On stability and periodic oscillations of an income-capital model with time delay and spatial diffusion

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Abstract

The paper aims to establish a realistic income-capital model and gives economic conclusions by some mathematical analysis. Firstly, unlike other known models which either neglect time delay or neglect spatial diffusion, our model includes both time delay and spatial diffusion. Secondly, taking the time delay as bifurcation parameter, the stability of positive equilibrium and periodic oscillations are studied by theoretical and numerical analysis under two different boundary conditions. At last, the theoretical results yield the following economic conclusions: 1) For the closed economy or the open economy, there exists a critical threshold of time delay. If the time delay is smaller than the critical threshold, then the economic system will keep balanced at the present state; If the time delay is lager than the critical threshold, the stability of present state will be destroyed, and the periodic oscillations will emerge; 2) The biggest difference between the critical threshold of open economy and that of closed economy is that the former is related to diffusion coefficients, while the latter is independent of diffusion coefficients; 3) The periodic oscillations are spatially homogeneous for closed economy,

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but are spatially inhomogeneous for open economy; 4) Regional income and capital disparities are more likely to occur in open economies than in closed economies; 5) Results reveal to some extent the causes of the gap between the rich and the poor and also provide insight into why developed economies are more likely to polarize than underdeveloped ones. Our theoretical analysis is based on the center manifold theorem, normal forms and Hopf bifurcation theory.

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1. Introduction

To investigate the trade cycle, Kaldor [1] first proposed a mathematical model, called the Kaldor model. Since then, the model has been used to describe various economic systems, one of which is the income-capital system:

$$\begin{cases} \frac{du}{dt} = I(u,v) - S(u,v), \\ \frac{dv}{dt} = I(u,v), \end{cases} \tag{1.1}$$

where u represents the total social income, v stands for the total social capital, I is the investment function, and S is the saving function. Strictly speaking, the model (1.1) only reveals the relation between the investment I, the saving S and the growth rate of the total social income and the total social capital per unit time, which cannot shed more light on the understanding of the trade cycle, due to the drawback that the exact expressions of I and S are unknown and the time delay for total social income to be converted into total social capital is also neglected.

Although it is impossible to find an analytical to determine the explicit expressions of I and S generally, based on the economical principles and the Keynesian theory, Ma [2] found that their expressions can be derived for some special cases. More precisely, if assuming that the model (1.1) has a positive equilibrium point (u_0, v_0), and in the neighborhood of which the function I and S in (1.1) satisfying the properties

$$\begin{aligned} &\frac{\partial I}{\partial u} > 0, \quad \frac{\partial S}{\partial u} > 0, \quad \frac{\partial S}{\partial v} > 0, \\ &\frac{\partial^2 I}{\partial u \partial v} \leq 0, \quad \frac{\partial^2 S}{\partial u^2} \propto (u - u_0), \quad \frac{\partial^2 S}{\partial v^2} \propto (v - v_0), \end{aligned}$$

he thus derived that the explicit expressions of I and S respectively take the forms as follows

$$\begin{aligned} &I = a(u-u_0) - \alpha u(v-v_0)^3, \\ &S = b(u-u_0) + c(v-v_0) + \gamma u(v-v_0)^3 + \beta v(u-u_0)^3, \end{aligned} \tag{1.2}$$

where $a, b, c, \gamma, \alpha, \beta$ are positive constants. Particularly, a presents the willingness to invest, and b is the willingness to save.

The time delay for total social income cannot be neglected. Indeed, Kalecki [3] argued that there exists a time delay for the total social income to be converted into the total social capital. To incorporate the time delay into Kaldor's model, Kraswiec and Szydlowski [4] proposed a new model, called the Kaldor-Kalecki model, given by

$$\begin{cases} \frac{du}{dt} = I(u(t), v(t)) - S(u(t), v(t)), \\ \frac{dv}{dt} = I(u(t-\tau), v(t)) - \delta v(t), \end{cases} \tag{1.3}$$

where δ is the depreciation rate of the total social capital, τ is the time delay.

Although the Kaldor-Kalecki model with I and S given by (1.2) is an analytical model to study the trade cycle, the inhomogenous distribution of the income and capital in space means that we have to incorporate the spatial diffusion terms into the above models. In other words, we employ the modified Kaldor-Kalecki (MKK) model, the partial differential model

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + k[u(x,t) - u_0] \\ \qquad -c[v(x,t) - v_0] - \rho u(x,t)[v(x,t) - v_0]^3 \\ \qquad -\beta v(x,t)[u(x,t) - u_0]^3, x \in \Omega, t \in (0, \infty), \\ \frac{\partial v}{\partial t} = d_2 \Delta v + a[u(x, t-\tau) - u_0] - \delta[v(x,t) - v_0] \\ \qquad -\alpha u(x, t-\tau)[v(x,t) - v_0]^3, \quad x \in \Omega, t \in (0, \infty), \\ u(x,t) = \varphi_1(x,t), v(x,t) = \varphi_2(x,t), x \in \bar{\Omega}, t \in [-\tau, 0], \end{cases} \tag{1.4}$$

to understand the economical phenomenon involved the trade cycle in the present work, where $d_1, d_2 > 0$ are diffusion coefficients, $k = a - b, \rho = \alpha + \gamma$, and the domain Ω takes

$$\Omega = (0, \pi).$$

The (1.4) is derived by adding the diffusion terms into the Kaldor-Kalecki model, and making use of the expressions of I and S given by (1.2).

We will discuss the stability and periodic oscillations of the MKK model (1.4) subjected to the following two different boundary conditions, respectively:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0,\ x \in \partial\Omega,\ t \in (0, \infty), \tag{1.5}$$

where n is the outward normal on the boundary $\partial\Omega$.

$$u = u_0,\ v = v_0,\ x \in \partial\Omega. \tag{1.6}$$

Please note that in general, (1.5) is called homogeneous Neumann boundary condition (corresponding to closed economy). (1.6) is called inhomogeneous Dirichlet boundary condition (corresponding to open economy).

Note that (u_0, v_0) is an equilibrium point of the MKK model (1.4) under boundary condition (1.5) or (1.6). By carrying out the linear transformation:

$$\begin{cases} \tilde{u} = u - u_0, \\ \tilde{v} = v - v_0, \end{cases} \tag{1.7}$$

and omitting the tildes, we obtain an equivalent system of (1.4)

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ku(x,t) - cv(x,t) - \rho u_0 v^3(x,t) \\ \qquad -\beta v_0 u^3(x,t) - \rho u(x,t) v^3(x,t) - \beta v(x,t) u^3(x,t), \\ \frac{\partial v}{\partial t} = d_2 \Delta v + au(x, t-\tau) - \delta v(x,t) \\ \qquad -\alpha u_0 v^3(x,t) - \alpha u(x, t-\tau) v^3(x,t), \\ u(x,t) = \varphi_1(x,t) - u_0, v(x,t) = \varphi_2(x,t) - v_0. \end{cases} \tag{1.8}$$

Correspondingly, the Eq.(1.8) should be equipped with the boundary conditions (1.5) and the homogeneous Dirichlet boundary condition

$$u = 0,\ v = 0,\ x \in \partial\Omega. \tag{1.9}$$

Therefore, the dynamics of (1.4) at equilibrium point (u_0, v_0) under boundary conditions (1.5) or (1.6) is equivalently converted to that of (1.8) at $(0, 0)$ under boundary conditions (1.5) or (1.9) via the linear transformation (1.7). Our analysis involved the dynamics of the MKK model (1.4) is based on the classical Hopf bifurcation theory, center manifold theorem and normal forms proposed by Hassard [5], Wu [6] and Faria [7].

The rest parts of this work are arranged as follows: Theoretical analysis under homogeneous Neumann boundary condition is carried out in Section 2. Theoretical analysis under inhomogeneous Dirichlet boundary condition is carried out in Section 3. Numerical analysis is provided in Section 4. And discussion and conclusions are illustrated in Section 5.

2. Stability and periodic oscillations with homogeneous Neumann boundary condition

2.1. Linear stable analysis

Since the stability of the positive equilibrium state (u_0, v_0) of the system (1.4) equipped with the boundary condition (1.5) is the same with that of the zero equilibrium point of the system (1.8) subjected to the condition (1.5), we thus focus on the stability of the zero solution $(0, 0)$ to the equation (1.8) supplemented with (1.5). To this end, let us define the function spaces

$$X_0 = \left\{ (u, v)^T \in H^2(\Omega, \mathbf{R}^2) \middle| \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0,\ x \in \partial\Omega \right\}, \quad X = L^2(\Omega, \mathbf{R}^2).$$

Denote

$$u_1(t) = u(., t),\ u_2(t) = v(., t),\ U = (u_1, u_2)^T \in X_0,$$

then model (1.8) can be rewritten into an abstract form

$$\frac{dU(t)}{dt} = D\Delta U + L(U_t) + F(U_t), \tag{2.10}$$

where the linear differential operators $D\Delta : C((-\tau, +\infty), X_0) \to X$, $L : C((-\tau, +\infty), X_0) \to X$ and the nonlinear operators $F : C((-\tau, +\infty), X_0) \to X$ are defined by, respectively,

$$\begin{aligned}
&D\Delta\phi = (d_1\Delta\phi_1(0), d_2\Delta\phi_2(0))^T, \quad \phi = (\phi_1, \phi_2) \in X_0,\\
&L(\phi) = \begin{pmatrix} k & -c \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix},\\
&F(\phi) = \begin{pmatrix} -\rho u_0\phi_2^3(0) - \beta v_0\phi_1^3(0) - \rho\phi_1(0)\phi_2^3(0) - \beta\phi_2(0)\phi_1^3(0) \\ -\alpha u_0\phi_2^3(0) - \alpha\phi_1(-\tau)\phi_2^3(0) \end{pmatrix}.
\end{aligned} \tag{2.11}$$

It is well known that the eigenvalues and eigenvectors of the Laplacian operator $\Delta : X_0 \to X$ are, respectively, given by

$$\rho_n = -n^2,\ \beta_n^1 = (cosnx, 0)^T,\ \beta_n^2 = (0, cosnx)^T,\ n \in \mathbf{N}_0 = \{0, 1, 2, \cdots\},$$

by which one can obtain the characteristic equation associated with the eigenvalue problem $D\Delta U + L(U_t) = \lambda U$ as follows

$$\lambda^2 + T_n\lambda + D_n + ace^{-\lambda\tau} = 0, \quad n \in \mathbf{N}_0, \tag{2.12}$$

where $T_n = (d_1 + d_2)n^2 + \delta - k,\ D_n = (d_1 n^2 - k)(d_2 n^2 + \delta)$.

For the convenience of stating some lemmas used to prove our main theorem, let us set

$$k < \delta \tag{2.13a}$$

$$k < \frac{ac}{\delta} \text{ and } \frac{d_2}{d_1}ac \leq \delta^2, \tag{2.13b}$$

$$k < 2\sqrt{\frac{acd_1}{d_2}} - \frac{d_1}{d_2}\delta \text{ and } \frac{d_2}{d_1}ac > \delta^2, \tag{2.13c}$$

$$|k|\delta < ac, \tag{2.13d}$$

$$k < d_1 - \frac{ac}{d_2 + \delta}. \tag{2.13e}$$

Then, we give the first lemma involving the stability of the system (1.8).

Lemma 2.1. *If* (2.13a)-(2.13b) *or* (2.13a),(2.13c) *holds true, then the zero point* $(0,0)^T$ *of the problem* (1.8) *subjected to* (1.9) *is asymptotically stable for the zero time delay* $\tau = 0$.

Proof. It is sufficient to show that all roots of the Eq. (2.12) with $\tau = 0$ have negative real parts. The Eq. (2.12) with $\tau = 0$ reads

$$\lambda^2 + T_n\lambda + D_n + ac = 0, \quad n \in \mathbf{N}_0, \tag{2.14}$$

whose all roots have negative real parts if and only if $T_n > 0$ and $D_n + ac > 0$ for any $n \in \mathbf{N}_0$, by the Routh-Hurwitz criterion [9, 10, 11]. Direct calculations find that if $k < \delta$, then $T_n > 0$ holds for any $n \in \mathbf{N}_0$. Let

$$f(m) = d_1 m + \frac{ac}{d_2 m + \delta} \ (m \geq 0),$$

then $f'(m) = 0$ has the unique zero point

$$m_0 = \sqrt{\frac{ac}{d_1 d_2}} - \frac{\delta}{d_2}.$$

If

$$m_0 \le 0 \text{ and } k < \min_{m\ge 0} f = f(0) = \frac{ac}{\delta},$$

then $D_n + ac > 0$ holds for any $n \in \mathbf{N}_0$. If

$$m_0 > 0 \text{ and } k < \min_{m\ge 0} f = f(m_0) = 2\sqrt{\frac{acd_1}{d_2}} - \frac{d_1}{d_2}\delta,$$

then $D_n + ac > 0$ holds for each $n \in \mathbf{N}_0$. In other words, if (2.13a)-(2.13b) or (2.13a),(2.13c) holds true, then $T_n > 0$ and $D_n + ac > 0$ hold for any $n \in \mathbf{N}_0$. □

Lemma 2.2. *If (2.13b) or (2.13c) holds, then 0 is not a root of the characteristic equation (2.12).*

Proof. It is not tough to check that if (2.13b) or (2.13c) holds, then $D_n + ac > 0$ holds for any $n \in \mathbf{N}_0$. □

Let us set

$$\begin{cases} k < \delta \text{ and } k < \frac{ac}{\delta} \text{ and } \frac{d_2}{d_1}ac \le \delta^2 \\ |k|\delta < ac \text{ and } k < d_1 - \frac{ac}{d_2+\delta} \end{cases}, \tag{2.15a}$$

$$\begin{cases} k < \delta \text{ and } k < 2\sqrt{\frac{acd_1}{d_2}} - \frac{d_1}{d_2}\delta \text{ and } \frac{d_2}{d_1}ac > \delta^2 \\ |k|\delta < ac \text{ and } k < d_1 - \frac{ac}{d_2+\delta} \end{cases}, \tag{2.15b}$$

and

$$\tau_{0,j}^{+} = \begin{cases} \frac{1}{\omega_0^{+}}[\arccos(\frac{k\delta+(\omega_0^{+})^2}{ac}) + 2j\pi], & \frac{(\delta-k)\omega_0^{+}}{ac} \ge 0, \\ \frac{1}{\omega_0^{+}}[2\pi - \arccos(\frac{k\delta+(\omega_0^{+})^2}{ac}) + 2j\pi], & \frac{(\delta-k)\omega_0^{+}}{ac} < 0, \quad j \in \mathbf{N}_0, \end{cases} \tag{2.16}$$

where

$$\omega_0^{+} = \frac{\sqrt{-\delta^2 - k^2 + \sqrt{(\delta^2-k^2)^2 + 4a^2c^2}}}{\sqrt{2}},$$

we thus have the following lemma

Lemma 2.3. *If the condition (2.15a) or (2.15b) holds, and $\tau = \tau_{0,j}^{+}$ ($j \in \mathbf{N}_0$), then the Eq. (2.12) has only one pair of purely imaginary roots $\pm i\omega_0^{+}$.*

Proof. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of the Eq. (2.12), then ω solves

$$-\omega^2 + [(d_1 + d_2)n^2 + \delta - k]i\omega + (d_1 n^2 - k)(d_2 n^2 + \delta) + ace^{-i\omega\tau} = 0,$$

whose real part and imaginary part are, respectively,

$$\begin{aligned} & -\omega^2 + (d_1 n^2 - k)(d_2 n^2 + \delta) + accos(\omega\tau) = 0, \\ & \left[(d_1 + d_2)n^2 + \delta - k\right]\omega - acsin(\omega\tau) = 0. \end{aligned} \tag{2.17}$$

From the identity (2.17) one can find that ω solves the quartic equation

$$\omega^4 + P_n\omega^2 + Q_n = 0, \tag{2.18}$$

where

$$\begin{aligned} P_n &= (d_1 n^2 - k)^2 + (d_2 n^2 + \delta)^2, \\ Q_n &= (d_1 n^2 - k)^2 (d_2 n^2 + \delta)^2 - a^2 c^2. \end{aligned}$$

It deduces from the inequality (2.13d) that $Q_0 < 0$, i.e., (2.18) has only one positive root for $n = 0$, implying that the Eq. (2.12) has one pair of purely imaginary roots $\pm i\omega_0^+$, given by

$$\omega_0^+ = \frac{\sqrt{-P_0 + \sqrt{P_0^2 - 4Q_0}}}{\sqrt{2}} = \frac{\sqrt{-\delta^2 - k^2 + \sqrt{(\delta^2 - k^2)^2 + 4a^2c^2}}}{\sqrt{2}}.$$

Besides, from (2.17), one can note that $\tau_{0,j}^+$ $(j \in \mathbf{N}_0)$ satisfy

$$\cos(\omega_0^+\tau_{0,j}^+) = \frac{k\delta + (\omega_0^+)^2}{ac}, \quad \sin(\omega_0^+\tau_{0,j}^+) = \frac{(\delta - k)\omega_0^+}{ac}.$$

Thanks to the condition (2.15a) or (2.15b), (2.13b) and (2.13e) or (2.13c) and (2.13e) hold, one thus deduces that $Q_n > 0$ for any $n \geq 1$. Owning to $P_n > 0$ for any $n \geq 1$, the Eq. (2.18) has no positive root when $n \geq 1$. Therefore, the characteristic equation (2.12) has no any purely imaginary root when $n \geq 1$. If (2.15a) or (2.15b) hold true, and $\tau = \tau_{0,j}^+$ $(j \in \mathbf{N}_0)$, the Eq. (2.12) has only one pair of simple pure imaginary roots $\pm i\omega_0^+$. $\square$

Lemma 2.4. *If* (2.15a) *or* (2.15b) *holds true,*

$$Re\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau=\tau_{0,j}^+} > 0, \ \forall j \in \mathbf{N}_0.$$

Proof. Differentiate on both sides of the Eq. (2.12) with respect to τ, we have the following equality

$$2\lambda d\lambda + [(d_1+d_2)n^2+\delta-k]d\lambda - ac(\lambda d\tau+\tau d\lambda)e^{-\lambda\tau}=0.$$

It derives from the preceding equation that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_{0,j}^{+}} = \frac{\delta-k+2i\omega_0^{+}}{aci\omega_0^{+}e^{-i\omega_0^{+}\tau_{0,j}^{+}}} - \frac{\tau_{0,j}^{+}}{i\omega_0^{+}},\ \forall j\in\mathbf{N}_0$$

by which we arrive at

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_{0,j}^{+}} = \frac{k^2+\delta^2+2(\omega_0^{+})^2}{a^2c^2}>0,\ \forall j\in\mathbf{N}_0.$$

□

Lemma 2.5. *If (2.15a) or (2.15b) holds true, then all roots of the Eq. (2.12) have negative real parts when $\tau\in[0,\tau_{0,0}^{+})$. In addition, there exists a root of the characteristic equation (2.12), which has positive real part when $\tau>\tau_{0,0}^{+}$.*

Proof. According to the Corollary 2.4 in [8] and Lemma 2.2-Lemma 2.4, all roots of the Eq. (2.12) have negative real parts when $\tau\in[0,\tau_{0,0}^{+})$, and there only exists a root of the Eq. (2.12) with positive real part when $\tau>\tau_{0,0}^{+}$. □

2.2. Main theorem

Let us define

$$\tau_{0,0}^{+}=\min_{j\in\mathbf{N}_0}\{\tau_{0,j}^{+}\}, \tag{2.19}$$

where $\tau_{0,j}^{+}$ are given in (2.16), and denote

$$\begin{aligned}
&c_0(0)=-3\bar{r}_0\tau_{0,j}^{+}[\beta v_0+(\rho+\alpha\bar{\eta}_0)u_0\xi_0^2\bar{\xi}_0],\ j\in\mathbf{N}_0,\\
&\mu_0=-\frac{Re(c_0(0))}{Re(\lambda'(\tau_{0,j}^{+}))},\quad \beta_0=2Re(c_0(0)),\\
&T_0=-\frac{1}{\omega_0^{+}\tau_{0,j}^{+}}[Im(c_0(0))+\mu_0 Im(\lambda'(\tau_{0,j}^{+}))],\quad \xi_0=\frac{k-i\omega_0^{+}}{c},\\
&\eta_0=\frac{-c}{\delta-i\omega_0^{+}},\quad r_0=\frac{1}{1+\xi_0\bar{\eta}_0+a\bar{\eta}_0e^{-i\omega_0^{+}\tau_{0,j}^{+}}},\\
&\lambda'(\tau_{0,j}^{+})=\frac{aci\omega_0^{+}e^{-i\omega_0^{+}\tau_{0,j}^{+}}}{\delta-k+2i\omega_0^{+}-ac\tau_{0,j}^{+}e^{-i\omega_0^{+}\tau_{0,j}^{+}}},
\end{aligned} \tag{2.20}$$

we thus have the following bifurcation theorem:

Theorem 2.1. *If (2.15a) or (2.15b) holds, then for the system (1.4) subjected to (1.5) the following assertions hold:*

(1) *The positive equilibrium $(u_0, v_0)^T$ is locally asymptotically stable for each $\tau \in [0, \tau_{0,0}^+)$, while it becomes unstable when $\tau > \tau_{0,0}^+$.*
(2) *For each $j \in \mathbf{N}_0$, there exists a Hopf bifurcation as τ crosses increasingly through the critical threshold $\tau_{0,j}^+$.*
(3) *If $\mu_0 < 0$ ($\mu_0 > 0$), the Hopf bifurcation is subcritical (supercritical), and the bifurcated periodic solution is unstable (stable) .*
(4) *If $T_0 < 0$ ($T_0 > 0$), the period of the corresponding bifurcated solution is decrease (increase);*
(5) *The bifurcated periodic solution is spatially homogeneous.*

Proof. If (2.15a) or (2.15b) hold, according to Lemma 2.1-Lemma 2.5 and the classical Hopf bifurcation theory one can obtain the assertions (1)-(2). For the system (1.4), Hopf bifurcation occurs at $\tau = \tau_{0,j}^+$ $(j \in \mathbf{N}_0)$, i.e., the Eq. (2.12) has one pair of purely imaginary roots $\pm i\omega_0^+$ when $\tau = \tau_{0,j}^+$ $(j \in \mathbf{N}_0)$. For simplicity, we denote $\tau_0 = \tau_{0,j}^+$ for any $j \in \mathbf{N}_0$ and $\omega_0 = \omega_0^+$, and define $C_{01} = C([-1, 0], X_0)$. Now, let us prove the the assertions (3)-(5). To this end, taking the following linear transformation

$$\tilde{t} = \frac{t}{\tau},\ h = \tau - \tau_0, \tag{2.21}$$

the system (2.10) thus can be rewritten as

$$\frac{dU(t)}{dt} = \tau D\Delta U + L(U_t, h) + f(U_t, h), \tag{2.22}$$

where $L : C_{01} \times \mathbf{R} \to X$ and $f : C_{01} \times \mathbf{R} \to X$ are given by

$$L(\phi, h) = (\tau_0 + h)\begin{pmatrix} k & -c \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \quad \phi = (\phi_1, \phi_2),$$
$$f(\phi, h) = (\tau_0 + h)\begin{pmatrix} -\rho u_0 \phi_2^3(0) - \beta v_0 \phi_1^3(0) - \rho\phi_1(0)\phi_2^3(0) - \beta\phi_2(0)\phi_1^3(0) \\ -\alpha u_0 \phi_2^3(0) - \alpha\phi_1(-1)\phi_2^3(0) \end{pmatrix}.$$

Under the linear transformation (2.21), apparently, there exists a Hopf bifurcation in the system (2.22) as h crosses 0. Then there is one pair of purely imaginary roots $\pm i\omega_0\tau_0$ of the characteristic equation for the following linearized equation of (2.22)

$$\frac{dU(t)}{dt} = \tau D\Delta U + L(U_t, h), \tag{2.23}$$

and all other roots have negative real parts for $\tau = \tau_0$, $h = 0$.

Consider the ordinary functional differential equation

$$\frac{dz}{dt} = L(z_t, h).$$

By the Riesz's representation theorem, there exists a bounded variation function $\eta(\theta, h)$ $(-1 \leq \theta \leq 0)$, such that for any $\phi \in C([-1, 0], \mathbf{R}^2)$,

$$L(\phi, h) = \int_{-1}^{0} d[\eta(\theta, h)]\phi(\theta),$$

where

$$\eta(\theta, h) = (\tau_0 + h)\left[\begin{pmatrix} k & -c \\ 0 & -\delta \end{pmatrix} \delta(\theta) + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \delta(\theta + 1) \right],$$

and

$$\delta(\theta) = \begin{cases} 1, & \theta = 0, \\ 0, & -1 \leq \theta < 0. \end{cases}$$

For any $\phi \in C([-1, 0], \mathbf{R}^2)$, $\psi \in C([0, 1], \mathbf{R}^2)$, we define A_0, A_0^* as follows

$$A_0(\phi(\theta)) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, \ \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta), \ \theta = 0, \end{cases}$$

$$A_0^*(\psi(s)) = \begin{cases} -\frac{d\psi(s)}{ds}, \ s \in (0, 1], \\ \int_{-1}^{0} d\eta(\theta, 0)\phi(-\theta), \ s = 0. \end{cases}$$

Owning to

$$(\psi, \phi) = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d[\eta(\theta, 0)]\phi(\xi)d\xi,$$

one can see that the operator A_0^* is the adjoint operator of A_0.

Let $q(\theta)$ $(\theta \in [-1,0])$ be the eigenvector of A_0 associated with eigenvalue $i\omega_0\tau_0$, and $q^*(s)$ $(s \in [0,1])$ be the eigenvector of A_0^* associated with eigenvalue $-i\omega_0\tau_0$, such that

$$(q^*, q) = 1,\ (q^*, \bar{q}) = 0.$$

Thus, upon performing some calculations, we have

$$q(\theta) = (1, \xi_0)^T e^{i\omega_0\tau_0\theta},\ \theta \in [-1, 0],$$

$$q^*(s) \ = r_0(1, \eta_0)e^{i\omega_0\tau_0 s},\ s \in [0, 1],$$

where

$$\xi_0 = \frac{k - i\omega_0}{c},\ \eta_0 = \frac{-c}{\delta - i\omega_0}\ , r_0 = \frac{1}{1 + \xi_0\bar{\eta}_0 + a\bar{\eta}_0 e^{-i\omega_0\tau_0}}.$$

Denote $\Phi = (q(\theta), \bar{q}(\theta)),\ \Psi = (q^*(s), \bar{q^*}(s))^T$, then

$$(\Psi, \Phi) = \begin{pmatrix} (q^*, q) & (q^*, \bar{q}) \\ (\bar{q^*}, q) & (\bar{q^*}, \bar{q}) \end{pmatrix} = I_{2\times 2}.$$

Introducing the central subspace P of the linearized system (2.23) and its adjoint space P^* as follows

$$P = \mathrm{span}\{q(\theta), \bar{q}(\theta)\}, \quad P^* = \mathrm{span}\{q^*(\theta), \bar{q^*}(\theta)\},$$

one can check that P is the generalized eigenspace associated with the eigenvalues set $\Lambda_0 = \{i\omega_0\tau_0, -i\omega_0\tau_0\}$, P^* is the adjoint eigenspace.

Define $f_0 \triangleq (\beta_0^1, \beta_0^2)$, $\beta_0^1 = (1, 0)^T$, $\beta_0^2 = (0, 1)^T$, and

$$c \cdot f_0 = c_1\beta_0^1 + c_2\beta_0^2,\ \forall c = (c_1, c_2)^T \in C_{01}.$$

Also, let us define $X_{0\mathbf{C}} \triangleq X_0 \bigoplus iX_0 = \{u + iv | u, v \in X_0\}$, then the corresponding complex value inner product is defined as

$$< U_1, U_2 >= \frac{1}{\pi}\int_0^\pi (u_1\bar{v_1} + u_2\bar{v_2})dx, \tag{2.24}$$

for any $U_1 = (u_1, u_2)^T, U_2 = (v_1, v_2)^T \in X_{0\mathbf{C}}$.

For any $\phi \in C_{01}$, let us define

$$< \phi, f_0 >= (< \phi, \beta_0^1 >, < \phi, \beta_0^2 >). \tag{2.25}$$

Then, the central subspace of the linear equation (2.23) at $h = 0$ is given by

$$P_{CN}C_{01} = \{(q(\theta)z + \bar{q}(\theta)\bar{z}) \cdot f_0,\ z \in \mathbf{C}\}, \tag{2.26}$$

and

$$P_{CN}C_{01}(\phi) = \Phi(\Psi, < \phi, f_0 > \cdot f_0),\ \phi \in C_{01}. \tag{2.27}$$

Note that the space C_{01} can be decomposed into

$$C_{01} = P_{CN}C_{01} \bigoplus P_S C_{01}, \tag{2.28}$$

where $P_S C_{01}$ is the complement space of $P_{CN}C_{01}$ in C_{01}.

Let A_U be the infinitesimal generator of the semigroup generated by linear system (2.23) at $h = 0$, then (2.22) can be rewritten as

$$\frac{dU_t}{dt} = A_U U_t + R(U_t, h), \tag{2.29}$$

where

$$R(\phi, h) = \begin{cases} 0,\ \theta \in [-1, 0), \\ f(\phi, h),\ \theta = 0. \end{cases} .$$

Since we only focus on the stability and the Hopf bifurcation at $h = 0$, it just needs to compute the coordinates on the central manifold at $h = 0$. According to (2.26) and (2.28), the solution to the Eq. (2.22) can be expressed as

$$U_t = (q(\theta)z(t) + \bar{q}(\theta)\bar{z}(t)) \cdot f_0 + W(z(t), \bar{z}(t)), \tag{2.30}$$

where $W(z(t), \bar{z}(t)) = (W^1(z(t), \bar{z}(t)), W^2(z(t), \bar{z}(t)))^T$.

On the center manifold, we have

$$W(z(t), \bar{z}(t)) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{2.31}$$

with which one can note that when $\tau = \tau_0$, z satisfies that

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0\tau_0 z + (q^*, < R(U_t, 0), f_0 >) \\ &= i\omega_0\tau_0 z + \bar{q}^*(0) < f(U_t, 0), f_0 > . \end{aligned} \tag{2.32}$$

Let us denote

$$g(z,\bar z) = \bar q^*(0) < f(U_t,0), f_0 >= g_{20}\frac{z^2}{2} + g_{11}z\bar z + g_{02}\frac{\bar z^2}{2} + g_{21}\frac{z^2\bar z}{2} + \cdots \quad (2.33)$$

It infers from (2.24), (2.25), (2.30) and (2.33) that $\bar q^*(0) = \bar r_0(1, \bar\eta_0)$ and $g_{20} = g_{11} = g_{02} = 0$. In what follows, we calculate g_{21}. Note that

$$< f(U_t,0), f_0 >= (< f(U_t,0), \beta_0^1 >, < f(U_t,0), \beta_0^2 >),$$

direct calculations give

$$\begin{aligned}
< f(U_t,0), \beta_0^1 >=& \frac{\tau_0}{\pi}\int_0^\pi -\rho u_0(\xi_0^3 z^3 + \bar\xi_0^3\bar z^3 + 3\xi_0^2\bar\xi_0 z^2\bar z + 3\xi_0\bar\xi_0^2 z\bar z^2)dx \\
& - \frac{\tau_0}{\pi}\int_0^\pi \beta v_0(z^3 + \bar z^3 + 3z^2\bar z + 3z\bar z^2)dx + h.o.t \\
=& -\rho\tau_0 u_0(\xi_0^3 z^3 + \bar\xi_0^3\bar z^3 + 3\xi_0^2\bar\xi_0 z^2\bar z + 3\xi_0\bar\xi_0^2 z\bar z^2) \\
& - \beta\tau_0 v_0(z^3 + \bar z^3 + 3z^2\bar z + 3z\bar z^2) + h.o.t, \\
< f(U_t,0), \beta_0^2 >=& \frac{\tau_0}{\pi}\int_0^\pi -\alpha u_0(\xi_0^3 z^3 + \bar\xi_0^3\bar z^3 + 3\xi_0^2\bar\xi_0 z^2\bar z + 3\xi_0\bar\xi_0^2 z\bar z^2)dx + h.o.t \\
=& -\alpha\tau_0 u_0(\xi_0^3 z^3 + \bar\xi_0^3\bar z^3 + 3\xi_0^2\bar\xi_0 z^2\bar z + 3\xi_0\bar\xi_0^2 z\bar z^2) + h.o.t.
\end{aligned}$$

Making use of the preceding equations, one can obtian

$$g_{21} = -6\tau_0\bar r_0[(\rho + \alpha\bar\eta_0)u_0\xi_0^2\bar\xi_0 + \beta v_0].$$

According to Theorem 2.1 in [6] and the linear transform (2.21), (2.32) can be transformed into the Poincare normal form

$$\frac{d\xi}{dt} = i\omega_0\xi + c_0(0)\xi|\xi|^2 + O(|\xi|^5),$$

where

$$\begin{aligned}
c_0(0) &= \frac{i}{2\omega_0\tau_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{1}{2}g_{21} \\
&= -3\bar r_0\tau_0[\beta v_0 + (\rho + \alpha\bar\eta_0)u_0\xi_0^2\bar\xi_0].
\end{aligned}$$

According to Theorem 2.2 in [6], if $\mu_0 < 0$ ($\mu_0 > 0$), the Hopf bifurcation is a subcritical (supercritical) bifurcation; if $\beta_0 > 0$ ($\beta_0 < 0$)(which is equivalent to $\mu_0 < 0$ ($\mu_0 > 0$) according to Lemma 2.4), the bifurcated periodic solution is unstable (stable); if $T_0 < 0$ ($T_0 > 0$), the period of the bifurcated solution is decrease (increase), where μ_0 and T_0 are given in (2.20). Furthermore, the expression of $U_t(h,\theta)$ implies that the bifurcated periodic solution is spatially homogeneous. □

3. Stability and periodic oscillations with inhomogeneous Dirichlet boundary condition

In this section, we consider the stability of the positive equilibrium point and the existence of Hopf bifurcation. To this end, let us firstly set

$$k < d_1 + d_2 + \delta, \tag{3.34a}$$

$$k < d_1 + \frac{ac}{d_2+\delta},\ \frac{d_2}{d_1}ac \le (d_2+\delta)^2, \tag{3.34b}$$

$$k < 2\sqrt{\frac{acd_1}{d_2}} - \frac{d_1}{d_2}\delta,\ \frac{d_2}{d_1}ac > (d_2+\delta)^2, \tag{3.34c}$$

$$|d_1 - k|(d_2+\delta) < ac, \tag{3.34d}$$

$$k < 4d_1 - \frac{ac}{4d_2+\delta}, \tag{3.34e}$$

$$\tau_{1,j}^+ = \begin{cases} \frac{1}{\omega_1^+}[\arccos(\frac{(\omega_1^+)^2-(d_1-k)(d_2+\delta)}{ac}) + 2j\pi],\ \frac{(d_1+d_2-\delta-k)\omega_1^+}{ac} \ge 0, \\ \frac{1}{\omega_1^+}[2\pi - \arccos(\frac{(\omega_1^+)^2-(d_1-k)(d_2+\delta)}{ac}) + 2j\pi],\ \frac{(d_1+d_2-\delta-k)\omega_1^+}{ac} < 0,\ j \in \mathbf{N}_0, \end{cases}$$

in which

$$\omega_1^+ = \frac{\sqrt{-(d_1-k)^2-(d_2+\delta)^2+\sqrt{[(d_1-k)^2-(d_2+\delta)^2]^2+4a^2c^2}}}{\sqrt{2}},$$

$$\tau_{1,0}^+ = \min_{j\in\mathbf{N}_0}\{\tau_{1,j}^+\}.$$

In what follows, we focus on the problems (1.8) subjected to (1.9). We define the function space

$$X_1 = \{(u,v)^T \in H^2(\Omega,\mathbf{R}^2) | u = v = 0,\ x \in \partial\Omega\},$$

and we let

$$u_1(t) = u(.,t),\ u_2(t) = v(.,t),\ U = (u_1,u_2)^T \in X_1,$$

then model (1.8) can be rewritten as an abstract form in $C_1 = C((-\tau,+\infty),X_1)$:

$$\frac{dU(t)}{dt} = D\Delta U + L(U_t) + F(U_t), \tag{3.35}$$

where $D\Delta : C_1 \to X$, $L : C_1 \to X$ and $F : C_1 \to X$ are given by

$$
\begin{aligned}
&D\Delta\phi = (d_1\Delta\phi_1(0), d_2\Delta\phi_2(0))^T,\\
&L(\phi) = \begin{pmatrix} k & -c \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix}, \quad \phi = (\phi_1, \phi_2),\\
&F(\phi) = \begin{pmatrix} -\rho u_0\phi_2^3(0) - \beta v_0\phi_1^3(0) - \rho\phi_1(0)\phi_2^3(0) - \beta\phi_2(0)\phi_1^3(0) \\ -\alpha u_0\phi_2^3(0) - \alpha\phi_1(-\tau)\phi_2^3(0) \end{pmatrix}.
\end{aligned}
$$

The eigenvalues and eigenvectors of the Laplacian operator $\Delta : C_1 \to X$ are given by, respectively,

$$\sigma_n = -n^2,\ \chi_n^1 = (sinnx, 0)^T,\ \chi_n^2 = (0, sinnx)^T,\ n \in \mathbf{N} = \{1, 2, \cdots\},$$

by which the characteristic equation of the linearized equation of (3.35) is

$$\lambda^2 + T_n\lambda + D_n + ace^{-\lambda\tau} = 0, \quad n \in \mathbf{N}, \tag{3.36}$$

where $T_n = (d_1+d_2)n^2+\delta-k,\ D_n = (d_1n^2-k)(d_2n^2+\delta)$. We thus obtain the following lemmas by employing the same approaches used in the preceding section.

Lemma 3.1. *If (3.34a) and (3.34b) or (3.34a) and (3.34c) hold true, then the positive equilibrium point $(u_0, v_0)^T$ of problems (1.4) subjected to (1.6) is asymptotically stable for $\tau = 0$.*

Lemma 3.2. *If (3.34b) or (3.34c) holds, then 0 is not a eigenvalue of the characteristic equation (3.36).*

Lemma 3.3. *If (3.34a)-(3.34b) and (3.34d)-(3.34e) or (3.34a), (3.34c) and (3.34d)-(3.34e) hold true, and $\tau = \tau_{1,j}^+$ $(j \in \mathbf{N})$, there is only one pair of purely imaginary roots $\pm i\omega_1^+$ of the Eq. (3.36).*

Lemma 3.4. *Under the conditions of the Lemma 3.3, we have*

$$Re\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_{1,j}^+} > 0,\ \forall j \in \mathbf{N}_0.$$

Lemma 3.5. *Under the conditions of the Lemma 3.3, all roots of the characteristic equation (3.36) have negative real parts when $\tau \in [0, \tau_{1,0}^+)$. In addition, there exists a root of the Eq. (3.36) with positive real part if $\tau > \tau_{1,0}^+$.*

Before stating our theorem involved the stabilities and periodic oscillations with the inhomogeneous Dirichlet boundary condition, we denote

$$\begin{aligned}
&\mu_1 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_1))}, \quad \beta_1 = 2Re(c_1(0)),\\
&T_1 = -\frac{1}{\omega_1\tau_1}[Im(c_1(0)) + \mu_1 Im(\lambda'(\tau_1))],\\
&\lambda'(\tau_1) = \frac{aci\omega_1 e^{-i\omega_1\tau_1}}{d_1 - k + d_2 + \delta + 2i\omega_1 - ac\tau_1 e^{-i\omega_1\tau_1}},\\
&\epsilon_1 = \sqrt{\frac{\tau-\tau_1}{\mu_1}} + o(|\sqrt{\frac{\tau-\tau_1}{\mu_1}}|),\\
&U_t(h,\theta) = \epsilon_1 Req(\theta)e^{i\omega_1\tau_1 t}\cdot f_1 + O(\epsilon_1^2).
\end{aligned}$$

Thus, the following theorem is obtained.

Theorem 3.1. *Under the conditions of the Lemma 3.3, then for the problems (1.4) subjected to (1.6) the following assertions hold:*

(1) *The positive equilibrium $(u_0, v_0)^T$ is locally asymptotically stable for $\tau \in [0, \tau_{1,0}^+)$, while it is unstable for $\tau \in (\tau_{1,0}^+, +\infty)$;*
(2) *For any $j \in \mathbf{N}_0$, Hopf bifurcation occurs as τ crosses increasingly through the critical threshold $\tau_{1,j}^+$;*
(3) *If $\mu_1 < 0$ $(\mu_1 > 0)$, the Hopf bifurcation is a subcritical* (supercritical) bifurcation, and the bifurcated periodic solution is unstable (stable);
(4) If $T_1 < 0$ $(T_1 > 0)$, the period of the bifurcated solution is decrease (increase);
(5) The bifurcated periodic solution is spatially inhomogeneous.

Proof. If (3.34a)-(3.34b) and (3.34d)-(3.34e) or (3.34a), (3.34c) and (3.34d)-(3.34e) hold true, according to Lemmas 3.1-3.5 and the well known Hopf bifurcation theory we can obtain the assertions (1)-(2) in Theorem 3.1. We thus only need to show the assertions (3)-(5). With the method used to prove Theorem 2.1, one can show that the Hopf bifurcation of the system (1.8) can be reduced to that of the low-order system of ODEs, given by

$$\frac{d\xi}{dt} = i\omega_1\xi + c_1(0)\xi|\xi|^2 + O(|\xi|^5),$$

where

$$c_1(0) = \frac{i}{2\omega_1\tau_1}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{1}{2}g_{21}.$$

$\tau_1 = \tau_{1,j}^+$ for any $j \in \mathbf{N}_0$ and $\omega_1 = \omega_1^+$,

Upon performing some calculations, we have

$$c_1(0) = -\frac{9}{8}\tau_1\bar{r}_1[(\rho + \alpha\bar{\eta}_1)u_0\xi_1^2\bar{\xi}_1 + \beta v_0].$$

With the Theorem 2.2 in [6], we obtain that if $\mu_1 < 0$ ($\mu_1 > 0$), the Hopf bifurcation is a subcritical (supercritical) bifurcation; if $\beta_1 > 0$ ($\beta_1 < 0$)(which is equivalent to $\mu_1 < 0$ ($\mu_1 > 0$) according to Lemma 3.4), the bifurcated periodic solution is unstable (stable); if $T_1 < 0$ ($T_1 > 0$), the periodic of the bifurcated solution is decrease (increase). Similarly, the bifurcated periodic solution is spatially inhomogeneous. □

4. Numerical analysis

In this section, we will give several examples to verify the reliability of theoretical analysis in the previous sections.

Example 4.1. *Let* $d_1 = 3, d_2 = 6, a = 3, b = 2.5, c = 1, \alpha = 2, \beta = 1, \gamma = 1, \rho = 3, u_0 = 60, v_0 = 80, \varphi_1(x,t) = 60.2 + 0.1cosx, \varphi_2(x,t) = 80.3 - 0.2cosx,$ *then* $k = a - b = 0.5, \delta = \alpha + \gamma = 3$, $\tau_{0,0}^+ = 0.9186$, $Re(c_0(0)) = -130.58 < 0$, $\mu_0 > 0$. *Note that conditions* (2.15a) *holds, according to Theorem 2.1, the positive equilibrium point* $(80,60)^T$ *is locally asymptotically stable for* $\tau \in [0, 0.9186)$*, shown in Figure 1. As* τ *crosses increasingly through the critical threshold* 0.9186*, the positive equilibrium point* $(80,60)^T$ *loses its stability, and a stable spatially homogeneous periodic solution appears, shown in Figure 2.*

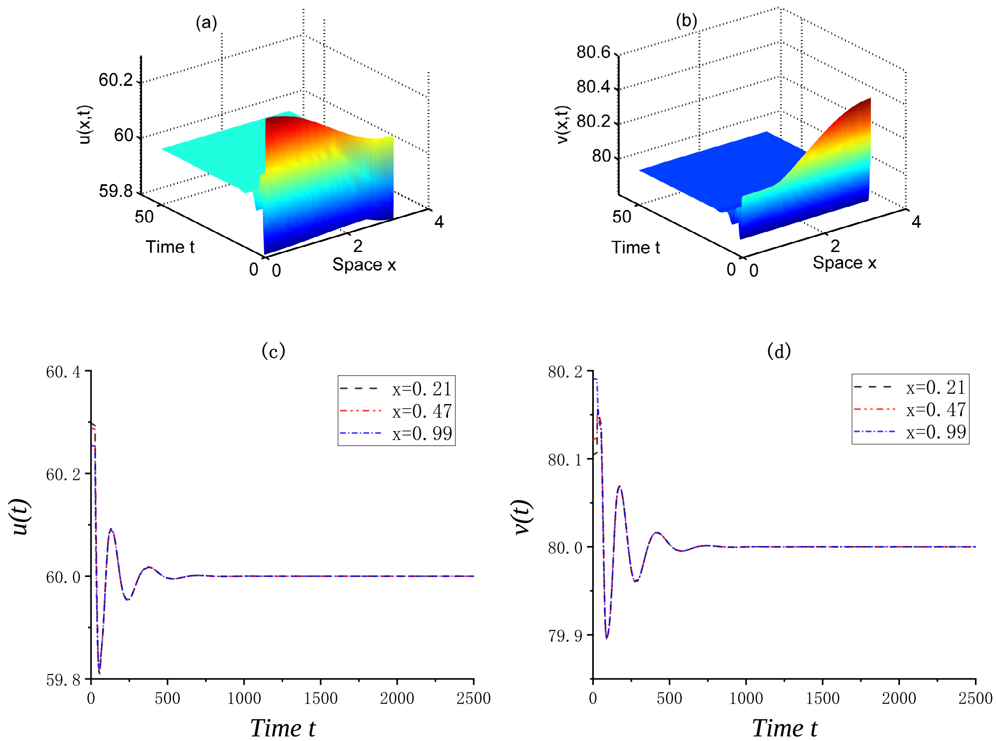

Figure 1: The positive equilibrium point $(u_0, v_0) = (60, 80)$ is asymptotically stable for $\tau = 0.5 < \tau_{0,0}^{+} = 0.9186$. (a): The graph of $u(x,t)$, (b): The graph of $v(x,t)$, (c): The graphs of $u(t)$ for $u(x,t)$ at $x = 0.21, 0.47, 0.99$, (d): The graphs of $v(t)$ for $v(x,t)$ at $x = 0.21, 0.47, 0.99$.

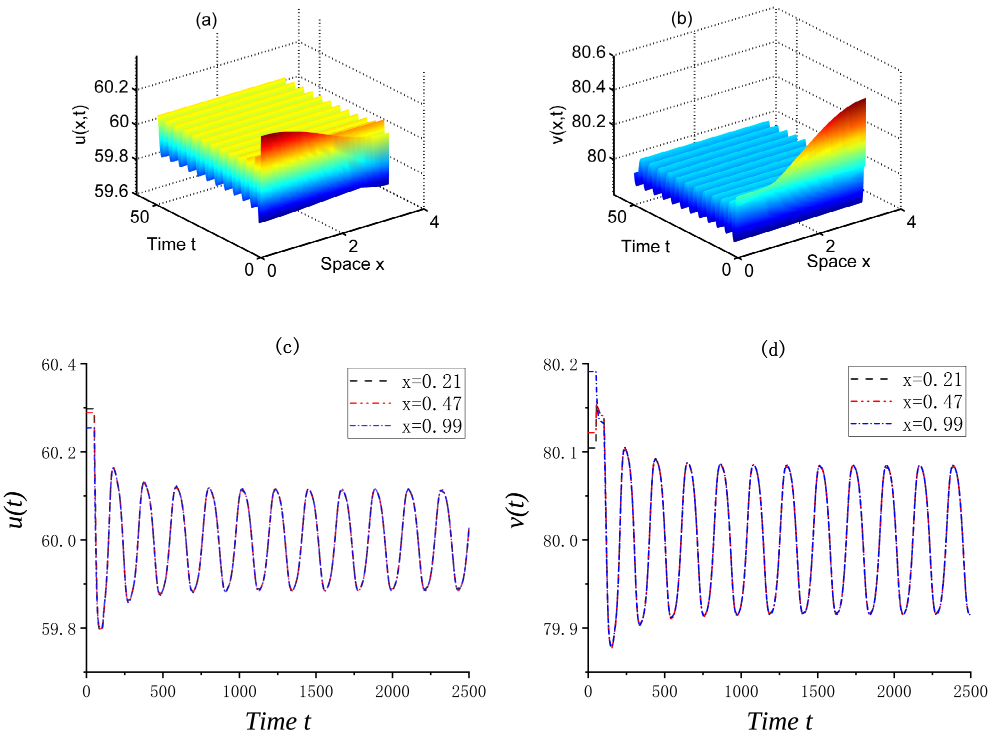

Figure 2: The positive equilibrium point $(u_0, v_0) = (60, 80)$ is unstable and there exists a stable spatially homogeneous periodic solution for $\tau = 1 > \tau_{0,0}^{+} = 0.9186$. (a): The graph of $u(x,t)$, (b): The graph of $v(x,t)$, (c): The graphs of $u(t)$ for $u(x,t)$ at $x = 0.21, 0.47, 0.99$, (d): The graphs of $v(t)$ for $v(x,t)$ at $x = 0.21, 0.47, 0.99$.

Example 4.2. *Let* $d_1 = 2, d_2 = 1, a = 3, b = 2, c = 2, \alpha = 1, \beta = 1, \gamma = 1, \rho = 2, u_0 = 60, v_0 = 80, \varphi_1(x,t) = 60 + 0.1 sinx, \varphi_2(x,t) = 80 + 0.2 sinx$, *then* $k = a - b = 1, \delta = \alpha + \gamma = 2$, $\tau_{1,0}^{+} = 1.145$, $Re(c_1(0)) = -13.741 < 0$, $\mu_1 > 0$. *Note that conditions* (3.34a)-(3.34b) *and* (3.34d)-(3.34e) *hold, according to Theorem 3.1, the positive equilibrium point* $(80, 60)^T$ *is locally asymptotically stable for* $\tau \in [0, 1.145)$, *shown in Figure 3. As* τ *crosses increasingly through the critical threshold* 1.145, *the positive equilibrium point*

$(80, 60)^T$ *loses its stability, and a stable spatially inhomogeneous periodic solution appears, shown in Figure 4.*

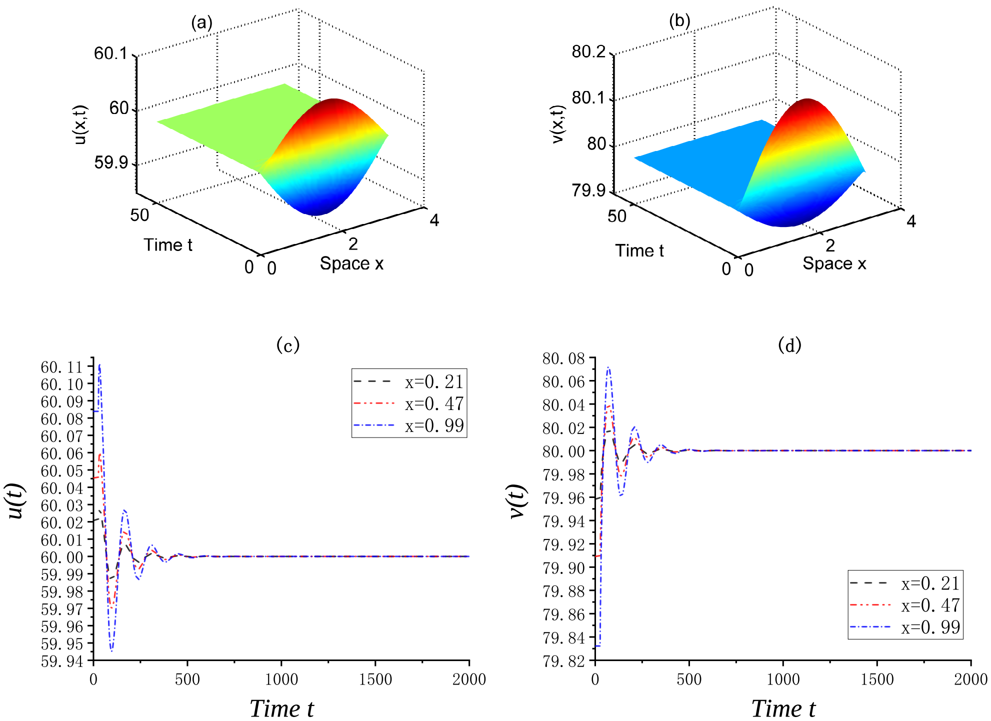

Figure 3: The positive equilibrium point $(u_0, v_0) = (60, 80)$ is asymptotically stable for $\tau = 0.5 < \tau_{1,0}^{+} = 1.145$. (a): The graph of $u(x,t)$, (b): The graph of $v(x,t)$, (c): The graphs of $u(t)$ for $u(x,t)$ at $x = 0.21, 0.47, 0.99$, (d): The graphs of $v(t)$ for $v(x,t)$ at $x = 0.21, 0.47, 0.99$.

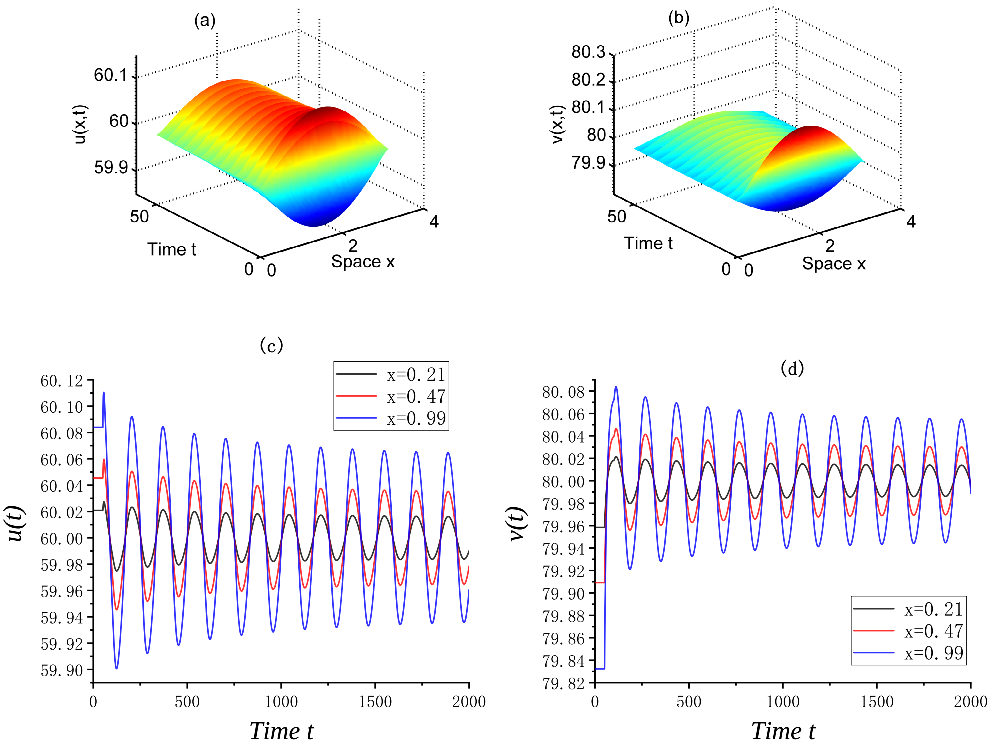

Figure 4: The positive equilibrium point $(u_0, v_0) = (60, 80)$ is unstable and there exists a stable spatially inhomogeneous periodic solution for $\tau = 1.25 > \tau_{1,0}^{+} = 1.145$. (a): The graph of $u(x,t)$, (b): The graph of $v(x,t)$, (c): The graphs of $u(t)$ for $u(x,t)$ at $x = 0.21, 0.47, 0.99$, (d): The graphs of $v(t)$ for $v(x,t)$ at $x = 0.21, 0.47, 0.99$.

5. Discussion and Conclusions

From boundary condition (1.5) we can infer that there is neither inflow nor outflow of income and capital on the boundary. It implies that the economy is a closed economy. On the contrary, boundary condition (1.6) implies that the economy is an open economy. According to Theorem 2.1 and Theorem 3.1, we have the following conclusions:

(1) For the closed economy or the open economy, there exists a critical threshold of time delay. If the time delay is smaller than the critical threshold, then the economic system will keep balanced at the present state; If the time delay is lager than the critical threshold, the stability of present state will be destroyed, and the periodic oscillations will emerge;

(2) The biggest difference between the critical threshold of open economy and that of closed economy is that the former is related to diffusion coefficients, while the latter is independent of diffusion coefficients.

(3) The periodic oscillations are spatially homogeneous for closed economy, while they are spatially inhomogeneous for open economy. The oscillations are spatially inhomogeneous implies that regional income and capital disparities occur. On the contrary, the oscillations being spatially homogeneous implies that there is no regional income or capital disparity. This result reveals to some extent the causes of the gap between the rich and the poor.

(4) Regional income and capital disparities are more likely to occur in open economy than in closed economy. The result provides insight into why developed economies are more likely to polarize than underdeveloped ones.

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