Received ...; | Revised ...; | Accepted ...

DOI: xxx/xxxx

ARTICLE TYPE

A Necessary and Sufficient Conditions for the Global Existence of Solutions to Reaction-Diffusion Equations on $\mathbb{R}^N$

Soon-Yeong Chung[1] | Jaeho Hwang[2]

[1]Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 04107, Republic of Korea
[2]Research Institute for Basic Science, Sogang University, Seoul 04107, Republic of Korea

Correspondence
*Soon-Yeong Chung. Email: sychung@sogang.ac.kr
ORCID : 0000-0001-5671-2806

Summary

A necessary-sufficient condition for the existence or nonexistence of global solutions to the following reaction-diffusion equations

$$\begin{cases} u_t = \Delta u + \psi(t)u^p, & \text{in } \mathbb{R}^N \times (0,t^*), \\ u(\cdot,0) = u_0 \geq 0, & \text{in } \mathbb{R}^N, \end{cases}$$

has not been known and remained as an open problem for a few decades. The purpose of this paper is to resolve this problem completely, even for more general source $\psi(t)f(u)$ as follows:

There is a global solution to the equation if and only if

$$\int_0^\infty \psi(t)\frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty}dt < \infty$$

for some nonnegative and nontrivial $u_0 \in C_0(\mathbb{R}^N)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup on $\mathbb{R}^N$.

KEYWORDS:
[MSC2020] 35B44; 35K57; 35K61
Semilinear parabolic equation, Fujita blow-up, Critical exponent

1 | INTRODUCTION

In this paper, we study the existence and nonexistence of global solutions to the following nonlinear reaction-diffusion equations

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \psi(t)f(u(x,t)), & (x,t) \in \mathbb{R}^N \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where the initial data u_0 is a nontrivial and nonnegative $C_0(\mathbb{R}^N)$-function, the function f is a locally Lipschitz continuous and nonnegative function on $(0,\infty)$ with $f(0) = 0$ and $f(u) > 0$ for $u > 0$, and the function ψ is a nonnegative and continuous function on $[0,\infty)$. Then it is known that the local existence of the solutions of the equation (1) is guaranteed (see[9]).

The reaction-diffusion equations (1) reflect rich practical backgrounds such as heat propagations, chemical processes, and so on (see[15] and references therein). Therefore, researchers in various fields have interested in the existence and nonexistence of the global solutions to the reaction-diffusion equations. In other words, studying a necessary and sufficient condition for the existence of the global solution is meaningful and applicable. Therefore, the studies on the existence and nonexistence of the global solutions to the reaction-diffusion equation have been attracted lots of researchers (see the survey articles [8,13]).

In 1966, Fujita [10] firstly studied the existence and nonexistence of the global solutions for the following reaction-diffusion equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^p(x,t), & (x,t) \in \mathbb{R}^N \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases}$$

where $p > 1$ and obtained that

(i) if $1 < p < p^*$, then there is no global solution for any initial data,

(ii) whereas, if $p > p^*$, then there exists a global solution whenever the initial data is sufficiently small (less than a small Gaussian),

where $p^* = 1 + \frac{2}{N}$ is called the critical exponent. In fact, researchers obtained that there is no global solution for $p = p^*$, after Fujita's result (see [11] for the case $N = 1$ or 2 and [1] for the case $N \geq 3$).

In his pioneering paper [17], Meier studied the existence and nonexistence of global solutions to the equations

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \psi(t)u^p(x,t), & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0 \geq 0, & x \in \Omega, \end{cases} \tag{2}$$

where Ω is general domain in $\mathbb{R}^N$, $p > 1$, and ψ is a nonnegative continuous function, under the Dirichlet boundary condition, and proved the following theorem:

Theorem 1 (See Meier [17])**.** Assume that $p > 1$ and $\psi \in C[0,\infty)$.

(i) If $\limsup_{t\to\infty} \|S(t)w_0\|_\infty^{p-1} \int_0^t \psi(\tau)d\tau = \infty$ for all $w_0 \in C_0(\Omega)$, then there is no global solution for any initial data.

(ii) If $\int_0^\infty \psi(\tau)\|S(\tau)w_0\|_\infty^{p-1}d\tau < \infty$ for some $u_0 \in C_0(\Omega)$, then there exists global positive solution for sufficiently small initial data.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition on $\partial\Omega$.

Meier give two sufficient conditions for the existence and nonexistence of the global solutions. However, a necessary and sufficient condition for the existence of the global solutions to the equation (1) is unknown, even for the case $f(u) = u^p$ and has remained as an open problem for a few decades. To our best knowledge, researches of necessary and sufficient conditions for the global existence of solutions for the reaction-diffusion equations in the current literature consider several specific source terms such as $t^\sigma u^p$, $e^{\beta t}u^p$, and so on (see [3,16,17]).

In fact, recent studies of the existence and nonexistence of the global solutions discuss only sufficient conditions for the blow-up solutions and global solutions (for example, see [4,5,14]). In conclusion, the open problem has faced methodological limitations and there has been no progress in research on necessary and sufficient conditions.

From the above point of view, the purpose of this paper is to give a necessary and sufficient condition for the existence of the global solutions, for more general source term $\psi(t)f(u)$.

In order to solve the open problem for more general reaction term $\psi(t)f(u)$ instead of $\psi(t)u^p$, we need to use the minorant function f_m and the majorant function f_M as follows:

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)} \text{ and } f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}.$$

In fact, the minorant function f_m and the majorant function f_M were firstly discussed in [7].

On the other hand, it is necessary to assume the growth conditions for the function f near the 0 and ∞, respectively, to discuss the existence and nonexistence of the global solutions. Regarding this fact, let us introduce related comments mentioned by Bandle and Brunner [2]:

"Heuristically, it is clear that for the Fujita property to hold, not only the growth of f at infinity plays a crucial role, but also its behavior near zero. It has to be sufficiently large, to prevail the dissipative effect of the Laplace operator on solutions with small initial data."

From this point of view, the following conditions are naturally necessary:

$$\int_1^\infty \frac{dw}{f_m(w)} < \infty \quad \text{and} \quad \lim_{w\to 0+} \frac{f_M(w)}{w} = 0. \tag{3}$$

Finally, we state the main theorem.

Theorem 2. Suppose that the function f is convex satisfying the condition (3) and the function ψ is a nonnegative and continuous function. Then the following statements are equivalent.

(i) $\int_0^\infty \psi(t)\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}dt = \infty$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\mathbb{R}^N)$.

(ii) $\int_1^\infty \psi(t)t^{\frac{N}{2}} f\left(\epsilon\, t^{-\frac{N}{2}}\right) dt = \infty$, for every $\epsilon > 0$.

(iii) There is no global solution to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\mathbb{R}^N)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup on $\mathbb{R}^N$.

Theorem 2 is the form of a necessary and sufficient condition for the existence of the global solutions of the equation (1). Therefore, we can see 'completely' whether or not we have global solutions. Also, the open problem mentioned above is solved with more general source term $\psi(t)f(u)$.

If $f(u) = u^p$, then the case $p = p^*$ and $p \neq p^*$ have been dealt in a different way and cannot be solved at the same time, in general (see [1,10,11]). However, we prove the cases all at once.

In the proof of the main theorem, the decay of $\|S(t)u_0\|_\infty$ plays an important role. From this fact, to discuss the existence and nonexistence of the global solutions on various domains such as the half space, cone, string, and so on, it may be necessary to estimate the decay of $\|S(t)u_0\|_\infty$ for each domains. From this point of view, our result is meaningful in mathematics and blow-up theory, because the result suggests research directions.

We organized this paper as follows: In Section 2, we discuss Meier's criterion. We introduce the multiplicative minorants and majorants and discuss main results in Section 3.

2 | DISCUSSION ON MEIER'S CONDITIONS

The purpose of this section is to discuss a necessary and sufficient condition for the existence of the global solutions, which has not been known and remained as an open problem. Let us deal with sufficient conditions for the blow-up solutions and global solutions to check that these conditions can be a necessary and sufficient condition.

From this point of view, let us discuss Meier's conditions on $\mathbb{R}^N$. First of all, the heat semigroup $(S(t))_{t\geq 0}$ on $\mathbb{R}^N$ is given by

$$S(t)f(x) := \int_{\mathbb{R}^N} \Gamma(x-y,t)f(y)dy,\ x \in \mathbb{R}^N,\ t \geq 0,$$

where f is a function in $C_0(\mathbb{R}^N)$. Here, the function Γ is the heat kernel defined by

$$\Gamma(x,t) := \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}},$$

which satisfies $\left(\frac{\partial}{\partial t} - \Delta\right)\Gamma(x,t) = 0$ for $t > 0$. Then it is known that $\|S(t)f\|_\infty \sim t^{-\frac{N}{2}}$ for a sufficiently large t, whenever $f \in C_0\left(\mathbb{R}^N\right) \cap L^1\left(\mathbb{R}^N\right)$ is a nonnegative and nontrivial function (we will discuss this in Remark 1 (iii)). Therefore, if the nonnegative and nontrivial initial data u_0 is in $C_0\left(\mathbb{R}^N\right) \cap L^1\left(\mathbb{R}^N\right)$, then Theorem 1 can be understood as follows:

- If

$$(C1)\ :\ \limsup_{t\to\infty} t^{-\frac{(p-1)N}{2}} \int_1^t \psi(\tau)d\tau = \infty,$$

then there is no global solution to the equation (2) for any nonnegative and nontrivial initial data.

· If

$$(C2)\ :\ \int_1^\infty \psi(t)t^{-\frac{(p-1)N}{2}}dt < \infty,$$

then there exists a global solution to the equation (2) for sufficiently small initial data.

Now, consider the function ψ defined by $\psi(t) := t^{-1+\frac{(p-1)N}{2}}(\ln(t+1))^{-\delta}$ for $0 \le \delta \le 1$. Then the condition $(C1)$ can be written as

$$\begin{aligned}\limsup_{t\to\infty} t^{-\frac{(p-1)N}{2}} \int_1^t \psi(\tau)d\tau =& \limsup_{t\to\infty} t^{-\frac{(p-1)N}{2}} \int_1^t \tau^{-1+\frac{(p-1)N}{2}}(\ln\tau)^{-\delta}d\tau \\ \le& 2\limsup_{t\to\infty} t^{-\frac{(p-1)N}{2}} \int_0^{\ln t} e^{\frac{(p-1)Ns}{2}} s^{-\delta}ds \\ \le& 2\limsup_{t\to\infty} t^{-\frac{(p-1)N}{2}} \int_0^{\ln t} e^{\frac{(p-1)Ns}{2}} ds \\ \le& (p-1)N < \infty\end{aligned}$$

and the condition $(C2)$ can be written as

$$\int_1^\infty \psi(t)t^{-\frac{(p-1)N}{2}}dt \ge \int_1^\infty (t+1)^{-1}(\ln(t+1))^{-\delta}dt = \int_{\ln 2}^\infty s^{-\delta}ds = \infty.$$

Therefore, we cannot determine whether the solution exists globally or not, by using Theorem 1. In fact, we show in this paper that the condition $(C1)$ is no longer necessary condition for the nonexistence of global solution. Therefore, the main part of this paper is focused on the condition $(C2)$ to see whether $(C2)$ is necessary and sufficient condition of the existence of the global solution.

3 | MAIN RESULTS

In this section, we firstly introduce the minorant function and the majorant function. Next, we prove the main theorem by using the minorant function and majorant function.

First of all, we introduce the definition of the minorant function and majorant function of the function f.

Definition 1. For a function f, the minorant function $f_m : [0,\infty) \to [0,\infty)$ and the majorant function $f_M : [0,\infty) \to [0,\infty)$ are defined by

$$\begin{aligned} f_m(u) &:= \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \quad u \ge 0, \\ f_M(u) &:= \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \quad u \ge 0. \end{aligned}$$

Then the functions f_m and f_M satisfy that $f(\alpha)f_m(u) \le f(\alpha u) \le f(\alpha)f_M(u)$, $0 < \alpha < 1$, $u > 0$. Then it is easy to see that the functions f, f_m, and f_M satisfy that

$$f_m(u)f(\alpha) \le f(u\alpha) \le f_M(u)f(\alpha),\ \ u > 0,\ \ 0 < \alpha < 1.$$

In fact, the properties of the minorant function and the majorant function were discussed in[7].

Now, we introduce the definition of the blow-up solutions and global solutions.

Definition 2. We say that a solution u exists globally, if $\|u(\cdot,t)\|_\infty$ exists for each time $t \ge 0$. On the other hand, a solution u blows up at finite time t^*, if $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*$.

Next, it is well-known that the solution u to the equation (1) has an integral representation:

$$\begin{aligned} u(x,t) =& S(t)u_0(x) + \int_0^t S(t-\tau)\psi(\tau)f(u(x,\tau))d\tau \\ =& \int_{\mathbb{R}^N} \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} \psi(\tau)\Gamma(x-y,t-\tau)f(u(y,\tau))dyd\tau, \end{aligned} \tag{4}$$

where Γ is the heat kernel defined by

$$\Gamma(x,t) := \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}.$$

Finally, we prove Theorem 2.

Proof of Theorem 2. **(i)⇒(ii)** : Let $\epsilon > 0$ be arbitrary given. Consider a nonnegative and nontrivial $u_0 \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ satisfying that $c_0 < \epsilon$, where $c_0 := (4\pi)^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}$. Then we have

$$\begin{aligned} S(t)u_0(x) =& \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \\ \le& (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u_0(y)dy \\ \le& c_0 t^{-\frac{N}{2}}, \end{aligned}$$

for every $t > 0$. This implies that $\|S(t)u_0\|_\infty \le c_0 t^{-\frac{N}{2}}$ for all $t \ge 0$. Since f is convex, $\frac{f(u)}{u}$ is nondecreasing in u. Then it follows that

$$\frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty} \le \frac{f\left(c_0 t^{-\frac{N}{2}}\right)}{c_0 t^{-\frac{N}{2}}} \le \frac{f\left(\epsilon t^{-\frac{N}{2}}\right)}{c_0 t^{-\frac{N}{2}}},$$

for every $t > 0$. Then this implies that

$$\int_0^\infty \psi(t)\frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty}dt \le \int_0^1 \psi(t)\frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty}dt + \int_1^\infty \psi(t)\frac{f\left(\epsilon t^{-\frac{N}{2}}\right)}{c_0 t^{-\frac{N}{2}}}dt,$$

which completes the proof.

(ii)⇒(iii) : Suppose that there exists a nonnegative and nontrivial initial data u_0 such that the solution u exists globally, on the contrary. Without loss of generality, we may assume that $u_0(y) \ge c_1 > 0$ for $|y| < 1$. Then we obtain from (4) and the semigroup property that

$$S(t)u(x,t) = u_1(x,t) + u_2(x,t), \tag{5}$$

where

$$u_1(x,t) = S(2t)u_0(x)$$

and

$$u_2(x,t) = \int_0^t S(2t-\tau)\psi(\tau)f(u(x,\tau))d\tau.$$

Firstly, we will show that

$$u_1(x,t) \ge C_1 t^{-\frac{N}{2}},\ t > 1, \tag{6}$$

where C_1 is a positive constant which depends on u_0 and N. We obtain

$$\begin{aligned} \|S(2t)u_0\|_\infty \ge S(2t)u_0(x)\big|_{x=0} =& \frac{1}{(8\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{8t}} u_0(y)dy \\ \ge& \frac{c_1}{(8\pi t)^{\frac{N}{2}}} \int_{|y|<1} e^{-\frac{|y|^2}{8}} dy \\ \ge& C_1 t^{-\frac{N}{2}}, \end{aligned} \tag{7}$$

where $C_1 = \frac{c_1}{(8\pi)^{\frac{N}{2}}} \int_{|y|<1} e^{-\frac{|y|^2}{8}} dy$, for all $t > 1$, which means that inequality (6) is true.

Next, it is clear that if $0 \le \tau \le t$, then $\tau \le 2t - \tau$ and so

$$(2t-\tau)^{\frac{N}{2}} \Gamma(x, 2t-\tau) \ge \tau^{\frac{N}{2}} \Gamma(x,\tau)$$

for all $x \in \mathbb{R}^N$, by the definition of the heat kernel Γ. It follows that

$$u_2(x,t) \ge \int_0^t \psi(\tau) \left(\frac{\tau}{2t-\tau}\right)^{\frac{N}{2}} S(\tau) f(u(x,\tau)) d\tau. \tag{8}$$

Then we apply Jensen's inequality to inequality (8) that

$$\begin{aligned} u_2(x,t) &\ge \int_0^t \psi(\tau) \left(\frac{\tau}{2t-\tau}\right)^{\frac{N}{2}} f\left(S(\tau)u(x,\tau)\right) d\tau \\ &\ge (2t)^{-\frac{N}{2}} \int_0^t \psi(\tau)\tau^{\frac{N}{2}} f\left(S(\tau)u(x,\tau)\right) d\tau. \end{aligned}$$

Put $v(x,t) := C_1 + 2^{-\frac{N}{2}} \int_0^t \psi(\tau)\tau^{\frac{N}{2}} f\left(S(\tau)u(x,\tau)\right) d\tau$. Then v exists globally, since u exists globally. Then inequalities (5), (6), and (8) imply that

$$S(t)u(x,t) \ge t^{-\frac{N}{2}} v(x,t),$$

for $t > 1$. It follows that

$$v'(x,t) = 2^{-\frac{N}{2}} \psi(t) t^{\frac{N}{2}} f\left(S(t)u(x,t)\right) \ge 2^{-\frac{N}{2}} \psi(t) t^{\frac{N}{2}} f\left(\frac{v(t)}{t^{\frac{N}{2}}}\right) \tag{9}$$

for $t > 1$, since f is nondecreasing. Now, take $0 < \epsilon < 1$ sufficiently small satisfying that $\epsilon < C_1$. We note that the value of the minorant function f_m can be 0 near the origin. However, $f_m(u) > 0$ for all $u \ge 1$. This implies that $f_m\left(\frac{C_1}{\epsilon}\right) > 0$. Then we obtain from $v(x,t) > C_1$ that $f_m\left(\frac{v(x,t)}{\epsilon}\right) > 0$ for all $t > 1$. Thanks to the minorant function f_m, we obtain from

$$f\left(v(t)t^{-\frac{N}{2}}\right) \ge f_m\left(\frac{v(x,t)}{\epsilon}\right) f\left(\epsilon\, t^{-\frac{N}{2}}\right)$$

that

$$\frac{v'(x,t)}{f_m\left(\frac{v(x,t)}{\epsilon}\right)} \ge 2^{-\frac{N}{2}} \psi(t) t^{\frac{N}{2}} f\left(\epsilon\, t^{-\frac{N}{2}}\right),$$

for $t > 1$. Integrating from 1 to t, it follows that

$$F(v(x,1)) - F(v(x,t)) \ge 2^{-\frac{N}{2}} \epsilon \int_1^t \psi(\tau)\tau^{\frac{N}{2}} f\left(\epsilon\, \tau^{-\frac{N}{2}}\right) d\tau,$$

where $F(v) := \int_{\frac{v}{\epsilon}}^{\infty} \frac{dw}{f_m(w)}$, $v > 0$. Then it is clear that F is positive, decreasing, and bijection. Also, $\lim_{v\to\infty} F(v) = 0$. Hence, we obtain

$$v(x,t) \ge F^{-1}\left[F(v(x,1)) - 2^{-\frac{N}{2}} \epsilon \int_1^t \psi(\tau)\tau^{\frac{N}{2}} f\left(\epsilon\, \tau^{-\frac{N}{2}}\right) d\tau\right]$$

for $t > 1$, which implies that v cannot be global. Hence, u blows up at finite time t^*.

(iii)⇒(i) : In this proof, we use the monotone sequence argument (see[4,14]). Suppose that there exist a nonnegative nontrivial function $w_0 \in C_0(\Omega)$ and $\epsilon > 0$ satisfying that

$$\alpha := \int_0^\infty \psi(t) \frac{f\left(\|S(t)w_0\|_\infty\right)}{\|S(t)w_0\|_\infty} dt < \infty.$$

Now, we will show that there exists a global solution u for some initial data. Take a nonnegative function $u_0 := \lambda w_0$, where the constant $\lambda > 0$ satisfies that

$$\lambda(1+\alpha) \le 1 \quad \text{and} \quad 0 < \frac{f_M\left(\lambda(1+\alpha)\right)}{\lambda} \le 1. \tag{10}$$

In fact, λ exists because the majorant function f_M satisfies $\lim_{w\to 0+}\frac{f_M(w)}{w}=0$. Also, $\lambda(1+\alpha)\le 1$ implies that $0<f_M(\lambda(1+\alpha))<\infty$. Now, we define a sequence $\{v_n\}_{(n\ge 0)}$ by

$$\begin{cases} v_0(x,t)=S(t)u_0(x), \\ v_n(x,t)=S(t)u_0(x)+\int_0^t \psi(\tau)S(t-\tau)f(v_{n-1}(x,\tau))d\tau,\ \ n\ge 1. \end{cases}$$

Then it is clear that $v_0(x,t)=S(t)u_0(x)\le(1+\alpha)S(t)u_0(x)$, for all $x\in\mathbb{R}^N$ and $t\ge 0$. Now, we will show by the induction that

$$0\le v_n(x,t)\le(1+\alpha)S(t)u_0, \tag{11}$$

for all $x\in\mathbb{R}^N$ and $t\ge 0$, for each $n\ge 0$. Assume that the inequality (11) holds for $n=k$. Then we obtain

$$\begin{aligned} v_{k+1}(x,t)=&S(t)u_0(x)+\int_0^t \psi(\tau)S(t-\tau)f(v_k(x,\tau))d\tau \\ \le&S(t)u_0(x)+\int_0^t \psi(\tau)S(t-\tau)f((1+\alpha)S(\tau)u_0(x))d\tau. \end{aligned}$$

Since $\frac{f(u)}{u}$ is nondecreasing, we obtain that

$$\begin{aligned} v_{k+1}(x,t)\le&S(t)u_0(x)+\int_0^t \psi(\tau)S(t-\tau)\frac{f((1+\alpha)S(\tau)u_0(x))}{S(\tau)u_0(x)}S(\tau)u_0(x)d\tau \\ \le&S(t)u_0(x)+S(t)u_0(x)\int_0^t \psi(\tau)\frac{f((1+\alpha)\|S(\tau)u_0\|_\infty)}{\|S(\tau)u_0\|_\infty}d\tau. \end{aligned}$$

It follows from the majorant function that

$$\begin{aligned} v_{k+1}(x,t)\le&S(t)u_0(x)+S(t)u_0(x)\int_0^t \psi(\tau)\frac{f(\lambda(1+\alpha)\|S(\tau)w_0\|_\infty)}{\lambda\|S(\tau)w_0\|_\infty}d\tau \\ \le&S(t)u_0(x)+\frac{f_M\left(\lambda(1+\alpha)\right)}{\lambda}S(t)u_0(x)\int_0^t \psi(\tau)\frac{f(\|S(\tau)w_0\|_\infty)}{\|S(\tau)w_0\|_\infty}d\tau \\ \le&S(t)u_0(x)+\alpha S(t)u_0(x), \end{aligned}$$

since $\frac{f(u)}{u}$ is nondecreasing and $u_0=\lambda w_0$. Since λ satisfies the inequality (10), we obtain that the inequality (11) is true for all $n\ge 0$. Also, it is easy to see that $v_n\le v_{n+1}$ for $n\ge 0$. By monotone convergence theorem, there exists $u(x,t)=\lim_{n\to\infty}v_n(x,t)$ which exists globally. Then we finally obtain that

$$u(x,t)=S(t)u_0(x)+\int_0^t \psi(\tau)S(t-\tau)f(u(x,\tau))d\tau,$$

which is desired. □

Remark 1. (i) One way to generalize the critical exponent p^* (for example, see [10,16]) is to introduce the critical set Λ_ψ:

$$\Lambda_\psi := \left\{ f : [0,\infty)\to[0,\infty) \;\middle|\; \int_1^\infty \psi(t)t^{\frac{N}{2}}f\left(t^{-\frac{N}{2}}\right)dt=\infty \right\}.$$

If $\psi(t)=t^\sigma$ and $f(u)=u^p$, then we see that $f\in\Lambda_\psi$ if and only if $1<p\le p^*=1+\frac{2+2s}{N}$, which was dealt in [16]. In fact, the critical set was firstly introduced in [6].

(ii) The case $1<p<p^*$ and the case $p=p^*$ have been derived in different ways in general (see [8,12]). From this point of view, our method is an elegant method in terms of an unified approach.

(iii) In view of proof of Theorem 2, we see that $\|S(t)u_0\|_\infty \sim t^{-\frac{N}{2}}$ for sufficiently large $t > 1$ for a limited class of u_0. More precisely, for a nonnegative and nontrivial $u_0 \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, it is true that $\|S(t)u_0\|_\infty \sim t^{-\frac{N}{2}}$ for sufficiently large $t > 1$. However, for $u_0 \in C_0(\mathbb{R}^N)$, $\|S(t)u_0\|_\infty \sim t^{-\frac{N}{2}}$ is not true, in general. For example, let us consider a nonnegative $C_0(\mathbb{R}^N)$-function u_0 such that

$$u_0(y) = \begin{cases} 1, & |y| \le 1, \\ \frac{1}{\sqrt{|y|}}, & |y| > 1, \end{cases}$$

where the norm is the Euclidean norm. Then it is clear that u_0 is not in $L^1(\mathbb{R}^N)$. Also, for $\mathbf{1} := (1, \cdots, 1)$, we have

$$\begin{aligned} \|S(t)u_0\|_\infty \ge S(t)u_0(x)\big|_{x=\mathbf{1}} &\ge \frac{1}{\pi^{\frac{N}{2}}} \left[\int_0^\infty e^{-r^2} \frac{1}{\sqrt{1 + r\sqrt{4t}}} dr \right]^N \\ &\ge \frac{1}{t^{\frac{N}{4}} \pi^{\frac{N}{2}}} \left[\int_0^\infty e^{-r^2} \frac{1}{\sqrt{1 + 2r}} dr \right]^N, \end{aligned}$$

for $t > 1$. Hence, we obtain by putting $\bar{c} := \pi^{-\frac{N}{2}} \left[\int_0^\infty e^{-r^2} \frac{1}{\sqrt{1+r}} dr \right]^N$ that $\|S(t)u_0\|_\infty \ge \bar{c} t^{-\frac{N}{4}}$ for $t > 1$, which implies that $\|S(t)u_0\|_\infty \sim t^{-\frac{N}{2}}$ for $t > 1$ is not true for $u_0 \in C_0(\mathbb{R}^N)$. In fact, this means that the equivalence of the condition (i) and (ii) in Theorem 2 is not trivial.

(iv) Suppose that f satisfies $f(\alpha u) \sim f(\alpha) f(u)$ for $0 < \alpha < 1$ and $u > 0$. Then it is clear that $f_m(u) = c_1 f(u)$ and $f_M(u) = c_2 f(u)$ for some positive constants c_1 and c_2 Then condition (3) can be understood as

$$\int_1^\infty \frac{dw}{f(w)} < \infty \quad \text{and} \quad \lim_{w \to 0+} \frac{f(w)}{w} = 0.$$

In fact, the condition $\int_1^\infty \frac{dw}{f(w)} < \infty$ is a necessary condition for the blow-up theory (see [12]) and $\lim_{w \to 0+} \frac{f(w)}{w} = 0$ is natural to discuss the nonexistence of the global solution (see [2]). From this point of view, the condition (3) is quite natural to discuss the existence and nonexistenc of the global solution using the minorant function and majorant function.

On the other hand, if the function f in the equation (1) is not multiplicative, then we cannot apply Meier's results. For example, let us consider the function $f(u) = \frac{u^3+u^2}{2}$. Then it is easy to see that $u^2 \le f(u) \le u^3$ for $u \ge 1$ and $u^3 \le f(u) \le u^2$ for $0 \le u \le 1$. Therefore, we cannot determine p in the case of $f(u) = \frac{u^3+u^2}{2}$ in Theorem 1. From this point of view, we have to consider a new method, so called the minorant method to deal with a function f which is not multiplicative. In conclusion, we provide a formula $\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}$ instead of $\|S(t)u_0\|_\infty^{p-1}$ to give a criterion of the existence of the global solution when the source term is $\psi(t) f(u)$.

Example 2. Let $\psi(t) := t^r + s^s$ and $f(u) := u^p + u^q$, where $r \ge s \ge 0$ and $p \ge q > 1$. Then we easily see that

$$f_m(u) = \min \left\{ u^q, \frac{u^p + u^q}{2} \right\} \text{ and } f_M(u) = \max \left\{ u^q, \frac{u^p + u^q}{2} \right\}.$$

This implies that $\int_1^\infty \frac{ds}{f_m(s)} < \infty$. On the other hand,

$$\begin{aligned} &\int_0^\infty (t^r + t^s)\, t^{\frac{N}{2}} \left(\epsilon^p t^{-\frac{pN}{2}} + \epsilon^q t^{-\frac{qN}{2}} \right) dt \\ &= \int_0^\infty \epsilon^p \left(t^{(r - \frac{(p-1)N}{2})t} + e^{(s - \frac{(p-1)N}{2})t} \right) + \epsilon^q \left(e^{(r - \frac{(q-1)N}{2})t} + e^{(s - \frac{(q-1)N}{2})t} \right) dt. \end{aligned}$$

Therefore, there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$ if and only if $r \ge \frac{(q-1)N}{2}$.

Remark 3. In 2014, Loayza and Paixão [14] studied the conditions for existence and nonexistence of the global solutions to the equation (1) under the general domain and obtained the following statements:

(i) for every $w_0 \in C_0(\Omega)$, there exist $\tau > 0$ such that

$$\int_{\|S(\tau)w_0\|_\infty}^{\infty} \frac{dw}{f(w)} \leq \int_0^{\tau} \psi(\sigma)d\sigma, \tag{12}$$

then there is no global solution u for every initial data,

(ii) the solution u exists globally for small initial data, whenever

$$\int_0^{\infty} \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}dt < 1, \tag{13}$$

for some $w_0 \in C_0(\Omega)$.

In fact, their results were not given in a form of a necessary-sufficient condition. Moreover, the conditions (12) and (13) don't seem to be lots of relevant. However, Theorem 2 implies that the conditions (12) and (13) have a strong relation, whenever f satisfies the condition (3) and $\Omega = \mathbb{R}^N$.

4 | FUNDING

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C1005348).

5 | CONFLICTS OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. 30 (1978), no. 1, 33–76.

2. C. Bandle and H. Brunner, *Blowup in diffusion equations: a survey*, J. Comput. Appl. Math. 97 (1998), no. 1-2, 3–22.

3. X. Bai, S. Zheng, and W. Wang, *Critical exponent for parabolic system with time-weighted sources in bounded domain*, J. Funct. Anal. 265 (2013), no. 6, 941–952.

4. R. Castillo and M. Loayza, *On the critical exponent for some semilinear reaction–diffusion systems on general domains*, J. Math. Anal. Appl 428 (2015), 1117–1134.

5. R. Castillo and M. Loayza, *Global existence and blow up for a coupled parabolic system with time-weighted sources on a general domain*, Z. Angew. Math. Phys. 70(2019), Art.57, 16pp.

6. S. -Y. Chung, M. -J. Choi, and J. -H. Park, *On the critical set for Fujita type blow-up of solutions to the discrete Laplacian parabolic equations with nonlinear source on networks*, Comput. Math. Appl. 78 (2019), no. 6, 1838–1850.

7. S. -Y. Chung and J. Hwang, *A necessary and sufficient condition for the existence of global solutions to discrete semilinear parabolic equations on networks*, Chaos Solitons Fractals 158 (2022), Paper No. 112055.

8. K. Deng and H.A. Levine, *The role of critical exponents in blow-up theorems: The sequel,* J. Math. Anal. Appl. 243 (2000), 85–126.

9. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964 xiv+347 pp.

10. H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966) 109–124.

11. K. Hayakawa, *On the nonexistence of global solutions of some semilinear parabolic equations*, Proc. Jpn. Acad. 49 (1973), 503–525.

12. B. Hu, *Blow-up Theories for Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 2018. Springer, Heidelberg, 2011.

13. H. A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. 32 (2) (1990) 262–288.

14. M. Loayza and C. S. da Paixão, *Existence and non-existence of global solutions for a semilinear heat equation on a general domain*, Electron. J. Differential Equations 2014, No. 168, 9 pp.

15. R.G. Pinsky, *Existence and nonexistence of global solutions for* $u_t = \Delta u + a(x)u^p$ *in* $\mathbb{R}^d$, J. Differential Equations 133 (1997) 152–177.

16. Y. Qi, *The critical exponents of parabolic equations and blow-up in* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh 128A (1998), 123–136.

17. P. Meier, *On the critical exponent for reaction-diffusion equations*, Arch. Rational Mech. Anal. 109 (1990), no. 1, 63–71.

$\square$