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A New Necessary and Sufficient Condition for the Existence of Global Solutions to Semilinear Parabolic Equations on bounded domains

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Summary

The purpose of this paper is to give a necessary and sufficient condition for the existence and non-existence of global solutions of the following semilinear parabolic equations

$$u_t = \Delta u + \psi(t) f(u), \quad \text{in } \Omega \times (0, t^*),$$

under the Dirichlet boundary condition on a bounded domain. In fact, this has remained as an open problem for a few decades, even for the case $f(u) = u^p$. As a matter of fact, we prove:

> there is no global solution for any initial data if and only if
> the function f satisfies
> $$\int_0^\infty \psi(t) \frac{f\left(\epsilon \left\|S(t)u_0\right\|_\infty\right)}{\left\|S(t)u_0\right\|_\infty} dt = \infty$$
> for every $\epsilon > 0$ and nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition.

KEYWORDS:
Semilinear parabolic equation, Fujita blow-up, Critical exponent

1 | INTRODUCTION

The phenomena of the global existence and nonexistence for nonlinear parabolic equations have received much attention due to their applications in physics, chemistry, and biology. Although many researchers have been studied long-time behaviors of the solutions to the following semilinear parabolic equations

$$u_t = \Delta u + \psi(t) u^p, \quad \text{in } \Omega \times (0, t^*), \tag{1}$$

a necessary and sufficient condition for the existence and non-existence of global solutions has remained as an open problem for a few decades (see survey articles[6,12]). In his seminal paper[7], Fujita firstly studied the equation (1) for the case $\psi \equiv 1$ and $\Omega = \mathbb{R}^N$ and obtained the following statements:

(i) if $1 < p < p^*$, then there is no global solution for any initial data,

(ii) whereas, if $p > p^*$, then there exists a global solution whenever the initial data is sufficiently small (less than a small Gaussian),

where $p^* = 1 + \frac{2}{N}$ is called the critical exponent.

In his pioneering paper[14], Meier considered the equations (1), where Ω is a general (bounded or unbounded) domain in $\mathbb{R}^N$, under the Dirichlet boundary condition and proved the following:

Theorem 1 ([14])**.** Assume that $p > 1$ and $\psi \in C[0, \infty)$.

(i) If $\limsup_{t\to\infty} \left\|S(t)u_0\right\|_\infty^{p-1} \int_0^t \psi(\tau)d\tau = \infty$ for all $u_0 \in C_0(\Omega)$, then there is no global solution for any initial data.

(ii) If $\int_0^\infty \psi(\tau) \left\|S(\tau)u_0\right\|_\infty^{p-1} d\tau < \infty$ for some $u_0 \in C_0(\Omega)$, then there exists global positive solution for sufficiently small initial data.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition on $\partial\Omega$.

Since it was not given in a form of a necessary and sufficient condition, the criterion in Theorem 1 is not satisfactory. More precisely, Meier's criterion cannot deal with every source terms, such as the critical case $p = p^*$ when $\psi \equiv 1$. Obviously, there are lots of ψ and p satisfying that

$$\int_0^\infty \psi(\tau) \left\|S(\tau)u_0\right\|_\infty^{p-1} d\tau = \infty \text{ and } \limsup_{t\to\infty} \left\|S(t)v_0\right\|_\infty^{p-1} \int_0^t \psi(\tau)d\tau < \infty \tag{2}$$

for every $u_0 \in C_0(\Omega)$ and for some $v_0 \in C_0(\Omega)$. That is to say, the existence and nonexistence of the global solutions for the case of (2) are crucial problems.

In fact, for the cases $\psi(t) \equiv 1$, $\psi(t) = e^{\beta t}$, or $\psi(t) = t^\sigma$, the necessary and sufficient condition for the existence of the global solution to the equation (1) have been studied (see [1,9,11,14,15]). However, for a general source term $\psi(t)u^p$, there is no paper which discuss the necessary and sufficient condition for the existence and nonexistence of the global solutions. Because of this fact, recent researches for the existence and nonexistence of global solutions have been studied based on Meier's criterion which were not the necessary and sufficient condition (for example, see [4,5]).

On the other hand, we note that the polynomial function $f(u) = u^p$ is multiplicative. *i.e.* $f(uv) = f(u)f(v)$ for $u > 0$ and $v > 0$. In general, multiplicative property of the source term is strongly used to obtain the existence and nonexistence of global solutions in the parabolic equations (for example, see [2,4,14]). For a general source term $\psi(t)f(u)$, there is no paper on necessary and sufficient condition for the existence of global solution. Here, one of our purpose is to deal with a general function $f(u)$ which is not multiplicative. In fact, we provide a formula $\frac{f(\left\|S(t)u_0\right\|_\infty)}{\left\|S(t)u_0\right\|_\infty}$ instead of $\left\|S(t)u_0\right\|_\infty^{p-1}$ to give a criterion of the existence of the global solution when the source term is $\psi(t)f(u)$.

In conclusion, there has been no progress in research on necessary and sufficient conditions for the general source term $\psi(t)f(u)$. That is to say, to give the necessary and sufficient condition for the general source term $\psi(t)f(u)$ has been unsolved problem for a few decades and so it is necessary to consider a new methodology.

From the above point of view, the purpose of this paper is twofold as follows:

(i) to obtain the necessary and sufficient condition for the existence and nonexistence of the global solutions for general source term $\psi(t)f(u)$.

(ii) to introduce a new method, so called a minorant method, to overcome a multiplicative property of source term.

Consequently, we discuss the following semilinear parabolic equations for more general source term $\psi(t)f(u)$ instead of $\psi(t)u^p$:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \psi(t)f(u(x,t)), & (x,t) \in \Omega \times (0,t^*), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, u_0 is a nonnegative and nontrivial $C_0(\Omega)$-function, and ψ is a nonnegative continuous function on $[0, \infty)$. Throughout this paper, we always assume that f satisfies the following conditions:

- $f : [0, \infty) \to [0, \infty)$ is a locally Lipschitz continuous function,
- $f(0) = 0$ and $f(s) > 0$ for $s > 0$,
- $\frac{f(u)}{u}$ is nondecrasing on $(0, \infty)$.

In fact, it is well-known that the local existence of the solutions of the equation (3) and the comparison principle are guaranteed by the first condition. Also, the third condition is the necessary condition for blow-up solutions (see[8,10]).

In order to solve the open problem mentioned above for more general source term $f(u)$ instead of u^p, we need to introduce the minorant function f_m and the majorant function f_M as follows:

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)} \text{ and } f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}.$$

Then the function f satisfies the following inequality:

$$f_m(u)f(\alpha) \leq f(u\alpha) \leq f_M(u)f(\alpha), \;\; u > 0, \;\; 0 < \alpha < 1.$$

To discuss blow-up and global phenomena using the minorant function f_m and the majorant function f_M, it is necessary to assume

$$\int_1^\infty \frac{ds}{f_m(s)} < \infty. \tag{4}$$

Finally, we obtained the following results to see 'completely' whether or not we have global solutions:

Theorem 2. Let ψ be a nonnegative nontrivial function and f satisfy the assumption (4). Then the there is no global solution u to the equation (3) for any initial data u_0 if and only if the function f satisfies

$$\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$

for every $\epsilon > 0$, where λ_0 is the first Dirichlet eigenvalue of the Laplace operator Δ.

Then Theorem 2 is the form of a necessary and sufficient condition for global solutions of the equation (3). It means that the open problem mentioned above is solved with more general source term $\psi(t)f(u)$.

As far as authors know, there is no paper which discuss the necessary and sufficient conditions (or Fujita's blow-up solutions) on the source term $f(u)$ instead of u^p. From this point of view, this paper and the minorant method will be clue to study long-time behaviors (especially, necessary and sufficient conditions) for solutions to PDEs with general type functions. Moreover, we hope that the minorant method will help to discuss the general functions in various research fields.

We organized this paper as follows: In Section 2, we discuss Meier's criterion. We introduce the multiplicative minorants and majorants and discuss main results in Section 3.

2 | DISCUSSION ON MEIER'S CONDITIONS

In this section, we discuss Meier's conditions for the existence and nonexistence of the global solution to the equation (1). If the domain Ω is bounded, then it is well-known that $\left\|S(t)u_0\right\|_\infty \sim e^{-\lambda_0 t}$ for $t > 1$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$. Therefore, Theorem 1 can be understood as follows:

- If

$$(C1) \; : \; \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t} \int_0^t \psi(\tau)d\tau = \infty,$$

  then there is no global solution to the equation (1) for any nonnegative and nontrivial initial data.

- If

$$(C2) \; : \; \int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}dt < \infty$$

then there exists a global solution to the equation (1) for sufficiently small initial data.

Then we are interested in the function ψ and the constant $p > 1$ don't satisfy both conditions $(C1)$ and $(C2)$. That is,

$$\limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)d\tau < \infty \quad \text{and} \quad \int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}dt = \infty. \tag{5}$$

Let us consider the function ψ defined by $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$. Then it follows that

$$\begin{aligned}\limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)d\tau &= \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t (\tau+1)^{-\delta}e^{(p-1)\lambda_0\tau}d\tau \\ &\le \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t e^{(p-1)\lambda_0\tau}d\tau \\ &= \frac{1}{(p-1)\lambda_0} < \infty\end{aligned}$$

and

$$\int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}dt = \int_0^\infty (t+1)^{-\delta}dt = \infty.$$

This implies that the function $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$, for which (5) is true, has not been dealt in Meier's results. Therefore, we don't know whether are global solution or not.

In particular, we consider the simple example of the functions ψ and f defined by $\psi(t) := (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}$ and $f(u) := u^2$ in the equation (3). Then the equation (3) follows that

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}u^2, & (x,t)\in\Omega\times(0,t^*), \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,t^*), \\ u(x,0) = u_0(x) \ge 0, & x\in\Omega. \end{cases} \tag{6}$$

Now, we consider the eigenfunction ϕ_0 to be $\sup_{x\in\Omega}\phi_0 dx = 1$, corresponding to the first Dirichlet eigenvalue λ_0. Suppose that the solution u to the equation (6) exists globally. Multiplying the equation (6) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\begin{aligned}&\int_\Omega u_t(x,t)\phi_0(x)dx \\ &= \int_\Omega \phi_0(x)\Delta u(x,t)dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\int_\Omega u^2\phi_0(x)dx \\ &= -\lambda_0\int_\Omega u(x,t)\phi_0(x)dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\left(\int_\Omega u(x,t)\phi_0(x)dx\right)^2,\end{aligned}$$

for all $t > 0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)dx$, for $t \ge 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \ge -\lambda_0 y(t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}y^2(t), \ t > 0, \\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)dx > 0. \end{cases} \tag{7}$$

Multiplying $e^{\lambda_0 t}$ by the inequality (7), then we have

$$\left[e^{\lambda_0 t}y(t)\right]' \ge (t+1)^{-\frac{1}{2}}\left[e^{\lambda_0 t}y(t)\right]^2 \ge 0,$$

for all $t > 0$, which implies that

$$\frac{d}{dt}\left[e^{\lambda_0 t}y(t)\right]^{-1} \le -(t+1)^{-\frac{1}{2}}$$

for all $t > 0$. Solving the differential inequality, then we obtain that

$$y(t) \geq \frac{e^{-\lambda_0 t}}{y_0^{-1} - \int_0^t (\tau+1)^{-\frac{1}{2}} d\tau},$$

for all $t > 0$, which leads a contradiction. Hence, the solution u to the equation (6) blows up in finite time.

The above example implies that the condition $(C1)$ is no longer necessary condition for the nonexistence of global solution. In fact, the main part of this paper is focused on the condition $(C2)$ to see whether $(C2)$ is necessary and sufficient condition of the existence of the global solution.

On the other hand, if the function f in the equation (3) is not multiplicative, then we cannot apply Meier's results. For example, let us consider the function $f(u) = \frac{u^2+u}{2}$. Then it is easy to see that $u \leq f(u) \leq u^2$ for $u \geq 1$ and $u^2 \leq f(u) \leq u$ for $0 \leq u \leq 1$. Therefore, we cannot determine p in the case of $f(u) = \frac{u^2+u}{2}$ in Theorem 1. From this point of view, we have to consider a new method, so called the minorant method to deal with a function f which is not multiplicative. In conclusion, we provide a formula $\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}$ instead of $\|S(t)u_0\|_\infty^{p-1}$ to give a criterion of the existence of the global solution when the source term is $\psi(t)f(u)$.

3 | MAIN RESULTS

In this section, we firstly introduce the minorant function and the majorant function. Next, we prove the main theorem by using the minorant function and majorant function.

First of all, we discuss multiplicative minorants and majorants of the function f, which will play an important role in this work.

Definition 1. For a function f, functions $f_m : [0,\infty) \to [0,\infty)$ and $f_M : [0,\infty) \to [0,\infty)$ are defined by

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \qquad u \geq 0,$$
$$f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \qquad u \geq 0.$$

Here, it is quite natural to f_m and f_M a multiplicative minorant and majorant of a function f respectively, considering the following properties:

- $f(\alpha)f_m(u) \leq f(\alpha u) \leq f(\alpha)f_M(u)$, $0 < \alpha < 1$, $u > 0$.
- If g and h be functions satisfying that

$$f(\alpha)g(u) \leq f(\alpha u) \leq f(\alpha)h(u), \;\; 0 < \alpha < 1, \;\; u > 0,$$

  then it follows that $g(u) \leq f_m(u)$ and $f_M(u) \leq h(u)$, $u > 0$.

In fact, the values of f_m and f_M are determined strongly depending on the value of f near zero, since for each $u \geq 1$,

$$f_m(u) \leq \inf_{0<\alpha<\frac{1}{u}} \frac{f(\alpha u)}{f(\alpha)} \quad \text{and} \quad f_M(u) \geq \sup_{0<\alpha<\frac{1}{u}} \frac{f(\alpha u)}{f(\alpha)}.$$

Also, if $\frac{f(u)}{u}$ is nondecreasing, then it is easy to see that the function f, the minorant f_m, and the majorant f_M satisfy the following properties:

(i) $f_m(u) \leq \frac{f(u)}{f(1)} \leq f_M(u)$ for $u \geq 0$.

(ii) $f_m\left(\frac{1}{\alpha}\right) \leq \frac{f(1)}{f(\alpha)} \leq f_M\left(\frac{1}{\alpha}\right)$ for $0 < \alpha \leq 1$.

(iii) $f_m(1) = f_M(1) = 1$.

(iv) $\frac{f_m(u)}{u}$ and $\frac{f_M(u)}{u}$ are nondecreasing in $(0, 1)$.

(v) $f_m(u) \leq u$ and $f_M(u) \leq u$ for $0 < u \leq 1$, since $\frac{f(\alpha u)}{f(\alpha)} = \frac{f(\alpha u)}{\alpha u}\frac{\alpha}{f(\alpha)}u \leq u$.

(vi) $\int_\eta^\infty \frac{ds}{f_M(s)} \leq f(1)\int_\eta^\infty \frac{ds}{f(s)} \leq \int_\eta^\infty \frac{ds}{f_m(s)}$ for $\eta \geq 1$.

(vii) $\int_0^1 \frac{ds}{f_m(s)} = \int_0^1 \frac{ds}{f(s)} = \int_0^1 \frac{ds}{f_M(s)} = \infty$.

We obtain from the property (vi) that $\int_\eta^\infty \frac{ds}{f(s)} < \infty$ implies $\int_\eta^\infty \frac{ds}{f_M(s)} < \infty$. However, the converse is not true, in general. In fact, examples and detailed properties the minorant function f_m and the majorant function f_M were discussed in[3].

Now, we introduce the definition of the blow-up solutions and global solutions.

Definition 2. We say that a solution u blows up at finite time t^*, if there exists (x_n, t_n) with $t_n \to t^*$, such that $|u(x_n, t_n)| \to \infty$. On the other hand, a solution u exists globally, if $\|u(\cdot, t)\|_\infty$ is bounded for each time $t \geq 0$.

Finally, we prove Theorem 2.

Proof. First of all, we consider the eigenfunction ϕ_0 to be $\sup_{x\in\Omega} \phi_0 dx = 1$, corresponding to the first Dirichlet eigenvalue λ_0. Suppose that the solution u exists globally, on the contrary. Multiplying the equation (3) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\begin{aligned}\int_\Omega u_t(x,t)\phi_0(x)dx &= \int_\Omega \phi_0(x)\Delta u(x,t)dx + \phi(t)\int_\Omega f(u(x,t))\phi_0(x)dx \\ &= -\lambda_0 \int_\Omega u(x,t)\phi_0(x)dx + \psi(t) f\left(\int_\Omega u(x,t)\phi_0(x)dx\right),\end{aligned}$$

for all $t > 0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)dx$, for $t \geq 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \geq -\lambda_0 y(t) + \psi(t) f(y(t)), \ t > 0, \\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)dx > 0. \end{cases}$$

Then the inequality can be written as

$$\left[e^{\lambda_0 t} y(t)\right]' \geq \psi(t) e^{\lambda_0 t} f(y(t)) \geq 0, \tag{8}$$

for $t > 0$ so that $e^{\lambda_0 t} y(t)$ is nondecreasing on $[0, \infty)$. On the other hand, by the definition of f_m, we can find $v_1 \in [0, 1]$ such that $f_m = 0$ on $[0, v_1)$ and $f_m > 0$ on (v_1, ∞). Then there exists $\epsilon > 0$ such that $y(0) > \epsilon v_1$. *i.e.* $v_1 < \frac{y(0)}{\epsilon} \leq \frac{e^{\lambda_0 t} y(t)}{\epsilon}$ for $t \geq 0$. Combining all these arguments, it follows from (8) and the definition of f_m that

$$\frac{\frac{e^{\lambda_0 t} y(t)}{\epsilon}}{f_m\left(\frac{e^{\lambda_0 t} y(t)}{\epsilon}\right)} \geq \frac{1}{\epsilon}\psi(t) e^{\lambda_0 t} f(\epsilon e^{-\lambda_0 t}),$$

for all $t > 0$. Now, define a function $F_m : (v_1, \infty) \to (0, v_\infty)$ by

$$F_m(v) := \int_v^\infty \frac{dw}{f_m(w)}, \ v > v_1$$

where $v_\infty := \lim_{v\to v_1} \int_v^\infty \frac{dw}{f_m(w)}$. Then it is easy to see that F_m is well-defined continuous function which is a strictly decreasing bijection with its inverse F_m^{-1} and $\lim_{v\to\infty} F_m(v) = 0$. Integrating the inequality (8) over $[0, t]$, we obtain

$$F_m\left(\frac{y(0)}{\epsilon}\right) - F_m\left(\frac{e^{\lambda_0 t} y(t)}{\epsilon}\right) \geq \frac{1}{\epsilon}\int_0^t \psi(\tau) e^{\lambda_0 \tau} f(\epsilon e^{-\lambda_0 \tau}) d\tau,$$

for all $t \geq 0$. Hence, we obtain

$$y(t) \geq \epsilon e^{-\lambda_0 t} F_m^{-1}\left[F_m\left(\frac{y(0)}{\epsilon}\right) - \frac{1}{\epsilon}\int_0^t \psi(\tau) e^{\lambda_0 \tau} f(\epsilon e^{-\lambda_0 \tau}) d\tau\right]$$

for all $t \geq 0$, which implies that $y(t)$ cannot be global.

We note that there exists a maximal interval $[0, m^*)$ on which f_M is finite. Then it is true that the integral $\int_v^{m^*} \frac{dw}{f_M(w)}$ is finite for each $v \in (0, m^*)$, $\lim_{v\to 0} \int_v^{m^*} \frac{dw}{f_M(w)} = \infty$, and $\lim_{v\to m^*} \int_v^{m^*} \frac{dw}{f_M(w)} = 0$. Then a function $F_M : (0, m^*) \to (0, \infty)$ defined by

$$F_M(v) := \int_v^{m^*} \frac{dw}{f_M(w)}, \; v \in (0, m^*)$$

is a well-defined continuous function which is a strictly decreasing bijection with its inverse F_M^{-1}. Now, take a number z_0 such that

$$0 < z_0 < F_M^{-1}\left[\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(e^{-\lambda_0 t}\right) dt\right]$$

and define a nondecreasing function $z : [0, \infty) \to [z_0, \infty)$ by

$$z(t) := F_M^{-1}\left[F_M(z_0) - \int_0^t \psi(\tau)e^{\lambda_0 \tau} f\left(e^{-\lambda_0 \tau}\right) d\tau\right], \; t \geq 0.$$

Then $z(t)$ is a bounded solution of the following ODE problem:

$$\begin{cases} z'(t) = \psi(t)e^{\lambda_0 t} f\left(e^{-\lambda_0 t}\right) f_M\left(z(t)\right), \; t > 0, \\ z(0) = z_0. \end{cases}$$

Now, consider a function $v(x, t) := e^{-\lambda_0 t}\phi_0(x)$ on $\overline{\Omega}\times[0, \infty)$ which is a solution to the heat equation $v_t = \Delta v$ under the Dirichlet boundary condition. Let $\overline{u}(x, t) := z(t)v(x, t)$ for $(x, t) \in \overline{\Omega} \times [0, \infty)$. Then since $\frac{f(u)}{u}$ is nondecrasing on $(0, 1)$, it follows that

$$\begin{aligned} \overline{u}_t(x, t) =& \Delta\overline{u}(x, t) + \psi(t)v(x, t)\left[\frac{f(e^{-\lambda_0 t})}{e^{-\lambda_0 t}}\right] f_M(z(t)) \\ \geq& \Delta\overline{u}(x, t) + \psi(t)v(x, t)\left[\frac{f(v(x, t))}{v(x, t)}\right] f_M(z(t)) \\ \geq& \Delta\overline{u}(x, t) + \psi(t)f(\overline{u}(x, t)) \end{aligned}$$

for all $(x, t) \in \Omega \times (0, \infty)$. It follows that $\overline{u}$ is the supersolution to the equation (3), which implies that u exists globally. □

Remark 1. Since $\left\|S(t)u_0\right\|_\infty \sim e^{-\lambda_0 t}$ for $t > 1$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$ when Ω is bounded, the statement

$$\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$

is equivalent to

$$\int_0^\infty \psi(t)\frac{f\left(\epsilon \left\|S(t)u_0\right\|_\infty\right)}{\left\|S(t)u_0\right\|_\infty} dt = \infty$$

for every $\epsilon > 0$ and nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$ when Ω is bounded. From this observation, we may expect that the form in the second condition of Theorem 1 is the form of the necessary and sufficient condition for the existence of the global solution on a general domain.

On the other hand, we can obtain the following valuable corollary immediately by using the fact $\left\|S(t)u_0\right\|_\infty \sim e^{-\lambda_0 t}$ for sufficiently large $t > 1$:

Corollary 1. Let the function ψ be a nonnegative continuous function and the function f be a nonnegative continuous and quasi-multiplitive function. *i.e.* there exist $\gamma_2 \geq \gamma_1 > 0$ such that

$$\gamma_1 f(\alpha)f(u) \leq f(\alpha u) \leq \gamma_2 f(\alpha)f(u), \tag{9}$$

for $0 < \alpha < 1$ and $u > 0$. Then the following statements are equivalent:

(i) $\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(e^{-\lambda_0 t}\right) dt = \infty$.

(ii) $\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}dt = \infty$ for every $w_0 \in C_0(\Omega)$.

(iii) $\int_1^\infty \psi(t)\frac{dt}{F(e^{-\lambda_0 t})} = \infty$, where $F(v) := \int_v^\infty \frac{dw}{f(w)}$.

(iv) There is no global solution to the equation (3) for any initial data.

Proof. Theorem 2 says that (i), (ii), and (iv) are equivalent, by using the following inequality:

$$\gamma_1 f(\epsilon) f(\|S(t)u_0\|_\infty) \le f(\epsilon \, \|S(t)u_0\|_\infty) \le \gamma_2 f(\epsilon) f(\|S(t)u_0\|_\infty),$$

for every $\epsilon > 0$. Therefore, we now discuss (iii).
(i)⇔(iii) : Let $F(v) = \int_v^\infty \frac{dw}{f(w)}$. Then the assumption (9) follows that

$$\int_1^\infty \frac{z}{\gamma_2 f(z) f(s)} ds \le \int_z^\infty \frac{dw}{f(w)} \le \int_1^\infty \frac{z}{\gamma_1 f(z) f(s)} ds,$$

for $z > 0$. This implies that

$$\frac{F(1)}{\gamma_2}\frac{z}{f(z)} \le F(z) \le \frac{F(1)}{\gamma_1}\frac{z}{f(z)}.$$

i.e. $F(z) \sim \frac{z}{f(z)}$, $z > 0$. Therefore, the proof is complete. □

Remark 2. In 2014, Loayza and Paixão[13] studied the conditions for existence and nonexistence of the global solutions to the equation (3) under the general domain and obtained the following statements:

(i) for every $w_0 \in C_0(\Omega)$, there exist $\tau > 0$ such that

$$\int_{\|S(\tau)w_0\|_\infty}^\infty \frac{dw}{f(w)} \le \int_0^\tau \psi(\sigma) d\sigma, \tag{10}$$

then there is no global solution u for every initial data,

(ii) the solution u exists globally for small initial data, whenever

$$\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}dt < 1, \tag{11}$$

for some $w_0 \in C_0(\Omega)$.

In fact, their results were not given in a form of a necessary and sufficient condition. Moreover, the conditions (10) and (11) don't seem to be lots of relevant. However, Corollary 1 imply that the conditions (10) and (11) have a strong relation, whenever f satisfies the condition (9) and Ω is bounded.

Also, by using Corollary 1, the example in Section 2 can be characterized completely as follows:

Remark 3. Let the domain Ω be bounded in $\mathbb{R}^N$, $\psi(t) := (t+1)^{-\sigma}e^{kt}$, and $f(u) := u^p$ where $\sigma \in \mathbb{R}$, $k \in \mathbb{R}$, and $p > 1$. Then the following statements are true.

(i) If $k > (p-1)\lambda_0$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(ii) If $k < (p-1)\lambda_0$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

(iii) If $k = (p-1)\lambda_0$ and $\sigma \le 1$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(iv) If $k = (p-1)\lambda_0$ and $\sigma > 1$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

1. X. Bai, S. Zheng, and W. Wang, *Critical exponent for parabolic system with time-weighted sources in bounded domain*, J. Funct. Anal. 265 (2013), no. 6, 941–952.

2. S. -Y. Chung, M. -J. Choi, and J. -H. Park, *On the critical set for Fujita type blow-up of solutions to the discrete Laplacian parabolic equations with nonlinear source on networks*, Comput. Math. Appl. 78 (2019), no. 6, 1838–1850.

3. S. -Y. Chung and J. Hwang, *A necessary and sufficient condition for the existence of global solutions to discrete semilinear parabolic equations on networks*, Chaos Solitons Fractals 158 (2022), Paper No. 112055.

4. R. Castillo and M. Loayza, *On the critical exponent for some semilinear reaction–diffusion systems on general domains*, J. Math. Anal. Appl 428(2015), 1117–1134.

5. R. Castillo and M. Loayza, *Global existence and blow up for a coupled parabolic system with time-weighted sources on a general domain*, Z. Angew. Math. Phys. 70(2019), Art.57, 16pp.

6. K. Deng, H.A. Levine, *The role of critical exponents in blow-up theorems: The sequel*, J. Math. Anal. Appl. 243 (2000) 85–126.

7. H. Fujita, *On the blowing up of solutions of the Cauchy problems for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac, Sci. Univ. Tokyo Sect. IA 13(1966), 109-124

8. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

9. K. Hayakawa, *On the nonexistence of global solutions of some semilinear parabolic equations*, Proc. Jpn. Acad. 49 (1973), 503–525.

10. B. Hu, *Blow-up Theories for Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 2018. Springer, Heidelberg, 2011.

11. K. Kobayashi, T. Siaro, and H. Tanaka, *On the blow-up problem for semilinear heat equations*, J. Math. Soc. Jpn. 29 (1977), 407–424.

12. H. A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. 32 (2) (1990) 262–288.

13. M. Loayza and C. S. da Paixão, *Existence and non-existence of global solutions for a semilinear heat equation on a general domain*, Electron. J. Differential Equations 2014, No. 168, 9 pp.

14. P. Meier, *On the critical exponent for reaction-diffusion equations*, Arch. Rational Mech. Anal. 109(1990), 63-71.

15. Y. Qi, *The critical exponents of parabolic equations and blow-up in* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh 128A (1998), 123–136.

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