GLOBAL WEIGHTED REGULARITY ESTIMATES FOR HIGHER ORDER ELLIPTIC AND PARABOLIC SYSTEMS

THE QUAN BUI AND XUAN TRUONG LE

ABSTRACT. In this paper, we prove the weighted Lorentz and weighted Orlicz estimates for the weak solutions to the higher order parabolic systems with the leading coefficients satisfying a small BMO norm condition. As a byproduct, we obtain the weighted estimates for the higher order elliptic systems.

CONTENTS

1. INTRODUCTION

In this paper, we are interested in the global regularity estimates for the higher-order parabolic system

$$\begin{cases} (\mathcal{L}u)^i := (u^i)_t + (-1)^m \sum_{j=1}^N \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{ij}^{\alpha\beta}(x,t) D^\beta u^j) = \sum_{|\alpha|=m} D^\alpha f_i^\alpha, \quad (x,t) \in \mathbb{R}^n_T, \\ u^i(x,0) = 0 \end{cases} \tag{1}$$

for all $i = 1, \dots, N$ with $N \in \mathbb{N}_+$, $m \in \mathbb{N}_+$, $u = (u_1, \dots, u_N)$ defined on $\mathbb{R}^n_T := \mathbb{R}^n \times (0,T)$ and $\mathbf{f} = \{f_i^\alpha\}$ with $f_i^\alpha \in L^2(\mathbb{R}^n_T)$ for all $1 \le i \le N$ and multi-indices α with $|\alpha| = m$.

As a byproduct, we obtain the regularity estimates for the corresponding higher-order elliptic system

$$(\mathcal{M}u)^i + \lambda u^i := \sum_{j=1}^N \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{ij}^{\alpha\beta}(x) D^\beta u^j) + \lambda u^i = \sum_{|\alpha|=m} D^\alpha f_i^\alpha \quad \text{in } \mathbb{R}^n, \tag{2}$$

for all $i = 1, \dots, N$, $u = (u_1, \dots, u_N)$ defined on $\mathbb{R}^n$ and $\mathbf{f} = \{f_i^\alpha\}$ with $f_i^\alpha \in L^2(\mathbb{R}^n)$ for all $1 \le i \le N$ and multi-indices α with $|\alpha| = m$, and $\lambda \ge 0$.

In this article, we assume that the coefficients $a_{ij}^{\alpha\beta}(x)$ and $a_{ij}^{\alpha\beta}(x,t)$ satisfy the following condition: there exist two positive constants $\Lambda_1, \Lambda_2 > 0$ such that

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$$|a_{ij}^{\alpha\beta}| \le \Lambda_1, \tag{3}$$

and

$$\sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=m} a_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \ge \Lambda_2 \sum_{i=1}^{N} \sum_{|\alpha|=m} |\xi_i^\alpha|^2, \quad \forall (\xi_i^\alpha), \tag{4}$$

where $a_{ij}^{\alpha\beta}$ is understood both $a_{ij}^{\alpha\beta}(x)$ and $a_{ij}^{\alpha\beta}(x,t)$.

In the paper, we use the following notations:

- For $m, N \in \mathbb{N}_+$ and $1 \le p \le \infty$, we denote by $W_N^{m,p}(\mathbb{R}^n)$ the cartesian product
$$W^{m,p}(\mathbb{R}^n) \times \ldots \times W^{m,p}(\mathbb{R}^n).$$
- For $u \in W^{m,p}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, we denote $D^k u = (D^\gamma u)_{|\gamma|=k}$.
- For $u = (u^1, \ldots, u^N) \in W_N^{m,p}(\mathbb{R}^n)$ and $0 \le k \le m$ we denote $D^k u = (D^k u^1, \ldots, D^k u^N)$ and
$$\left|D^k u\right|^2 = \sum_{i=1}^{N} \left|D^k u^i\right|^2 = \sum_{i=1}^{N} \sum_{|\gamma|=k} \left|D^\gamma u^i\right|^2.$$

We recall the definition of weak solutions to the systems (1) and (2).

Definition 1.1. *(a) A function $u = (u^1, \ldots, u^N)$ with $u^i \in C(0,T; L^2(\mathbb{R}^n)) \cap L^2(0,T; W^{m,2}(\mathbb{R}^n)), i = 1, \ldots, N$ is said to be a weak solution to the system (2) if*

$$\begin{aligned} \sum_{i=1}^{N} \int_{\mathbb{R}^n_T} u^i \varphi^i_t dxdt - \int_{\mathbb{R}^n_T} \sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=m} a_{ij}^{\alpha\beta}(x,t) D^\beta u^j D^\alpha \varphi^i dxdt \\ = (-1)^{m+1} \int_{\mathbb{R}^n_T} \sum_{i=1}^{N} \sum_{|\alpha|=m} f_i^\alpha D^\alpha \varphi^i dxdt, \end{aligned} \tag{5}$$

for all $\varphi^i \in C_0^\infty(\mathbb{R}^n_T)$, $i = 1, \ldots, N$.

(b) A function $u = (u^1, \ldots, u^N) \in W_N^{m,2}(\mathbb{R}^n)$ is said to be a weak solution to the system (2) if

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=m} a_{ij}^{\alpha\beta}(x) D^\beta u^j D^\alpha \varphi^i dx + \lambda \int_{\mathbb{R}^n} \sum_{i=1}^{N} u^i \varphi^i dx = \int_{\mathbb{R}^n} \sum_{i=1}^{N} \sum_{|\alpha|=m} f_i^\alpha D^\alpha \varphi^i dx, \tag{6}$$

for all $\varphi^i \in C_0^\infty(\mathbb{R}^n)$, $i = 1, \ldots, N$.

The regularity estimate problem for elliptic and parabolic systems (1) and (2) has received a great deal of attention from many mathematicians. This topic plays an important role in the theory of partial differential equations. In particular, the regularity estimates for the second-order elliptic and parabolic equations have been investigated intensively so far. See for example [24, 25, 26, 9, 17, 20, 21, 22, 19, 16, 6, 11] for the standard second-order linear elliptic and parabolic equations corresponding to $m = N = 1$, and [12, 7] for the second-order linear elliptic systems corresponding to $m = 1$ and the references therein. However, there are not many studies on the regularity estimates for the higher order equations. Here, we would like to list certain results in this research direction.

(a) The regularity results for weak solutions to higher-order nonlinear elliptic systems on a bounded domain in $\mathbb{R}^n$ were obtained by [13]. Some interesting results on differentiability theory concerning the higher-order elliptic systems were proved in [14].

(b) In [15], the authors proved the L^p-type regularity for the higher order elliptic equations with VMO coefficients. The L^p-theory of higher-order parabolic and elliptic systems in the whole space $\mathbb{R}^n$, on the half space $\mathbb{R}^n_+$ and on a bounded domain in $\mathbb{R}^n$ can be found in [10].

(c) In [8], the authors established optimal gradient estimates in the Orlicz space for solutions of a nonhomogeneous elliptic equation of higher order with discontinuous coefficients on a nonsmooth domain.

(d) Recently, the authors in [28] proved the regularity estimates for the weak solutions to the equations (1) and (2) corresponding to $N = 1$ in the Orlicz settings by using the free maximal function technique introduced by Acerbi and Mingione in [1].

Before introducing the main results we wouls like to set up the assumptions we will work in the paper.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ the open ball centered x with radius r. For $z = (x,t) \in \mathbb{R}^{n+1}$, we shall mean $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For $z = (x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ we define the cylinder $Q_r(z) = B_r(x) \times (t - r^{2m}, t + r^{2m}]$. *From now on, by a cylinder Q, we shall mean $Q = Q_r(z)$ for some $z \in \mathbb{R}^{n+1}$ and $r > 0$.*

Let E be a measurable subset in $\mathbb{R}^{n+1}$ (or in $\mathbb{R}^n$). For a measurable function f defined on E we denote

$$\bar{f}_E = \fint_E f = \frac{1}{|E|}\int_E f.$$

Throughout this paper, apart from (3) and (4) we additionally assume that the coefficients $a^{\alpha\beta}_{ij}(x)$ and $a^{\alpha\beta}_{ij}(x,t)$ satisfy the small BMO norm condition as follows.

Definition 1.2. (a) *Parabolic case: Let $R, \delta > 0$. The coefficients $\{a^{\alpha\beta}_{ij}(x,t)\}$ are said to satisfy the small (δ, R)-BMO condition if*

$$\sup_{z:=(x,t)\in\mathbb{R}^{n+1}} \sup_{0<r\le R} \fint_{Q_r(z)} |a^{\alpha\beta}_{ij}(y,s) - \overline{a^{\alpha\beta}_{ij}}_{B_r(x)}(s)|^2\, dyds \le \delta^2, \tag{7}$$

for all i, j, α, β as in (1), where

$$\overline{a^{\alpha\beta}_{ij}}_{B_r(x)}(s) = \frac{1}{|B_r(x)|}\int_{B_r(x)} a^{\alpha\beta}_{ij}(y,s)dy.$$

(b) *Elliptic case: Let $R, \delta > 0$. The coefficients $\{a^{\alpha\beta}_{ij}(x)\}$ are said to satisfy the small (δ, R)-BMO condition if*

$$\sup_{x\in\mathbb{R}^n, 0<r\le R} \fint_{B_r(x)} |a^{\alpha\beta}_{ij}(y) - \overline{a^{\alpha\beta}_{ij}}_{B_r(x)}|^2\, dy \le \delta^2, \tag{8}$$

for all i, j, α, β as in (2).

Remark 1.3. *Note that under the conditions (3), (4) and (7), it is easy to see that for any $\tau \in [1, \infty)$ there exists $\epsilon > 0$ so that*

$$\sup_{z:=(x,t)\in\mathbb{R}^{n+1}} \sup_{0<r\le R} \fint_{Q_r(z)} |a^{\alpha\beta}_{ij}(y,s) - \overline{a^{\alpha\beta}_{ij}}_{B_r(x)}(s)|^\tau\, dyds \lesssim \delta^\epsilon, \tag{9}$$

for all i, j, α, β as in (1).

The main aim of this paper is to prove the regularity estimates for the higher order parabolic systems (1) in the settings of the weighted Lorentz spaces and the weighted Orlicz spaces, and then the regularity estimates for the higher order elliptic systems (2) are obtained as a byproduct. More precisely, Theorems 1.4 and 1.5 give regularity estimates in the weighted Lorentz spaces for the systems (1) and (2). Meanwhile, Theorems 1.6 and 1.7 give regularity estimates in the weighted Orlicz spaces for the systems (1) and (2). In these theorems, we refer to Section 2 for the definitions of Muckenhoupt weights, and the weighted Lorentz and Orlicz spaces.

Theorem 1.4. *Assume that the coefficients $\{a^{\alpha\beta}_{ij}(x,t)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (7), $p \in (2, \infty)$, $q \in (0, \infty]$ and $w(x,t) \equiv w(x) \in A_{p/2}(\mathbb{R}^{n+1})$. Then there exists* *positive constant δ_0 such that if $\delta < \delta_0$, $f_i^\alpha \in L_w^{p,q}(\mathbb{R}^n_T)$ for all $|\alpha| = m$ and $i = 1, \dots, N$, and u is a weak solution to the system (1), then we have*

$$\sum_{k=0}^{m} \left\| \, |D^k u| \, \right\|_{L_w^{p,q}(\mathbb{R}^n_T)} \lesssim \| |\mathbf{f}| \|_{L_w^{p,q}(\mathbb{R}^n_T)} . \tag{10}$$

The estimate (14) is nothing, but implies the $W^{m,p}$ regularity estimates for the weak solution to the system (2), i.e,

$$\sum_{k=0}^{m} \left\| \, |D^k u| \, \right\|_{L^p(\mathbb{R}^n_T)} \lesssim \| |\mathbf{f}| \|_{L^p(\mathbb{R}^n_T)} , \quad p > 2. \tag{11}$$

By the standard duality argument, we imply that (11) is valid for $1 < p < 2$, and hence, (11) holds true for all $1 < p < \infty$.

Theorem 1.5. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (8), $p \in (2, \infty)$, $q \in (0, \infty]$ and $w \in A_{p/2}(\mathbb{R}^n)$. Then there exist positive constants δ_0 and λ_0 such that if $\delta < \delta_0, \lambda \geq \lambda_0$, $f_i^\alpha \in L_w^{p,q}(\mathbb{R}^n)$ for all $|\alpha| = m$ and $i = 1, \dots, N$, and u is a weak solution to the system (2), then we have*

$$\sum_{k=0}^{m} \left\| |D^k u| \right\|_{L_w^{p,q}(\mathbb{R}^n)} \lesssim \| |\mathbf{f}| \|_{L_w^{p,q}(\mathbb{R}^n)} . \tag{12}$$

Theorem 1.6. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (7), and assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w(x,t) \equiv w(x) \in A_{i_\Phi}(\mathbb{R}^{n+1})$. Then there exists positive constant δ_0 such that if $\delta < \delta_0$, $f_i^\alpha \in L_w^\Phi(\mathbb{R}^n_T)$ for all $|\alpha| = m$ and $i = 1, \dots, N$, and u is a weak solution to the system (1), then we have*

$$\sum_{k=0}^{m} \int_{\mathbb{R}^n_T} \Phi\left(|D^k u|^2 \right) w(x,t) dx dt \lesssim \int_{\mathbb{R}^n_T} \Phi\left(|\mathbf{f}|^2 \right) w(x,t) dx dt. \tag{13}$$

Theorem 1.7. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (8), and assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i_\Phi}(\mathbb{R}^n)$. Then there exist positive constants δ_0 and λ_0 such that if $0 < \delta < \delta_0, \lambda \geq \lambda_0$, $f_i^\alpha \in L_w^\Phi(\mathbb{R}^n)$ for all $|\alpha| = m$ and $i = 1, \dots, N$, and u is a weak solution to the system (2), then we have*

$$\sum_{k=0}^{m} \int_{\mathbb{R}^n} \Phi\left(|D^k u|^2 \right) w(x) dx \lesssim \int_{\mathbb{R}^n} \Phi\left(|\mathbf{f}|^2 \right) w(x) dx. \tag{14}$$

Note that the results in Theorems 1.6 and 1.7 extend those in [28] to weighted Orlicz settings and to the systems, whereas the estimates in Theorems 1.4 and 1.5 are new. Moreover, the small BMO condition (7) is weaker than that in [28]. Indeed, the small BMO condition allows the coefficients to be bounded with respect to t and have a small BMO seminorm with respect to x. This is contrast to the small BMO condition in [28] which requires the coefficients to have a small BMO seminorm with respect to both x and t. Our small BMO norm condition (7) is similar to those used in [22].

Moreover, our approach relies on those in [9] which makes use of approximation scheme, the Vitali type covering lemma and the Hardy-Littlewood maximal function. This approach is different from those in [28] which make use of the free maximal function technique, respectively. It is not clear that these approach can be applicable to our setting.

The organization of the paper is as follows. In Section 2, we give some definitions of Muckenhoupt weights and the weighted Lorentz and Orlicz spaces. Some approximation results are given in Section **??**. Finally, Section 3 is devoted to the proofs of the main results.

Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write $A \lesssim B$ if there is a universal constant C so that $A \le CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We denote by $\mathcal{O}(\mathtt{data})$ the small quantity such that $\lim_{\mathtt{data}\to 0} \mathcal{O}(\mathtt{data}) = 0$.

2. Muckenhoupt weights, weighted Lorentz spaces and weighted Orlicz spaces

2.1. **Muckenhoupt weights.**

In what follows, by a cylinder Q, we shall mean $Q = Q_r(z)$ for some $z \in \mathbb{R}^{n+1}$ and $r > 0$.

For $z_i = (x_i, t_i) \in \mathbb{R}^{n+1}, i = 1, 2$, we define the distance

$$d(z_1, z_2) = \max\{|x_1 - x_2|, |t_1 - t_2|^{\frac{1}{2m}}\}.$$

Hence, following [27] we can define the class of Muckenhoupt weights as follows. Let $1 \le p < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class* $A_p(\mathbb{R}^{n+1})$, say $w \in A_p(\mathbb{R}^{n+1})$, if there exists a positive constant C so that

$$[w]_{A_p(\mathbb{R}^{n+1})} := \sup_Q \Big(\fint_Q w(z)dz\Big)\Big(\fint_Q w^{-1/(p-1)}(x)dz\Big)^{p-1} \le C, \quad \text{if } 1 < p < \infty, \tag{15}$$

and

$$\fint_Q w(z)dz \le C \operatorname*{ess\text{-}inf}_{z\in Q} w(z), \quad \text{if } p = 1, \tag{16}$$

where the supremum is taken over all cylinders Q in $\mathbb{R}^{n+1}$. We say that $w \in A_\infty(\mathbb{R}^{n+1})$ if $w \in A_p(\mathbb{R}^{n+1})$ for some $p \in [1, \infty)$. We shall denote $w(E) := \int_E w(z)dz$ for any measurable set $E \subset \mathbb{R}^{n+1}$.

Lemma 2.1 ([27]). *Let $w \in A_p(\mathbb{R}^{n+1}), 1 \le p < \infty$. Then, there exist $\kappa_w > 0$, and a constant $C > 1$ such that for any cyinder Q, and any measurable subset $E \subset Q$,*

$$C^{-1}\Big(\frac{|E|}{|Q|}\Big)^p \le \frac{w(E)}{w(Q)} \le C\Big(\frac{|E|}{|Q|}\Big)^{\kappa_w}.$$

The class of Muckenhoupt weight $A_p(\mathbb{R}^n)$ can be defined similarly with the balls taking place of the cylinders in (15) and (16).

2.2. **The weighted Lorentz spaces.**

Let $w \in A_\infty(\mathbb{R}^{n+1})$, $0 < p < \infty$ and $0 < q \le \infty$. The weighted Lorentz space $L^{p,q}_w(\mathbb{R}^{n+1})$ is defined as the set of all measurable functions f on $\mathbb{R}^{n+1}$ such that

$$\|f\|_{L^{p,q}_w(\mathbb{R}^{n+1})} := \left\{p\int_0^\infty \left[t^p w\left(\{z \in \mathbb{R}^{n+1} : |f(z)| > t\}\right)\right]^{q/p} \frac{dt}{t}\right\}^{1/q} < \infty$$

In the particular case $p = q$, the weighted Lorentz spaces $L^{p,p}_w(\mathbb{R}^{n+1})$ coincide with the weighted Lebesgue spaces $L^p_w(\mathbb{R}^{n+1})$ which is defined as all measurable functions f on $\mathbb{R}^{n+1}$ such that

$$\|f\|_{L^p_w(\mathbb{R}^{n+1})} = \Big(\int_{\mathbb{R}^{n+1}} |f(x)|^p w(x,t)dxdt\Big)^{1/p}.$$

The weighted Lorentz spaces $L^{p,q}_w(E)$ and $L^{p,q}_w(\mathbb{R}^n)$ with $w \in A_\infty(\mathbb{R}^n)$ are defined similarly with some appropriate modifications.

For $r > 0$, the Hardy-Littlewood maximal function $\mathbb{M}_r$ is defined by

$$\mathbb{M}_r f(z) = \sup_{Q\ni z}\Big(\frac{1}{|Q|}\int_Q |f(y,s)|^r\, dyds\Big)^{1/r}, \ z \in \mathbb{R}^{n+1},$$

where the supremum is taken over all cylinders containing z. When $r = 1$, we write $\mathbb{M}$ instead of $\mathbb{M}_1$.

Arguing similarly to the proof of [23, Lemma 3.11], we can show that:

Lemma 2.2. *Let $r < p < \infty$, $0 < q \le \infty$ and $w \in A_{p/r}(\mathbb{R}^{n+1})$. Then we have*

$$\|\mathbb{M}_r f\|_{L^{p,q}_w(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^{p,q}_w(\mathbb{R}^{n+1})}.$$

2.3. **The weighted Orlicz spaces.** Let us recall the definition of the weighted Orlicz spaces.

A convex function $\Phi : \mathbb{R} \to [0, \infty)$ is said to be a Young function if it is even, nonnegative, nondecreasing on $[0, \infty)$, and satisfy

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0.$$

Definition 2.3. *(a) A Young function Φ is said to satisfy the global Δ_2 condition, denoted by $\Phi \in \Delta_2$, if there exists a positive constant a_1 such that*

$$\Phi(2t) \le a_1\Phi(t), \quad \forall t > 0.$$

(b) A Young function Φ is said to satisfy the global ∇_2 condition, denoted by $\Phi \in \nabla_2$, if there exists a positive constant $a_2 > 1$ such that

$$2a_2\Phi(t) \le \Phi(a_2 t), \quad \forall t > 0.$$

For $\Phi \in \Delta_2 \cap \nabla_2$, observe that there exists a constant $q \ge 1$ so that

$$\lambda^q\Phi(t) \le c\Phi(\lambda t), \quad \text{for all } t > 0 \text{ and } \lambda \ge 1, \tag{17}$$

where c is a constant independent of λ and t. Then the lower index $i(\Phi)$ is defined as the supremum of all q satisfying (17). Note that in the particular case $\Phi(t) = t^q, 1 \le q < \infty$, we have $i_\Phi = q$.

Let $w \in A_\infty(\mathbb{R}^{n+1})$ and $\Phi \in \Delta_2 \cap \nabla_2$. The weighted Orlicz space $L^\Phi_w(\mathbb{R}^{n+1})$ is defined as all Lebesgue measurable functions f on $\mathbb{R}^{n+1}$ such that

$$\int_{\mathbb{R}^{n+1}} \Phi(f(x,t))w(x,t)dxdt < \infty$$

with respect to the norm

$$\|f\|_{L^\Phi_w(\mathbb{R}^{n+1})} = \inf\left\{\lambda : \int_{\mathbb{R}^{n+1}} \Phi\left(\frac{f(x,t)}{\lambda}\right) w(x,t)dxdt \le 1\right\}.$$

Note that if $f \in L^\Phi_w(\mathbb{R}^{n+1})$, then we have

$$\int_{\mathbb{R}^{n+1}} \Phi(f(x,t))w(x,t)dxdt \sim \int_{\mathbb{R}^{n+1}} w\left(\{(x,t) : |f(x,t)| > \lambda\}\right) d\Phi(\lambda). \tag{18}$$

When $\Phi(t) = t^q, 1 \le q < \infty$, then space $L^\Phi_w(\mathbb{R}^{n+1})$ coincides with the weighted Lebesgue space $L^q_w(\mathbb{R}^{n+1})$.

The weighted orlicz spaces $L^\Phi_w(E)$ with E is a measurable subset in $\mathbb{R}^{n+1}$ and $L^\Phi_w(\mathbb{R}^n)$ are defined in the same manner with some appropriate modifications.

It is interesting to note that the maximal function is bounded on the weighted Orlicz spaces.

Lemma 2.4 ([18])**.** *Let $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}(\mathbb{R}^{n+1})$. Then we have*

$$\int_{\mathbb{R}^{n+1}} \Phi(f(x,t))w(x,t)dxdt \lesssim \int_{\mathbb{R}^{n+1}} \Phi(\mathbb{M}f(x,t))w(x,t)dxdt \lesssim \int_{\mathbb{R}^{n+1}} \Phi(f(x,t))w(x,t)dxdt.$$

3. Weighted regularity estimates for elliptic and parabolic equations

Before giving the proofs of the main results we would like to recall approximation results in [4].

Fix $z_0 = (x_0, t_0) \in \mathbb{R}^n \times (0, T)$, and $0 < R < R_0/2$. Let $u = (u^1, \ldots, u^N)$ with $u^i \in C(0, T; L^2(\mathbb{R}^n)) \cap L^p(0, T; W^{m,2}(\mathbb{R}^n))$ be a weak solution to the system (1). We now consider the following Dirichlet problem

$$\tag{19} \begin{cases} (w^i)_t + (-1)^m \sum_{j=1}^N \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{ij}^{\alpha\beta}(x,t) D^\beta w^j) = 0 & \text{in} \quad Q_{2R} \equiv Q_{2R}(z_0), \\ |u^i - w^i| + \ldots + |D^{m-1}(u^i - w^i)| = 0 & \text{on} \quad \partial_p Q_{2R}, \end{cases}$$

for all $i = 1, \ldots, N$, where $\partial_p Q_r(z) := \partial Q_r(z) \backslash (B_r(x) \times \{t + r^{2m}\})$ for all $r > 0$ and $z = (x, t) \in \mathbb{R}^{n+1}$.

With a such w, we next consider the following problem:

$$\tag{20} \begin{cases} (v^i)_t + (-1)^m \sum_{j=1}^N \sum_{|\alpha|=|\beta|=m} D^\alpha (\overline{a_{ij}^{\alpha\beta}}_{B_R}(t) D^\beta v^j) = 0 & \text{in} \quad Q_R, \\ |v^i - w^i| + \ldots + |D^{m-1}(v^i - w^i)| = 0 & \text{on} \quad \partial_p Q_R, \end{cases}$$

for all $i = 1, \ldots, N$.

Proposition 3.1 ([4])**.** *Let v be a weak solution to the problem (20). Then there exists $\epsilon_1 > 0$ such that*

$$\tag{21} \left(\fint_{Q_R} |D^m(v - w)|^2 dz \right)^{1/2} \lesssim \delta^{\epsilon_1} \left(\fint_{Q_{2R}} |D^m w|^2 dz \right)^{1/2}.$$

Proposition 3.2 ([4])**.** *If $u \in C(0, T; L^2(\mathbb{R}^n)) \cap L^p(0, T; W^{m,2}(\mathbb{R}^n))$ is a weak solution to the system (1) and v is a weak solution to (20), then we have*

$$\tag{22} \| \, |D^m v| \, \|_{L^\infty(Q_{R/2})} \lesssim \left(\fint_{Q_{2R}} |D^m u|^2 \right)^{1/2} + \left(\fint_{Q_{2R}} |\mathbf{f}|^2 \right)^{1/2},$$

and

$$\tag{23} \fint_{Q_R} |D^m(u - v)|^2 \lesssim \mathcal{O}(\delta) \fint_{Q_{2R}} |D^m u|^2 + \fint_{Q_{2R}} |\mathbf{f}|^2.$$

We are now ready to prove the main results. We first give the proofs of Theorems 1.4 and 1.5.

3.1. **Weighted Lorentz estimates.**

The proofs of Theorems 1.4 and 1.5 rely on the following proposition.

Proposition 3.3. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (7), $p \in (2, \infty)$, $q \in (0, \infty]$ and $w \in A_{p/2}(\mathbb{R}^{n+1})$. Then there exist positive constants δ_0 and λ_0 such that if $\delta < \delta_0, \lambda \geq \lambda_0$, g_i^α and $h^i \in L_w^{p,q}(\mathbb{R}_T^n)$ for all $|\alpha| = m$ and $i = 1, \ldots, N$, and u is a weak solution to the system*

$$\tag{24} \begin{cases} (\mathcal{L}u)^i + \lambda u^i = \sum_{|\alpha|=m} D^\alpha g_i^\alpha + h^i, & \text{in } \mathbb{R}_T^n, \\ u^i(x, 0) = 0 \end{cases}$$

for all $i = 1, \ldots, N$.

$$\tag{25} \sum_{0 \leq k \leq m} \lambda^{\frac{1}{2} - \frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(\mathbb{R}_T^n)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|g_i^\alpha\|_{L_w^{p,q}(\mathbb{R}_T^n)} + \sum_{i=1}^N \|h^i\|_{L_w^{p,q}(\mathbb{R}_T^n)}$$

for all $\lambda \geq \lambda_0$.

We now postpone the proof of this technical proposition. First we give the proof of Theorem 1.4 and Theorem 1.5 assuming that the proposition holds.

Proof of Theorem 1.4: Assume that u is a weak solution to the system (1). Let λ_0 be a constant as in Proposition 3.3. We define

$$\tilde{u}(x,t) = u(x,t)e^{-\lambda_0 t}.$$

Then we have $\tilde{u}$ is a weak solution to the system

$$\begin{cases} (\mathcal{L}\tilde{u})^i + \lambda_0 \tilde{u}^i = \sum_{|\alpha|=m} D^\alpha(e^{-\lambda_0 t} f_i^\alpha), & \text{in } \mathbb{R}^n_T, \\ \tilde{u}^i(x,0) = 0 \end{cases}$$

Applying Proposition 3.3 we get that

$$\sum_{0\le k\le m} \Big\| |D^k \tilde{u}| \Big\|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|e^{-\lambda_0 t} f_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

Since $e^{-T\lambda_0} \le e^{-\lambda_0 t} \le 1$, from the above inequality we imply that

$$\sum_{0\le k\le m} \Big\| |D^k u| \Big\|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

□

Proof of Theorem 1.5: Let $\phi \in C_0^\infty(0,T)$ such that $0 \le \phi \le 1$, $\phi = 1$ in $[T/4, T/2]$ and $|\phi_t| \lesssim 1$. We now define

$$v^i(x,t) = u^i(x)\phi(t), \quad i = 1, \dots, N,$$

where u is a weak solution to the system (2).

Observe that, for $i = 1, \dots, N$,

$$\begin{aligned} (\mathcal{L}v)^i(x,t) + \lambda v^i(x) &= u^i(x)\phi_t(t) + (-1)^m (\mathcal{M}u)^i(x)\phi(t) \\ &= \sum_{|\alpha|=m} D^\alpha(f_i^\alpha(x)\phi(t)) + u^i(x)\phi_t(t), \end{aligned}$$

and $v(x,0) = 0$.

Hence, by Proposition 3.3 we have, for $\lambda \ge \lambda_0$,

$$\sum_{0\le k\le m} \lambda^{\frac{m-k}{2m}} \Big\| |D^k u(x)| \phi(t) \Big\|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha(x)\phi(t)\|_{L^{p,q}_w(\mathbb{R}^n_T)} + \sum_{i=1}^N \|u^i(x)\phi_t(t)\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

Since $|\phi| + |\phi_t| \lesssim 1$, we have

$$\sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha(x)\phi(t)\|_{L^{p,q}_w(\mathbb{R}^n_T)} + \sum_{i=1}^N \|u^i(x)\phi_t(t)\|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n)} + \sum_{i=1}^N \|u^i\|_{L^{p,q}_w(\mathbb{R}^n)}.$$

On the other hand, for a fixed $p_1 \in [1,p)$ we have

$$\Big\| |D^k u| \phi(t) \Big\|_{L^{p,q}_w(\mathbb{R}^n_T)} \ge C \Big\| |D^k u| \Big\|_{L^{p,q}_w(\mathbb{R}^n)} \|\phi\|_{L^{p_1}(0,T)} \ge C \Big\| |D^k u| \Big\|_{L^{p,q}_w(\mathbb{R}^n)}.$$

As a consequence,

$$\sum_{0\le k\le m} \lambda^{\frac{m-k}{2m}} \Big\| |D^k u| \Big\|_{L^{p,q}_w(\mathbb{R}^n)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n)} + \sum_{i=1}^N \|u^i\|_{L^{p,q}_w(\mathbb{R}^n)},$$

as long as $\lambda \ge \lambda_0$.

This implies

$$\sum_{0\le k\le m} \Big\| |D^k u| \Big\|_{L^{p,q}_w(\mathbb{R}^n)} \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|f_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n)},$$

for sufficiently large λ. □

To prove Proposition 3.3, we need the following result which gives a regularity estimate for the weak solution to the system (1) with compact support. But we first recall the following useful whose proof can be obtained by a similar argument to [23].

Lemma 3.4. *Let $Q := Q_{R_0}(z_0)$ for some $z_0 \in \mathbb{R}^{n+1}$, $w \in A_\infty(\mathbb{R}^{n+1})$ and $r < R_0/4$. Suppose that $E \subset F \subset Q$ are measurable and satisfy the following conditions:*

(a) *$w(E) < \epsilon w(Q_r(z))$, for some $\epsilon \in (0,1)$ and for every $z \in Q$;*
(b) *for any cylinder $Q_\rho(z)$ with $\rho \in (0, 2r)$ and $y \in Q$, if $w(E \cap Q_\rho(z)) \geq \epsilon w(Q_\rho(z))$ then $Q \cap Q_\rho(z) \subset F$.*

Then there exists $c := c(n, w)$ such that

$$w(E) \leq c\epsilon w(F).$$

Proposition 3.5. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (7), and assume that $w \in A_\infty(\mathbb{R}^{n+1})$ and $u \in C_0^\infty(Q_{R_0}(z_0)), z_0 \in \mathbb{R}^{n+1}$, is a weak solution to (1). Then there exists $A_0 = A_0(n, \Lambda_1, \Lambda_2) > 1$ so that the following holds true. For any $\epsilon > 0$, there exist $\delta = \delta(n, \Lambda_1, \Lambda_2, \epsilon, w)$ and $\gamma = \gamma(n, \Lambda_1, \Lambda_2, \epsilon, w) \in (0,1)$ such that for all $\lambda > 0$,*

(26)
$$w\left(\left\{z \in Q_{R_0}(z_0) : \mathbb{M}(|D^m u|^2) > A_0\lambda, \mathbb{M}(|\mathbf{f}|^2) \leq \gamma\lambda\right\}\right) \leq B\epsilon w\left(\left\{z \in Q_{R_0}(z_0) : \mathbb{M}(|D^m u|^2) > \lambda\right\}\right),$$

where B is a constant independing on ϵ.

Proof. For convenience, we write Q_r for $Q_r(z_0)$ for all $r > 0$. We set

$$E = \left\{z \in Q_{R_0} : \mathbb{M}\left(|D^m u|^2\right)(z) > A_0\lambda, \mathbb{M}\left(|\mathbf{f}|^2\right)(x) \leq \gamma\lambda\right\},$$

and

$$F = \left\{z \in Q_{R_0} : \mathbb{M}(|D^m u|^2)(z) > \lambda\right\}.$$

We now fix $\epsilon \in (0,1)$. Since the Hardy-Littlewood maximal function $\mathbb{M}$ is weak type $(1,1)$, from the standard L^2-estimate for weak solution to the equation (1) we have

$$\begin{aligned}
|E| &\leq \frac{c}{A_0\lambda}\int_{Q_{R_0}} |D^m u(z)|^2 dz \leq \frac{c}{A_0\lambda}\int_{Q_{R_0}} |\mathbf{f}|^2 dz \\
&\leq \frac{c}{A_0\lambda}\gamma\lambda|Q_{R_0}| \leq c\gamma|Q_{R_0}| \\
&\leq c_1\alpha|Q_{R_0/16}(z)|,
\end{aligned}$$

for all $z \in Q_{R_0}$.

Hence, from Lemma 2.1 there exists a constant $\gamma_0 > 0$ so that for any $0 < \gamma < \gamma_0$, we have

$$w(E) \leq \epsilon w(Q_{R_0/16}(z)), \quad \forall z \in Q_{R_0}.$$

We now prove that for any cylinder $Q_\rho(z)$ with $\rho \in (0, R_0/8)$ and $z \in Q_{R_0}$, if $w(E \cap Q_\rho(z)) \geq \epsilon w(Q_\rho(z))$ then $Q_{R_0} \cap Q_\rho(z) \subset F$. Once this is proved the desired estimate (26) follows immediately by applying Lemma 3.4.

Assume, for a contrary, that $\Omega \cap Q_\rho(\bar z) \cap F^c \neq \emptyset$ for some $\bar z = (\bar x, \bar t) \in Q_{R_0}$ and $\rho \in (0, R_0/8)$. Due to Lemma 2.1 again, it suffices to prove that

$$|E \cap Q_\rho(\bar z))| < \epsilon |Q_\rho(\bar z)|. \tag{27}$$

Let $z_1 \in Q_\rho(\bar z) \cap F^c$ and $z_2 \in E \cap Q_\rho(\bar z)$. Hence, we have

$$\mathbb{M}(|D^m u|^2)(z_1) \leq \lambda, \quad \mathbb{M}(|\mathbf{f}|^2)(z_2) \leq \gamma\lambda. \tag{28}$$

Observe that for $z \in Q_\rho(\bar z)$, we have

$$\mathbb{M}(|D^m u|^2)(z) \leq \max\left\{3^{n+2m}\lambda, \mathbb{M}(|D^m u|^2\chi_{Q_{2\rho(\bar z)}})(z)\right\}. \tag{29}$$

We now consider the following equations:

$$(30)\qquad \begin{cases} (w^i)_t + (-1)^m \sum_{j=1}^{N} \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{ij}^{\alpha\beta}(x,t) D^\beta w^j) = 0 & \text{in} \quad Q_{8\rho}(\bar z), \\ |u^i - w^i| + \ldots + |D^{m-1}(u^i - w^i)| = 0 & \text{on} \quad \partial_p Q_{8\rho}(\bar z), \end{cases}$$

for all $i = 1, \ldots, N$, and

$$(31)\qquad \begin{cases} (v^i)_t + (-1)^m \sum_{j=1}^{N} \sum_{|\alpha|=|\beta|=m} D^\alpha (\overline{a_{ij}^{\alpha\beta}}_{B_{4\rho}}(t) D^\beta v^j) = 0 & \text{in} \quad Q_{4\rho}(\bar z), \\ |v^i - w^i| + \ldots + |D^{m-1}(v^i - w^i)| = 0 & \text{on} \quad \partial_p Q_{4\rho}(\bar z), \end{cases}$$

for all $i = 1, \ldots, N$.

Then by Proposition 3.2, we have

$$\| \, |D^m v| \, \|_{L^\infty(Q_{2\rho}(\bar z))} \lesssim \Big(\fint_{Q_{8\rho}(\bar z)} |D^m u|^2 \Big)^{1/2} + \Big(\fint_{Q_{8\rho}(\bar z)} |\mathbf{f}|^2 \Big)^{1/2},$$

and

$$\fint_{Q_{2\rho}(\bar z)} |D^m(u-v)|^2 \lesssim \mathcal{O}(\delta) \fint_{Q_{8\rho}(\bar z)} |D^m u|^2 + \fint_{Q_{8\rho}(\bar z)} |\mathbf{f}|^2.$$

This along with (28) implies that there exist $c_2, c_3 > 0$ so that

$$(32)\qquad \| \, |D^m v| \, \|^2_{L^\infty(Q_{2\rho}(\bar z))} \le c_2 \lambda,$$

and

$$(33)\qquad \fint_{Q_{2\rho}(\bar z)} |D^m(u-v)|^2 \le c_3(\mathcal{O}(\delta) + \gamma)\lambda.$$

Therefore, taking $A_0 = \max\{3^{n+2m}, 4c_2\} + 1$ we have

$$\begin{aligned} |E \cap Q_\rho(\bar z)| &\le \big| \{ z \in Q_\rho(\bar z) : \mathbb{M}(|D^m u|^2 \chi_{Q_{2\rho}(\bar z)})(z) > A_0 \lambda \} \big| \\ &\le \big| \{ z \in Q_\rho(\bar z) : \mathbb{M}(|D^m (u-v)|^2 \chi_{Q_{2\rho}(\bar z)})(z) > A_0 \lambda/4 \} \big| \\ &\quad + \big| \{ z \in Q_\rho(\bar z) : \mathbb{M}(|D^m v|^2 \chi_{Q_{2\rho}(\bar z)})(z) > A_0 \lambda/4 \} \big| . \end{aligned}$$

Due to (32), we have

$$\big| \{ z \in Q_\rho(\bar z) : \mathbb{M}(|D^m v|^2 \chi_{Q_{2\rho}(\bar z)})(z) > A_0 \lambda/4 \} \big| = 0.$$

Hence, this along with (33) implies

$$\begin{aligned} |E \cap Q_\rho(\bar z)| &\le \big| \{ z \in Q_\rho(\bar z) : \mathbb{M}(|D^m (u-v)|^2 \chi_{Q_{2\rho}(\bar z)})(z) > A_0 \lambda/4 \} \big| \\ &\le \frac{C|Q_{2\rho}(\bar z)|}{A_0 \lambda} \fint_{Q_{2\rho}(\bar z)} |D^m(u-v)|^2 dz \\ &\lesssim c_3(\mathcal{O}(\delta) + \gamma)|Q_{2\rho}(\bar z)| \\ &\le c_4(\mathcal{O}(\delta) + \gamma)|Q_{\rho}(\bar z)|. \end{aligned}$$

This yields (27) by taking δ and γ sufficiently small. □

From the good-λ inequality in Proposition 3.5, we obtain the following estimate.

Proposition 3.6. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ, R_0)-BMO condition (7), $p \in (2,\infty)$, $q \in (0,\infty]$ and $w \in A_{p/2}(\mathbb{R}^{n+1})$. Then there exists positive constant δ_0 such that if $\delta < \delta_0$, $f_i^\alpha \in L^{p,q}_w(\mathbb{R}^n_T)$ for all $|\alpha| = m$ and $i = 1, \ldots, N$, and $u \in C_0^\infty(Q_{R_0}(z_0), \mathbb{R}^n)$ is a weak solution to the system (1), then we have*

$$(34)\qquad \| \, |D^m u| \, \|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \| \, |\mathbf{f}| \, \|_{L^{p,q}_w(\mathbb{R}^n_T)} .$$

Proof. By Proposition 3.5, we have

$$\begin{aligned}\left\|\mathbb{M}\left(|D^m u|^2\right)\right\|_{L^{p,q}_w(\mathbb{R}^n_T)} &= \left\{p\int_0^\infty \left[A_0^p\lambda^p w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|D^m u|^2)(z)>A_0\lambda\}\right)\right]^{q/p}\frac{d\lambda}{\lambda}\right\}^{1/q}\\ &\lesssim \left\{p\int_0^\infty \left[A_0^p\lambda^p w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|\mathbf{f}|^2)(z)>\gamma\lambda\}\right)\right]^{q/p}\frac{d\lambda}{\lambda}\right\}^{1/q}\\ &\quad + B\epsilon\left\{p\int_0^\infty \left[A_0^p\lambda^p w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|D^m u|^2)(z)>\lambda\}\right)\right]^{q/p}\frac{d\lambda}{\lambda}\right\}^{1/q}\\ &\le C(p,q)A_0\gamma^{-1}\left\|\mathbb{M}\left(|\mathbf{f}|^2\right)\right\|_{L^{p,q}_w(\mathbb{R}^n_T)} + C(q,r)BA_0\epsilon\left\|\mathbb{M}\left(|D^m u|^2\right)\right\|_{L^{p,q}_w(Q_{R_0}(z_0))}.\end{aligned}$$

Taking ϵ such that $C(q,r)BA_0\epsilon<1$, we find that

$$\left\|\mathbb{M}\left(|D^m u|^2\right)\right\|_{L^{p,q}_w(\mathbb{R}^n_T)} \le C\left\|\mathbb{M}\left(|\mathbf{f}|^2\right)\right\|_{L^{p,q}_w(\mathbb{R}^n_T)},$$

which along with Lemma 2.2 implies

$$\||D^m u|\|_{L^{p,q}_w(\mathbb{R}^n_T)} \le C\,\||\mathbf{f}|\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

□

Proposition 3.3 can be deduced from Proposition 3.6, by using an idea by S. Agmon [3]. Although this ideas was used in [10, 20, 21, 28] in various settings such as second order elliptic and parabolic equations and higher-order parabolic equations, more complicated analysis and calculations would be carefully examined in our setting.

Proposition 3.7. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ,R_0)-BMO condition (7), $p\in(2,\infty)$, $q\in(0,\infty]$ and $w\in A_{p/2}(\mathbb{R}^{n+1})$. Then there exists a positive constant δ_0 such that if $\delta<\delta_0$ and $f_i^\alpha\in L^{p,q}_w(\mathbb{R}^n_T)$ for all $|\alpha|=m$ and $i=1,\dots,N$, then for any weak solution $u\in C_0^\infty(Q_{R_0/2}(z_0),\mathbb{R}^n)$ to the following system*

$$\begin{cases}(\mathcal{L}u)^i+\lambda u^i=\displaystyle\sum_{|\alpha|\le m}D^\alpha g_i^\alpha, & \text{in } \mathbb{R}^n_T,\\ u^i(x,0)=0\end{cases}\tag{35}$$

for all $i=1,\dots,N$, we have

$$\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}\left\||D^k u|\right\|_{L^{p,q}_w(\mathbb{R}^n_T)} \lesssim \sum_{i=1}^N\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)},\tag{36}$$

for all $\lambda>\lambda_0$.

Proof. Fix $\phi(y)\in C_0^\infty(-R_0/2,R_0/2)$. For $(x,y,t)\in\mathbb{R}^{n+1}\times\mathbb{R}$, we define

$$\tilde{u}^i(x,y,t)=u^i(x,t)D_y^m\psi(y),$$

for $i=1,\dots,N$, where $\psi(y)=\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}\right)$.

We now define

$$(\tilde{\mathcal{L}}\tilde{u})^i(x,y,t)=(\mathcal{L}\tilde{u}(x,t))^i+(-1)^m D_y^{2m}\tilde{u}^i,$$

for all $i=1,\dots,N$.

Setting $\tilde{Q}_R=Q_R((x_0,0),t_0)$ to be a cylinder in $\mathbb{R}^{n+1}\times\mathbb{R}$, since $u\in C_0^\infty(Q_{R_0/2}(z_0))$ is a weak solution to (35), $\tilde{u}\in C_0^\infty(\tilde{Q}_R)$ is a weak solution to the following equation

$$\begin{cases}(\tilde{\mathcal{L}}\tilde{u})^i=\displaystyle\sum_{|\alpha|=m}D_x^\alpha(g_i^\alpha D_y^m\psi(y))+\sum_{|\alpha|\le m-1}D_x^\alpha D_y^{m-|\alpha|}(g_i^\alpha D_y^{|\alpha|}\psi(y))\\ \qquad -D_y^m[(\lambda u^i(x,t)\psi(y))-(-1)^m(D_y^m\tilde{u}^i)] \quad \text{in } \mathbb{R}^{n+1}_T:=\mathbb{R}^{n+1}\times(0,T),\\ \tilde{u}^i(x,y,0)=0,\end{cases}\tag{37}$$

for all $i = 1, \dots, N$.

Applying Proposition 3.6 we obtain

$$\||D^m_{(x,y)}\tilde{u}|\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)} \lesssim I_1 + I_2 + I_3,$$

where

$$I_1 := \sum_{i=1}^N \sum_{|\alpha|=m} \|g_i^\alpha D_y^m \psi(y)\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}, \qquad I_2 := \sum_{i=1}^N \sum_{|\alpha|\le m-1} \|g_i^\alpha D_y^{|\alpha|} \psi(y)\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)},$$

$$I_3 := \sum_{i=1}^N \|(\lambda u^i(x,t)\psi(y)) - (-1)^m (D_y^m \tilde{u}^i)\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}.$$

Let $\lambda_0 > 0$ be a fixed number which will be determined later. By a straightforward calculation we obtain, for $\lambda \ge \lambda_0$,

$$I_1 \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)}, \qquad I_2 \lesssim \sum_{i=1}^N \sum_{|\alpha|\le m-1} \lambda^{\frac{|\alpha|}{2m}} \|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)},$$

and

$$I_3 \lesssim \lambda^{1-\frac{1}{2m}} \sum_{i=1}^N \|u^i\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

Therefore,

$$\||D^m_{(x,y)}\tilde{u}|\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)} \lesssim \sum_{i=1}^N \sum_{|\alpha|\le m} \lambda^{\frac{|\alpha|}{2m}} \|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)} + \lambda^{1-\frac{1}{2m}} \sum_{i=1}^N \|u^i\|_{L^{p,q}_w(\mathbb{R}^n_T)}. \tag{38}$$

Fix a multi-index $|\alpha| = m$ and $i = 1, \dots, N$. We then have

$$\begin{aligned}
D^\alpha_{(x,y)}\tilde{u} &= \sum_{\beta\le\alpha} c_{\alpha,\beta} D_x^\beta u^i \phi(y) D_y^{2m-|\beta|} \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4}\right) \\
&\quad + \sum_{k=0}^{2m-|\beta|-1} \sum_{\beta\le\alpha} c_{\alpha,\beta,k} D_x^\beta u^i D_y^{2m-|\beta|-k} \phi(y) D^k \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4}\right) \\
&= \sum_{\beta\le\alpha} c_{\alpha,\beta} \lambda^{\frac{2m-|\beta|}{2m}} D_x^\beta u^i \phi(y) \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4} - \frac{(2m-|\beta|)\pi}{2}\right) \\
&\quad + \sum_{k=0}^{2m-|\beta|-1} \sum_{\beta\le\alpha} c_{\alpha,\beta,k} \lambda^{\frac{k}{2m}} D_x^\beta u^i D_y^{2m-|\beta|-k} \phi(y) \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4} - \frac{k\pi}{2}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\sum_{\beta\le\alpha} c_{\alpha,\beta} \lambda^{\frac{2m-|\beta|}{2m}} D_x^\beta u^i \phi(y) \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4} - \frac{(2m-|\beta|)\pi}{2}\right) \\
&= D^\alpha_{(x,y)}\tilde{u} - \sum_{k=0}^{2m-|\beta|-1} \sum_{\beta\le\alpha} c_{\alpha,\beta,k} \lambda^{\frac{k}{2m}} D_x^\beta u^i D_y^{2m-|\beta|-k} \phi(y) \cos\left(\lambda^{\frac{1}{2m}} y + \frac{\pi}{4} - \frac{k\pi}{2}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}&\left\||D_x^k u|\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{(2m-k)\pi}{2}\right)\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\\&\lesssim \||D^m_{(x,y)}\tilde{u}|\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\\&\quad+\sum_{k=0}^{2m-\ell-1}\sum_{\ell\le m}\lambda^{\frac{k}{2m}}\left\||D_x^\ell u|D_y^{2m-\ell-k}\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{k\pi}{2}\right)\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}.\end{aligned}\tag{39}$$

Fix $1\le p_1<p<p_2$. Then we have, for $0\le k\le m$,

$$\begin{aligned}&\left\||D_x^k u|\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{(2m-k)\pi}{2}\right)\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\\&\quad\ge C\left\||D_x^k u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}\left\|\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{(2m-k)\pi}{2}\right)\right\|_{L^{p_1}(dy)}.\end{aligned}\tag{40}$$

On the other hand, by Lebesgue differentiation theorem, there exists λ_1 so that for all $\lambda>\lambda_1$ we have

$$\left\|\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{(2m-k)\pi}{2}\right)\right\|_{L^{p_1}(dy)}\ge 1/10.\tag{41}$$

This together with (40) implies

$$\left\||D_x^k u|\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{(2m-k)\pi}{2}\right)\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\ge C\left\||D_x^k u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}.\tag{42}$$

for all $\lambda>\lambda_1$.

On the other hand, we have

$$\begin{aligned}&\left\||D_x^\ell u|D_y^{2m-\ell-k}\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{k\pi}{2}\right)\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\\&\quad\lesssim\left\||D_x^\ell u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}\left\|D_y^{2m-\ell-k}\phi(y)\cos\left(\lambda^{\frac{1}{2m}}y+\frac{\pi}{4}-\frac{k\pi}{2}\right)\right\|_{L^{p_2}(dy)}\\&\quad\lesssim\left\||D_x^\ell u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)},\end{aligned}\tag{43}$$

for all $0\le\ell\le m$.

From (38), (40), (42) and (43) we have

$$\begin{aligned}\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}\left\||D_x^k u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}&\lesssim\sum_{k=0}^{2m-\ell-1}\sum_{\ell\le m}\lambda^{\frac{k}{2m}}\left\||D_x^\ell u|\right\|_{L^{p,q}_w(\mathbb{R}^{n+1}_T)}\\&\quad+\sum_{i=1}^N\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}+\lambda^{1-\frac{1}{2m}}\sum_{i=1}^N\|u^i\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}\\&\lesssim\sum_{0\le\ell\le m}\lambda_0^{-\frac{1}{2m}}\lambda^{1-\frac{\ell}{2m}}\left\||D_x^\ell u|\right\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}\\&\quad+\sum_{i=1}^N\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}+\lambda_0^{-\frac{1}{2m}}\lambda\sum_{i=1}^N\|u^i\|_{L^{p,q}_w(\mathbb{R}^{n}_T)}.\end{aligned}$$

Hence, we can rewrite the inequality above as follows:

$$\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}\Big\||D^k_x u|\Big\|_{L^{p,q}_w(\mathbb{R}^n_T)}\lesssim\sum_{i=1}^N\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|g^\alpha_i\|_{L^{p,q}_w(\mathbb{R}^n_T)} + \lambda_0^{-\frac{1}{2m}}\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}\Big\||D^k_x u|\Big\|_{L^{p,q}_w(\mathbb{R}^n_T)}. \tag{44}$$

This implies there exists λ_0 so that

$$\sum_{0\le k\le m}\lambda^{1-\frac{k}{2m}}\Big\||D^k_x u|\Big\|_{L^{p,q}_w(\mathbb{R}^n_T)}\lesssim\sum_{i=1}^N\sum_{|\alpha|\le m}\lambda^{\frac{|\alpha|}{2m}}\|g^\alpha_i\|_{L^{p,q}_w(\mathbb{R}^n_T)},$$

for all $\lambda>\lambda_0$ as desired. ☐

We now give the proof of Proposition 3.3.

Proof of Proposition 3.3: Let $\phi\in C^\infty_0(Q_{R_0/2}(z_0)$ such that

$$0\le\phi\le 1, \phi=1 \text{ in } Q_{R_0/4}(z_0),\ |\phi_t|\lesssim R_0^{-1} \text{ and } |D^\alpha\phi|\lesssim R_0^{-|\alpha|},\quad \forall|\alpha|\le m. \tag{45}$$

We set

$$\hat{u}^i(x,t)=u^i(x,t)\phi(x-x_0,t-t_0):=u^i(x,t)\phi_{z_0}(x,t).$$

Observe that

$$(\mathcal{L}\hat{u})^i+\lambda\hat{u}^i=(-1)^m\sum_{j=1}^N\sum_{|\alpha|=|\beta|=m}D^\alpha(a^{\alpha\beta}_{ij}(x,t)D^\beta(u^j\phi_{z_0}))+(u^i_t+\lambda u^i)\phi_{z_0}+u^i(\phi_{z_0})_t$$

Since u is a weak solution to the equation (24), we obtain further

$$\begin{aligned}(\mathcal{L}\hat{u})^i+\lambda\hat{u}^i =&(-1)^m\sum_{j=1}^N\sum_{|\alpha|=|\beta|=m}D^\alpha\Big(a^{\alpha\beta}_{ij}(x,t)\sum_{\gamma\le\beta}C^\gamma_\beta D^\gamma u^jD^{\beta-\gamma}\phi_{z_0}\Big)\\ &-(-1)^m\sum_{j=1}^N\sum_{|\alpha|=|\beta|=m}D^\alpha(a^{\alpha\beta}_{ij}(x,t)D^\beta u^j)\phi_{z_0}+\sum_{|\alpha|=m}D^\alpha g^\alpha_i\phi_{z_0}+h^i\phi_{z_0}+u^i(\phi_{z_0})_t\\ =&I_1+I_2+(+h^i\phi_{z_0}+u^i(\phi_{z_0})_t),\end{aligned} \tag{46}$$

where

$$I_1=(-1)^m\sum_{j=1}^N\sum_{|\alpha|=|\beta|=m}\left[D^\alpha\Big(a^{\alpha\beta}_{ij}(x,t)\sum_{\gamma\le\beta}C^\gamma_\beta D^\gamma u^jD^{\beta-\gamma}\phi_{z_0}\Big)-D^\alpha(a^{\alpha\beta}_{ij}(x,t)D^\beta u^j)\phi_{z_0}\right],$$

and

$$I_2=\sum_{|\alpha|=m}D^\alpha g^\alpha_i\phi_{z_0}.$$

Note that we have

$$D^\alpha(fg)=gD^\alpha f-\sum_{\beta<\alpha}(-1)^{|\alpha-\beta|}C^\beta_\alpha D^\beta(fD^{\alpha-\beta}g). \tag{47}$$

This implies

$$gD^\alpha f=D^\alpha(fg)+\sum_{\beta<\alpha}(-1)^{|\alpha-\beta|}C^\beta_\alpha D^\beta(fD^{\alpha-\beta}g). \tag{48}$$

Applying the formulas (47) and (48) we obtain

$$
\begin{aligned}
I_1 =& \sum_{j=1}^{N} \sum_{|\alpha|\le m-1} D^\alpha \Big(\sum_{|\beta|=m} a_{ij}^{\alpha\beta}(x,t) \sum_{\gamma<\beta} C_\beta^\gamma D^\gamma u^j D^{\beta-\gamma}\phi_{z_0} \Big) \\
&- \sum_{j=1}^{N} \sum_{|\alpha|\le m-1} D^\alpha \Big(\sum_{|\beta|=m} \sum_{\gamma<\alpha} (-1)^{\alpha-\gamma} C_\alpha^\gamma a_{ij}^{\alpha\beta}(x,t) D^\beta u^j D^{\alpha-\gamma}\phi_{z_0} \Big),
\end{aligned}
$$

and

$$
I_2 = \sum_{|\alpha|=m} \sum_{\gamma\le\alpha} (-1)^{|\alpha-\gamma|} C_\alpha^\gamma D^\gamma (D^\alpha g_i^\alpha D^{\alpha-\gamma}\phi_{z_0}).
$$

Substituting the identities above into (46) then then applying Proposition 3.7, (45) and (3) we obtain

$$
\begin{aligned}
&\sum_{0\le k\le m} \lambda^{1-\frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/4}(z_0))} \\
&\quad \lesssim \sum_{|\alpha|\le m-1} \lambda^{\frac{|\alpha|}{2m}} \left(\sum_{0\le k\le m-1} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/2}(z_0))} \right) + \sum_{i=1}^{N} \sum_{|\alpha|=m} \lambda^{\frac{|\alpha|}{2m}} \|g_i^\alpha\|_{L_w^{p,q}(Q_{R_0/2}(z_0))} \\
&\quad\quad + \sum_{i=1}^{N} \|h^i\|_{L_w^{p,q}(Q_{R_0/2}(z_0))} + \sum_{i=1}^{N} \|u^i\|_{L_w^{p,q}(Q_{R_0/2}(z_0))}.
\end{aligned}
$$

for all $\lambda > \lambda_0 \ge 1$.

Since $\lambda > 1$, this implies, for all $\lambda \ge \lambda_0$,

$$
\begin{aligned}
&\sum_{0\le k\le m} \lambda^{\frac12-\frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/4}(z_0))} \\
&\quad \lesssim \lambda_0^{-\frac{1}{2m}} \left(\sum_{0\le k\le m-1} \lambda^{\frac12-\frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/2}(z_0))} \right) + \sum_{i=1}^{N} \sum_{|\alpha|=m} \|g_i^\alpha\|_{L_w^{p,q}(Q_{R_0/2}(z_0))} \\
&\quad\quad + \sum_{i=1}^{N} \|h^i\|_{L_w^{p,q}(Q_{R_0/2}(z_0))}.
\end{aligned}
$$

Observe that we can pick a disjoint family of cylinder $\{Q_{R_0/4}(z_i)\}_{i=1}^\infty$ so that

(i) $\bigcup_i Q_{R_0/4}(z_i) = \mathbb{R}^n_T$;
(ii) there exists $C > 0$ so that $\sum_{i=1}^\infty \chi_{Q_{R_0/2}(z_i)} \le C$.

Hence, from the above inequality we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{0\le k\le m} \lambda^{\frac12-\frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/4}(z_j))} \\
&\quad \lesssim \lambda_0^{-\frac{1}{2m}} \sum_{j=1}^{\infty} \sum_{0\le k\le m} \lambda^{\frac12-\frac{k}{2m}} \left\| |D^k u| \right\|_{L_w^{p,q}(Q_{R_0/4}(z_j))} \\
&\quad\quad + \sum_{j=1}^{\infty} \sum_{i=1}^{N} \sum_{|\alpha|=m} \|g_i^\alpha\|_{L_w^{p,q}(Q_{R_0/2}(z_j))} + \sum_{j=1}^{\infty} \sum_{i=1}^{N} \|h^i\|_{L_w^{p,q}(Q_{R_0/2}(z_j))}.
\end{aligned}
$$

This follows that

$$\sum_{0\le k\le m}\lambda^{\frac{1}{2}-\frac{k}{2m}}\left\||D^k u|\right\|_{L^{p,q}_w(\mathbb{R}^n_T)}\lesssim\lambda_0^{-\frac{1}{2m}}\sum_{0\le k\le m}\lambda^{\frac{1}{2}-\frac{k}{2m}}\left\||D^k u|\right\|_{L^{p,q}_w(\mathbb{R}^n_T)}+\sum_{i=1}^N\sum_{|\alpha|=m}\|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)}$$
$$+\sum_{i=1}^N\left\|h^i\right\|_{L^{p,q}_w(\mathbb{R}^n_T)}.$$

Therefore, for sufficiently large λ, we have

$$\sum_{0\le k\le m}\lambda^{\frac{1}{2}-\frac{k}{2m}}\left\||D^k u|\right\|_{L^{p,q}_w(\mathbb{R}^n_T)}\lesssim\sum_{i=1}^N\sum_{|\alpha|=m}\|g_i^\alpha\|_{L^{p,q}_w(\mathbb{R}^n_T)}+\sum_{i=1}^N\left\|h^i\right\|_{L^{p,q}_w(\mathbb{R}^n_T)}$$

as desired. □

3.2. **Weighted Orlicz estimates.** In this section, we provide the proof of Theorems 1.6 and 1.7. Since the proofs of these two theorems are similar to those of Theorems 1.4 and 1.5, we just sketch the proofs.

We first prove the following result.

Proposition 3.8. *Assume that the coefficients $\{a_{ij}^{\alpha\beta}(x,t)\}$ satisfy (3), (4) and small (δ,R_0)-BMO condition (7), $\Phi\in\Delta_2\cap\nabla_2$ and $w\in A_{i_\Phi}(\mathbb{R}^{n+1})$. Then there exists positive constant δ_0 such that if $\delta<\delta_0$, $f_i^\alpha\in L^\Phi_w(\mathbb{R}^n_T)$ for all $|\alpha|=m$ and $i=1,\dots,N$, and $u\in C_0^\infty(Q_{R_0}(z_0),\mathbb{R}^n)$ is a weak solution to the system (1), then we have*

$$\int_{\mathbb{R}^n_T}\Phi\left(|D^m u|^2\right)w(z)dz\lesssim\int_{\mathbb{R}^n_T}\Phi\left(|\mathbf{f}|^2\right)w(z)dz. \tag{49}$$

Proof. Applying Theorem 3.5 and (18), we have

$$\begin{aligned}\int_{\mathbb{R}^n_T}\Phi\left(\mathbb{M}\left(|D^m u|^2\right)(z)\right)w(z)dz&\sim\int_0^\infty w\left(\left\{z\in Q_{R_0}(z_0):\mathbb{M}\left(|D^m u|^2\right)(z)>\lambda\right\}\right)d\Phi(\lambda)\\&\sim\int_0^\infty w\left(\left\{z\in Q_{R_0}(z_0):\mathbb{M}\left(|D^m u|^2\right)(x)>A_0\lambda\right\}\right)d\Phi(A_0\lambda)\\&\lesssim\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|\mathbf{f}|>\gamma\lambda\}\right)d\Phi(A_0\lambda)\\&\quad+B\epsilon\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|D^m u|^2)(x)>\lambda\}\right)d\Phi(A_0\lambda)\\&\lesssim\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|\mathbf{f}|^2(z)>\lambda\}\right)d\Phi(\gamma^{-1}A_0\lambda)\\&\quad+B\epsilon\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|D^m u|^2)(z)>\lambda\}\right)d\Phi(A_0\lambda).\end{aligned}\tag{50}$$

Since $\Phi\in\Delta_2\cap\nabla_2$, we have

$$\Phi(At)\le Ca_1^{\log_2 A}\Phi(t),\forall t>0,A\ge1,$$

where C is a constant independing on t and A.

From this and (50), we have

$$\begin{aligned}\int_{\mathbb{R}^n_T}\Phi\left(\mathbb{M}\left(|D^m u|^2\right)(z)\right)w(z)dz\lesssim&a_1^{\log_2\gamma^{-1}A_0}\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|\mathbf{f}|^2)(z)>\lambda\}\right)d\Phi(\lambda)\\&+Ba_1^{\log_2 A_0}\epsilon\int_0^\infty w\left(\{z\in Q_{R_0}(z_0):\mathbb{M}(|D^m u|^2)(z)>\lambda\}\right)d\Phi(\lambda).\end{aligned}$$

Applying (18) again, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n_T} \Phi\left(\mathbb{M}\left(|D^m u|^2\right)(z)\right) w(z)dx \leq & c_\Phi a_1^{\log_2 \gamma^{-1} A_0} \int_\Omega \Phi\left(\mathbb{M}\left(|\mathbf{f}|^2\right)(z)\right) w(z)dz \\ & + c_\Phi B a_1^{\log_2 A_0} \epsilon \int_{\mathbb{R}^n_T} \Phi\left(\mathbb{M}\left(|D^m u|^2\right)(z)\right) w(z)dz.\end{aligned}$$

Taking ϵ such that $c_\Phi B a_1^{\log_2 A_0} \epsilon < 1$, and then using Lemma 2.4 we can conclude that

$$\int_{\mathbb{R}^n_T} \Phi\left(\mathbb{M}\left(|D^m u|^2\right)(z)\right) w(z)dz \leq C \int_{\mathbb{R}^n_T} \Phi\left(\mathbb{M}\left(|\mathbf{f}|^2\right)(z)\right) w(z)dz.$$

□

With Proposition 3.8 in hand, by the argument used to prove Proposition 3.3, we can prove the following result.

Proposition 3.9. *Assume that the coefficients* $\{a_{ij}^{\alpha\beta}\}$ *satisfy* (3), (4) *and small* (δ, R_0)*-BMO condition* (7), $\Phi \in \Delta_2 \cap \nabla_2$ *and* $w \in A_{i_\Phi}(\mathbb{R}^{n+1})$. *Then there exist positive constants* δ_0 *and* λ_0 *such that if* $\delta < \delta_0, \lambda \geq \lambda_0$, $g_i^\alpha, h^i \in L^\Phi_w(\mathbb{R}^{n+1})$ *for all* $|\alpha| = m$ *and* $i = 1, \ldots, N$, *and* u *is a weak solution to the system*

$$\begin{cases} (\mathcal{L}u)^i + \lambda u^i = \sum_{|\alpha|=m} D^\alpha g_i^\alpha + h^i, & \text{in } \mathbb{R}^n_T, \\ u^i(x,0) = 0 \end{cases} \tag{51}$$

for all $i = 1, \ldots, N$.

$$\sum_{0\leq k\leq m} \int_{\mathbb{R}^n_T} \Phi\left(\lambda^{1-\frac{k}{m}} |D^k u|^2\right) w(z)dz \lesssim \sum_{i=1}^N \sum_{|\alpha|=m} \int_{\mathbb{R}^n_T} \Phi\left(|g_i^\alpha|^2\right) w(z)dz + \sum_{i=1}^N \int_{\mathbb{R}^n_T} \Phi\left(|h^i|^2\right) w(z)dz, \tag{52}$$

for all $\lambda \geq \lambda_0$.

At this stage, repeating exactly the arguments in the proof of Theorem 1.4 and Theorem 1.5, we obtain Theorem 1.6 and Theorem 1.7.

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Dong Nai University, 4 Le Quy Don, Bien Hoa, Dong Nai, Viet Nam
Email address: buithequan2002@yahoo.com

University of Economics Ho Chi Minh City, Ho Chi Minh City, Viet Nam
Email address: lxuantruong@ueh.edu.vn