekker Inc, New York, 1980.

[6] I. Benedetti, N.V. Loi, L. Malaguti, Nonlocal problems for differential inclusions in Hilbert spaces, *Set-Valued Var. Anal.*, **22** (2014), 639–656.

[7] X.J. Chen, Z.Y. Wang, Differential variational inequality approach to dynamic games with shared constraints, *Math. Program.,* **146** (2014), 379–408.

[8] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.

[9] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[10] S. Hu, N.S. Papageorgiou, *Handbook of multivalued Analysis (Theory)*, Kluwer Academic Publishers, Dordrecht Boston, London, 1997.

[11] T.D. Ke, N.V. Loi, V. Obukhovskii, Decay solutions for a class of fractional differential variational inequalities, *Fract. Calc. Appl. Anal.,* **18** (2015), 531–553.

[12] M. Kamemskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Space*, Water de Gruyter, Berlin, 2001.

[13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elservier Science B.V., Amsterdam, 2006.

[14] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677–2682.

[15] X.W. Li, Y.X. Li, Z.H. Liu, J. Li, Sensitivity analysis for optimal control problems described by nonlinear fractional evolution inclusions, *Fract. Calc. Appl. Anal.,* **21(6)** (2018), 1439–1470.

[16] X.W. Li, Z.H. Liu, Sensitivity analysis of optimal control problems described by differential hemivariational inequalities, *SIAM J. Control Optim.,* **56(5)** (2018), 3569–3597.

[17] X.W. Li, Z.H. Liu, M. Sofonea, Unique solvability and exponential stability of differential hemivariational inequalities,*Appl. Anal.,* 2019 (Inpress).

[18] Y.X. Li, X.W. Li, Y.L Liu, On the approximate controllability for fractional evolution hemivariational inequalities, *Math. Meth. Appl. Sci.,* **39** (2016), 3088–3101.

[19] Z.H. Liu, N.V. Loi, V. Obukhovskii, Existence and global bifurcation of periodic solutions to a class of differential variational inequalities, *Int. J. Bifurcat. Chaos,* **23** (2013).

[20] Z.H. Liu, S. Migóski, S.D. Zeng, Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces, *J. Differ. Equat.*, **263** (2017), 3989–4006.

[21] Z.H. Liu, S.D. Zeng, D. Motreanu, Evolutionary problems driven by variational inequalities, *J. Differ. Equat.*, **260** (2016), 6787–6799.

[22] Z.H. Liu, S.D. Zeng, D. Motreanu, Partial differential hemivariational inequalities, *Advances in Nonlinear Analysis*, **7** (4)(2018),571-586.

[23] Z.H. Liu, S.D. Zeng, Y. Bai, Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications, *Fract. Calc. Appl. Anal.*, 19(1)(2016) 188-211.

[24] Z.H. Liu, B. Zeng, Existence and controllability for fractional evolution inclusions of Clarke's subdifferential type, *Appl. Math. Comput.*, **257** (2015), 178–189.

[25] N.V. Loi, On two parameter global bifurcation of periodic solutions to a class of differential variational inequalities, *Nonlinear Anal.,* **122** (2015), 83–99.

[26] N.V. Loi, T.D. Ke, V. Obukhovskii, P. Zecca, Topological methods for some classes of differential variational inequalities, *J. Nonlinear Convex Anal.,* **17** (2016), 403–419.

[27] S. Migórski, A. Ochal, Quasi-static hemivariational inequality via vanishing acceleration approach, *SIAM J. Math. Anal.,* **41** (2009), 1415–1435.

[28] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.

[29] S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elast.*, **127** (2017), 151–178.

[30] Z. Naniewicz, P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, vol.188, CRC Press, 1994.

[31] T.V.A. Nguyen, D.K. Tran, On the differential variational inequalities of parabolic elliptic type, *Math. Meth. Appl. Sci.,* **40** (2017), 4683–4695.

[32] J.S. Pang, D.E. Stewart, Differential variational inequalities, *Math. Program.,* **113** (2008), 345–424.

[33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[35] R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Comput. Math. Appl.*, **62** (2011), 1451–1459

[36] A. Strauss, J.A. Yorke, Perturbing Uniform Asymptotically Stable Nonlinear Systems,*J. Differ. Equat.*, **6** (1969), 452–483.

[37] H.C. Zhou, B.Z. Guo, Boundary feedback stabilization for an unstable time fractional reaction diffusion equation, *SIAM J. Control Optim.*, **56(1)** (2018), 75–101.

[38] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, **59** (2010), 1063–1077.