# Some results about multifactor uncertain differential equations with applications to extreme values and time integral

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**Abstract**

Previous literature has proved that there exists a unique solution about multifactor uncertain differential equation (MUDE for short) when their coefficients satisfy strict global Lipschitz continuous condition. In this paper, firstly, we consider new existence and uniqueness theorem under the weaker local Lipschitz continuous condition. The next, when the coefficients do not satisfy the Lipschitz condition, we just showcase existence theorem under continuous and linear growth conditions. Once more, we establish the inverse uncertainty distributions (IUDs for short) of supremum, infimum and time integral about the uncertain process $Z_k$, meanwhile, we design some numerical algorithms for solving these IUDs. In the end, some numerical examples are presented to verify the effectiveness of algorithms.

*Keywords:* MUDEs, existence and uniqueness, the local Lipschitz condition, extreme values and time integral

## 1. Introduction

Except for random and fuzzy phenomenon, human uncertainty associated with belief degrees is another antagonistic type of indeterminate phenomenon. For describing human uncertainty, Liu [1] formed uncertainty theory that is a new axiomatic system in 2010, and updated the 4th edition in 2015 [2], which cause great repercussions in mathematics. Until now, a number of scholars have considered the applications of uncertainty theory in various fields (for example, Liu [2, 3]; Gao et al. [4]; Wang et al. [5]; Li et al. [6]).

In 2008, Liu [7] first proposed uncertain differential equation (UDE for short) which is driven by canonical Liu process. Later in 2009, Liu [8] claimed Liu process is an uncertain process with stationary and independent normal uncertain increments. Chen and Liu [9] considered the analytic solution of a linear UDE in 2010. After that, Gao [10] proved the existence and uniqueness theorem of UDE. Furthermore, the special nonlinear UDEs were solved in 2013(for example, Liu et al. [11]; Yao et al. [12]; Liu et al. [13]; Wang et al. [14]). More importantly, Yao and Chen [15, 16] designed an Euler numerical method to obtain the IUDs of solution of the UDEs. After that, various improved numerical algorithms were designed to obtain the IUDs, (for example, Liu et al. [17]; Yao et al. [18]; Liu et al. [19]; Wang et al. [20]; Wang et al. [21]). Based on above theoretical development, UDEs have been successfully applied in many fields (for example, Yang et al. [22, 23]; Gao et al. [24, 25, 26, 27]; Jia et al. [28]).

However, in practice, considering the case that there usually exist multiple canonical Liu processes influencing dynamical systems, it is natural that the UDEs can be extended to MUDEs. Then the existence and uniqueness theorem and $\alpha$-path of MUDE were discussed by Li et al. [29]. After that, Zhang et al. [30] investigated stability in mean and stability in measure for the solutions of MUDE. Sheng et al. [31] presented a sufficient condition for a MUDE being almost surely stable. Following that, Ma et al. obtained a sufficient condition for a MUDE being stable in distribution [32] and in p-th moment [33]. Sun et al. [34] investigated the optimal control problem for the multifactor uncertain system and applied MUDE to differential portfolio game model. Roughly speaking, our paper has the following innovations. First, the local Lipschitz continuous condition always exist while strict global

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Lipschitz continuous condition only exist in relatively few situations. In this sense, we can say the local Lipschitz continuous condition are more available than global Lipschitz condition. This is of particular vital in this paper since global Lipschitz conditions are subject to more restrictions. Second, when the coefficients do not satisfy the Lipschitz condition, we prove that MUDE has at least one solution under some suitable conditions. Third, we give the IUDs of extreme values and time integral, which become more significant to future applications of MUDE. Last but not least, we design some numerical algorithms for solving these IUDs.

The rest of this paper is arranged as follows. Section 2 reviews some basic concepts and $\alpha$-path of MUDE. Section 3 studies new existence and uniqueness theorem of MUDE. Section 4 proves existence of MUDE under continuous and linear growth conditions. Section 5 investigates extreme values and time integral of MUDE. Section 6 gives conclusion and prospect.

## 2. Preliminaries

In this section, we review some preliminaries including canonical Liu process, and MUDEs, more details see the book [2].

**Definition 2.1.** *[8] An uncertain process $C_k$ is a canonical Liu process with respect with time $k$ if it satisfies the following (a),(b),(c)*
*(a) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,*
*(b) $C_k$ is independent increments and stationary,*
*(c) every increment $C_{r+k} - C_r$ is a normal uncertain variable with expected value $0$ and variance $k^2$, its uncertainty distribution is*

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}k}\right)\right)^{-1}, \quad x \in \Re.$$

**Theorem 2.1.** *[9] Suppose that $C_k$ is a canonical process, and $Z_k$ is an integrable uncertain process with respect to time $k$ on $[a, b]$. Then the inequality*

$$|\int_a^b Z_k(\gamma) dC_k| \le K(\gamma) \int_a^b |Z_k(\gamma)| dk$$

*holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $Z_k(\gamma)$.*

**Definition 2.2.** *[29] Suppose $C_{1k}, C_{2k}, \cdots, C_{mk}$ are independent canonical Liu processes, and*

$$h, p_1, p_2, \cdots, p_m$$

*are some given functions. Then*

$$dZ_k = h(k, Z_k)dk + \sum_{i=1}^{m} p_i(k, Z_k) dC_{ik}, \tag{1}$$

*is called a MUDE with respect to $C_{1k}, C_{2k}, \cdots, C_{mk}$. A solution is an uncertain process $Z_k$ that satisfies (1) identically in time $k$.*

**Definition 2.3.** *[29] Let $C_{1k}, C_{2k}, \cdots, C_{mk}$ be independent canonical processes. Let $\alpha$ be a number with $(0 < \alpha < 1)$. The $\alpha$-path of the MUDE*

$$dZ_k = h(k, Z_k)dk + \sum_{i=1}^{m} p_i(k, Z_k) dC_{ik},$$

*with initial value $Z_0$ is a deterministic function $Z_k^\alpha$ with respect to time $k$ which solves the following equation*

$$dZ_k^\alpha = h(k, Z_k^\alpha)dk + \sum_{i=1}^{m} |p_i(k, Z_k^\alpha)| \Phi^{-1}(\alpha) dk,$$

*where $\Phi^{-1}(\alpha)$ is the IUD of standard normal uncertain variable, i.e.,*

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1.$$

**Theorem 2.2.** *[29] Assume that $h, p_1, p_2, \cdots, p_m$ are continuous functions of two variables and $C_{1k}, C_{2k}, \cdots, C_{mk}$ are independent canonical processes. Let $Z_k$ and $Z_k^\alpha$ be the solution and $\alpha$-path of the MUDE*

$$dZ_k = h(k, Z_k)dk + \sum_{i=1}^{m} p_i(k, Z_k)dC_{ik},$$

*Then we get*

$$\mathcal{M}\{Z_k \le Z_k^\alpha, \forall k\} = \alpha,$$

$$\mathcal{M}\{Z_k > Z_k^\alpha, \forall k\} = 1 - \alpha.$$

**Theorem 2.3.** *[29] Assume that $h, p_1, p_2, \cdots, p_m$ are continuous functions of two variables and $C_{1k}, C_{2k}, \cdots, C_{mk}$ are independent canonical processes. Let $Z_k$ and $Z_k^\alpha$ be the solution and $\alpha$-path of the MUDE*

$$dZ_k = h(k, Z_k)dk + \sum_{i=1}^{m} p_i(k, Z_k)dC_{ik},$$

*Then $Z_k$ has an IUD*

$$\Psi_k^{-1}(\alpha) = Z_k^\alpha, \quad 0 < \alpha < 1.$$

## 3. Existence and uniqueness theorem with the local Lipschitz condition

### *3.1. Multifactor uncertain differential equations*

**Remark 3.1.** *According to the definition of uncertain canonical $C_k(\gamma)$, where $\gamma \in \Gamma$ defined in Definition 2.2 [9],almost all sample paths of $C_k$ are Lipschitz continuous functions. That is, there exists a set $\Gamma_0$ in $\Gamma$ with $\mathcal{M}\{\Gamma_0\} = 1$ such that for any $\gamma \in \Gamma_0$, $C_k(\gamma)$is Lipschitz continuous. To do this simply, we set $\Gamma_0 = \Gamma$. Thus, for each $\gamma$, by Lemma 4.1 in [9], there exists a positive number $K(\gamma)$ such that*

$$|C_r(\gamma) - C_k(\gamma)| \le K(\gamma)|r - k|, \forall r, k \ge 0.$$

*Besides, the uncertain integrals of each $C_{ik}$ is equivalent to Riemann-Stieltjes integral from the point of each sample path. Hence, we can just focus on the following multifactor uncertain integral equation*

$$Z_k(\gamma) = Z_0(\gamma) + \int_0^k h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_0^k p_i(r, Z_r(\gamma))dC_{ir}(\gamma). \tag{2}$$

*Our goal is to prove that, for each sample path $\gamma$, the MUDE (2) has a unique solution on $[0, +\infty)$ under certain reasonable conditions.*

*First of all, we discuss the existence and uniqueness for MUDE in a local interval $[k_0, k_0 + \alpha]$ for some positive $\alpha$, The equation (2) becomes*

$$Z_k(\gamma) = Z_{k_0}(\gamma) + \int_{k_0}^k h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_{k_0}^k p_i(r, Z_r(\gamma))dC_{ir}(\gamma) \tag{3}$$

*and the following Theorem 3.1 will give the result of existence and uniqueness of multifactor uncertain integral equation (3).*

*3.2. Existence and uniqueness of the solution*

**Theorem 3.1.** *Fixing $\gamma \in \Gamma$, the multifactor uncertain integral equation* (3) *has a unique solution in* $[k_0, k_0 + \alpha]$ *if the coefficients $h$ and $p_i$ are locally Lipschitz continuous of z, where $i = 1, \cdots, m$. In other words, for each*

$$D = \{(k, z) | k \in [k_0, k_0 + a], z \in [Z_{k_0}(\gamma) - b, Z_{k_0}(\gamma) + b]\},$$

*there exists a positive constant $L_D$ such that*

$$|h(k, z_1) - h(k, z_2)| + \sum_{i=1}^{m} |p_i(k, z_1) - p_i(k, z_2)| \leq L_D |z_1 - z_2|,$$

*where $a > 0, b > 0, (k, z_1) \in D, (k, z_2) \in D$, and*

$$Q = \max_{D} \left\{ |h(k, z)| + K(\gamma) \sum_{i=1}^{m} |p_i(k, z)| \right\} + 1,$$

*$K(\gamma)$ is the maximum Lipschitz constant to $C_{ik}(\gamma)$, and $\alpha = \min\{a, b/Q\}$.*

*Proof* By using successive approximations, we will prove this theorem in three steps.

$$\begin{cases} Z_k^{(0)}(\gamma) = Z_{k_0}(\gamma) \\ Z_k^{(n+1)}(\gamma) = Z_{k_0}(\gamma) + \displaystyle\int_{k_0}^{k} h(r, Z_r^{(n)}(\gamma)) dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, Z_r^{(n)}(\gamma)) dC_{ir}(\gamma). \end{cases} \tag{4}$$

It's easy to find that $\{Z_k^{(n)}(\gamma)\}$ is continuous in time $k$ for any $n \geq 0$.

**Step 1.**(existence) In this step, we will prove that

$$\left(k, Z_k^{(n)}(\gamma)\right) \in D, n \geq 0$$

when $k \in [k_0, k_0 + a]$.

Here, we use mathematical induction. When $n = 0$,

$$\begin{cases} k \in [k_0, k_0 + a] \\ Z_k^{(0)}(\gamma) = Z_{k_0}(\gamma) \in [Z_{k_0}(\gamma) - b, Z_{k_0}(\gamma) + b] \end{cases} \tag{5}$$

Thus the conclusion is obviously established. Assume that

$$\left(k, Z_k^{(n)}(\gamma)\right) \in D, n \geq 0$$

when $k \in [k_0, k_0 + a]$, we have

$$\begin{aligned} |Z_k^{(n+1)}(\gamma) - Z_{k_0}(\gamma)| &= \left| \int_{k_0}^{k} h(r, Z_r^{(n)}(\gamma)) dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, Z_r^{(n)}(\gamma)) dC_{ir}(\gamma) \right| \\ &\leq \left| \int_{k_0}^{k} h(r, Z_r^{(n)}(\gamma)) dr \right| + \sum_{i=1}^{m} \left| \int_{k_0}^{k} p_i(r, Z_r^{(n)}(\gamma)) dC_{ir}(\gamma) \right| \\ &\leq \int_{k_0}^{k} \left| h(r, Z_r^{(n)}(\gamma)) \right| + K(\gamma) \sum_{i=1}^{m} \left| p_i(r, Z_r^{(n)}(\gamma)) \right| dr \\ &\leq Q \cdot |k - k_0| \\ &\leq Q \cdot \alpha \leq b, \end{aligned}$$

This indicates that $(k, Z_k^{(n)}(\gamma)) \in D$ for $n = 0, 1, 2, \cdots$, when $k \in [k_0, k_0 + \alpha]$.

**Step 2.** In this step, we will prove that the sequence $\{Z_k^{(n)}(\gamma)\}_{n=0}^{+\infty}$ given by (4) converges uniformly to the solution of the equation (3) on $[k_0, k_0+\alpha]$ as $n \to \infty$.

First, we will prove

$$|Z_k^{(n+1)}(\gamma) - Z_k^{(n)}(\gamma)| \leq \frac{Q(L_D + mK(\gamma)L_D)^n}{(n+1)!}|k-k_0|^{n+1}.$$

Similar to **Step 1**, in this step, we also use mathematical induction. When $n=0$,

$$\begin{aligned}
&|Z_k^{(1)}(\gamma) - Z_k^{(0)}(\gamma)| \\
=&\left|\int_{k_0}^k h(r, Z_r^{(0)}(\gamma))dr + \sum_{i=1}^n \int_{k_0}^k p_i(r, Z_r^{(0)}(\gamma))dC_{ir}(\gamma)\right| \\
=&\left|\int_{k_0}^k h(r, Z_r^{(0)}(\gamma))dr\right| + \left|K(\gamma)\sum_{i=1}^n \int_{k_0}^k p_i(r, Z_r^{(0)}(\gamma))dr\right| \\
\leq& \int_{k_0}^k |h(r, Z_r^{(0)}(\gamma))| + K(\gamma)\sum_{i=1}^n |p_i(r, Z_r^{(0)}(\gamma))| \\
\leq& Q \cdot |k-k_0|,
\end{aligned}$$

Assume that

$$\begin{aligned}
&|Z_k^{(n)}(\gamma) - Z_k^{(n-1)}(\gamma)| \\
\leq&\frac{Q(L_D + mK(\gamma)L_D)^{n-1}}{n!}|k-k_0|^n,
\end{aligned}$$

when $k \in [k_0, k_0+\alpha]$, we have

$$\begin{aligned}
|Z_k^{(n+1)}(\gamma) - Z_k^{(n)}(\gamma)| =& \left|\int_{k_0}^k h(r, Z_r^{(n)}(\gamma))dr + \sum_{i=1}^n \int_{k_0}^k p_i(r, Z_r^{(n)}(\gamma))dC_{ir}(\gamma)\right. \\
&\left. - \int_{k_0}^k h(r, Z_r^{(n-1)}(\gamma))dr - \sum_{i=1}^m \int_{k_0}^k p_i(r, Z_r^{(n-1)}(\gamma))dC_{ir}(\gamma)\right| \\
\leq& \int_{k_0}^k \left|h(r, Z_r^{(n)}(\gamma))dr - \int_{k_0}^k h(r, Z_r^{(n-1)}(\gamma))\right| dr \\
&+ \sum_{i=1}^m \int_{k_0}^k \left|p_i(k, Z_r^{(n)}(\gamma))dC_{ir}(\gamma) - \int_{k_0}^k p_i(k, Z_r^{(n-1)}(\gamma))\right| dC_{ir}(\gamma) \\
\leq& \int_{k_0}^k L_D|Z_r^{(n)}(\gamma) - Z_r^{(n-1)}(\gamma)|dr + mK(\gamma)\int_{k_0}^k L_D|Z_r^{(n)}(\gamma) - Z_r^{(n-1)}(\gamma)|dr \\
\leq& L_D(1+mK(\gamma))\int_{k_0}^k |Z_r^{(n)}(\gamma) - Z_r^{(n-1)}(\gamma)|dr \\
\leq& L_D(1+mK(\gamma))\int_{k_0}^k \frac{Q(L_D + K(\gamma)L_D)^{n-1}}{n!}|k-k_0|^n dr \\
\leq& \frac{Q(L_D + mK(\gamma)L_D)^n}{n!}\int_{k_0}^k |r-k_0|^n dr. \\
\leq& \frac{Q(L_D + mK(\gamma)L_D)^n}{(n+1)!}|k-k_0|^{n+1}.
\end{aligned}$$

The above inequality gives an upper bound of

$$Z_k^{(n+1)}(\gamma) - Z_k^{(n)}(\gamma)$$

on $[k_0, k_0 + \alpha]$ for $n = 0, 1, 2, \cdots$. Obviously, for any $\epsilon > 0$, there exists a integer N ($N > 0$) such that

$$\begin{aligned}
&\sum_{n\geq N} |Z_k^{(n+1)}(\gamma) - Z_k^{(n)}(\gamma)| \\
\leq &\sum_{n\geq N} \frac{Q(L_D + mK(\gamma)L_D)^n}{(n+1)!}|k - k_0|^{n+1}, \\
= &\frac{Q}{L_D + mK(\gamma)L_D} \sum_{n\geq N} \frac{(L_D + mK(\gamma)L_D)^{n+1}}{(n+1)!}|k - k_0|^{n+1}, \\
\leq &\frac{Q}{L_D + mK(\gamma)L_D} \sum_{n\geq N} \frac{(L_D + mK(\gamma)L_D)^{n+1}}{(n+1)!}\alpha^{n+1}, \\
\leq &\frac{Q}{L_D + mK(\gamma)L_D} \sum_{n\geq N} \frac{(L_D + mK(\gamma)L_D)^{n+1}}{(n+1)!} \\
< &\epsilon.
\end{aligned}$$

where the last inequality from the fact

$$\lim_{n\to+\infty} \frac{a^{n+1}}{(n+1)!} = 0.$$

Because

$$Z_k^n(\gamma) = Z_k^0(\gamma) + \sum_{i=1}^{m}(Z_k^i(\gamma) - Z_k^{i-1}(\gamma)),$$

the above inequality indicates that $Z_k^n(\gamma)$ converges uniformly on $[k_0, k_0 + \alpha]$ as $n \to +\infty$.

$$Z_k^{(n+1)}(\gamma) = Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, Z_r^{(n)}(\gamma))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, Z_r^{(n)}(\gamma))dC_{ir}(\gamma).$$

Denote $Z_k(\gamma) = \lim_{n\to+\infty} Z_k^n(\gamma)$. Taking the limit on both sides of the above equation, it holds that

$$Z_k(\gamma) = Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, Z_r(\gamma))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, Z_r(\gamma))dC_{ir}(\gamma)$$

That is, the sequence $\{Z_k^n(\gamma)\}$ given by (4) converges uniformly to the solution of the equation (3) on $[k_0, k_0 + \alpha]$ as $n \to +\infty$.

Because each $\{Z_k^n(\gamma)\}$ is continuous, $Z_k(\gamma)$ is also continuous on $[k_0, k_0 + \alpha]$. The proof of existence is completed.

**Step 3.**(uniqueness) Step 3 will prove that $Z_k(\gamma)$ obtained in Step 2 is the unique solution of the equation (3) on $[k_0, k_0 + \alpha]$.

Assume that $\tilde{Z}_k(\gamma)$ is another solution of the equation (3), i.e.,

$$\tilde{Z}_k(\gamma) = Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, \tilde{Z}_r(\gamma))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, \tilde{Z}_r(\gamma))dC_{ir}(\gamma), \tag{6}$$

where $0 < \beta \leq \alpha$.

Following the local Lipchitz condition, we have

$$|Z_k(\gamma) - \tilde{Z}_k(\gamma)|$$

$$
\begin{aligned}
&=\left|\int_{k_0}^{k} h(r,Z_r(\gamma))dr+\sum_{i=1}^{m}\int_{k_0}^{k} p_i(r,Z_r(\gamma))dC_{ir}(\gamma)\right.\\
&\quad\left.-\int_{k_0}^{k} h(r,\tilde{Z}_r(\gamma))dr-\sum_{i=1}^{m}\int_{k_0}^{k} p_i(r,\tilde{Z}_r(\gamma))dC_{ir}(\gamma)\right|\\
&\le\left|\int_{k_0}^{k} h(r,Z_r(\gamma))dr-\int_{k_0}^{k} h(r,\tilde{Z}_r(\gamma))dr\right|\\
&\quad+\sum_{i=1}^{m}\left|\int_{k_0}^{k} p_i(r,Z_r(\gamma))dC_{ir}(\gamma)-\int_{k_0}^{k} p_i(r,\tilde{Z}_r(\gamma))dC_{ir}(\gamma)\right|\\
&\le\int_{k_0}^{k} L_D\left|Z_r(\gamma)-\tilde{Z}_r(\gamma)\right|dr+mK(\gamma)\int_{k_0}^{k} L_D\left|Z_r(\gamma)-\tilde{Z}_r(\gamma)\right|dr\\
&=L_D(1+mK(\gamma))\int_{k_0}^{k}\left|Z_r(\gamma)-\tilde{Z}_r(\gamma)\right|dr.
\end{aligned}
$$

By Gronwall inequality, we have

$$
|Z_k(\gamma)-\tilde{Z}_k(\gamma)|\le 0\cdot\exp\left(k(L_D(1+mK(\gamma)))\right)=0.
$$

That is to say, $Z_k(\gamma)=\tilde{Z}_k(\gamma)$, for any $[k_0,k_0+\alpha]$. The proof of uniqueness is completed. Until now, we complete the proof of Theorem 3.1. □

According to Theorem 3.1, multifactor uncertain integral equation (3) has a unique solution on the local interval $[k_0,k_0+\alpha]$. Next, Theorem 3.2 will show that the solution of multifactor uncertain integral equation (3) can be extended to the infinite domain $[0,+\infty)$.

**Theorem 3.2.** *Fixing $\gamma\in\Gamma$, multifactor uncertain differential equation* (3) *has a unique solution on* $[0,+\infty)$ *if the coefficients $h$, and $p_i$ satisfy one-sided local Lipschitz condition of Theorem 3.1 and the local linear growth condition, in other words, for each $T>0$,there exists a constant $M_T$ such that*

$$
|h(k,z)|\vee\sum_{i=1}^{m}|p(k,z)|\le M_T(1+|z|),\forall z\in\mathbb{R},k\in[0,T].
$$

*Proof* Define $\varrho=\{k|$ MUDE (2) has a unique continuous solution on $[0,k)\}$, and $\rho=\sup\varrho$. According to Theorem 3.1, the set $\rho$ is nonempty.

We will prove that $\rho=+\infty$. Assume that $\rho<+\infty$. By the definition, $Z_k(\gamma)$ is the unique solution of the equation (2) on $[0,\rho)$. Then, we have

$$
\begin{aligned}
|Z_k(\gamma)|&=\left|Z_0(\gamma)+\int_0^k h(r,Z_r(\gamma))dr+\sum_{i=1}^{m}\int_0^k p_i(r,Z_r(\gamma))dC_{ir}(\gamma)\right|\\
&\le|Z_0(\gamma)|+\left|\int_0^k h(r,Z_r(\gamma))dr\right|+K(\gamma)\sum_{i=1}^{m}\left|\int_0^k p_i(r,Z_r(\gamma))dr\right|\\
&\le|Z_0(\gamma)|+M_\rho(1+mK(\gamma))\int_0^k 1+|Z_r(\gamma)|dr\\
&\le|Z_0(\gamma)|+\rho M_\rho(1+mK(\gamma))+M_\rho(1+K(\gamma))\int_0^k|Z_r(\gamma)|dr
\end{aligned}
$$

for any $k\in[0,\rho)$. Set

$$
A=|Z_0(\gamma)|+\rho M_\rho(1+mK(\gamma)).
$$

By Gronwall inequality, we have

$$
|Z_k(\gamma)|\le A\cdot\exp(M_\rho(1+mK(\gamma)))\rho=N_0<+\infty,\forall k\in[0,\rho).
$$

That is to say, $|Z_k(\gamma)|$ is bounded on $[0, \rho)$.

Thus, we have

$$\begin{aligned}
&|Z_{k_1}(\gamma) - Z_{k_2}(\gamma)| \\
=&\left|Z_0(\gamma) + \int_0^{k_1} h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_0^{k_1} p_i(r, Z_r(\gamma))dC_{ir}(\gamma)\right. \\
&\left. - Z_0(\gamma) - \int_0^{k_2} h(r, Z_r(\gamma))dr - \sum_{i=1}^{m}\int_0^{k_2} p_i(r, Z_r(\gamma))dC_{ir}(\gamma)\right| \\
\leq&\left|\int_{k_1}^{k_2} h(r, Z_r(\gamma))dr\right| + \sum_{i=1}^{m}\left|\int_{k_1}^{k_2} p_i(r, Z_r(\gamma))dC_{ir}(\gamma)\right| \\
\leq&\int_{k_1}^{k_2} |h(r, Z_r(\gamma))|dr + K(\gamma)\sum_{i=1}^{m}\int_{k_1}^{k_2} |p_i(r, Z_r(\gamma))|dr \\
&\leq M_\rho \int_{k_1}^{k_2} 1 + |Z_r(\gamma)|dr + M_\rho m K(\gamma)\int_{k_1}^{k_2} 1 + |Z_r(\gamma)|dr \\
&= M_\rho(1 + mK(\gamma))\int_{k_1}^{k_2} 1 + |Z_r(\gamma)|dr \\
&\leq M_\rho(1 + mK(\gamma))\int_{k_1}^{k_2} (1 + N_0)dr \\
&\leq M_\rho(1 + mK(\gamma))(1 + N_0)|k_1 - k_2|, \forall k_1, k_2 \in [0, \rho).
\end{aligned}$$

It holds that $\lim_{k\to\rho^-} Z_k(\gamma)$ exists. Set $Z_\rho(\gamma) = \lim_{k\to\rho^-} Z_k(\gamma)$. Thus $Z_k(\gamma)$ is continuous on the interval $[0, \rho]$, and

$$Z_k(\gamma) = Z_0(\gamma) + \int_0^k h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_0^k p_i(r, Z_r(\gamma))dC_{ir}(\gamma). \tag{7}$$

Consider the following multifactor uncertain differential equation

$$\begin{cases} Z_k(\gamma) = Z_\rho(\gamma) + \int_\rho^k h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_\rho^k p_i(r, Z_r(\gamma))dC_{ir}(\gamma), k \in (\rho, +\infty) \\ Z_k(\gamma) = Z_0(\gamma) + \int_0^k h(r, Z_r(\gamma))dr + \sum_{i=1}^{m}\int_0^k p_i(r, Z_r(\gamma))dC_{ir}(\gamma), k \in [0, \rho]. \end{cases} \tag{8}$$

Theorem 3.1 means that there exists a positive number $\alpha$ such that multifactor uncertain integral equation (8) has a unique continuous solution $\tilde{Z}_k(\gamma)$ on the interval $[\rho, \rho + \alpha]$.

Thus, setting the function

$$\hat{Z}_k(\gamma) = \begin{cases} Z_k(\gamma), if\ k \in [0, \rho] \\ \tilde{Z}_k(\gamma), if\ k \in (\rho, \rho + \alpha], \end{cases} \tag{9}$$

$\hat{Z}_k(\gamma)$ is the unique continuous solution of the equation (3) on the interval $[0, \rho + \alpha]$. It is a contradiction from $\rho = \sup \varrho < +\infty$. So, $\rho = +\infty$, and the solution of multifactor uncertain integral equation (2) can be extended uniquely to $[0, +\infty)$. So we complete the proof of Theorem 3.2. □

**Example 3.1.** *Consider the MUDE*

$$dZ_k = kZ_k dk + \sum_{i=1}^{m} kZ_k dC_{ik}.$$

*It is obvious that $h(k,z) = kz$ and $p_i(k,z) = kz$ are local Lipschitz continuous rather than global Lipschitz continuous. So Li and Peng's result does not work for this MUDE. However, by Theorem* (3.1) *and* (3.2)*, it has a unique continuous solution. Actually, the solution is*

$$Z_k = Z_0 \exp(\frac{k^2}{2} + \sum_{i=1}^{m} \int_0^k r dC_{ir}).$$

## 4. Existence theorem without the Lipschitz condition

**Theorem 4.1.** *Fixing $\gamma \in \Gamma$, the multifactor uncertain integral equation* (3) *has at least one sample-continuous solution in $[k_0, k_0 + \alpha]$ if the coefficients $h(k,z)$,$p_i(k,z)$ are continuous on $(k,z) \in [0, +\infty) \times D$. The $D$ is given as follow:*

$$D = \{(k,z) | k \in [k_0, k_0 + a], z \in [Z_{k_0}(\gamma) - b, Z_{k_0}(\gamma) + b]\},$$

*where $a > 0, b > 0$ and*

$$Q = \max_D \left\{ |h(k,z)| + K(\gamma) \sum_{i=1}^{m} |p_i(k,z)| \right\} + 1,$$

*$K(\gamma)$ is the maximum Lipschitz constant to $C_{ik}(\gamma)$, and $\alpha = \min\{a, b/Q\}$.*

*Proof* Define a continuous functions sequence $\{\tilde{Z}_n(k)\}_{n=1}^{\infty}$

$$\tilde{Z}_n(k) = \begin{cases} Z_{k_0}(\gamma), if\ k \le k_0 \\ Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma), if\ k \in [k_0, k_0 + \alpha] \end{cases} \tag{10}$$

In fact, if $k_0 \le k \le k_0 + \frac{1}{n}$, then $k - \frac{1}{n} \le k_0$ and $\tilde{Z}_n(k - \frac{1}{n}) = Z_{k_0}(\gamma)$. And we also have $\tilde{Z}_n(k)$ is continuous on $[k_0, k_0 + 1/n]$. Furthermore, for $k_0 \le k \le k_0 + 1/n$,

$$\begin{aligned}
|\tilde{Z}_n(k) - Z_{k_0}(\gamma)| =& \left| \int_{k_0}^{k} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma) \right| \\
\le& \left| \int_{k_0}^{k} h(r, Z_{k_0}(\gamma))dr \right| + \sum_{i=1}^{m} \left| \int_{k_0}^{k} p_i(r, Z_{k_0}(\gamma))dC_{ir}(\gamma) \right| \\
\le& \int_{k_0}^{k} \left| h(r, Z_{k_0}(\gamma)) \right| + K(\gamma) \sum_{i=1}^{m} \left| p_i(r, Z_{k_0}(\gamma)) \right| dr \\
\le& Q \cdot |k - k_0| \\
\le& Q \cdot \alpha \le b,
\end{aligned}$$

So when $k_0 \le k \le k_0 + 1/n.$, $(k, \tilde{Z}_n(k)) \in D$. If $k_0 + 1/n \le k \le k_0 + 2/n$, then $k_0 \le k - 1/n \le k_0 + 1/n$. Thus, $\tilde{Z}_n(k - \frac{1}{n})$ is controlled by the above step and $|\tilde{Z}_n(k - \frac{1}{n}) - Z_{k_0}(\gamma)| \le b$. Also, $\tilde{Z}_n(k)$ is continuous on $[k_0, k_0 + 2/n]$. Furthermore, for $k_0 + 1/n \le k \le k_0 + 2/n$,

$$\begin{aligned}
|\tilde{Z}_n(k) - Z_{k_0}(\gamma)| =& \left| \int_{k_0}^{k} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma) \right| \\
\le& \left| \int_{k_0}^{k} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr \right| + \sum_{i=1}^{m} \left| \int_{k_0}^{k} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma) \right| \\
\le& \int_{k_0}^{k} \left| h(r, \tilde{Z}_n(r - \frac{1}{n})) \right| + K(\gamma) \sum_{i=1}^{m} \left| p_i(r, \tilde{Z}_n(r - \frac{1}{n})) \right| dr
\end{aligned}$$

$$\leq Q \cdot |k - k_0|$$
$$\leq Q \cdot \alpha \leq b,$$

After limited steps, $\tilde{Z}_n(k)$ is constructed. Also, $\tilde{Z}_n(k)$ is bounded on $[k_0, k_0 + \alpha]$, due to $|\tilde{Z}_n(k) - Z_{k_0}(\gamma)| \leq b$.

It is easy to see that $\tilde{Z}_n(k)$ is Lipschtiz continuous on $[k_0, k_0 + \alpha]$. In fact,

$$\begin{aligned}
|\tilde{Z}_n(k_1) - \tilde{Z}_n(k_2)| &= \left| \int_{k_1}^{k_2} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr + \sum_{i=1}^{m} \int_{k_1}^{k_2} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma) \right| \\
&\leq \left| \int_{k_1}^{k_2} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr \right| + \sum_{i=1}^{m} \left| \int_{k_1}^{k_2} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma) \right| \\
&\leq \int_{k_1}^{k_2} \left| h(r, \tilde{Z}_n(r - \frac{1}{n})) \right| + K(\gamma) \sum_{i=1}^{m} \left| p_i(r, \tilde{Z}_n(r - \frac{1}{n})) \right| dr \\
&\leq Q \cdot |k_1 - k_2|, \forall n \geq 1.
\end{aligned} \tag{11}$$

So, $\{\tilde{Z}_n(k)\}_{n=1}^{\infty}$ is bounded and equicontinuous. It is easy to see that there also exists a subsequence of $\{\tilde{Z}_n(k)\}_{n=1}^{\infty}$ that converges uniformly on $[k_0, k_0 + \alpha]$. For convenience, the subsequence is still denoted by $\{\tilde{Z}_n(k)\}_{n=1}^{\infty}$.

Denote $Z_k(\gamma) = \lim_{n\to\infty} \tilde{Z}_n(k)$. $Z_k(\gamma)$ is continuous on $[k_0, k_0 + \alpha]$. It holds from $\tilde{Z}_n(k)$ and (11) that

$$\begin{aligned}
|\tilde{Z}_n(k - \frac{1}{n}) - Z_k(\gamma)| &= |\tilde{Z}_n(k - \frac{1}{n}) - \tilde{Z}_n(k)| + |\tilde{Z}_n(k) - Z_k(\gamma)| \\
&\leq \frac{Q}{n} + |\tilde{Z}_n(k) - Z_k(\gamma)|, \forall n \geq 1, \forall k \in [k_0, k_0 + \alpha].
\end{aligned}$$

So, $\tilde{Z}_n(k - \frac{1}{n})$ converges uniformly to $Z_k(\gamma)$.

Because $h(k, z)$ and $p_i(k, z)$ are continuous, by the concept of $\tilde{Z}_n(k)$,

$$\tilde{Z}_n(k) = Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, \tilde{Z}_n(r - \frac{1}{n}))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, \tilde{Z}_n(r - \frac{1}{n}))dC_{ir}(\gamma). \tag{12}$$

We take limit on both sides of expression, we can get

$$Z_k(\gamma) = Z_{k_0}(\gamma) + \int_{k_0}^{k} h(r, Z_r(\gamma))dr + \sum_{i=1}^{m} \int_{k_0}^{k} p_i(r, Z_r(\gamma))dC_{ir}(\gamma). \tag{13}$$

So, the multifactor uncertain integral equation (3) has at least one sample-continuous solution. Similar to Section 3, the solution can be extended to $[0, +\infty)$. □

## 5. Applications

Extreme values are often used in American call option, American put option. Time integral is often used in Asian call option, Asian put option. Next, we will give the IUDs of extreme values and time integral.

### 5.1. IUD of supremum

**Theorem 5.1.** *Let $Z_k$ and $Z_k^\alpha$ be the solution and $\alpha$-path of the MUDE,respectively. Then for a strictly increasing payoff function $C(z)$, the supremum*

$$\sup_{0\leq k\leq T} C(Z_k)$$

*have the IUD*

$$\Psi_T^{-1}(\alpha) = \sup_{0\leq k\leq T} C(Z_k^\alpha)$$

*for a strictly decreasing payoff function $C(z)$, the supremum*

$$\sup_{0\le k\le T} C(Z_k)$$

*have the IUD*

$$\Psi_T^{-1}(\alpha) = \sup_{0\le k\le T} C(Z_k^{1-\alpha}).$$

*Proof* When $C(z)$ is a strictly increasing payoff function related to $z$, we have

$$\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{\alpha})\} \supset \{Z_k \le Z_k^{\alpha}, \forall k\}$$

and

$$\{\sup_{0\le k\le T} C(Z_k) > \sup_{0\le k\le T} C(Z_k^{\alpha})\} \supset \{Z_k > Z_k^{\alpha}, \forall k\}.$$

By the monotonicity of uncertain measure and Theorem 2.2, we have

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{\alpha})\} \ge \mathcal{M}\{Z_k \le Z_k^{\alpha}, \forall k\} = \alpha$$

and

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) > \sup_{0\le k\le T} C(Z_k^{\alpha})\} \ge \mathcal{M}\{Z_k > Z_k^{\alpha}, \forall k\} = 1-\alpha.$$

It follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{\alpha})\} = \alpha.$$

By Theorem 2.3, the supremum $\sup_{0\le k\le T} C(Z_k)$ has an IUD $\Psi_T^{-1}(\alpha) = \sup_{0\le k\le T} C(Z_k^{\alpha})$.

Similarly, for a strictly decreasing payoff function $C(z)$, we have

$$\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{1-\alpha})\} \supset \{Z_k \ge Z_k^{1-\alpha}, \forall k\}$$

and

$$\{\sup_{0\le k\le T} C(Z_k) > \sup_{0\le k\le T} C(Z_k^{1-\alpha})\} \supset \{Z_k < Z_k^{1-\alpha}, \forall k\}.$$

By the monotonicity of uncertain measure and Theorem 2.2, we have

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{1-\alpha})\} \ge \mathcal{M}\{Z_k \ge Z_k^{1-\alpha}, \forall k\} = \alpha$$

and

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) > \sup_{0\le k\le T} C(Z_k^{1-\alpha})\} \ge \mathcal{M}\{Z_k < Z_k^{1-\alpha}, \forall k\} = 1-\alpha.$$

It follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{\sup_{0\le k\le T} C(Z_k) \le \sup_{0\le k\le T} C(Z_k^{1-\alpha})\} = \alpha.$$

By Theorem 2.3, the supremum $\sup_{0\le k\le T} C(Z_k)$ has an IUD $\Psi_T^{-1}(\alpha) = \sup_{0\le k\le T} C(Z_k^{1-\alpha})$. we can get the conclusion. □

To calculate the IUD of the supremum, we design numerical algorithms for both cases as follow.

Table 1: The situation about a strictly increasing payoff function

| **Algorithm 1.1** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$,and $Q = C(Z_0)$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^\alpha = Z_j^\alpha + h(k_j, Z_j^\alpha)l + \sum_{i=1}^{m} \lvert g_i(k_j, Z_j^\alpha)\rvert \Phi^{-1}(\alpha)l$ and calculate $Z_{j+1}^\alpha$ |
| Step 5 | Set $Q \leftarrow \max(Q, C(Z_{j+1}^\alpha))$, $j \leftarrow j+1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\sup\limits_{0\le k\le T} C(Z_k) = Q$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Psi_T^{-1}(\alpha) = Q$ at each $\alpha$ in (0,1). |

Table 2: The situation about a strictly decreasing payoff function

| **Algorithm 1.2** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$,and $Q = C(Z_0)$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^{1-\alpha} = Z_j^{1-\alpha} + h(k_j, Z_j^{1-\alpha})l + \sum_{i=1}^{m} \lvert g_i(k_j, Z_j^{1-\alpha})\rvert \Phi^{-1}(\alpha)l$ and calculate $Z_{j+1}^{1-\alpha}$ |
| Step 5 | Set $Q \leftarrow \max(Q, C(Z_{j+1}^{1-\alpha}))$, $j \leftarrow j+1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\sup\limits_{0\le k\le T} C(Z_k) = Q$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Psi_T^{-1}(\alpha) = Q$ at each $\alpha$ in (0,1). |

### *5.2. IUD of infimum*

**Theorem 5.2.** *Let $Z_k$ and $Z_k^\alpha$ be the solution and $\alpha$-path of the MUDE,respectively. Then for a strictly increasing payoff function $C(z)$, the supremum*

$$\inf_{0\le k\le T} C(Z_k)$$

*have the IUD*

$$\Upsilon_T^{-1}(\alpha) = \inf_{0\le k\le T} C(Z_k^\alpha)$$

*for a strictly decreasing payoff function $C(z)$, the supremum*

$$\inf_{0\le k\le T} C(Z_k)$$

*have the IUD*

$$\Upsilon_T^{-1}(\alpha) = \inf_{0\le k\le T} C(Z_k^{1-\alpha}).$$

*Proof* When $C(z)$ is a strictly increasing payoff function, we have

$$\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^\alpha)\} \supset \{Z_k \le Z_k^\alpha, \forall k\}$$

and

$$\{\inf_{0\le k\le T} C(Z_k) > \inf_{0\le k\le T} C(Z_k^\alpha)\} \supset \{Z_k > Z_k^\alpha, \forall k\}.$$

By the monotonicity of uncertain measure and Theorem 2.2, we have

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^\alpha)\} \ge \mathcal{M}\{Z_k \le Z_k^\alpha, \forall k\} = \alpha$$

and

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) > \inf_{0\le k\le T} C(Z_k^\alpha)\} \ge \mathcal{M}\{Z_k > Z_k^\alpha, \forall k\} = 1-\alpha.$$

It follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^\alpha)\} = \alpha.$$

By Theorem 2.3, the supremum $\inf_{0\le k\le T} C(Z_k)$ has an IUD $\Upsilon_T^{-1}(\alpha) = \inf_{0\le k\le T} C(Z_k^\alpha)$.

Similarly, for a strictly decreasing payoff function $C(z)$, we have

$$\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^{1-\alpha})\} \supset \{Z_k \ge Z_k^{1-\alpha}, \forall k\}$$

and

$$\{\inf_{0\le k\le T} C(Z_k) > \inf_{0\le k\le T} C(Z_k^{1-\alpha})\} \supset \{Z_k < Z_k^{1-\alpha}, \forall k\}.$$

By the monotonicity of uncertain measure and Theorem 2.2, we have

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^{1-\alpha})\} \ge \mathcal{M}\{Z_k \ge Z_k^{1-\alpha}, \forall k\} = \alpha$$

and

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) > \inf_{0\le k\le T} C(Z_k^{1-\alpha})\} \ge \mathcal{M}\{Z_k < Z_k^{1-\alpha}, \forall k\} = 1-\alpha.$$

It follows from the duality axiom of uncertain measure that

$$\mathcal{M}\{\inf_{0\le k\le T} C(Z_k) \le \inf_{0\le k\le T} C(Z_k^{1-\alpha})\} = \alpha.$$

By Theorem 2.3, the supremum $\inf_{0\le k\le T} C(Z_k)$ has an IUD $\Upsilon_T^{-1}(\alpha) = \inf_{0\le k\le T} C(Z_k^{1-\alpha})$, we can get the conclusion. □

To calculate the IUD of the infimum, we design numerical algorithms for both cases as follow.

Table 3: The situation about a strictly increasing payoff function

| **Algorithm 2.1** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$,and $Q = C(Z_0)$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^\alpha = Z_j^\alpha + h(k_j, Z_j^\alpha)l + \sum_{i=1}^{m} \lvert g_i(k_j, Z_j^\alpha)\rvert \Phi^{-1}(\alpha) l$ and calculate $Z_{j+1}^\alpha$ |
| Step 5 | Set $Q \leftarrow \min(Q, C(Z_{j+1}^\alpha))$, $j \leftarrow j+1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\inf_{0\le k\le T} C(Z_k) = Q$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Upsilon_T^{-1}(\alpha) = Q$ at each $\alpha$ in (0,1). |

Table 4: The situation about a strictly decreasing payoff function,

| **Algorithm 2.2** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$,and $Q = C(Z_0)$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^{1-\alpha} = Z_j^{1-\alpha} + h(k_j, Z_j^{1-\alpha})l + \sum_{i=1}^{m} \lvert g_i(k_j, Z_j^{1-\alpha})\rvert\Phi^{-1}(\alpha)l$ and calculate $Z_{j+1}^{1-\alpha}$ |
| Step 5 | Set $Q \leftarrow \min(Q, C(Z_{j+1}^{1-\alpha}))$, $j \leftarrow j + 1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\inf_{0\le k\le T} C(Z_k) = Q$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Upsilon_T^{-1}(\alpha) = Q$ at each $\alpha$ in (0,1). |

### *5.3. IUD of time integral*

**Theorem 5.3.** *Let $Z_k$ and $Z_k^\alpha$ be the solution and $\alpha$-path of the MUDE,respectively. For $C(z)$ is a strictly increasing payoff function. Then the integral*

$$\int_0^T C(Z_k)dk$$

*have the IUD*

$$\Upsilon_T^{-1}(\alpha) = \int_0^T C(Z_k^\alpha)dk,$$

*for $C(z)$ is a strictly decreasing payoff function, then the integral*

$$\int_0^T C(Z_k)dk$$

*have the IUD*

$$\Upsilon_T^{-1}(\alpha) = \int_0^T C(Z_k^{1-\alpha})dk.$$

*Proof* When $C(z)$ is a strictly increasing payoff function with respect to $z$, it is always true that

$$\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^\alpha)dk\right\} \supset \{C(Z_k) \le C(Z_k^\alpha),\ \forall k\} = \{Z_k \le Z_k^\alpha,\ \forall k\}.$$

and

$$\left\{\int_0^T C(Z_k)dk > \int_0^T C(Z_k^\alpha)dk\right\} \supset \{C(Z_k) > C(Z_k^\alpha),\ \forall k\} = \{Z_k > Z_k^\alpha,\ \forall k\}.$$

By Theorem 2.2, we obtain

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^\alpha)dk\right\} \ge \mathcal{M}\{Z_k \le Z_k^\alpha,\ \forall k\} = \alpha.$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk > \int_0^T C(Z_k^\alpha)dk\right\} \ge \mathcal{M}\{Z_k > Z_k^\alpha,\ \forall k\} = 1 - \alpha.$$

It follows that

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^\alpha)dk\right\} = \alpha.$$

In other words, $\int_0^T C(Z_k)dk$ has an IUD $\Upsilon_T^{-1}(\alpha) = \int_0^T C(Z_k^\alpha)dk$.

Similarly, for a strictly decreasing payoff function $C(z)$, it is always true that

$$\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^{1-\alpha})dk\right\} \supset \{C(Z_k) \le C(Z_k^\alpha),\ \forall k\} = \{Z_k \ge Z_k^{1-\alpha},\ \forall k\}.$$

and

$$\left\{\int_0^T C(Z_k)dk > \int_0^T C(Z_k^{1-\alpha})dk\right\} \supset \{C(Z_k) > C(Z_k^{1-\alpha}),\ \forall k\} = \{Z_k < Z_k^{1-\alpha},\ \forall k\}.$$

By Theorem 2.2, we obtain

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^{1-\alpha})dk\right\} \ge \mathcal{M}\{Z_k \ge Z_k^{1-\alpha},\ \forall k\} = \alpha.$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk > \int_0^T C(Z_k^{1-\alpha})dk\right\} \ge \mathcal{M}\{Z_k < Z_k^{1-\alpha},\ \forall k\} = 1-\alpha.$$

It follows that

$$\mathcal{M}\left\{\int_0^T C(Z_k)dk \le \int_0^T C(Z_k^{1-\alpha})dk\right\} = \alpha.$$

In other words, $\int_0^T C(Z_k)dk$ has an IUD $\Upsilon_T^{-1}(\alpha) = \int_0^T C(Z_k^{1-\alpha})dk$, we can get the conclusion.

To calculate the IUD of the integral of $C(Z_k)$, we design numerical algorithms for both cases as follow.

Table 5: The situation about a strictly increasing payoff function

| **Algorithm 3.1** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^\alpha = Z_j^\alpha + h(k_j, Z_j^\alpha)l + \sum_{i=1}^m \lvert g_i(k_j, Z_j^\alpha)\rvert \Phi^{-1}(\alpha)l$ and calculate $Z_{j+1}^\alpha$ and $C(Z_{j+1}^\alpha)$ |
| Step 5 | Set $j \leftarrow j + 1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\int_0^T C(Z_k)dk = \sum_{j=1}^N C(Z_j^\alpha)l$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Upsilon_T^{-1}(\alpha) = \sum_{j=1}^N C(Z_j^\alpha)l$ at each $\alpha$ in (0,1). |

Table 6: The situation about a strictly decreasing payoff function,

| **Algorithm 3.2** | Let $Z_0^\alpha = Z_0$ be an initial guess. |
|---|---|
| Step 1 | Set $\alpha = 0$, step length $\Delta\alpha$. |
| Step 2 | $\alpha = \alpha + \Delta\alpha$ |
| Step 3 | Fix $l$ as the step length. Set $j = 0$, $N = \frac{T}{l}$ for a fixed $\alpha$ in (0,1) |
| Step 4 | Use the recursion formula $Z_{j+1}^{1-\alpha} = Z_j^{1-\alpha} + h(k_j, Z_j^{1-\alpha})l + \sum_{i=1}^{m} \lvert g_i(k_j, Z_j^{1-\alpha})\rvert\Phi^{-1}(\alpha)l$ and calculate $Z_{j+1}^{1-\alpha}$ and $C(Z_{j+1}^{1-\alpha})$ |
| Step 5 | Set $j \leftarrow j + 1$. |
| Step 6 | Repeat Step 4 and Step 5 for $N$ times. |
| Step 7 | Obtain the IUD $\int_0^T C(Z_k)dk = \sum_{j=1}^{N} C(Z_j^{1-\alpha})l$ for $\alpha$. |
| Step 8 | Return to step 2, while $\alpha + \Delta\alpha < 1$. |
| Step 9 | Output the iterative results. Then the IUD $\Upsilon_T^{-1}(\alpha) = \sum_{j=1}^{N} C(Z_j^{1-\alpha})l$ at each $\alpha$ in (0,1). |

□

*5.4. Numerical examples*

**Example 5.1.** *Let $\eta, a, b$ be real number. For the following nonlinear MUDE,*

$$dZ_k = \frac{\eta - Z_k}{1-k}dk + adC_{1k} + b\sqrt{Z_k}dC_{2k}, 0 \le k < 1 \tag{14}$$

*with given initial value $Z_0 = 1$. The $\alpha$-path equivalent equation of Eq. (14) is*

$$dZ_k^\alpha = \left(\frac{\eta - Z_k^\alpha}{1-k} + \left((|a| + |b|)\sqrt{Z_k^\alpha}\Phi^{-1}(\alpha)\right)\right)dk$$

*(i)Consider the supremum*

$$\sup_{0\le k\le T} \exp(-\delta k)(Z_k - H) \tag{15}$$

*where $\delta$ and $H$ are real numbers. The $\alpha$-path of Eq. (15) is*

$$\sup_{0\le k\le T} \exp(-\delta k)(Z_k^\alpha - H) \tag{16}$$

*for given times $T > 0$. We set $\eta = 2, a = 1, b = 1, k = 0.9$ and $N = 1,000$. in Eq. (14) and choose the parameters $\delta = 0.02$ and $H = 1$ in Eq. (16). Based on the* **Algorithm 1.1** *we design, the IUD of supremum at $T = 0.9$ is shown in Figure 1. And we can get*

$$E\left[\sup_{0\le k\le T} \exp(-\delta k)(Z_k - H)\right] = 12.6572$$

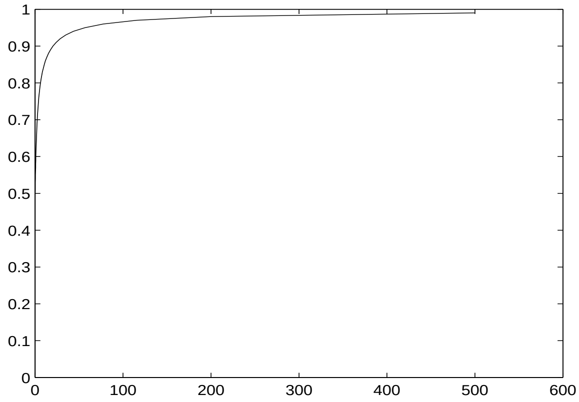


Figure 1: The IUDs cure of supremum at T = 0.9

*(ii)Consider the infimum*

$$\inf_{0\leq k\leq T} \exp(-\delta k)(Z_k - H) \tag{17}$$

*where $\delta$ and $H$ are real numbers. The $\alpha$-path of Eq.* (17) *is*

$$\inf_{0\leq k\leq T} \exp(-\delta k)(Z_k^\alpha - H) \tag{18}$$

*for given times $T > 0$. We set $\eta = 2, a = 1, b = 1, k = 0.9$ and $N = 1,000$. in Eq.* (14) *and choose the parameters $\delta = 0.02$ and $H = 1$ in Eq.* (18)*. Based on the* **Algorithm 2.1** *we design, the IUD of infimum at $T = 0.9$ is shown in Figure 2. And we can get*

$$E\left[\inf_{0\leq k\leq T} \exp(-\delta k)(Z_k - H)\right] = -0.3205$$

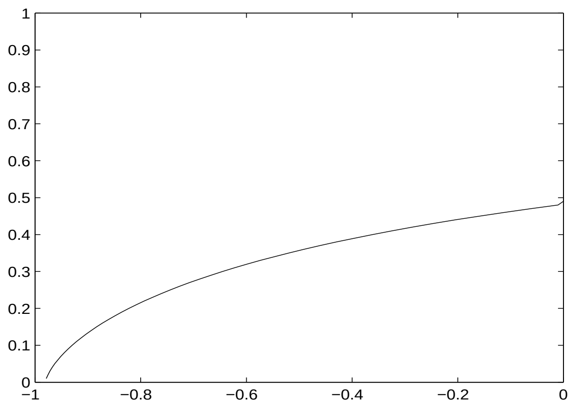


Figure 2: The IUDs cure of infimum at T = 0.9

*(iii)Consider the time integral*

$$\int_0^T \exp(-\delta k)(Z_k - H)dk \tag{19}$$

*where $\delta$ and $H$ are real numbers. The $\alpha$-path of Eq.* (19) *is*

$$\int_0^T \exp(-\delta k)(Z_k^\alpha - H)dk \tag{20}$$

*for given times $T > 0$. We set $\eta = 2, a = 1, b = 1, k = 0.9$ and $N = 1,000$. in Eq.* (14) *and choose the parameters $\delta = 0.02$ and $H = 1$ in Eq.* (20). *Based on the* **Algorithm 3.1** *we design, the IUD of time integral at $T = 0.9$ is shown in Figure 3. And we can get*

$$E\left[\int_0^T \exp(-\delta k)(Z_k - H)dk\right] = 11.2282$$

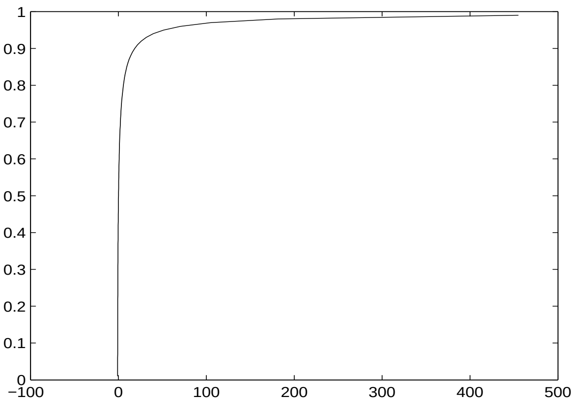


Figure 3: The IUDs cure of time integral at T = 0.9

**Remark 5.1.** *We above just present an example for strictly increasing payoff function, similarly, the situation for strictly decreasing payoff function can be simulated, we omit here.*

### 6. Conclusion and prospect

The subject of MUDE has been only investigated by a few scholars, such as Li et al. (2015), Zhang, Gao and Yang (2016), etc. By contrast, we mainly study new existence and uniqueness theorem, which subsumes the study of the local Lipschitz continuous condition in the context of certain MUDEs. Meanwhile, when the coefficients do not satisfy the Lipschitz condition, some new conditions are needed here to overcome this difficult. this article also establish IUDs of MUDEs. Hopefully we may inspire some potential practical applications in finance in the future.

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