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RESEARCH ARTICLE

Multiple solutions for critical nonhomogeneous elliptic systems in non-contractible domain

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The paper is concerned with the multiple solutions of a nonhomogeneous elliptic system with critical exponent over a non-contractible domain, precisely, a smooth bounded annular domain. We prove the existence of four solutions using variational methods and Lusternik-Schnirelmann theorey, when the inner hole of the annulus is sufficiently small.

KEYWORDS:
Nehari manifold, critical Sobolev exponent, non-contractible domain, nonhomogeneous problem

1 | INTRODUCTION

The paper deals with the following nonhomogeneous elliptic system with critical exponent

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \frac{\alpha}{2^*}|u|^{\alpha-2}u|v|^{\beta} + \epsilon f \quad in\ \Omega, \\ -\Delta v = |v|^{2^*-2}v + \frac{\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + \epsilon g \quad in\ \Omega, \\ u = v = 0 \qquad\qquad\qquad\qquad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$, $\alpha, \beta > 1$ and $\alpha + \beta = 2^* = \frac{2N}{N-2}$. Functions $f(x), g(x)$ satisfy $0 \leq f, g \in L^\infty(\Omega)$ and $f, g \not\equiv 0$. Equation (1) arises from many physical problems, especially in describing some phenomena in nonlinear optics[1,2]. It is also a model in Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$[3,4]. For more physical background of coupled elliptic system, we refer the readers to Cheng and Zou[5,6].

Problem (1) can be seen as a counterpart of the following scalar equation

$$-\Delta u = |u|^{2^*-2}u + f \quad in\ \Omega, \qquad u = 0 \quad in\ \partial\Omega, \tag{2}$$

where Ω is a smooth bounded domain. A remarkable result by Tarantello[7] established that there exist at least two solutions of (2) by splitting Nehari manifold into three parts. For a non-contractible domain Ω, where Ω satisfies:
(V) Ω is a smooth bounded domain in $\mathbb{R}^N$ and there exist constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega, \qquad \{x \in \mathbb{R}^N : |x| < R_1\} \nsubseteq \bar{\Omega},$$

[8]it is shown that there exist at least four solutions of (2) by using the splitting Nehari manifold method and Lusternik-Schnirelmann theorey.

Recently, significant effort has been focused on coupled elliptic system with critical exponent. Peng et al[9] showed that, in the case $\epsilon = 0$ and $\Omega = \mathbb{R}^N$, (1) has a kind of uniqueness result on the least energy solutions and a non-degeneracy result on a special family of positive solutions. Moreover, they investigated the existence of positive vector solutions of (1) with $\epsilon = 0$ when

Ω satisfies condition (V). The multiplicity results of (1) with $\epsilon = 0$ by Clapp and Faya[10] established the existence of a prescribed number of nontrivial solutions under suitable symmetry assumptions on smooth bounded domain Ω and the existence of infinitely many solutions on $\mathbb{R}^N$. The literatures above mainly focus on problem (1) with $\epsilon = 0$. It is natural to consider what happens if $\epsilon \neq 0$. The result as follows:

Theorem 1. Assume that Ω satisfies condition (V). Then there exists a $\epsilon' > 0$ such that, for $0 < \epsilon < \epsilon'$, (1) has at least three solutions, one of which is a positive least energy solution. Furthermore, if R_1 is small enough, then there exists a $\epsilon'' > 0$ such that (1) has at least four solutions whenever $0 < \epsilon < \epsilon''$.

It is known that the established method to deal with nonhomogeneous problems is the splitting Nehari manifold method introduced from Tarantello[7]. This idea was also used to study other nonhomogeneous problems, for instance, Qi and Zhang[4], Cao and Zhou[11], Clapp et al[12] and Shen et al[13]. However, to the best of our knowledge, there is almost no research applying the idea to study nonhomogeneous elliptic systems with critical exponents. In fact, the energy functional associated to (1) does not satisfy the global $(PS)_c$ condition since it includes critical exponents. We have to find the range of c where the $(PS)_c$ condition holds for the energy functional.

The proof of Theorem 1 mainly takes inspiration from He[8] and Peng[9]. To prove Theorem 1, we follow the idea of Qi and Zhang[4] to split the Nehari manifold into three parts, where the Nehari manifold is defined by

$$\mathcal{N}_\epsilon := \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : \langle \mathcal{J}_\epsilon'(u, v), (u, v)\rangle = 0\} \tag{3}$$

and its three parts

$$\begin{aligned} &\{\mathcal{N}_\epsilon^+ := (u, v) \in \mathcal{N}_\epsilon : \langle \xi_\epsilon'(u, v), (u, v)\rangle > 0\},\\ &\{\mathcal{N}_\epsilon^0 := (u, v) \in \mathcal{N}_\epsilon : \langle \xi_\epsilon'(u, v), (u, v)\rangle = 0\},\\ &\{\mathcal{N}_\epsilon^- := (u, v) \in \mathcal{N}_\epsilon : \langle \xi_\epsilon'(u, v), (u, v)\rangle < 0\}, \end{aligned} \tag{4}$$

where $\mathcal{J}_\epsilon$ is the energy functional associated to (1) (given by (6)) and

$$\xi_\epsilon(u, v) := \langle \mathcal{J}_\epsilon'(u, v), (u, v)\rangle. \tag{5}$$

For the first solution, we seek the help of Nehari manifold method to prove the existence of positive least energy solution $(u_1, v_1) \in \mathcal{N}_\epsilon^+$. To proceed further, we prove some estimates of the energy functional. With the help of these estimates we find that there exists a $t_0 > 0$ such that $(u_1 + t_0 u_\delta^{\sigma,\rho}, v_1 + t_0 v_\delta^{\sigma,\rho}) \in \mathcal{N}_\epsilon^-$, where $u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}$ are related to the minimizers of the Sobolev constant S. Moreover, $\mathcal{J}_\epsilon(u_1 + t_0 u_\delta^{\sigma,\rho}, v_1 + t_0 v_\delta^{\sigma,\rho})$ is below the first critical level and satisfies the Palais-Smale condition, where the first critical level is

$$\mathcal{J}_\epsilon(u_1, v_1) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$$

and $S_{\alpha,\beta}$ is defined in (10). Subsequently, by using Lusternik-Schnirelmann theorey and the well-known result of Ambraosetti[14], we prove the existence of the second and third solutions of (1) in $\mathcal{N}_\epsilon^-$. In order to prove the existence of the fourth solution, a high energy solution in $\mathcal{N}_\epsilon^-$, we use a version of global compactness lemma from Peng et al[9] to prove that the energy functional $\mathcal{J}_\epsilon$ satisfies the Palais-Smale condition between the first and second critical levels, where the second critical level is

$$\inf_{(u,v)\in\mathcal{N}_\epsilon^-} \mathcal{J}_\epsilon(u, v) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

On the other hand, applying the minimax Lemma of Brezis and Nirenberg[15, Theorem 1], we find a Palais-Smale sequence (u_n, v_n) of $\check{\mathcal{J}}_\epsilon$, where $\check{\mathcal{J}}_\epsilon(u, v) = \mathcal{J}_\epsilon(t_{(u,v)}^-(u, v))$ and $t_{(u,v)}^-(u, v) \in \mathcal{N}_\epsilon^-$. Then it follows from the idea of Szulkin and Weth[16, Corollary 2.10] that $t_{(u_n,v_n)}^-(u_n, v_n)$ is a Palais-Smale sequence of $\mathcal{J}_\epsilon$, which on using the obtained Palais-Smale condition yields the desired result.

The paper is organized by the following way. In Section 2, we give some preliminary results and the variational framework. We prove the existence of the first-fourth solutions in Sections 3-5 respectively.

2 | PRELIMINARIES

We denote some basic notations used in the paper. We first denote $H_0^1(\Omega)$ with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$ and $E := H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm

$$\|(u, v)\|^2 := \int\limits_\Omega |\nabla u|^2 + |\nabla v|^2 dx.$$

Throughout the paper we use $|\cdot|_p$ to denote the $L^p(\Omega)$-norm. The energy functional $\mathcal{J}_\epsilon : E \to \mathbb{R}$ is defined by

$$\mathcal{J}_\epsilon(u,v) := \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta dx - \epsilon\int_\Omega fu + gv dx. \tag{6}$$

It is known that the critical points of $\mathcal{J}_\epsilon$ correspond to the weak solutions of (1). We shall constraint the energy functional on the Nehari manifold (3). It is clear that only $\mathcal{N}_\epsilon^0$ contains the element $(0,0)$. Obviously, $\mathcal{N}_\epsilon^+ \cup \mathcal{N}_\epsilon^0$ and $\mathcal{N}_\epsilon^- \cup \mathcal{N}_\epsilon^0$ are both closed subset of E. Next we give an explanation of the three parts of $\mathcal{N}_\epsilon$. Before doing this we denote

$$A(u,v) := \|(u,v)\|^2, \quad B(u,v) := \textstyle\int_\Omega |u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta dx, \quad D(u,v) := \int_\Omega fu + gv dx. \tag{7}$$

The Nehari manifold $\mathcal{N}_\epsilon$ is closely linked to the behaviour of $\varphi_\epsilon(t) : t \to \mathcal{J}_\epsilon(tu,tv)$, where $\varphi_\epsilon(t)$ is defined by

$$\varphi_\epsilon(t) = \mathcal{J}_\epsilon(tu,tv) = \tfrac{A(u,v)}{2}t^2 - \tfrac{B(u,v)}{2^*}t^{2^*} - \epsilon D(u,v)t \quad for\ t > 0. \tag{8}$$

Obviously, $(tu,tv) \in \mathcal{N}_\epsilon$ if and only if

$$\varphi_\epsilon'(t) = \frac{1}{t}\langle \mathcal{J}_\epsilon'(tu,tv),(tu,tv)\rangle = 0.$$

Furthermore, one easily checks that, for $(tu,tv) \in \mathcal{N}_\epsilon$, there holds

$$\varphi_\epsilon''(t) = \frac{1}{t^2}[\langle \xi_\epsilon'(tu,tv),(tu,tv)\rangle - \xi_\epsilon(tu,tv)] = \frac{1}{t^2}\langle \xi_\epsilon'(tu,tv),(tu,tv)\rangle.$$

It follows from (4) that

$$\begin{aligned} &\{(tu,tv) \in \mathcal{N}_\epsilon^+, t > 0 \Leftrightarrow \varphi_\epsilon'(t) = 0, \varphi_\epsilon''(t) > 0\}, \\ &\{(tu,tv) \in \mathcal{N}_\epsilon^0, t > 0 \Leftrightarrow \varphi_\epsilon'(t) = 0, \varphi_\epsilon''(t) = 0\}, \\ &\{(tu,tv) \in \mathcal{N}_\epsilon^-, t > 0 \Leftrightarrow \varphi_\epsilon'(t) = 0, \varphi_\epsilon''(t) < 0\}. \end{aligned} \tag{9}$$

We denote the constant

$$S_{\alpha,\beta}(\Omega) := \inf_{(u,v)\in Y(\Omega)\times Y(\Omega)\backslash\{(0,0)\}} \frac{\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx}{(\int_\Omega |u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta dx)^{2/2^*}}, \tag{10}$$

where $Y(\mathbb{R}^N) = D^{1,2}(\mathbb{R}^N)$ if $\Omega = \mathbb{R}^N$ and $Y(\Omega) = H_0^1(\Omega)$ if Ω is a smooth bounded domain. We recall the Sobolev constant

$$S(\Omega) = \inf_{u\in Y(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u|^{2^*} dx)^{2/2^*}}.$$

It is known that $S(\mathbb{R}^N)$ is achieved by the function[17, Theorem 1.42]

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}},$$

which is a solution of $-\Delta u = |u|^{2^*-2}u$ for $x \in \mathbb{R}^N$ with

$$\int_{\mathbb{R}^N} |\nabla U|^2 dx = \int_{\mathbb{R}^N} U^{2^*} dx = S^{\frac{N}{2}}. \tag{11}$$

Then, we have the following result.

Lemma 1. Let Ω be $\mathbb{R}^N$ or a bounded domain of $\mathbb{R}^N$. Then
(i) $S_{\alpha,\beta}(\Omega) = F(\tau_{min})S(\Omega)$ and $\frac{1}{2^{2/2^*}} \le F(\tau_{min}) := \min\limits_{\tau\ge 0} F(\tau) \le 1$, where $F(\tau) : [0,+\infty) \to \mathbb{R}^+$ and $F(\tau) = \frac{1+\tau^2}{(1+\tau^\beta+\tau^{2^*})^{2/2^*}}$;
(ii) $S_{\alpha,\beta}(\mathbb{R}^N)$ has the minimizers $\{(U_\delta^{x_0}, \tau_{min}U_\delta^{x_0})\}$, where $U_\delta^{x_0}(x) := \delta^{\frac{2-N}{2}}U(\frac{x-x_0}{\delta})$, $x_0 \in \mathbb{R}^N$ and $\delta > 0$.

Proof. By the definition of $F(\tau_{min})$, we find that $F(\tau_{min}) \le 1$. Moreover, since $\tau^\beta \le 1 + \tau^{2^*}$, we have that

$$\frac{1+\tau^2}{(1+\tau^\beta+\tau^{2^*})^{2/2^*}} \ge \frac{1+\tau^2}{2^{2/2^*}(1+\tau^{2^*})^{2/2^*}} \ge \frac{1}{2^{2/2^*}} \quad for\ \ any\ \ \tau \ge 0.$$

In the following we only need to check that part (i) holds for a bounded domain of $\mathbb{R}^N$. For the case $\Omega = \mathbb{R}^N$, parts (i) and (ii) were proved by Peng et al[9, Lemma 2.1]. Let $\{w_n\} \subset H_0^1(\Omega)\backslash\{0\}$ be a minimizing sequence for $S(\Omega)$. Define $u_n = w_n$ and $v_n = \tau_{min}w_n$. By the definition of $S_{\alpha,\beta}(\Omega)$, we have that $S_{\alpha,\beta}(\Omega) \le F(\tau_{min})S(\Omega)$. Moreover, in a fashion similar to the argument of Peng et al[9, Lemma 2.1(i)], $S_{\alpha,\beta}(\Omega) \ge F(\tau_{min})S(\Omega)$. □

[18]It is known that $S(\Omega) = S(\mathbb{R}^N)$ if Ω is bounded, which implies by Lemma 1 that $S_{\alpha,\beta}(\Omega) = S_{\alpha,\beta}(\mathbb{R}^N)$. So, in the following we may write $S = S(\Omega) = S(\mathbb{R}^N)$ and $S_{\alpha,\beta} = S_{\alpha,\beta}(\Omega) = S_{\alpha,\beta}(\mathbb{R}^N)$. Moreover, it follows from the arguments of Willem[17, Proposition 1.43] that $S(\Omega)$ is never achieved in a domain $\Omega \neq \mathbb{R}^N$. In a standard way, we see that $S_{\alpha,\beta}(\Omega) = F(\tau_{min})S(\Omega)$ is never achieved in a domain $\Omega \neq \mathbb{R}^N$.

Lemma 2. Assume that Ω satisfies (V). Then
(i) $\mathcal{J}_\epsilon$ is coercive and bounded from below on $\mathcal{N}_\epsilon$(thus on $\mathcal{N}_\epsilon^+$ and $\mathcal{N}_\epsilon^-$);
(ii) there exists a $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, $\mathcal{N}_\epsilon^0 = \{(0,0)\}$ and $\mathcal{N}_\epsilon^\pm \neq \emptyset$. Furthermore, for any $(u, v) \in E\backslash\{(0,0)\}$, if $D(u, v) > 0$, then there exists a unique number $t^+_{(u,v)}$ and a unique number $t^-_{(u,v)}$ satisfying $0 < t^+_{(u,v)} < t_{max} < t^-_{(u,v)}$ such that $t^+_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^+$ and $t^-_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^-$; if $D(u, v) \leq 0$, then there exists a unique number $t^-_{(u,v)}$ satisfying $0 < t_{max} < t^-_{(u,v)}$ such that $t^-_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^-$, where $t_{max} = [\frac{A(u,v)}{(2^*-1)B(u,v)}]^{\frac{1}{2^*-2}}$. Moreover, $\mathcal{J}_\epsilon(t^+_{(u,v)}(u, v)) = \inf\limits_{0\leq t\leq t^-_{(u,v)}} \mathcal{J}_\epsilon(t(u, v))$ and $\mathcal{J}_\epsilon(t^-_{(u,v)}(u, v)) = \max\limits_{t\geq t_{max}} \mathcal{J}_\epsilon(t(u, v))$;
(iii) $\mathcal{N}_\epsilon^-$ is closed.

Proof. (i) It is clear that, for any $(u, v) \in \mathcal{N}_\epsilon$, there holds

$$\mathcal{J}_\epsilon(u, v) = (\frac{1}{2} - \frac{1}{2^*})\|(u, v)\|^2 - (1 - \frac{1}{2^*})\epsilon \int\limits_\Omega fu + gv dx.$$

By a direct calculation, we get that

$$\int_\Omega fu + gv dx \leq \max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}(\|u\| + \|v\|) \leq \sqrt{2}\max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}\|(u, v)\|. \tag{12}$$

Thus,

$$\mathcal{J}_\epsilon(u, v) \geq (\frac{1}{2} - \frac{1}{2^*})\|(u, v)\|^2 - \sqrt{2}\epsilon(1 - \frac{1}{2^*})\max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}\|(u, v)\|,$$

which implies that $\mathcal{J}_\epsilon$ is coercive and bounded from below on $\mathcal{N}_\epsilon$.
(ii) From the definition of φ_ϵ in (8), we get that

$$\varphi'_\epsilon(t) = r(t) - \epsilon D(u, v),$$

where $r(t) := A(u, v)t - B(u, v)t^{2^*-1}$ for $t > 0$. For any $(u, v) \in E\backslash\{(0,0)\}$, we have that $r''(t) < 0$, $r(0) = 0$, $r(t) \to -\infty$ as $t \to +\infty$ and $r(t) > 0$ for a small $t > 0$. So, $r(t)$ has a unique global maximum value $r(t_{max}) = \frac{(2^*-2)A(u,v)}{2^*-1}[\frac{A(u,v)}{(2^*-1)B(u,v)}]^{\frac{1}{2^*-2}}$. If $0 < \epsilon D(u, v) < r(t_{max})$, $\varphi'_\epsilon(t) = 0$ has two solutions $t^+_{(u,v)}, t^-_{(u,v)}$ satisfying $0 < t^+_{(u,v)} < t_{max} < t^-_{(u,v)}$. Since $\varphi''_\epsilon(t^+_{(u,v)}) > 0$ and $\varphi''_\epsilon(t^-_{(u,v)}) < 0$, we infer that $t^+_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^+$ and $t^-_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^-$. Moreover, if $\epsilon D(u, v) \leq 0$, $\varphi'_\epsilon(t) = 0$ has only one solution $t^-_{(u,v)}$ satisfying $t^-_{(u,v)} > t_{max}$. Obviously, $\varphi''_\epsilon(t^-_{(u,v)}) < 0$ and $t^-_{(u,v)}(u, v) \in \mathcal{N}_\epsilon^-$. It follows from the analysis above that

$$\mathcal{J}_\epsilon(t^+_{(u,v)}(u, v)) = \inf_{0\leq t\leq t^-_{(u,v)}} \mathcal{J}_\epsilon(t(u, v)), \quad \mathcal{J}_\epsilon(t^-_{(u,v)}(u, v)) = \max_{t\geq t_{max}} \mathcal{J}_\epsilon(t(u, v)).$$

To prove $\mathcal{N}_\epsilon^0 = \{(0,0)\}$, we only need to check that $\varphi''_\epsilon(t) > 0$ or $\varphi''_\epsilon(t) < 0$ for any $\varphi'_\epsilon(t) = 0$ and $(u, v) \in E\backslash\{(0,0)\}$. We assume, without loss of generality, that $\|(u, v)\| = 1$. By the analysis above, we find that $\mathcal{N}_\epsilon^0 = \{(0,0)\}$ if the following inequality holds,

$$\epsilon D(u, v) < r(t_{max}) = \frac{2^*-2}{2^*-1}[\frac{1}{(2^*-1)B(u,v)}]^{\frac{1}{2^*-2}}. \tag{13}$$

Next we shall find a constant $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, (13) holds. From (12), we get that $D(u, v) \leq \sqrt{2}\max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}$. Since $A(u, v) = 1$, $B(u, v)$ is bounded from above. There exists a $\epsilon_0 > 0$ such that

$$\sqrt{2}\epsilon_0 \max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\} \leq \frac{2^*-2}{2^*-1}[\frac{1}{(2^*-1)\sup\limits_{\|(u,v)\|=1} B(u,v)}]^{\frac{1}{2^*-2}}. \tag{14}$$

It is easily seen that (13) holds if $0 < \epsilon < \epsilon_0$. Hence, $\mathcal{N}_\epsilon^0 = \{(0,0)\}$. Moreover, the sets $\{(u, v) \in E : \|(u, v)\| = 1, \epsilon D(u, v) \leq 0\}$ and $\{(u, v) \in E : \|(u, v)\| = 1, 0 < \epsilon D(u, v) < r(t_{max})\}$ are nonempty, which implies that $\mathcal{N}_\epsilon^\pm \neq \emptyset$.

(iii) It is clear that $(0, 0) \notin \mathcal{N}_\epsilon^-$ and $cl(\mathcal{N}_\epsilon^-) \subset \mathcal{N}_\epsilon^- \cup \{(0,0)\}$, where $cl(\mathcal{N}_\epsilon^-)$ denotes the closure of $\mathcal{N}_\epsilon^-$. So, to prove that $\mathcal{N}_\epsilon^-$ is closed, we only need to check that $dist((0,0), \mathcal{N}_\epsilon^-) > 0$. For any $(u, v) \in \mathcal{N}_\epsilon^-$, we denote $(u_0, v_0) = (\frac{u}{\|(u,v)\|}, \frac{v}{\|(u,v)\|})$. Applying the proof of (ii), we get that $\varphi'_\epsilon(t) = 0$ has a solution $t^-_{(u_0,v_0)}$ satisfying $t^-_{(u_0,v_0)} > t_{max}$ such that $t^-_{(u_0,v_0)}(u_0, v_0) = (u, v) \in \mathcal{N}_\epsilon^-$. Thus $t^-_{(u_0,v_0)} = \|(u, v)\| > t_{max} = (\frac{1}{(2^*-1)B(u_0,v_0)})^{\frac{1}{2^*-2}}$. Moreover, $B(u_0, v_0)$ is bounded from above since $A(u_0, v_0) = 1$. So, there exists a $\sigma > 0$ such that $\|(u, v)\| > \sigma$. In conclusion, we have that $dist((0,0), \mathcal{N}_\epsilon^-) = \inf\limits_{(u,v)\in\mathcal{N}_\epsilon^-} \|(u, v)\| \geq \sigma > 0$. □

In the following we may always assume that $\epsilon < \epsilon_0$. We denote the minimization problems

$$c_\epsilon(\Omega) := \inf_{(u,v)\in\mathcal{N}_\epsilon} \mathcal{J}_\epsilon(u,v), \quad c_\epsilon^+(\Omega) := \inf_{(u,v)\in\mathcal{N}_\epsilon^+} \mathcal{J}_\epsilon(u,v), \quad c_\epsilon^-(\Omega) := \inf_{(u,v)\in\mathcal{N}_\epsilon^-} \mathcal{J}_\epsilon(u,v). \tag{15}$$

Note that $c_0(\Omega)$ is independent of Ω and $c_0(\Omega) = c_0(\mathbb{R}^N) = \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$.

Lemma 3. For each $(u,v) \in \mathcal{N}_\epsilon^+$, one has $D(u,v) > 0$ and $\mathcal{J}_\epsilon(u,v) < 0$. In particular, $c_\epsilon^+(\Omega) < 0$. Moreover, there exists a ϵ_1 satisfying $0 < \epsilon_1 \le \epsilon_0$ such that $c_\epsilon^-(\Omega) > 0$ for $0 < \epsilon < \epsilon_1$.

Proof. For each $(u,v) \in \mathcal{N}_\epsilon^+$, we have that $A(u,v) - (2^* - 1)B(u,v) > 0$. Hence,

$$\epsilon D(u,v) = A(u,v) - B(u,v) > (2^* - 2)B(u,v) > 0$$

and

$$\begin{aligned} \mathcal{J}_\epsilon(u,v) &= \tfrac{1}{2}A(u,v) - \tfrac{1}{2^*}B(u,v) - \epsilon D(u,v) \\ &= (\tfrac{1}{2} - \tfrac{1}{2^*})B(u,v) - (1 - \tfrac{1}{2^*})\epsilon D(u,v) < -[2\cdot 2^* - 3]\tfrac{2^*-2}{2\cdot 2^*}B(u,v) < 0. \end{aligned}$$

This inequality implies that $c_\epsilon^+(\Omega) < 0$.

Recalling Lemma 2 and its proof, we find that, for any $(u,v) \in E\backslash\{(0,0)\}$, there holds $\mathcal{J}_\epsilon(t_{(u,v)}^-(u,v)) \ge \mathcal{J}_\epsilon(t_{max}(u,v))$. So, to prove $c_\epsilon^-(\Omega) > 0$, we only need to prove that there exists a $C > 0$ such that $\varphi_\epsilon(t_{max}) \ge C > 0$. As the proof of Lemma 2 (ii), we take $(u,v) \in E\backslash\{(0,0)\}$ such that $\|(u,v)\| = 1$. For $0 < \epsilon < \epsilon_0$, equation (13) holds. In a fashion similar to the arguments for (13), we find that there exists a ϵ_1 satisfying $0 < \epsilon_1 \le \epsilon_0$ such that, for $0 < \epsilon < \epsilon_1$, there holds

$$\epsilon D(u,v) < \frac{2^* - 2}{2\cdot 2^* \cdot (2^* - 1)}\left[\frac{1}{(2^* - 1)B(u,v)}\right]^{\frac{1}{2^*-2}}.$$

We use this inequality to deduce that

$$\begin{aligned} \varphi_\epsilon(t_{max}) &= \tfrac{1}{2}t_{max}^2 - \tfrac{B(u,v)}{2^*}t_{max}^{2^*} - \epsilon D(u,v)t_{max} \\ &\ge \tfrac{(2^*-2)(2^*+1)}{2\cdot 2^*\cdot(2^*-1)^{\frac{2^*}{2^*-2}}}\left[\tfrac{1}{B(u,v)}\right]^{\frac{2}{2^*-2}} - \tfrac{2^*-2}{2\cdot 2^*\cdot(2^*-1)}\left[\tfrac{1}{(2^*-1)B(u,v)}\right]^{\frac{2}{2^*-2}} \\ &= \tfrac{2^*\cdot(2^*-2)}{2\cdot 2^*\cdot(2^*-1)^{\frac{2^*}{2^*-2}}}\left[\tfrac{1}{B(u,v)}\right]^{\frac{2}{2^*-2}}. \end{aligned}$$

Since $A(u,v) = 1$ and $B(u,v)$ has its upper bound, we get that

$$\varphi_\epsilon(t_{max}) \ge \frac{2^*\cdot(2^* - 2)}{2\cdot 2^*\cdot(2^* - 1)^{\frac{2^*}{2^*-2}}}\left[\frac{1}{\sup\limits_{\|(u,v)\|=1} B(u,v)}\right]^{\frac{2}{2^*-2}} := C > 0.$$

Hence, $c_\epsilon^-(\Omega) > 0$. □

Lemma 4. If $c_\epsilon(\Omega)$ is achieved by $(u_0,v_0) \in \mathcal{N}_\epsilon$, then $(u_0,v_0) \in \mathcal{N}_\epsilon^+$ and $\mathcal{J}_\epsilon(u_0,v_0) = c_\epsilon(\Omega) = c_\epsilon^+(\Omega) < 0$. Moreover, if $c_\epsilon^+(\Omega)$ (or $c_\epsilon^-(\Omega)$) is achieved by $(u_0,v_0) \in \mathcal{N}_\epsilon^+$ (or $(u_0,v_0) \in \mathcal{N}_\epsilon^-$), then (u_0,v_0) is a nontrivial solution of (1).

Proof. Let $(u_0,v_0) \in \mathcal{N}_\epsilon$ be such that $\mathcal{J}_\epsilon(u_0,v_0) = c_\epsilon(\Omega)$. It follows from Lemma 3 that $c_\epsilon(\Omega) \le c_\epsilon^+(\Omega) < 0$. We suppose, by contradiction, that $(u_0,v_0) \in \mathcal{N}_\epsilon^-$. Reviewing Lemma 2 (ii) and its proof, we get that there exists a unique number $t_{(u_0,v_0)}^- = 1 > t_{max} > t_{(u_0,v_0)}^+ > 0$ such that

$$c_\epsilon(\Omega) \le c_\epsilon^+(\Omega) \le \mathcal{J}_\epsilon(t_{(u_0,v_0)}^+(u_0,v_0)) < \mathcal{J}_\epsilon(t_{(u_0,v_0)}^-(u_0,v_0)) = c_\epsilon(\Omega),$$

a contradiction. So, $(u_0,v_0) \in \mathcal{N}_\epsilon^+$ and $c_\epsilon^+(\Omega) \le \mathcal{J}_\epsilon(u_0,v_0) = c_\epsilon(\Omega) \le c_\epsilon^+(\Omega)$. The proof of the second assertion follows from Qi and Zhang[4, Lemma 3.2]. □

Lemma 5. There exists a bounded sequence $(u_n,v_n) \subset \mathcal{N}_\epsilon^+(\mathcal{N}_\epsilon^-$ or $\mathcal{N}_\epsilon)$ such that $\mathcal{J}_\epsilon(u_n,v_n) \to c_\epsilon^+(\Omega)(c_\epsilon^-(\Omega)$ or $c_\epsilon(\Omega))$ and $\mathcal{J}_\epsilon'(u_n,v_n) \to 0$ as $n \to \infty$.

The proof of Lemma 5 follows from the arguments of Qi and Zhang[4, Lemma 4.6 and Lemma 4.7].

3 | EXISTENCE OF FIRST SOLUTION

In this section we shall prove the existence of the first solution in $\mathcal{N}_\epsilon^+$.

Proposition 1. Let $0 < \epsilon < \epsilon_1$, where ϵ_1 is given by Lemma 3. Then there exists a $(u_1, v_1) \in \mathcal{N}_\epsilon^+$ such that $\mathcal{J}_\epsilon(u_1, v_1) = c_\epsilon^+(\Omega) = c_\epsilon(\Omega) < 0$ and $(u_1, v_1), u_1, v_1 > 0$ is a positive least energy solution of (1).

Proof. It follows from Lemma 5 that we find a bounded $(PS)_{c_\epsilon(\Omega)}$ sequence $\{(u_n, v_n)\}$ of $\mathcal{J}_\epsilon$ on $\mathcal{N}_\epsilon$. We may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. Passing to a subsequence, $(u_n, v_n) \to (u_0, v_0)$ for a.e. $x \in \Omega$. Recalling that the sequences

$$y_n := |u_n|^{\alpha-2}u_n|v_n|^\beta \quad \textit{and} \quad z_n := |u_n|^\alpha|v_n|^{\beta-2}v_n, \ \alpha + \beta = 2^*$$

are uniformly bounded in $L^{(2^*)'}(\Omega)$ and converge pointwisely to $y_0 = |u_0|^{\alpha-2}u_0|v_0|^\beta$ and $z_0 = |u_0|^\alpha|v_0|^{\beta-2}v_0$ respectively, we get that $(y_n, z_n) \rightharpoonup (y_0, z_0)$ weakly in $L^{(2^*)'}(\Omega) \times L^{(2^*)'}(\Omega)$. So, $\langle \mathcal{J}_\epsilon'(u_n, v_n), (\varphi, \psi)\rangle \to \langle \mathcal{J}_\epsilon'(u_0, v_0), (\varphi, \psi)\rangle$ for any $(\varphi, \psi) \in E$. We get that $\mathcal{J}_\epsilon'(u_0, v_0) = 0$ and $(u_0, v_0) \in \mathcal{N}_\epsilon$.

Since

$$\mathcal{J}_\epsilon(u_n, v_n) = c_\epsilon(\Omega) + o_n(1) \quad \textit{and} \quad \langle \mathcal{J}_\epsilon'(u_n, v_n), (u_n, v_n)\rangle = o_n(1),$$

we apply the weakly lower semi continuous of $A(u_n, v_n)$ and the fact $D(u_n, v_n) \to D(u_0, v_0)$ to obtain that

$$c_\epsilon(\Omega) = \liminf_{n\to\infty}(\frac{1}{2} - \frac{1}{2^*})A(u_n, v_n) - \lim_{n\to\infty}(1 - \frac{1}{2^*})\epsilon D(u_n, v_n) \geq \mathcal{J}_\epsilon(u_0, v_0) \geq c_\epsilon(\Omega).$$

Applying Lemma 4, we get that $(u_0, v_0) \in \mathcal{N}_\epsilon^+$ and $\mathcal{J}_\epsilon(u_0, v_0) = c_\epsilon(\Omega) = c_\epsilon^+(\Omega) < 0$. Moreover, (u_0, v_0) is a nontrivial solution of (1).

In the following we prove that $(u_1, v_1) := t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|) \in \mathcal{N}_\epsilon^+$ is a positive least energy solution of (1). Let $(|u'|, |v'|) := \frac{(|u_0|,|v_0|)}{\|(|u_0|,|v_0|)\|}$ and $(u', v') := \frac{(u_0,v_0)}{\|(u_0,v_0)\|}$. It follows from Lemma 3 that $D(|u_0|, |v_0|) \geq D(u_0, v_0) > 0$. By Lemma 2, we get that there exists a unique number $t^+_{(|u_0|,|v_0|)} > 0$ such that $t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|) \in \mathcal{N}_\epsilon^+$. It is clear that $\|(u_0, v_0)\| = \|(|u_0|, |v_0|)\|$. We infer that

$$t^+_{(|u_0|,|v_0|)}\|(u_0, v_0)\|(|u'|, |v'|) = t^+_{(|u_0|,|v_0|)}\|(|u_0|, |v_0|)\|(|u'|, |v'|) = t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|) \in \mathcal{N}_\epsilon^+. \tag{16}$$

Moreover,

$$\|(u_0, v_0)\|(u', v') \in \mathcal{N}_\epsilon^+. \tag{17}$$

Next we compare $t^+_{(|u_0|,|v_0|)}\|(u_0, v_0)\|$ with $\|(u_0, v_0)\|$. Equivalently, recalling the proof in Lemma 2 (ii), we compare the first solution of $\varphi_\epsilon'(t) = 0$ under the case $(u, v) = (|u'|, |v'|)$ with its first solution under the case $(u, v) = (u', v')$. Since $D(|u'|, |v'|) \geq D(u', v') > 0$, $A(|u'|, |v'|) = A(u', v')$ and $B(|u'|, |v'|) = B(u', v')$, taking account of the graph of $\varphi_\epsilon'(t) = 0$, we get that $t^+_{(|u_0|,|v_0|)}\|(u_0, v_0)\| \geq \|(u_0, v_0)\|$. Equivalently,

$$t^+_{(|u_0|,|v_0|)} \geq 1. \tag{18}$$

Since $t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|) \in \mathcal{N}_\epsilon^+$, from Lemma 2, we get that

$$\mathcal{J}_\epsilon(t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|)) \leq \mathcal{J}_\epsilon(|u_0|, |v_0|).$$

Hence,

$$c_\epsilon(\Omega) \leq c_\epsilon^+(\Omega) \leq \mathcal{J}_\epsilon(t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|)) \leq \mathcal{J}_\epsilon(|u_0|, |v_0|) \leq \mathcal{J}_\epsilon(u_0, v_0) = c_\epsilon(\Omega).$$

From Lemma 4, we get that $(u_1, v_1) = t^+_{(|u_0|,|v_0|)}(|u_0|, |v_0|), u_1, v_1 \geq 0$ is a nonnegative solution of (1). In view of (1), we get that $u_1, v_1 \not\equiv 0$ since $f, g \not\equiv 0$. Applying the maximum principle to each equality of (1), we get that $(u_1, v_1), u_1, v_1 > 0$ is a positive least energy solution of (1). □

4 | EXISTENCE OF SECOND AND THIRD SOLUTIONS

In this section we shall find two solutions of (1) in $\mathcal{N}_\epsilon^-$ under suitable range of critical level.

Lemma 6. If $\{(u_n, v_n)\}$ is a $(PS)_c$-sequence of $\mathcal{J}_\epsilon$ with

$$c < c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.$$

Then $\{(u_n, v_n)\}$ has a convergent subsequence.

Proof. Let (u_0, v_0) be the weak limit of (u_n, v_n) and $(\eta_n, \mu_n) := (u_n - u_0, v_n - v_0)$. Then $(\eta_n, \mu_n) \rightharpoonup (0,0)$ in E. We introduce the following version of Brezis-Lieb Lemma from Han[19, Lemma 3.4],

$$\int_\Omega |u_n|^\alpha |v_n|^\beta dx = \int_\Omega |\eta_n|^\alpha |\mu_n|^\beta + |u_0|^\alpha |v_0|^\beta dx + o_n(1),\ \alpha+\beta = 2^*,$$

and the Brezis-Lieb Lemma for the other terms,

$$\textstyle\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla \eta_n|^2 + |\nabla u_0|^2 dx + o_n(1), \quad \int_\Omega |u_n|^{2^*} dx = \int_\Omega |\eta_n|^{2^*} + |u_0|^{2^*} dx + o_n(1).$$

Moreover, $\int_\Omega f u_n + g v_n dx = \int_\Omega f u_0 + g v_0 dx + o_n(1)$. We have that

$$c \leftarrow \mathcal{J}_\epsilon(u_n, v_n) = \mathcal{J}_\epsilon(u_0, v_0) + \tfrac{1}{2}\|(\eta_n, \mu_n)\|^2 - \tfrac{1}{2^*}\textstyle\int_\Omega |\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta dx + o_n(1). \tag{19}$$

In a fashion similar to the proof in Lemma 1 that $\langle \mathcal{J}_\epsilon'(u_n, v_n), (\varphi, \psi)\rangle \to \langle \mathcal{J}_\epsilon'(u_0, v_0), (\varphi, \psi)\rangle$ for any $(\varphi, \psi) \in E$. We get that $\mathcal{J}_\epsilon'(u_0, v_0) = 0$ and $(u_0, v_0) \in \mathcal{N}_\epsilon$. Hence

$$o_n(1) = \langle \mathcal{J}_\epsilon'(u_n, v_n), (u_n, v_n)\rangle = \|(\eta_n, \mu_n)\|^2 - \textstyle\int_\Omega |\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta dx + o_n(1). \tag{20}$$

We may assume that there exists a constant $b \geq 0$ such that

$$\|(\eta_n, \mu_n)\|^2 \to b \ \ and \ \ \int_\Omega |\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta dx \to b.$$

By the definition of $S_{\alpha,\beta}$, we get that $S_{\alpha,\beta} b^{\frac{2}{2^*}} \leq b$. Hence, we have either $b = 0$ or $b \geq S_{\alpha,\beta}^{\frac{N}{2}}$. If $b = 0$, the proof is completed. Otherwise $b \geq S_{\alpha,\beta}^{\frac{N}{2}}$. From (19), we get that

$$c = \mathcal{J}_\epsilon(u_0, v_0) + \frac{1}{N} b \geq c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

This contradicts to $c < c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. Thus, $b = 0$. □

In the following we may assume that

$$R_1 = \rho \quad and \quad R_2 = \frac{1}{\rho} \quad for \quad \rho \in (0, \frac{1}{2}).$$

Now, we denote the radially symmetric function $\varphi_\rho \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi_\rho \leq 1$ for all $x \in \mathbb{R}^N$ and

$$\varphi_\rho(x) = \begin{cases} 0, & 0 \leq |x| \leq \frac{3\rho}{2}, \\ 1, & 2\rho \leq |x| \leq \frac{1}{2\rho}, \\ 0, & |x| \geq \frac{3}{4\rho}. \end{cases}$$

Moreover, for $\sigma \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and $0 < \delta < 1$, we denote

$$U_\delta^\sigma(x) = \frac{[N(N-2)\delta^2]^{\frac{N-2}{4}}}{(\delta^2 + |x - (1-\delta)\sigma|^2)^{\frac{N-2}{2}}},$$

which is a form of the translation and dilation of $U(x)$ (in (11)). From Lemma 1, we know that $S_{\alpha,\beta}(\mathbb{R}^N)$ is realized by $(U_\delta^\sigma(x), \tau_{min} U_\delta^\sigma(x))$. Let

$$u_\delta^{\sigma,\rho}(x) := \varphi_\rho(x) U_\delta^\sigma(x) \in H_0^1(\Omega) \ \ and \ \ v_\delta^{\sigma,\rho}(x) := \tau_{min} \varphi_\rho(x) U_\delta^\sigma(x) \in H_0^1(\Omega). \tag{21}$$

Then we have the following estimates.

Lemma 7. There hold
(i) $A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \leq (1+\tau_{min}^2) S^{\frac{N}{2}} + O(\delta^{N-2})$ and $A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \geq (1+\tau_{min}^2) S^{\frac{N}{2}} - O(\delta^{N-2})$. Moreover, $A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) = (1+\tau_{min}^2) S^{\frac{N}{2}} + o_\delta(1)$ uniformly in σ as $\delta \to 0$;
(ii) $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \leq (1 + \tau_{min}^\beta + \tau_{min}^{2^*}) S^{\frac{N}{2}} + O(\delta^N)$ and $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \geq (1 + \tau_{min}^\beta + \tau_{min}^{2^*}) S^{\frac{N}{2}} - O(\delta^N)$. Moreover, $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) = (1 + \tau_{min}^\beta + \tau_{min}^{2^*}) S^{\frac{N}{2}} + o_\delta(1)$ uniformly in σ as $\delta \to 0$;
(iii) $(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \rightharpoonup (0,0)$ weakly in E uniformly in σ as $\delta \to 0$;
(iv) $\mathcal{J}_0(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \leq \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + O(\delta^{N-2})$.

Proof. (i) It is clear that there exist constants $d_1, d_2 > 0$ such that $d_1 < |x-(1-\delta)\sigma| < d_2$ for all $x \in B_{2\rho}$ whenever $\delta < 1-2\rho$. By direct calculations, we get that

$$\begin{aligned}
&|\int_{\mathbb{R}^N} |\nabla u_\delta^{\sigma,\rho}|^2 - |\nabla U_\delta^\sigma|^2 dx| \\
&\le \int_{(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}) \cup B_{2\rho}} |\nabla U_\delta^\sigma|^2 dx + C\rho^{-2} \int_{B_{2\rho}} |U_\delta^\sigma|^2 dx + C\rho^2 \int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} |U_\delta^\sigma|^2 dx \\
&\le C\delta^{N-2} \int_{(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}) \cup B_{2\rho}} \frac{|x-(1-\delta)\sigma|^2}{|x-(1-\delta)\sigma|^{2N}} dx + C\delta^{N-2} \int_{B_{2\rho} \cup (B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}})} \frac{dx}{|x-(1-\delta)\sigma|^{2(N-2)}} dx = O(\delta^{N-2}).
\end{aligned} \tag{22}$$

From (11), we find that $\int_{\mathbb{R}^N} |\nabla u_\delta^{\sigma,\rho}|^2 dx \le \int_{\mathbb{R}^N} |\nabla U_\delta^\sigma|^2 dx + O(\delta^{N-2}) = S^{\frac{N}{2}} + O(\delta^{N-2})$. Hence, $A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) = (1+\tau_{min}^2)\int_{\mathbb{R}^N} |\nabla u_\delta^{\sigma,\rho}|^2 dx \le (1+\tau_{min}^2) S^{\frac{N}{2}} + O(\delta^{N-2})$. By a similar way, we obtain $A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \ge (1+\tau_{min}^2) S^{\frac{N}{2}} - O(\delta^{N-2})$. So, part (i) holds.

(ii) It is clear that $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) = (1+\tau_{min}^\beta+\tau_{min}^{2^*})\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} dx$. We have that $\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} dx \le \int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{3\rho}{2}}} |U_\delta^\sigma|^{2^*} dx = \{\int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} + \int_{B_{\frac{1}{2\rho}} \setminus B_{\frac{3\rho}{2}}}\} |U_\delta^\sigma|^{2^*} dx$. Moreover, $\int_{B_{\frac{1}{2\rho}} \setminus B_{\frac{3\rho}{2}}} |U_\delta^\sigma|^{2^*} dx \le \int_{\mathbb{R}^N} |U_\delta^\sigma|^{2^*} dx = S^{\frac{N}{2}}$ and $\int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} |U_\delta^\sigma|^{2^*} dx \le C\delta^N \int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\delta)\sigma|^{2N}} = O(\delta^N)$. Hence, $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \le (1+\tau_{min}^\beta+\tau_{min}^{2^*}) S^{\frac{N}{2}} + O(\delta^N)$.

Now we prove the second assertion. It is clear that

$$\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} dx \ge \int_{B_{\frac{1}{2\rho}} \setminus B_{2\rho}} |U_\delta^\sigma|^{2^*} dx = \{\int_{\mathbb{R}^N} - \int_{\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}} - \int_{B_{2\rho}}\} |U_\delta^\sigma|^{2^*} dx.$$

By direct calculations, we get that $\int_{\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}} |U_\delta^\sigma|^{2^*} dx \le C\delta^N \int_{\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\delta)\sigma|^{2N}} = O(\delta^N)$. Moreover, there exist constants $d_1, d_2 > 0$ such that $d_1 < |x-(1-\delta)\sigma| < d_2$ for all $x \in B_{2\rho}$ when $\delta < 1-2\rho$. Hence, $\int_{B_{2\rho}} |U_\delta^\sigma|^{2^*} dx \le C\delta^N \int_{B_{2\rho}} \frac{dx}{|x-(1-\delta)\sigma|^{2N}} = O(\delta^N)$ and $B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) = (1+\tau_{min}^\beta+\tau_{min}^{2^*})\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} dx \ge (1+\tau_{min}^\beta+\tau_{min}^{2^*}) S^{\frac{N}{2}} - O(\delta^N)$. The third assertion follows from the first and second assertions.

(iii) It follows from the arguments of He and Yang[8, Lemma 4.2(iii)] that $(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \rightharpoonup (0,0)$ weakly in E as $\delta \to 0$.

(iv) For $t > 0$, we denote that

$$K(t) := \frac{t^2}{2}[(1+\tau_{min}^2)S^{\frac{N}{2}} + O(\delta^{N-2})] - \frac{t^{2^*}}{2^*}[(1+\tau_{min}^\beta+\tau_{min}^{2^*})S^{\frac{N}{2}} - O(\delta^N)].$$

Then $K(t) \to -\infty$ as $t \to \infty$ and $K(t) > 0$ for a sufficiently small $t > 0$. So, there exists a $t_\delta > 0$ such that $\max_{t>0} K(t)$ is attained and $t_\delta = [\frac{(1+\tau_{min}^2)S^{\frac{N}{2}} + O(\delta^{N-2})}{(1+\tau_{min}^\beta+\tau_{min}^{2^*})S^{\frac{N}{2}} - O(\delta^N)}]^{\frac{1}{2^*-2}}$. Moreover, there exist $t_1, t_2 > 0$ such that $t_1 < t_\delta < t_2$ for a small $\delta > 0$. Clearly, by parts (i) and (ii), $K(t)$ is an increasing function in $(0, t_\delta]$ and $\mathcal{J}_0(tu_\delta^{\sigma,\rho}, tv_\delta^{\sigma,\rho}) \le K(t)$. Hence,

$$\mathcal{J}_0(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \le \max_{t>0} \mathcal{J}_0(tu_\delta^{\sigma,\rho}, tv_\delta^{\sigma,\rho}) \le K(t_\delta) = \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + O(\delta^{N-2}). \tag{23}$$

□

Lemma 8. There exists a $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$,

$$\sup_{t \ge 0} \mathcal{J}_\epsilon(u_1 + tu_\delta^{\sigma,\rho}, v_1 + tv_\delta^{\sigma,\rho}) < c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} \tag{24}$$

uniformly in $\sigma \in \mathbb{S}^{N-1}$, where $(u_1, v_1), u_1, v_1 > 0$ is given by Lemma 1.

Proof. It follows from Lemma 1 that $\mathcal{J}_\epsilon(u_1, v_1) = c_\epsilon(\Omega)$ and

$$\begin{aligned}
&t[\langle u_1, u_\delta^{\sigma,\rho}\rangle_{H_0^1(\Omega)} + \langle v_1, v_\delta^{\sigma,\rho}\rangle_{H_0^1(\Omega)} - \epsilon D(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})] \\
&= t\int_\Omega u_1^{2^*-1} u_\delta^{\sigma,\rho} + v_1^{2^*-1} v_\delta^{\sigma,\rho} + \frac{\alpha}{2^*} u_1^{\alpha-1} v_1^\beta u_\delta^{\sigma,\rho} + \frac{\beta}{2^*} u_1^\alpha v_1^{\beta-1} v_\delta^{\sigma,\rho} dx.
\end{aligned}$$

So, we have that

$$
\begin{aligned}
&\mathcal{J}_\epsilon(u_1+tu_\delta^{\sigma,\rho},v_1+tv_\delta^{\sigma,\rho})=\tfrac{1}{2}A(u_1,v_1)+\tfrac{1}{2}A(tu_\delta^{\sigma,\rho},tv_\delta^{\sigma,\rho})\\
&+\langle u_1,tu_\delta^{\sigma,\rho}\rangle_{H_0^1(\Omega)}+\langle v_1,tv_\delta^{\sigma,\rho}\rangle_{H_0^1(\Omega)}-\tfrac{1}{2^*}B(u_1+tu_\delta^{\sigma,\rho},v_1+tv_\delta^{\sigma,\rho})-\epsilon D(u_1,v_1)-\epsilon D(tu_\delta^{\sigma,\rho},tv_\delta^{\sigma,\rho})\\
&=\mathcal{J}_\epsilon(u_1,v_1)+\tfrac{t^2}{2}A(u_\delta^{\sigma,\rho},v_\delta^{\sigma,\rho})-\tfrac{t^{2^*}}{2^*}B(u_\delta^{\sigma,\rho},v_\delta^{\sigma,\rho})\\
&-\tfrac{1}{2^*}\int_\Omega[(u_1+tu_\delta^{\sigma,\rho})^\alpha(v_1+tv_\delta^{\sigma,\rho})^\beta-u_1^\alpha v_1^\beta-(tu_\delta^{\sigma,\rho})^\alpha(tv_\delta^{\sigma,\rho})^\beta-\alpha u_1^{\alpha-1}v_1^\beta tu_\delta^{\sigma,\rho}-\beta u_1^\alpha v_1^{\beta-1}tv_\delta^{\sigma,\rho}]dx\\
&-\tfrac{1}{2^*}\int_\Omega[(u_1+tu_\delta^{\sigma,\rho})^{2^*}-u_1^{2^*}-(tu_\delta^{\sigma,\rho})^{2^*}-2^*u_1^{2^*-1}tu_\delta^{\sigma,\rho}]dx\\
&-\tfrac{1}{2^*}\int_\Omega[(v_1+tu_\delta^{\sigma,\rho})^{2^*}-v_1^{2^*}-(tu_\delta^{\sigma,\rho})^{2^*}-2^*v_1^{2^*-1}tv_\delta^{\sigma,\rho}]dx.
\end{aligned}
\tag{25}
$$

Moreover, the following claims 1-2 hold.

Claim 1 : $\int_\Omega[(u_1+tu_\delta^{\sigma,\rho})^\alpha(v_1+tv_\delta^{\sigma,\rho})^\beta-u_1^\alpha v_1^\beta-(tu_\delta^{\sigma,\rho})^\alpha(tv_\delta^{\sigma,\rho})^\beta-\alpha u_1^{\alpha-1}v_1^\beta tu_\delta^{\sigma,\rho}-\beta u_1^\alpha v_1^{\beta-1}tv_\delta^{\sigma,\rho}]dx\geq 0$.

To prove Claim 1, we define $f(x,y):[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ by

$$f(x,y)=(1+x)^\alpha(1+y)^\beta-1-x^\alpha y^\beta-\alpha x-\beta y.$$

By direct calculations, we have that

$$
\begin{aligned}
\tfrac{\partial f(x,y)}{\partial x}&=\alpha(1+x)^{\alpha-1}(1+y)^\beta-\alpha x^{\alpha-1}y^\beta-\alpha\\
&\geq\alpha(1+x)^{\alpha-1}(1+y^\beta)-\alpha x^{\alpha-1}y^\beta-\alpha\\
&=\alpha(1+x)^{\alpha-1}-\alpha+\alpha(1+x)^{\alpha-1}y^\beta-\alpha x^{\alpha-1}y^\beta\geq 0.
\end{aligned}
$$

Similarly, $\frac{\partial f(x,y)}{\partial y}\geq 0$. Moreover, $f(0,0)=0$. So, we get that $f(x,y)\geq 0$ for any $x\geq 0$ and $y\geq 0$. Applying the inequality $f(x,y)\geq 0$ and the fact $u_1,v_1>0$, we obtain claim 1.

Claim 2 : $\int_\Omega(u_1+tu_\delta^{\sigma,\rho})^{2^*}-u_1^{2^*}-(tu_\delta^{\sigma,\rho})^{2^*}-2^*u_1^{2^*-1}tu_\delta^{\sigma,\rho}dx\geq O(\delta^{\frac{N-2}{2}})$ and $\int_\Omega(v_1+tv_\delta^{\sigma,\rho})^{2^*}-v_1^{2^*}-(tv_\delta^{\sigma,\rho})^{2^*}-2^*v_1^{2^*-1}tv_\delta^{\sigma,\rho}dx\geq O(\delta^{\frac{N-2}{2}})$.

The proof of Claim 2 follows from He and Yang[8, Equations (4.7) and (4.8)]. By using Claims 1-2 and Lemma 7, we infer from (25) that

$$
\begin{aligned}
&\mathcal{J}_\epsilon(u_1+tu_\delta^{\sigma,\rho},v_1+tv_\delta^{\sigma,\rho})\leq\mathcal{J}_\epsilon(u_1,v_1)+\tfrac{t^2}{2}[(1+\tau_{min}^2)S^{\frac{N}{2}}+O(\delta^{N-2})]\\
&-\tfrac{t^{2^*}}{2^*}[(1+\tau_{min}^\beta+\tau_{min}^{2^*})S^{\frac{N}{2}}-O(\delta^N)]-O(\delta^{\frac{N-2}{2}}).
\end{aligned}
\tag{26}
$$

Let $K(t):=\frac{t^2}{2}[(1+\tau_{min}^2)S^{\frac{N}{2}}+O(\delta^{N-2})]-\frac{t^{2^*}}{2^*}[(1+\tau_{min}^\beta+\tau_{min}^{2^*})S^{\frac{N}{2}}-O(\delta^N)]$. Then $\lim\limits_{t\to 0^+}K(t)>0$ and $K(t)\to-\infty$ as $t\to+\infty$. So, there exists a $t_\delta>0$ such that $\sup\limits_{t>0}K(t)$ is attained and $t_\delta=[\frac{(1+\tau_{min}^2)S^{\frac{N}{2}}+O(\delta^{N-2})}{(1+\tau_{min}^\beta+\tau_{min}^{2^*})S^{\frac{N}{2}}-O(\delta^N)}]^{\frac{1}{2^*-2}}$. Moreover,

$$\sup_{t>0}K(t)=K(t_\delta)=\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}+O(\delta^{N-2}).$$

We infer from (26) that

$$\mathcal{J}_\epsilon(u_1+tu_\delta^{\sigma,\rho},v_1+tv_\delta^{\sigma,\rho})\leq c_\epsilon(\Omega)+\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}+O(\delta^{N-2})-O(\delta^{\frac{N-2}{2}}).$$

So, there exists a $\delta_0>0$ such that, for $0<\delta<\delta_0$, the result holds. □

Lemma 9. There exists a $t_0>0$ such that $(u_1+t_0u_\delta^{\sigma,\rho},v_1+t_0v_\delta^{\sigma,\rho})\in\mathcal{N}_\epsilon^-$ for $0<\delta<\delta_0$. Moreover, $c_\epsilon^-(\Omega)<c_\epsilon(\Omega)+\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$.

Proof. By the definition of $\mathcal{N}_\epsilon^-$, we have that $\mathcal{N}_\epsilon^-=\{(u,v)\in E\backslash\{(0,0)\}:\frac{1}{\|(u,v)\|}t^-_{\frac{(u,v)}{\|(u,v)\|}}=1\}$. Moreover, $E\backslash\mathcal{N}_\epsilon^-=U_1\cup U_2$, where

$$
\begin{aligned}
U_1&:=\{(u,v)\in E\backslash\{(0,0)\}:\|(u,v)\|<t^-_{\frac{(u,v)}{\|(u,v)\|}}\}\cup\{(0,0)\},\\
U_2&:=\{(u,v)\in E\backslash\{(0,0)\}:\|(u,v)\|>t^-_{\frac{(u,v)}{\|(u,v)\|}}\}.
\end{aligned}
$$

We claim that $\mathcal{N}_\epsilon^+\subset U_1$. Indeed, let $(u,v)\in\mathcal{N}_\epsilon^+$, we have that $1=t^+_{(u,v)}<t_{max}<t^-_{(u,v)}=\frac{1}{\|(u,v)\|}t^-_{\frac{(u,v)}{\|(u,v)\|}}$.

Next we prove that there exists a $s_0>1$ such that $(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})\in U_2$ for $0<\delta<\delta_0$. It follows from Lemma 2 that there exists a unique number $t^-_{\frac{(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})}{\|(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})\|}}>0$ such that $t^-_{\frac{(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})}{\|(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})\|}}\frac{(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})}{\|(u_1+s_0u_\delta^{\sigma,\rho},v_1+s_0v_\delta^{\sigma,\rho})\|}\in\mathcal{N}_\epsilon^-$. Since $\mathcal{J}_\epsilon$ is coercive on $\mathcal{N}_\epsilon^-$, there exists a $c > 0$ such that $0 < t^-_{\frac{(u_1+s_0u_\delta^{\sigma,\rho}, v_1+s_0v_\delta^{\sigma,\rho})}{\|(u_1+s_0u_\delta^{\sigma,\rho}, v_1+s_0v_\delta^{\sigma,\rho})\|}} < c$. In view of Lemma 7 (*i*), for a sufficiently small δ, there holds $\|(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})\| \geq S^{\frac{N}{4}}$. Let $s_0 = \frac{|c^2-\|(u_1,v_1)\|^2|^{\frac{1}{2}}}{S^{\frac{N}{4}}} + 1$. We have that $s_0 \geq \frac{|c^2-\|(u_1,v_1)\|^2|^{\frac{1}{2}}}{\|(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})\|} + 1$ and

$$\begin{aligned}
&\|(u_1 + s_0u_\delta^{\sigma,\rho}, v_1 + s_0v_\delta^{\sigma,\rho})\|^2 \\
&= \|(u_1, v_1)\|^2 + s_0^2\|(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})\|^2 + 2s_0(\langle u_1, u_\delta^{\sigma,\rho}\rangle + \langle v_1, v_\delta^{\sigma,\rho}\rangle) \\
&\geq \|(u_1, v_1)\|^2 + |c^2 - \|(u_1, v_1)\|^2| \geq c^2 > (t^-_{\frac{(u_1+s_0u_\delta^{\sigma,\rho}, v_1+s_0v_\delta^{\sigma,\rho})}{\|(u_1+s_0u_\delta^{\sigma,\rho}, v_1+s_0v_\delta^{\sigma,\rho})\|}})^2,
\end{aligned}$$

which implies $(u_1 + s_0u_\delta^{\sigma,\rho}, v_1 + s_0v_\delta^{\sigma,\rho}) \in U_2$.

For each $0 < \delta < \delta_0$, we define a path $\xi_\delta(r) = (u_1, v_1) + rs_0(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})$ for $r \in [0, 1]$. Then

$$\xi_\delta(0) = (u_1, v_1) \quad and \quad \xi_\delta(1) = (u_1 + s_0u_\delta^{\sigma,\rho}, v_1 + s_0v_\delta^{\sigma,\rho}).$$

It is clear that $(u_1, v_1) \in \mathcal{N}_\epsilon^+ \subset U_1$ and $(u_1 + s_0u_\delta^{\sigma,\rho}, v_1 + s_0v_\delta^{\sigma,\rho}) \in U_2$. Moreover, $\frac{1}{\|(u,v)\|}t^-_{\frac{(u,v)}{\|(u,v)\|}}$ is a continuous function and $\xi_\delta([0, 1])$ is connected. So, there exists a $r_0 \in (0, 1)$ such that $\xi_\delta(r_0) = (u_1 + r_0s_0u_\delta^{\sigma,\rho}, v_1 + r_0s_0v_\delta^{\sigma,\rho}) \in \mathcal{N}_\epsilon^-$. Let $t_0 = r_0s_0$, we have that $(u_1 + t_0u_\delta^{\sigma,\rho}, v_1 + t_0v_\delta^{\sigma,\rho}) \in \mathcal{N}_\epsilon^-$. Applying Lemma 8, we get that

$$c_\epsilon^-(\Omega) \leq \mathcal{J}_\epsilon(u_1 + t_0u_\delta^{\sigma,\rho}, v_1 + t_0v_\delta^{\sigma,\rho}) < c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.$$

□

In the following we shall show that, for a sufficiently small $\lambda > 0$,

$$cat(\{(u, v) \in \mathcal{N}_\epsilon^- : \mathcal{J}_\epsilon(u, v) \leq c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} - \lambda\}) \geq 2,$$

where

$$\begin{aligned}
cat(X) :=& \min\{k \in \mathbb{N} : \textit{there exist closed subsets } X_1, ..., X_k \subset X \textit{ such} \\
&\textit{that } X_j \textit{ is contractible to a point in } X \textit{ for all } j \textit{ and } \cup_{j=1}^k X_j = X\}.
\end{aligned}$$

To start with, we introduce the following two lemmas for our proof.

Lemma 10. [14] Suppose that X is a Hilbert manifold and $G \in C^1(X, \mathbb{R})$. Assume that for $c' \in \mathbb{R}$ and $k \in \mathbb{N}$

1. G satisfies the Palais-Smale condition for energy level $c \leq c'$;
2. $cat(\{x \in X : G(x) \leq c'\}) \geq k$.

Then G has at least k critical points in $\{x \in X : G(x) \leq c'\}$.

Lemma 11. [20, Theorem 2.5] Let X be a topological space. Suppose that there are two continuous maps $\Phi : \mathbb{S}^{N-1} \to X$ and $\Psi : X \to \mathbb{S}^{N-1}$ such that $\Psi \circ \Phi$ is homotopic to the identity map of $\mathbb{S}^{N-1}$. Then $cat(X) \geq 2$.

Note that, for each $(u, v) \in E\backslash\{(0, 0)\}$, there exists a unique number $t^* > 0$ such that $t^*(u, v) \in \mathcal{N}_0$.

Lemma 12. For each $(u, v) \in E\backslash\{(0, 0)\}$ and $0 < \kappa < 1$, there holds

$$(1 - \epsilon\kappa)^{\frac{N}{2}}\mathcal{J}_0(t^*(u, v)) - \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \leq \mathcal{J}_\epsilon(t^-_{(u,v)}(u, v)) \leq (1 + \epsilon\kappa)^{\frac{N}{2}}\mathcal{J}_0(t^*(u, v)) + \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).$$

Proof. For $c \in \mathbb{R}$, we denote that

$$\begin{aligned}
&B_c(u, v) = cB(u, v), \quad \mathcal{I}_c(u, v) = \tfrac{1}{2}A(u, v) - \tfrac{1}{2^*}B_c(u, v), \\
&\mathcal{M}_c = \{(u, v) \in E\backslash\{(0, 0)\} : \langle \mathcal{I}_c'(u, v), (u, v)\rangle = 0\}.
\end{aligned}$$

Now we study the relationship between $\mathcal{J}_\epsilon$ and $\mathcal{I}_c$. For each $(u, v) \in E\backslash\{(0, 0)\}$ and $0 < \kappa < 1$, we have that

$$|\int_\Omega fu + gv dx| \leq \|f\|_{H^{-1}}\|u\| + \|g\|_{H^{-1}}\|v\| \leq \frac{\kappa}{2}\|(u, v)\|^2 + \frac{1}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).$$

So, there holds

$$\tfrac{1-\epsilon\kappa}{2}A(u, v) - \tfrac{1}{2^*}B(u, v) - \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \leq \mathcal{J}_\epsilon(u, v) \leq \tfrac{1+\epsilon\kappa}{2}A(u, v) - \tfrac{1}{2^*}B(u, v) + \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).$$

Equivalently,

$$(1 - \epsilon\kappa)\mathcal{I}_{\frac{1}{1-\epsilon\kappa}}(u, v) - \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \leq \mathcal{J}_\epsilon(u, v) \leq (1 + \epsilon\kappa)\mathcal{I}_{\frac{1}{1+\epsilon\kappa}}(u, v) + \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2). \tag{27}$$

Next we seek the help of function $\mathcal{I}_c$ to prove the lemma. For each $(u,v) \in E\backslash\{(0,0)\}$, we denote $k(t) := \mathcal{I}_c(t(u,v)) = \frac{1}{2}A(u,v)t^2 - \frac{1}{2^*}B_c(u,v)t^{2^*}$. Let $t_c = (\frac{A(u,v)}{B_c(u,v)})^{\frac{1}{2^*-2}} > 0$. By direct calculations, we get that $t_c(u,v) \in \mathcal{M}_c$ and

$$\max_{t\geq 0} \mathcal{I}_c(t(u,v)) = \mathcal{I}_c(t_c(u,v)) = \frac{1}{N}\frac{A(u,v)^{\frac{N}{2}}}{B_c(u,v)^{\frac{N-2}{2}}}.$$

From Lemma 3, we get that $c_\epsilon^-(\Omega) > 0$. So, $\max_{t>0} \mathcal{J}_\epsilon(tu,tv) = \mathcal{J}_\epsilon(t^-_{(u,v)}(u,v))$. This fact together with (27) yield that

$$\begin{aligned}&(1-\epsilon\kappa)\mathcal{I}_{\frac{1}{1-\epsilon\kappa}}(t_{\frac{1}{1-\epsilon\kappa}}(u,v)) - \tfrac{\epsilon}{2\kappa}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}) \leq \mathcal{J}_\epsilon(t^-_{(u,v)}(u,v))\\ &\leq (1+\epsilon\kappa)\mathcal{I}_{\frac{1}{1+\epsilon\kappa}}(t_{\frac{1}{1+\epsilon\kappa}}(u,v)) + \tfrac{\epsilon}{2\kappa}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}).\end{aligned} \tag{28}$$

Moreover, we find that

$$\mathcal{I}_{\frac{1}{1-\epsilon\kappa}}(t_{\frac{1}{1-\epsilon\kappa}}(u,v)) = \frac{(1-\epsilon\kappa)^{\frac{N-2}{2}}}{N}\frac{A(u,v)^{\frac{N}{2}}}{B(u,v)^{\frac{N-2}{2}}} = (1-\epsilon\kappa)^{\frac{N-2}{2}}\mathcal{J}_0(t^*(u,v)),$$

where $t^*(u,v) \in \mathcal{N}_0$. The result follows from (28). □

Lemma 13. Assume that Ω satisfies condition (V). Then there exists a $\mu_0 > 0$ such that if $(u,v) \in \mathcal{N}_0$ with $\mathcal{J}_0(u,v) \leq \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + \mu_0$, then $|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx| \neq 0$.

Proof. Let $\{(u_n,v_n)\} \subset \mathcal{N}_0$ be such that $\mathcal{J}_0(u_n,v_n) = \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + o_n(1)$. In a fashion similar to the argument for the second assertion in Lemma 5 that we infer $\mathcal{J}_0'(u_n,v_n) \to 0$. So, $\{(u_n,v_n)\}$ is a Palais-Smale sequence of $\mathcal{J}_0$ at the level $\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$. Note that $S_{\alpha,\beta} = F(\tau_{min})S$ is never achieved in a bounded domain Ω. In other words, if (u_0,v_0) is a solution of (1) with $\epsilon = 0$, then $\mathcal{J}_0(u_0,v_0) > \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$. Now, using the global compactness lemma from Peng et al[9, Theorem 1.7], we get that

$$(u_n,v_n) = (r_n^1)^{\frac{2-N}{2}}(U_1(\frac{x-x_n^1}{r_n^1}), V_1(\frac{x-x_n^1}{r_n^1})) + o_n(1)$$

in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$, where $r_n^1 \to 0$ as $n\to\infty$, $x_n^1 \in \bar{\Omega}$ and $(U_1,V_1) \neq (0,0)$ is a solution of (1) with $\epsilon = 0$ and $\Omega = \mathbb{R}^N$. Suppose, up to a subsequence, that $\frac{x_n^1}{|x_n^1|} \to y_0$ as $n\to\infty$, where y_0 is a unit vector in $\mathbb{R}^N$. We have that

$$\begin{aligned}&|\textstyle\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla u_n|^2 + |\nabla v_n|^2)dx|\\ &= |\textstyle\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla (r_n^1)^{\frac{2-N}{2}}U_1(\frac{x-x_n^1}{r_n^1})|^2 + |\nabla (r_n^1)^{\frac{2-N}{2}}V_1(\frac{x-x_n^1}{r_n^1})|^2)dx| + o_n(1)\\ &= |\textstyle\int_{\mathbb{R}^N}\frac{x_n^1+r_n^1 z}{|x_n^1+r_n^1 z|}(|\nabla U_1(z)|^2 + |\nabla V_1(z)|^2)dz| + o_n(1)\\ &= |y_0\textstyle\int_{\mathbb{R}^N}(|\nabla U_1(z)|^2 + |\nabla V_1(z)|^2)dz| + o_n(1) \neq 0.\end{aligned}$$

□

For $0<\delta<\delta_0$ (given by Lemma 9), we define $H_\delta : \mathbb{S}^{N-1} \to E$ by

$$H_\delta(\sigma) = (u_1 + t_0u_\delta^{\sigma,\rho}, v_1 + t_0v_\delta^{\sigma,\rho}) \in \mathcal{N}_\epsilon^-, \tag{29}$$

where $(u_1 + t_0u_\delta^{\sigma,\rho}, v_1 + t_0v_\delta^{\sigma,\rho})$ is given by Lemma 9. From Lemma 8, we find that there exists a $\lambda_\delta > 0$ such that

$$\mathcal{J}_\epsilon(u_1 + t_0u_\delta^{\sigma,\rho}, v_1 + t_0v_\delta^{\sigma,\rho}) \leq c_\epsilon(\Omega) + \tfrac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} - \lambda_\delta,$$

which implies that

$$H_\delta(\mathbb{S}^{N-1}) \subset \{(u,v) \in \mathcal{N}_\epsilon^- : \mathcal{J}_\epsilon(u,v) \leq c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} - \lambda_\delta\}. \tag{30}$$

Lemma 14. There exists a $\epsilon_2 > 0$ such that, for $0<\epsilon<\epsilon_2$ and any

$$(u_0,v_0) \in \{(u,v) \in \mathcal{N}_\epsilon^- : \mathcal{J}_\epsilon(u,v) \leq c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}\},$$

there holds $|\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla u_0|^2 + |\nabla v_0|^2)dx| \neq 0$.

Proof. Let $(u_0, v_0) \in \{(u,v) \in \mathcal{N}_\epsilon^- : \mathcal{J}_\epsilon(u,v) \le c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}\}$. Then $t^-_{(u_0,v_0)} = 1$. It is clear that $c_\epsilon(\Omega) < 0$. So,

$$\mathcal{J}_\epsilon(u_0, v_0) \le \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

From Lemma 12, we have that there exists a $0 < \kappa_0 < 1$ such that

$$(1-\epsilon\kappa_0)^{\frac{N}{2}} \mathcal{J}_0(t^*(u_0,v_0)) - \frac{\epsilon}{2\kappa_0}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}) \le \mathcal{J}_\epsilon(t^-_{(u_0,v_0)}(u_0,v_0)) = \mathcal{J}_\epsilon(u_0,v_0),$$

where $t^*(u_0,v_0) \in \mathcal{N}_0$. Hence,

$$\begin{aligned}
\mathcal{J}_0(t^*(u_0,v_0)) &\le (1-\epsilon\kappa_0)^{-\frac{N}{2}}[\mathcal{J}_\epsilon(u_0,v_0) + \tfrac{\epsilon}{2\kappa_0}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}})] \\
&\le (1-\epsilon\kappa_0)^{-\frac{N}{2}}[\tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \tfrac{\epsilon}{2\kappa_0}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}})] \\
&= [(1-\epsilon\kappa_0)^{-\frac{N}{2}} - 1]\tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + [\tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \tfrac{\epsilon}{2\kappa_0(1-\epsilon\kappa_0)^{\frac{N}{2}}}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}})].
\end{aligned}$$

We easily find that there exists a $\epsilon_2 > 0$ such that, for $0 < \epsilon < \epsilon_2$, there hold $[(1-\epsilon\kappa_0)^{-\frac{N}{2}} - 1]\frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < \frac{\mu_0}{2}$ and $\frac{\epsilon}{2\kappa_0(1-\epsilon\kappa_0)^{\frac{N}{2}}}(\|f\|^2_{H^{-1}} + \|g\|^2_{H^{-1}}) < \frac{\mu_0}{2}$, where μ_0 is defined in Lemma 13. Hence, we obtain that

$$\mathcal{J}_0(t^*(u_0,v_0)) \le \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \mu_0.$$

This inequality together with Lemma 13 yield the result. □

We define $G : \{(u,v) \in \mathcal{N}_\epsilon^- : \mathcal{J}_\epsilon(u,v) \le c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}\} \to \mathbb{S}^{N-1}$ by

$$G(u,v) = \frac{\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx}{|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx|}.$$

Note that G is well defined since Lemma 14.

Lemma 15. For $0 < \epsilon < \epsilon_2$ and $0 < \delta < \delta_0$, the map

$$G \circ H_\delta : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$$

is homotopic to the identity, where $H_\delta(\sigma)$ is defined in (29).

Proof. We define

$$\mathcal{K} = \{(u,v) \in E \backslash \{(0,0)\} : |\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx| \ne 0\}$$

and $\bar{G} : \mathcal{K} \to \mathbb{S}^{N-1}$ by

$$\bar{G}(u,v) = \frac{\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx}{|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + |\nabla v|^2)dx|}$$

as an extension of G.

It is clear that there exists a $t^* > 0$ such that $t^*(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \in \mathcal{N}_0$. From (23), we find that

$$\mathcal{J}_0(t^* u_\delta^{\sigma,\rho}, t^* v_\delta^{\sigma,\rho}) \le \max_{t>0} \mathcal{J}_0(t u_\delta^{\sigma,\rho}, t v_\delta^{\sigma,\rho}) \le \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + O(\delta^{N-2}).$$

For a sufficiently small δ, this inequality together with Lemma 13 yield that $|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u_\delta^{\sigma,\rho}|^2 + |\nabla v_\delta^{\sigma,\rho}|^2)dx| \ne 0$. Thus, $\bar{G}(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})$ is well defined. Let $y : [s_1, s_2] \to \mathbb{S}^{N-1}$ be a regular geodesic between $\bar{G}(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})$ and $\bar{G}(H_\delta(\sigma))$ such that

$$y(s_1) = \bar{G}(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho}) \ \ and \ \ y(s_2) = \bar{G}(H_\delta(\sigma)).$$

Moreover, in a fashion similar to the argument in Lemma 7 and the analysis above, there exists a $t' > 0$ such that $t'(u_{2(1-k)\delta}^{\sigma,\rho}, v_{2(1-k)\delta}^{\sigma,\rho}) \in \mathcal{N}_0$ and

$$\mathcal{J}_0(t' u_{2(1-k)\delta}^{\sigma,\rho}, t' v_{2(1-k)\delta}^{\sigma,\rho}) \le \max_{t>0} \mathcal{J}_0(t u_{2(1-k)\delta}^{\sigma,\rho}, t v_{2(1-k)\delta}^{\sigma,\rho}) \le \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \mu_0$$

for a sufficiently small δ and $k \in [\frac{1}{2}, 1)$, where μ_0 is defined in Lemma 13. So, $|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u^{\sigma,\rho}_{2(1-k)\delta}|^2 + |\nabla v^{\sigma,\rho}_{2(1-k)\delta}|^2)dx| \neq 0$ and $\bar{G}(u^{\sigma,\rho}_{2(1-k)\delta}, v^{\sigma,\rho}_{2(1-k)\delta})$ is well defined for $k \in [\frac{1}{2}, 1)$. Now we define $\varsigma_\delta(k,\sigma) : [0,1] \times \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ by

$$\varsigma_\delta(k,\sigma) = \begin{cases} y(2k(s_1 - s_2) + s_2), & if\ k \in [0, \frac{1}{2}), \\ \bar{G}(u^{\sigma,\rho}_{2(1-k)\delta}, v^{\sigma,\rho}_{2(1-k)\delta}), & if\ k \in [\frac{1}{2}, 1), \\ \sigma, & if\ k = 1. \end{cases}$$

We claim that $\lim_{k\to 1^-} \varsigma_\delta(k,\sigma) = \sigma$ and $\lim_{k\to \frac{1}{2}^-} \varsigma_\delta(k,\sigma) = \bar{G}(u^{\sigma,\rho}_\delta, v^{\sigma,\rho}_\delta)$.

(i) $\lim_{k\to 1^-} \varsigma_\delta(k,\sigma) = \sigma$: let $k_n \to 1^-$ as $n \to \infty$. Since $u^{\sigma,\rho}_\delta(x) = \delta^{\frac{2-N}{2}} \varphi_\rho(x) U(\frac{x-(1-\delta)\sigma}{\delta})$, we use the equality to deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} \tfrac{x}{|x|}(|\nabla u^{\sigma,\rho}_{2(1-k_n)\delta}|^2 + |\nabla v^{\sigma,\rho}_{2(1-k_n)\delta}|^2)dx \\ &= (1+\tau^2_{min}) \int_{\mathbb{R}^N} \tfrac{x}{|x|}([2(1-k_n)\delta]^{\frac{2-N}{2}} |\nabla(\varphi_\rho(x) U(\tfrac{x-[1-2(1-k_n)\delta]\sigma}{2(1-k_n)\delta}))|^2 dx \\ &= (1+\tau^2_{min}) \int_{\mathbb{R}^N} \tfrac{2(1-k_n)\delta z + [1-2(1-k_n)\delta]\sigma}{|2(1-k_n)\delta z + [1-2(1-k_n)\delta]\sigma|} |\nabla(\varphi_\rho(x) U(z))|^2 dz. \end{aligned}$$

Moreover, $\frac{2(1-k_n)\delta z + [1-2(1-k_n)\delta]\sigma}{|2(1-k_n)\delta z + [1-2(1-k_n)\delta]\sigma|} \to \sigma$ as $k_n \to 1^-$ and

$$\int_{\mathbb{R}^N} |\nabla(\varphi_\rho(x)U(z))|^2 dz = \int_{\mathbb{R}^N} |\nabla u^{\sigma,\rho}_{2(1-k_n)\delta}|^2 dx \to S^{\frac{N}{2}} \quad as\ \ k_n \to 1^-.$$

Hence,

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u^{\sigma,\rho}_{2(1-k_n)\delta}|^2 + |\nabla v^{\sigma,\rho}_{2(1-k_n)\delta}|^2)dx \to (1+\tau^2_{min}) S^{\frac{N}{2}} \sigma$$

and $\lim_{k\to 1^-} \varsigma_\delta(k,\sigma) = \sigma$.

(ii) $\lim_{k\to \frac{1}{2}^-} \varsigma_\delta(k,\sigma) = \bar{G}(u^{\sigma,\rho}_\delta, v^{\sigma,\rho}_\delta)$: one has that

$$\lim_{k\to \frac{1}{2}^-} \varsigma_\delta(k,\sigma) = \lim_{k\to \frac{1}{2}^-} y(2k(s_1 - s_2) + s_2) = y(s_1) = \bar{G}(u^{\sigma,\rho}_\delta, v^{\sigma,\rho}_\delta).$$

Moreover, $\varsigma_\delta \in C([0,1] \times \mathbb{S}^{N-1}, \mathbb{S}^{N-1})$, $\varsigma_\delta(0,\sigma) = \bar{G}(H_\delta(\sigma))$ and $\varsigma_\delta(1,\sigma) = \sigma$ for $\sigma \in \mathbb{S}^{N-1}$ provided $0 < \delta < \delta_0$ and $0 < \epsilon < \epsilon_2$. Thus the result follows. □

Proposition 2. Let $0 < \epsilon < \epsilon' = \min\{\epsilon_1, \epsilon_2\}$, where $\epsilon_i, i = 1, 2$ are defined in Lemmas 3 and 14 respectively. Then $\mathcal{J}_\epsilon$ has two critical points in

$$\{(u,v) \in \mathcal{N}^-_\epsilon : \mathcal{J}_\epsilon(u,v) \le c_\epsilon(\Omega) + \frac{1}{N} S^{\frac{N}{2}}_{\alpha,\beta}\}.$$

Equivalently, (1) has solutions $(u_i, v_i) \in \mathcal{N}^-_\epsilon, i = 2, 3$ with $\mathcal{J}_\epsilon(u_i, v_i) \le c_\epsilon(\Omega) + \frac{1}{N} S^{\frac{N}{2}}_{\alpha,\beta}$.

Proof. Applying Lemma 11, Lemma 15 and (30), we have that

$$cat(\{(u,v) \in \mathcal{N}^-_\epsilon : \mathcal{J}_\epsilon(u,v) \le c_\epsilon(\Omega) + \frac{1}{N} S^{\frac{N}{2}}_{\alpha,\beta} - \lambda_\delta\}) \ge 2.$$

Now from Lemma 6 and Lemma 10, we find two solutions of (1) in $\mathcal{N}^-_\epsilon$ with $\mathcal{J}_\epsilon(u,v) \le c_\epsilon(\Omega) + \frac{1}{N} S^{\frac{N}{2}}_{\alpha,\beta}$. □

5 | EXISTENCE OF FOURTH SOLUTION

In this section, we shall prove the existence of a high energy solution in $\mathcal{N}^-_\epsilon$ by using the minimax lemma of Brezis and Nirenberg[15, Theorem 1].

Lemma 16. Let $0 < \epsilon \le \epsilon_3$, where

$$\epsilon_3 = [\frac{2^*}{2 \cdot 2^*(2^*-1) + \alpha(\alpha-1) + \beta(\beta-1)}]^{\frac{1}{2^*-2}} \frac{2\sqrt{2} S^{\frac{N}{4}}}{(N+2)\max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}}.$$

Then (u_1, v_1) is the unique critical point of $\mathcal{J}_\epsilon$ in $\mathcal{N}^+_\epsilon$.

Proof. Let $r_0 = [\frac{2^*}{2\cdot 2^*(2^*-1)+\alpha(\alpha-1)+\beta(\beta-1)}]^{\frac{1}{2^*-2}} S^{\frac{N}{4}}$. Then,
Claim 1: $\mathcal{N}_\epsilon^+ \subset B_{r_0}(0) = \{(u,v) \in E\backslash\{(0,0)\} : \|(u,v)\| < r_0\}$.

Indeed, if $(u,v) \in \mathcal{N}_\epsilon^+$, then $A(u,v) > (2^*-1)B(u,v)$. By using (12), we get that

$$A(u,v) = B(u,v) + \epsilon D(u,v) < \frac{1}{2^*-1}A(u,v) + \sqrt{2}\epsilon \max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}\|(u,v)\|,$$

which implies that, for $0 < \epsilon < \epsilon_3$,

$$\|(u,v)\| < \frac{\sqrt{2}(N+2)\epsilon_3 \max\{\|f\|_{H^{-1}}, \|g\|_{H^{-1}}\}}{4} = r_0.$$

Claim 2: $\mathcal{J}_\epsilon(u,v)$ is strictly convex in $B_{r_0}(0)$.
For all $(u,v) \in B_{r_0}(0)$, one has that

$$\begin{aligned}
&\mathcal{J}_\epsilon''(u,v)((\varphi,\psi),(\varphi,\psi)) = \|(\varphi,\psi)\|^2 - (2^*-1)\int_\Omega |u|^{2^*-2}\varphi^2 + |v|^{2^*-2}\psi^2 dx \\
&-\tfrac{\alpha(\alpha-1)}{2^*}\int_\Omega |u|^{\alpha-2}|v|^\beta\varphi^2 dx - \tfrac{\beta(\beta-1)}{2^*}\int_\Omega |u|^\alpha|v|^{\beta-2}\psi^2 dx \\
&\geq \|(\varphi,\psi)\|^2 - (2^*-1)|u|_{2^*}^{2^*-2}|\varphi|_{2^*}^2 - (2^*-1)|v|_{2^*}^{2^*-2}|\psi|_{2^*}^2 \\
&-\tfrac{\alpha(\alpha-1)}{2^*}|u|_{2^*}^{\alpha-2}|v|_{2^*}^\beta|\varphi|_{2^*}^2 - \tfrac{\beta(\beta-1)}{2^*}|u|_{2^*}^\alpha|v|_{2^*}^{\beta-2}|\psi|_{2^*}^2 \\
&\geq \|(\varphi,\psi)\|^2\{1 - [2(2^*-1) + \tfrac{\alpha(\alpha-1)}{2^*} + \tfrac{\beta(\beta-1)}{2^*}]S^{-\frac{N}{N-2}}\|(u,v)\|^{2^*-2}\} > 0,
\end{aligned}$$

where $(\varphi,\psi) \in E\backslash\{(0,0)\}$. Claims 1-2 imply that (u_1,v_1) is the unique critical point of $\mathcal{J}_\epsilon$ in $\mathcal{N}_\epsilon^+$. ☐

The following global compactness lemma is a version of Peng et al[9, Theorem 1.7].

Lemma 17. Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $\{(u_n,v_n)\}$ be a Palais-Smale sequence for $\mathcal{J}_\epsilon$ at level c, i.e. $\mathcal{J}_\epsilon(u_n,v_n) \to c$ and $\mathcal{J}_\epsilon'(u_n,v_n) \to 0$ in H^{-1} as $n \to +\infty$. Then there exists a solution (u_0,v_0) of (1), l sequences of positive numbers $\{r_n^j\}_n$ $1 \leq j \leq l$ and l sequences of points $\{x_n^j\}_n$ $1 \leq j \leq l$ in $\bar{\Omega}$, such that up to a subsequence,

(i) $(u_n,v_n) = (u_0,v_0) + \sum_{j=1}^{l}(r_n^j)^{\frac{N-2}{2}}(U_j(r_n^j(x-x_n^j)), V_j(r_n^j(x-x_n^j))) + (\sigma_n^1,\sigma_n^2)$, in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, where $\|(\sigma_n^1,\sigma_n^2)\| \to 0$, $r_n^j \to \infty$ as $n \to \infty$ and (U_j,V_j) are nonzero critical points of

$$\mathcal{J}_\infty(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta dx;$$

(ii) $\mathcal{J}_\epsilon(u_n,v_n) = \mathcal{J}_\epsilon(u_0,v_0) + \sum_{j=1}^{l}\mathcal{J}_\infty(U_j,V_j) + o_n(1)$,
where $o_n(1) \to 0$ as $n \to \infty$.

Lemma 18. Let $\{(u_n,v_n)\} \subset \mathcal{N}_\epsilon^-$ be a $(PS)_c$ sequence for $\mathcal{J}_\epsilon$ with

$$c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} < c < c_\epsilon^-(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.$$

Then there exists s subsequence still denoted by $\{(u_n,v_n)\}$ and a nonzero $(u_0,v_0) \in \mathcal{N}_\epsilon^-$ such that $(u_n,v_n) \to (u_0,v_0)$ strongly in E and $\mathcal{J}_\epsilon(u_0,v_0) = c$.

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence. Then by standard arguments, (u_n,v_n) is bounded in E, and there exists a subsequence still denoted by $\{(u_n,v_n)\}$ and (u_0,v_0) such that $(u_n,v_n) \rightharpoonup (u_0,v_0)$ in E. In a fashion similar to the arguments in Proposition 1, we get that $\mathcal{J}_\epsilon'(u_0,v_0) = 0$ and $(u_0,v_0) \in \mathcal{N}_\epsilon$. It is clear that $(0,0)$ is not a solution of (1). So, we infer from Lemma 16 that either $(u_0,v_0) \in \mathcal{N}_\epsilon^-$ or $(u_0,v_0) = (u_1,v_1)$. By Lemma 3 and Lemma 9, we find that $c > c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} > c_\epsilon^-(\Omega) > 0$. It follows from Lemma 17 that

$$c_\epsilon(\Omega) + \frac{l}{N}S_{\alpha,\beta}^{\frac{N}{2}} \leq c = \mathcal{J}_\epsilon(u_0,v_0) + \sum_{j=1}^{l}\mathcal{J}_\infty(U_j,V_j) < c_\epsilon^-(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.$$

So, we have that $l \leq 1$. If $l = 0$, then we are done. If $l = 1$ and $(u_0,v_0) = (u_1,v_1)$, then

$$c = \mathcal{J}_\epsilon(u_1,v_1) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} = c_\epsilon(\Omega) + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}},$$

a contradiction. If $l = 1$ and $(u_0, v_0) \in \mathcal{N}_\epsilon^-$, then

$$c = \mathcal{J}_\epsilon(u_0, v_0) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} \geq c_\epsilon^-(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}},$$

again a contradiction. Hence $l = 0$ and the result follows from Lemma 17. □

Let

$$(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) = \frac{(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})}{(\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} + |v_\delta^{\sigma,\rho}|^{2^*} + |u_\delta^{\sigma,\rho}|^\alpha |v_\delta^{\sigma,\rho}|^\beta dx)^{\frac{1}{2^*}}}, \tag{31}$$

where $(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})$ is defined in (21).

Lemma 19. There hold:
(i) $A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) \to S_{\alpha,\beta}$ as $\delta \to 0$ uniformly in $\sigma \in \mathbb{S}^{N-1}$;
(ii) There exists a $\rho_0 > 0$ such that $\sup\limits_{\sigma \in \mathbb{S}^{N-1}, \delta \in (0,1)} A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) < 2^{\frac{2}{N}} S_{\alpha,\beta}$ for $0 < \rho < \rho_0$.

Proof. (i) It is clear that $A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) = \frac{A(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})}{B(u_\delta^{\sigma,\rho}, v_\delta^{\sigma,\rho})^{\frac{2}{2^*}}}$. The result follows from Lemma 1 and Lemma 7.
(ii) In a fashion similar to the arguments of Goel and Sreenadh[21, Lemma 4.2], we find that

$$\lim_{\rho \to 0} \sup_{\sigma \in \mathbb{S}^{N-1}, \delta \in (0,1)} \int_{\mathbb{R}^N} |\nabla u_\delta^{\sigma,\rho}|^2 - |\nabla U_\delta^\sigma|^2 dx = 0, \quad \lim_{\rho \to 0} \sup_{\sigma \in \mathbb{S}^{N-1}, \delta \in (0,1)} \int_{\mathbb{R}^N} |u_\delta^{\sigma,\rho}|^{2^*} - |U_\delta^\sigma|^{2^*} dx = 0.$$

Since $\int_{\mathbb{R}^N} |\nabla U_\delta^\sigma|^2 dx = \int_{\mathbb{R}^N} |U_\delta^\sigma|^{2^*} dx = S^{\frac{N}{2}}$, we have that

$$\sup_{\sigma \in \mathbb{S}^{N-1}, \delta \in (0,1)} A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) \to F(\tau_{min}) S = S_{\alpha,\beta} \quad as \quad \rho \to 0,$$

where $F(\tau_{min})$ is given by Lemma 1. So, there exists a $\rho_0 > 0$ such that $\sup\limits_{\sigma \in \mathbb{S}^{N-1}, \delta \in (0,1)} A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) < 2^{\frac{2}{N}} S_{\alpha,\beta}$ for $0 < \rho < \rho_0$. □

Let

$$M = \{(u, v) \in E : \int_\Omega |u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta dx = 1\}. \tag{32}$$

Now, for any $(u, v) \in E$, we denote the function $H : M \to \mathbb{R}^N$ by

$$H(u, v) = \int_\Omega x(|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx$$

and also let

$$M_0 = \{(u, v) \in M : H(u, v) = 0\}. \tag{33}$$

Proposition 3. There hold:
(a) $\lim\limits_{\delta \to 0} H(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) = \sigma$, where $(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho})$ is defined by (31);
(b) Let $m_0 = \inf\limits_{(u,v) \in M_0} A(u, v)$, then $S_{\alpha,\beta} < m_0$;
(c) There exists a $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$ and $|\sigma| = 1$, we have $S_{\alpha,\beta} < A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) < \frac{m_0 + S_{\alpha,\beta}}{2}$.

Proof. (a) We have that

$$\begin{aligned}
&\int_\Omega (x - \sigma)(|u_\delta^{\sigma,\rho}|^{2^*} + |v_\delta^{\sigma,\rho}|^{2^*} + |u_\delta^{\sigma,\rho}|^\alpha |v_\delta^{\sigma,\rho}|^\beta) dx \\
&= (1 + \tau_{min}^\beta + \tau_{min}^{2^*}) \int_\Omega (x - \sigma) |u_\delta^{\sigma,\rho}|^{2^*} dx \\
&\leq (1 + \tau_{min}^\beta + \tau_{min}^{2^*}) [| \int_{\mathbb{R}^N} (x - \sigma) |U_\delta^\sigma|^{2^*} dx| + | \int_{\mathbb{R}^N} (x - \sigma)(\varphi_\rho^{2^*}(x) - 1) |U_\delta^\sigma|^{2^*} dx|],
\end{aligned}$$

where τ_{min} is defined by Lemma 1. It follows from the argument of He and Yang[22, Lemma 3.4] that

$$| \int_{\mathbb{R}^N} (x - \sigma) |U_\delta^\sigma|^{2^*} dx| + | \int_{\mathbb{R}^N} (x - \sigma)(\varphi_\rho^{2^*}(x) - 1) |U_\delta^\sigma|^{2^*} dx| \to 0 \ \ as \ \ \delta \to 0.$$

Hence,

$$H(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) - \sigma = \frac{\int_\Omega (x - \sigma)(|u_\delta^{\sigma,\rho}|^{2^*} + |v_\delta^{\sigma,\rho}|^{2^*} + |u_\delta^{\sigma,\rho}|^\alpha |v_\delta^{\sigma,\rho}|^\beta) dx}{\int_\Omega |u_\delta^{\sigma,\rho}|^{2^*} + |v_\delta^{\sigma,\rho}|^{2^*} + |u_\delta^{\sigma,\rho}|^\alpha |v_\delta^{\sigma,\rho}|^\beta dx} \to 0.$$

(b) In a fashion similar to the proof of He and Yang[22, Lemma 3.3], we obtain (b).
(c) Apparently $S_{\alpha,\beta} \leq A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho})$. Note that $S_{\alpha,\beta}$ is not attained in a domain $\Omega \neq \mathbb{R}^N$. So, $S_{\alpha,\beta} < A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho})$. Moreover, by Lemma 19$(i)$ and Part (b), we get the desired result. □

Let $r_0 = 1 - \delta_0$, where δ_0 is given by Proposition 3 (c). For $\sigma \in \mathbb{S}^{N-1}$ and $|(1-\delta)\sigma| \geq r_0$, i.e. $0 < \delta \leq \delta_0$. We denote

$$\bar{B}_{r_0} = \{(1-\delta)\sigma \in \mathbb{R}^N \,:\, |(1-\delta)\sigma| \leq r_0, \sigma \in \mathbb{S}^{N-1}, 0 < \delta < 1\}.$$

and a subset Σ of E by

$$\Sigma = \{(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) \,:\, (1-\delta)\sigma \in \bar{B}_{r_0}\}.$$

Let

$$\mathcal{H} = \{h \in C(M,M) \,:\, h(u,v) = (u,v) \; for \; (u,v) \; such \; that \; A(u,v) < \frac{m_0 + S_{\alpha,\beta}}{2}\}$$

and $\Lambda = \{T \subset M \,:\, T = h(\Sigma), h \in \mathcal{H}\}$. Since $id \in \mathcal{H}$, $\mathcal{H}$ is nonempty. Then we introduce the following result of He and Yang[22, Lemma 3.6].

Lemma 20. If $T \in \Lambda$, then $T \cap M_0 \neq \emptyset$, where M_0 is defined in (33).

It follows from Lemma 3 that $c_\epsilon^-(\Omega) > 0$ for $0 < \epsilon < \epsilon_1$. Now we define $\check{\mathcal{J}}_\epsilon \,:\, E \to \mathbb{R}^N$ by

$$\check{\mathcal{J}}_\epsilon(u,v) := \max_{t>0} \mathcal{J}_\epsilon(tu,tv) = \mathcal{J}_\epsilon(t^-_{(u,v)}(u,v)), \tag{34}$$

where $t^-_{(u,v)}$ is given by Lemma 2. For each $(u,v) \in E$, $\langle \mathcal{J}_\epsilon'(t^-_{(u,v)}(u,v)), (u,v)\rangle = 0$ and $\frac{d^2}{dt^2}|_{t=t^-_{(u,v)}} \mathcal{J}_\epsilon(t(u,v)) < 0$. By implicit function theorem, we get that $t^-_{(u,v)} \in C^1(E,(0,\infty))$. As a result, $\check{\mathcal{J}}_\epsilon(u,v) = \mathcal{J}_\epsilon(t^-_{(u,v)}(u,v)) \in C^1(E,\mathbb{R})$. Then we follow the idea of Szulkin and Weth[16, Corollary 2.10] to get the following lemma.

Lemma 21. The following holds:
(a) If $\{(u_n,v_n)\} \subset E\backslash\{(0,0)\}$ is a $(PS)_c$ sequence of $\check{\mathcal{J}}_\epsilon$, then $\{t^-_{(u_n,v_n)}(u_n,v_n)\} \subset \mathcal{N}_\epsilon^-$ is a $(PS)_c$ sequence of $\mathcal{J}_\epsilon$;
(b) If $(u,v) \in E\backslash\{(0,0)\}$ is a critical point of $\check{\mathcal{J}}_\epsilon$, then $t^-_{(u,v)}(u,v) \in \mathcal{N}_\epsilon^-$ is a critical point of $\mathcal{J}_\epsilon$.

Proof. (a) For each $(u,v) \in E\backslash\{(0,0)\}$, we have that $\check{\mathcal{J}}_\epsilon(u,v) = \mathcal{J}_\epsilon(t^-_{(u,v)}(u,v)) \in C^1(E,\mathbb{R})$. Let $\hat{m} \,:\, E\backslash\{(0,0)\} \to \mathcal{N}$ be a map given by $\hat{m}(u,v) := t^-_{(u,v)}(u,v)$. Next we check that $\hat{m}$ is a continuous map. Let $(u_n,v_n) \to (u',v')$ and $t_n = t^-_{(u_n,v_n)}$, then $\hat{m}(u_n,v_n) = t_n(u_n,v_n)$. If $t_n \to +\infty$, then

$$\circ(1) = \frac{\mathcal{J}_\epsilon(u_n,v_n)}{t_n^2} \leq \frac{\mathcal{J}_\epsilon(\hat{m}(u_n,v_n))}{t_n^2} \to -\infty,$$

a contradiction. We may assume that $t_n \to t_0 \geq 0$. Hence,

$$\mathcal{J}_\epsilon(t^-_{(u',v')}(u',v')) \geq \mathcal{J}_\epsilon(t_0(u',v')) = \lim_{n\to\infty} \mathcal{J}_\epsilon(t_n(u_n,v_n)) \geq \lim_{n\to\infty} \mathcal{J}_\epsilon(t^-_{(u',v')}(u_n,v_n)) = \mathcal{J}_\epsilon(t^-_{(u',v')}(u',v')).$$

As a result, $t_0 = t^-_{(u',v')} > 0$, which implies that $\hat{m}$ is continuous. Thus $m = \hat{m}|_{S^1}$, where $S^1 := \{(u,v) \in E \,:\, \|(u,v)\| = 1\}$ is the unit sphere in E, is a homeomorphism. The inverse function is given by $m^{-1}(u,v) = \frac{(u,v)}{\|(u,v)\|}$ for $(u,v) \in \mathcal{N}_\epsilon^-$. Applying the result of Szulkin and Weth[16, Proposition 2.9], we get that $\mathcal{J}_\epsilon \circ \hat{m} \in C^1(E\backslash\{(0,0)\},\mathbb{R})$ and

$$\langle (\mathcal{J}_\epsilon \circ \hat{m})'(u,v), (\varphi,\psi)\rangle = \frac{\|\hat{m}(u,v)\|}{\|(u,v)\|} \langle \mathcal{J}_\epsilon'(\hat{m}(u,v)), (\varphi,\psi)\rangle \tag{35}$$

for $(u,v), (\varphi,\psi) \in E$ and $(u,v) \neq (0,0)$. Moreover, $\mathcal{J}_\epsilon \circ m \,:\, S^1 \to \mathbb{R}$ is of C^1-class[16, Corollary 2.10]. Let $(\bar{u}_n,\bar{v}_n) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|} \in S^1$. Then

$$m(\bar{u}_n,\bar{v}_n) = t^-_{(u_n,v_n)}\|(u_n,v_n)\|(\bar{u}_n,\bar{v}_n) \in \mathcal{N}_\epsilon^-.$$

Since $\{(u_n,v_n)\}$ is a $(PS)_c$ sequence of $\check{\mathcal{J}}_\epsilon$, we have that

$$\check{\mathcal{J}}_\epsilon(u_n,v_n) = \mathcal{J}_\epsilon(t^-_{(u_n,v_n)}(u_n,v_n)) = (\mathcal{J}_\epsilon \circ m)(\bar{u}_n,\bar{v}_n) \to c$$

and $(\mathcal{J}_\epsilon \circ m)'(\bar{u}_n,\bar{v}_n) \to 0$. It is clear that $m(\bar{u}_n,\bar{v}_n) = t^-_{(u_n,v_n)}(u_n,v_n) \in \mathcal{N}_\epsilon^-$ and

$$\|t^-_{(u_n,v_n)}(u_n,v_n)\|\langle \mathcal{J}_\epsilon'(t^-_{(u_n,v_n)}(u_n,v_n)), (\bar{u}_n,\bar{v}_n)\rangle = \langle \mathcal{J}_\epsilon'(t^-_{(u_n,v_n)}(u_n,v_n)), t^-_{(u_n,v_n)}(u_n,v_n)\rangle = 0.$$

It follows from (35) that

$$\langle (\mathcal{J}_\epsilon \circ m)'(\bar{u}_n,\bar{v}_n),(\varphi,\psi)\rangle = \|t^-_{(u_n,v_n)}(u_n,v_n)\|\langle \mathcal{J}'_\epsilon(t^-_{(u_n,v_n)}(u_n,v_n)),(\varphi,\psi)\rangle$$
$$= \|t^-_{(u_n,v_n)}(u_n,v_n)\|\langle \mathcal{J}'_\epsilon(t^-_{(u_n,v_n)}(u_n,v_n)),((\varphi,\psi)+t(\bar{u}_n,\bar{v}_n))\rangle$$

for $(\varphi,\psi)\in T_{(\bar{u}_n,\bar{v}_n)}S^1$ and $t\in\mathbb{R}$, where $T_{(\bar{u}_n,\bar{v}_n)}S^1$ stands for the tangent space S^1 at $(\bar{u}_n,\bar{v}_n)$. Hence

$$0 \leftarrow \|(\mathcal{J}_\epsilon\circ m)'(\bar{u}_n,\bar{v}_n)\| = \sup_{(\varphi,\psi)\in T_{(\bar{u}_n,\bar{v}_n)}S^1,\|(\varphi,\psi)\|=1} \langle (\mathcal{J}_\epsilon\circ m)'((\bar{u}_n,\bar{v}_n)),(\varphi,\psi)\rangle = \|t^-_{(u_n,v_n)}(u_n,v_n)\|\|\mathcal{J}'_\epsilon(t^-_{(u_n,v_n)}(u_n,v_n))\|.$$

Since $t^-_{(u_n,v_n)}(u_n,v_n)\in\mathcal{N}^-_\epsilon$, we have that $\|t^-_{(u_n,v_n)}(u_n,v_n)\|\geq\beta>0$. We infer from the equation above that $\mathcal{J}'_\epsilon(t^-_{(u_n,v_n)}(u_n,v_n))\to 0$. Hence $\{t^-_{(u_n,v_n)}(u_n,v_n)\}$ is a $(PS)_c$ sequence of $\mathcal{J}_\epsilon$.
(b) The proof is similar as it in part (a). □

We introduce the following minimax lemma of Brezis and Nirenberg [15, Theorem 1].

Lemma 22. Let E be a Banach space and $\check{\mathcal{J}}_\epsilon\in C^1(E,\mathbb{R})$. Let K be a compact metric space, $K_0\subset K$ be a closed set and $y\in C(K_0,E)$. Define

$$\Gamma=\{g\in C(K,E):\ g(s)=y(s)\ if\ s\in K_0\},\quad c^*_\epsilon=\inf_{g\in\Gamma}\sup_{s\in K}\check{\mathcal{J}}_\epsilon(g(s)),\ c^{**}_\epsilon=\sup_{y(K_0)}\check{\mathcal{J}}_\epsilon.$$

If $c^*_\epsilon>c^{**}_\epsilon$ then there exists a sequence $\{\omega_n\}\subset E$ satisfying $\check{\mathcal{J}}_\epsilon(\omega_n)\to c^*_\epsilon$ and $\check{\mathcal{J}}'_\epsilon(\omega_n)\to 0$. Further, if $\check{\mathcal{J}}_\epsilon$ satisfies $(PS)_{c^*_\epsilon}$ condition then there exists a $\omega_0\in E$ such that $\check{\mathcal{J}}_\epsilon(\omega_0)=c^*_\epsilon$ and $\check{\mathcal{J}}'_\epsilon(\omega_0)=0$.

It follows from (32) that $M\subset E$. We set that

$$F=\{q\in C(\bar{B}_{r_0},M):\ q_{|\partial B_{r_0}}=(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\}.$$

and

$$\bar{c}=\inf_{q\in F}\sup_{(1-\delta)\sigma\in\bar{B}_{r_0}}\|q((1-\delta)\sigma)\|^2,\quad \hat{c}=\sup_{\partial B_{r_0}}\|(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2. \tag{36}$$

Obviously, $q((1-\delta)\sigma)=(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\in F$ for $(1-\delta)\sigma\in\bar{B}_{r_0}$. It follows from Lemma 19 (ii) that, for $0<\rho<\rho_0$, there holds

$$\bar{c}\leq\sup_{(1-\delta)\sigma\in\bar{B}_{r_0}}\|(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2\leq\sup_{\sigma\in\mathbb{S}^{N-1},\delta\in(0,1)}\|(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2<2^{\frac{2}{N}}S_{\alpha,\beta}. \tag{37}$$

By Lemma 20, we have that, for $h\in\mathcal{H}$, there exists $(1-\bar{\delta})\bar{\sigma}\in\bar{B}_{r_0}$ such that $h(\bar{u}^{\bar{\sigma},\rho}_{\bar{\delta}},\bar{v}^{\bar{\sigma},\rho}_{\bar{\delta}})\in M_0$. So

$$m_0\leq\|h(\bar{u}^{\bar{\sigma},\rho}_{\bar{\delta}},\bar{v}^{\bar{\sigma},\rho}_{\bar{\delta}})\|^2\leq\sup_{(1-\delta)\sigma\in\bar{B}_{r_0}}\|h(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2,$$

where m_0 is given by Proposition 3 (b). Moreover, for $h\in\mathcal{H}$, $h(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)=(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\in F$. Hence,

$$m_0\leq\inf_{h(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\in F}\sup_{(1-\delta)\sigma\in\bar{B}_{r_0}}\|h(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2=\bar{c}.$$

This inequality together with Proposition 3 (b) and (37) yield that

$$S_{\alpha,\beta}<m_0\leq\bar{c}<2^{\frac{2}{N}}S_{\alpha,\beta}\ \ for\ \ 0<\rho<\rho_0. \tag{38}$$

Moreover, from Proposition 3 (c), we find that

$$\hat{c}=\sup_{\partial B_{r_0}}\|(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)\|^2<\frac{m_0+S_{\alpha,\beta}}{2}<m_0\leq\bar{c}.$$

Let

$$c^*_\epsilon=\inf_{q\in F}\sup_{(1-\delta)\sigma\in\bar{B}_{r_0}}\check{\mathcal{J}}_\epsilon(q((1-\delta)\sigma)). \tag{39}$$

Lemma 23. There hold:
(i) $\check{\mathcal{J}}_\epsilon(\bar{u}^{\sigma,\rho}_\delta,\bar{v}^{\sigma,\rho}_\delta)=\frac{1}{N}S^{\frac{N}{2}}_{\alpha,\beta}+o(1)$ as $\delta\to0$;
(ii) For $0<\rho<\rho_0$ (given by Lemma 19), there exists a $\epsilon_4>0$ such that if $0<\epsilon<\epsilon_4$, then

$$c_\epsilon(\Omega)+\frac{1}{N}S^{\frac{N}{2}}_{\alpha,\beta}<c^*_\epsilon<c^-_\epsilon(\Omega)+\frac{1}{N}S^{\frac{N}{2}}_{\alpha,\beta}.$$

Proof. (i) It follows from Lemma 7 (iii) that $(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) \rightharpoonup (0,0)$ weakly in E uniformly in σ as $\delta \to 0$. Let $\varphi_\epsilon(t) = \mathcal{J}_\epsilon(t(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}))$. Solving

$$\varphi_\epsilon'(t) = A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho})t - t^{2^*-1} - \epsilon \int_\Omega f\bar{u}_\delta^{\sigma,\rho} + g\bar{v}_\delta^{\sigma,\rho} dx = 0,$$

We conclude $t_\delta = (A(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}))^{\frac{1}{2^*-2}} + o(1)$ as $\delta \to 0$. Combining Lemma 19 (i) and (39), we get that

$$\check{\mathcal{J}}_\epsilon(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) = \max_{t>0} \mathcal{J}_\epsilon(t(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho})) = \max_{t>0} \varphi_\epsilon(t) = \varphi_\epsilon(t_\delta) = \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + o(1) \ \ as\ \delta \to 0.$$

(ii) It follows from (34) that

$$\check{\mathcal{J}}_0(u,v) = \max_{t>0} \mathcal{J}_0(tu,tv) = \frac{1}{N} \frac{A(u,v)^{\frac{N}{2}}}{B(u,v)^{\frac{N-2}{2}}}.$$

For each $q \in F$ and $(1-\delta)\sigma \in \bar{B}_{r_0}$, we have that

$$\check{\mathcal{J}}_0(q((1-\delta)\sigma)) = \frac{1}{N} \|q((1-\delta)\sigma)\|^N.$$

This equality together with (36) and (39) yield that

$$c_0^* = \frac{1}{N} \inf_{q\in F} \sup_{(1-\delta)\sigma\in\bar{B}_{r_0}} \|q((1-\delta)\sigma)\|^N = \frac{1}{N} \bar{c}^{\frac{N}{2}}. \tag{40}$$

It follows from (38) that

$$\tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < c_0^* < \tfrac{2}{N} S_{\alpha,\beta}^{\frac{N}{2}} \ \ for\ 0 < \rho < \rho_0. \tag{41}$$

In view of Lemma 12 and (34) , for a fix $0 < \kappa < 1$, there holds

$$\begin{aligned}&(1-\epsilon\kappa)^{\frac{N}{2}} \check{\mathcal{J}}_0(q((1-\delta)\sigma)) - \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \le \check{\mathcal{J}}_\epsilon(q((1-\delta)\sigma))\\ &\le (1+\epsilon\kappa)^{\frac{N}{2}} \check{\mathcal{J}}_0(q((1-\delta)\sigma)) + \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).\end{aligned}$$

and

$$(1-\epsilon\kappa)^{\frac{N}{2}} c_0^* - \frac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \le c_\epsilon^* \le (1+\epsilon\kappa)^{\frac{N}{2}} c_0^* + \frac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).$$

So, for any $\eta > 0$, there exists a $\epsilon_1(\eta) > 0$ such that if $0 < \epsilon < \epsilon_1(\eta)$ then

$$c_0^* - \eta < c_\epsilon^* < c_0^* + \eta. \tag{42}$$

Moreover, from Lemma 12, we have that

$$(1-\epsilon\kappa)^{\frac{N}{2}} \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} - \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2) \le c_\epsilon^-(\Omega) \le (1+\epsilon\kappa)^{\frac{N}{2}} \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \tfrac{\epsilon}{2\kappa}(\|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2).$$

So, for any $\eta > 0$ there exists a $\epsilon_2(\eta) > 0$ such that if $0 < \epsilon < \epsilon_2(\eta)$ then $\frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} - \eta < c_\epsilon^-(\Omega) < \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \eta$. Equivalently,

$$\tfrac{2}{N} S_{\alpha,\beta}^{\frac{N}{2}} - \eta < c_\epsilon^-(\Omega) + \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < \tfrac{2}{N} S_{\alpha,\beta}^{\frac{N}{2}} + \eta. \tag{43}$$

In view of (41), for a fix $0 < \eta \le \min\{\frac{\frac{2}{N} S_{\alpha,\beta}^{\frac{N}{2}} - c_0^*}{2}, c_0^* - \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}\}$, if $0 < \epsilon < \epsilon_4 = \min\{\epsilon_1(\eta), \epsilon_2(\eta)\}$, then applying (41), (42) and (43), we get

$$c_\epsilon(\Omega) + \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} \le c_0^* - \eta < c_\epsilon^* < c_0^* + 2\eta - \eta \le \tfrac{2}{N} S_{\alpha,\beta}^{\frac{N}{2}} - \eta < c_\epsilon^-(\Omega) + \tfrac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}. \tag{44}$$

So, the result follows. □

Proposition 4. If $0 < \rho < \rho_0$ and $0 < \epsilon < \epsilon'' = \min\{\epsilon', \epsilon_3, \epsilon_4\}$, where ρ_0, ϵ', ϵ_3 and ϵ_4 are given by Lemma 19, Proposition 2, Lemma 16 and Lemma 23 respectively, then there exists a solution $(u_4, v_4) \in \mathcal{N}_\epsilon^-$ of (1) with

$$c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < \mathcal{J}_\epsilon(u_4, v_4) < c_\epsilon^-(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

Proof. It follows from (44) that $c_\epsilon^* > \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. From Lemma 23 (i), we have that $c_\epsilon^* > \check{\mathcal{J}}_\epsilon(\bar{u}_\delta^{\sigma,\rho}, \bar{v}_\delta^{\sigma,\rho}) = \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + o(1)$ whenever δ is sufficiently small. So, applying Lemma 22 and (39), we get that there exists a sequence $\{(u_n, v_n)\} \subset E$ such that $\check{\mathcal{J}}_\epsilon(u_n, v_n) \to c_\epsilon^*$ and $\check{\mathcal{J}}_\epsilon'(u_n, v_n) \to 0$. Then by Lemma 21, we find that $\{t^-_{(u_n,v_n)}(u_n, v_n)\} \subset \mathcal{N}_\epsilon^-$ is a $(PS)_{c_\epsilon^*}$ of $\mathcal{J}_\epsilon$, which on using Lemma 18 gives that

$$t^-_{(u_n,v_n)}(u_n, v_n) \to t^-_{(u_0,v_0)}(u_0, v_0) := (u_4, v_4) \in \mathcal{N}_\epsilon^- \ \ strongly\ in\ E$$

and $\mathcal{J}_\epsilon(u_4, v_4) = c_\epsilon^*$. So, the result follows from Lemma 23 (ii). $\square$

proof of Theorem 1 : By Proposition 1, we find the first solution $(u_1, v_1) \in \mathcal{N}_\epsilon^+$ with $u_1, v_1 > 0$ and $\mathcal{J}_\epsilon(u_1, v_1) = c_\epsilon(\Omega)$ whenever $0 < \epsilon < \epsilon_1$. Let $0 < \epsilon < \epsilon'$, then by Proposition 2, we get two solutions $(u_2, v_2), (u_3, v_3) \in \mathcal{N}_\epsilon^-$ of (1) with $\mathcal{J}_\epsilon(u_2, v_2), \mathcal{J}_\epsilon(u_3, v_3) \le c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. Finally, if $0 < \rho < \rho_0$ and $0 < \epsilon < \epsilon''$, as a result of Proposition 4, we have that $(u_4, v_4) \in \mathcal{N}_\epsilon^-$ is a solution of (1) with $c_\epsilon(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} < \mathcal{J}_\epsilon(u_4, v_4) < c_\epsilon^-(\Omega) + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$.

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