An eigenvalue problem for nonlinear Schrödinger-Poisson system with steep potential well

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Abstract

In this paper, we study an eigenvalue problem for Schrödinger-Poisson system with indefinite nonlinearity and potential well as follows:

$$\begin{cases} -\Delta u + \mu V(x)u + K(x)\phi u = \lambda f(x)u + g(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $4 \leq p < 6$, the parameters $\mu, \lambda > 0$, $V \in C(\mathbb{R}^3)$ is a potential well, and the functions $f \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $g \in L^{\infty}(\mathbb{R}^3)$ are allowed to be sign-changing. It is well known that such a system with the potential being positive constant has two positive solutions when $\lim_{|x|\to\infty} g(x) = g_\infty < 0$, $K = 0$ in the set $\{x \in \mathbb{R}^3 : g(x) = 0\}$ and $\lambda > \lambda_1(f)$ with near $\lambda_1(f)$, where $\lambda_1(f)$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ (see e.g. Huang et al., J. Differential Equations **255**, 2463 (2013)). The main purpose is to obtain the existence and multiplicity of positive solutions without the above assumptions for g and K. The results are obtained via variational method and steep potential. Furthermore, we also consider the concentration of solutions as $\mu \to \infty$.

Keywords. Schrödinger-Poisson system, eigenvalue problem, steep potential well, mountain pass theory.
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1 Introduction

In the paper, we are concerned the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = h(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

This system has been first derived from Benci and Fortunato [1] in order to study the stationary solutions of Schrödinger equations coupled with Maxwell equations. It describes the interaction of a charged particle with its own electrostatic field. The unknown functions u and ϕ represent the wave functions associated to the particle and electric potential, respectively. The functions V and K denote an external potential and nonnegative density charge, respectively. The presence of the nonlinear term $h(x,u)$ simulates the interaction effect among many particles. We refer the reader to [1] and [2] for more details.

In recent years, system (1.1) has been widely studied under variant assumptions on V, K and h. See, for example, [1–7] for the autonomous case and [4, 8–20] for the non-autonomous case. In

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particular, many papers have been devoted to the nonlinear term $h(x,u) = g(x)|u|^{p-2}u, 2 < p < 6$. For which some assumptions on V, K and g are considered in order to overcome the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ into $L^r(\mathbb{R}^3)$, $2 \leq r < 6$, since system (1.1) is on the whole space $\mathbb{R}^3$.

In the work [9], Cerami and Vaira studied the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = g(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\lambda \equiv 1, 4 < p < 6$, g and K are nonnegative real functions,

$$\lim_{|x|\to\infty} g(x) = g_\infty > 0, \quad \lim_{|x|\to\infty} K(x) = K_\infty = 0 \text{ and } K \in L^2(\mathbb{R}^3).$$

They obtained the existence of bound and ground state positive solutions by using the Nehari manifold method and establishing a global compactness lemma to overcome the lack of compactness. Later, Vaira [10] considered system (1.2) with $\lambda \in \mathbb{R}$, g and K being nonnegative functions,

$$\lim_{|x|\to\infty} g(x) = g_\infty > 0, \quad \lim_{|x|\to\infty} K(x) = K_\infty > 0 \text{ and } K - K_\infty \in L^2(\mathbb{R}^3).$$

For which the positive ground state solutions were obtained in the cases where $2 < p < 6$ if $\lambda < 0$ and $4 < p < 6$ if $\lambda > 0$ by using the Nehari manifold method.

Recently, Huang et al. [11] studied Schrödinger-Poisson system with indefinite nonlinearity, namely

$$\begin{cases} -\Delta u + u + K(x)\phi u = \lambda f(x)u + g(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $4 < p < 6$, K and f are nonnegative functions, $K \in L^2(\mathbb{R}^3), f \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $g \in C(\mathbb{R}^3)$ which changes sign in $\mathbb{R}^3$,

$$\lim_{|x|\to\infty} g(x) = g_\infty < 0 \quad \text{and} \quad K = 0 \text{ a.e. in the set } \{x \in \mathbb{R}^3 : g(x) = 0\}. \tag{1.4}$$

In which they proved the existence of two positive solutions in $\lambda > \lambda_1(f)$ and near $\lambda_1(f)$, where $\lambda_1(f)$ is the first eigenvalue of $-\Delta u + u = \lambda f(x)u$ in $H^1(\mathbb{R}^3)$, whose corresponding eigenfunction is denoted by ϕ_1. Besides the condition about g in (1.4) is different from [9, 10], it is worth noting that they do not need the sign condition $\int_{\mathbb{R}^3} g(x)\phi_1^p dx < 0$, which has been shown to be a necessary condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearity, see [21–23]. Later, Chen [12] studied the case of $K \in L^\infty(\mathbb{R}^3)$ (more precisely, $K(x) \equiv 1$) for system (1.3). Since the condition about K in (1.4) is not established now, the author assume the additional condition

$$|\{x \in \mathbb{R}^3 : g(x) = 0\}| = 0 \tag{1.5}$$

to ensure that the existence of two positive solutions can be obtained in $\lambda > \lambda_1(f)$ and near $\lambda_1(f)$.

Very recently, Shen and Han [16] studied system (1.3) with $p = 4$, f and K being nonnegative functions, $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, $f \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $g \in C(\mathbb{R}^3)$ to be sign-changing and $\lim_{|x|\to\infty} g(x) = g_\infty < 0$. The authors do not need the condition about K in (1.4) that $K = 0$ a.e. in $\{x \in \mathbb{R}^3 : g(x) = 0\}$, and assume the following condition:

$$\int_{\mathbb{R}^3} g(x)\phi_1^4(x)dx - \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)\phi_1^2(x)\phi_1^2(y)}{|x-y|}dydx < 0. \tag{1.6}$$

For which the existence of two positive solutions in $\lambda > \lambda_1(f)$ and near $\lambda_1(f)$ was proved via the Nehari manifold method.

In recent year, many papers study Schrödinger-Poisson system with steep potential well, namely

$$\begin{cases} -\Delta u + \mu V(x)u + K(x)\phi u = h(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.7}$$

where $\mu > 0$ is a parameter and the potential V satisfies the following conditions:

(V1) V is a non-negative continuous function in $\mathbb{R}^3$.

(V2) There exists $c_0 > 0$ such that the set $\{V < c_0\} := \{x \in \mathbb{R}^3 : V(x) < c_0\}$ is nonempty and has finite positive measure.

(V3) $\Omega = int\{x \in \mathbb{R}^3 : V(x) = 0\}$ is nonempty bounded domain and has a smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^3 : V(x) = 0\}$.

The conditions (V1)-(V3) were first introduced by Bartsch and Wang [24] in the study of nonlinear Schrödinger equations. Here μV represents a potential well whose depth is controlled by the parameter μ, and it is called a steep potential well if μ large. Later, steep potential well was first applied into Schrödinger-Poisson system in [25], and then has been widely applied to the study of Schrödinger-Poisson system. We refer the reader to [25–32] and the references therein.

In the paper [26], Zhao et al. studied system (1.7) with $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3), K \geq 0$ and $h(x,u) = |u|^{p-2}u$. In which the existence of nontrivial solution and concentration results are obtained via variational methods when $3 < p < 6$. In particular, the potential V is allowed to be sign-changing for the case $4 < p < 6$. Later, Du et al. [27] considered system (1.7) with $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3), K \geq 0$ and $h(x,u) = g(x)F(u)$, where g is a positive bounded function and F is either asymptotically linear or asymptotically 3-linear at infinity. The existence and asymptotic behavior of solutions are proved via variational methods. As far as I know, system (1.7) with indefinite nonlinearity has not been studied.

Motivated by the above works [11, 12, 16, 26, 27], we consider Schrödinger-Poisson system in the following form:

$$\begin{cases} -\Delta u + \mu V(x)u + K(x)\phi u = \lambda f(x)u + g(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$SP_{\mu,\lambda}$}$$

where $4 \leq p < 6$, the parameters $\mu, \lambda > 0$, and the potential V satisfies conditions (V1)-(V3). The aim of this paper is to answer the following questions:

(I) Does system $(SP_{\mu,\lambda})$ with $4 \leq p < 6$ still admit multiple positive solutions without the conditions (1.4) and (1.5)?

(II) Can we consider system $(SP_{\mu,\lambda})$ with the weight functions f and g to be sign-changing?

(III) Can we deal with the case where $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ as in [16, 26, 27]?

In order to give the definite answer to questions (I)-(III), we assume that the functions K, f and g satisfy the following conditions:

(F) $f \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $|\{x \in \Omega : f(x) > 0\}| > 0$;

(G) $g \in L^\infty(\mathbb{R}^3)$ which changes sign in Ω;

(K1) $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3), K \geq 0$ and $K \not\equiv 0$ in Ω;

(K2) $K > 0$ a.e. in the set $\{x \in \Omega : g(x) = 0\}$.

Remark 1.1. *If f is bounded in $\overline{\Omega}$ and $|\{x \in \Omega : f(x) > 0\}| > 0$, then there exists a sequence of eigenvalues $\{\lambda_n(f_{\overline{\Omega}})\}$ of the problem*

$$-\Delta u = \lambda f_{\overline{\Omega}}(x)u, \quad u \in H_0^1(\Omega), \tag{1.8}$$

with $0 < \lambda_1(f_{\overline{\Omega}}) < \lambda_2(f_{\overline{\Omega}}) \leq \cdots$ and each eigenvalue being of finite multiplicity, where $f_{\overline{\Omega}}$ is a restriction of f on $\overline{\Omega}$. Denote by e_1 the positive principal eigenfunction with

$$\lambda_1(f_{\overline{\Omega}}) = \int_\Omega |\nabla e_1|^2 dx = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega f_{\overline{\Omega}} u^2 dx = 1\right\}.$$

Moreover, we have

$$\lambda_2(f_{\overline{\Omega}}) = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega f_{\overline{\Omega}} u^2 dx = 1, \int_\Omega \nabla u \nabla e_1 dx = 0\right\}.$$

In the present paper, we shall prove the existence and multiplicity of positive solutions for system $(SP_{\mu,\lambda})$ via the mountain pass theory. In order to show the mountain pass geometry of the related functional, we follow the argument in [23] to decompose each $u \in X$ (defined later) as $u = te_{1,\mu} + w$, where $w \in \{\text{span}\{e_{1,\mu}\}\}^\perp$ and $e_{1,\mu}$ is the positive principal eigenfunction corresponding to the positive principal eigenvalue $\lambda_{1,\mu}(f)$ of the problem

$$-\Delta u + \mu V(x)u = \lambda f(x)u.$$

Then by the approximation estimate

$$\lambda_{1,\mu}(f) \to \lambda_1^-(f_{\overline{\Omega}}) \text{ as } \mu \to \infty,$$

we can deduce the mountain pass geometry of the related functional in $\lambda > \lambda_1(f_{\overline{\Omega}})$ and near $\lambda_1(f_{\overline{\Omega}})$. Moreover, we apply the concentration compactness principle [33] and a variant version of the steep well method [24] to overcome the difficulty that the lack of compactness. Furthermore, we also consider the concentration of solutions.

We state the main results.

Theorem 1.2. *Suppose that $4 \leq p < 6$ and conditions (V1)-(V3), (F), (G), (K1) and (K2) hold. In addition, for $p = 4$, we assume the following:*

(K3) $K = 0$ a.e. in the set $\{x \in \Omega : g(x) > 0\}$;

(D1) $\|g\|_{L^\infty}|\{V < c_0\}|^{\frac{1}{3}} \leq \dfrac{S^4}{64\pi\lambda_1(f_{\overline{\Omega}})^2}\displaystyle\int_\Omega\int_\Omega \frac{K(x)K(y)e_1^2(x)e_1^2(y)}{|x-y|}dydx.$

Then we have the following results.

(i) For every $0 < \lambda \leq \lambda_1(f_{\overline{\Omega}})$, system $(SP_{\mu,\lambda})$ has a positive solution for $\mu > 0$ large enough.

(ii) There exists $\delta_0 > 0$ such that for every $\lambda_1(f_{\overline{\Omega}}) < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, system $(SP_{\mu,\lambda})$ has two positive solutions for $\mu > 0$ large enough.

Theorem 1.3. *Let (u_μ, ϕ_{u_μ}) be the solutions obtained in Theorem 1.2. Then we have $u_\mu \to u_\infty$ in X as $\mu \to \infty$, where $u_\infty \in H_0^1(\Omega)$ is a positive solution of equation:*

$$\begin{cases} -\Delta u + K(x)\left(\displaystyle\int_\Omega \frac{K(y)u^2(y)}{4\pi|x-y|}dy\right)u = \lambda f(x)u + g(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$S_{\infty,\lambda}$}$$

The outline of the paper is as follows. In next section, we introduce the variational setting and some basic estimates. In section 3, we consider the eigenvalue problem $-\Delta u + \mu V(x)u = \lambda f(x)u$. In section 4, we prove the corresponding energy functional having the mountain pass geometry and satisfying the Palais-Smale condition. In section 5, we prove Theorem 1.2. In last section, we give the proof of Theorem 1.3.

2 Notations and variational setting

We denote the $L^q(\mathbb{R}^3)$-norm for $1 \le q \le \infty$ by $\|\cdot\|_{L^q}$. As we take a subsequence for a sequence $\{u_n\}$, we shall still use $\{u_n\}$ to denote it. We use $o(1)$ to denote a quantity that depends on $n \in \mathbb{N}$ and goes to zero as $n \to \infty$. Let S be the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, where $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}} = \|\nabla u\|^2_{L^2}$.

Next, we consider the variational setting for system $(SP_{\mu,\lambda})$. It is well known that for any $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1.2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = \int_{\mathbb{R}^3} K(x)u^2 v dx \quad \text{for all } v \in D^{1,2}(\mathbb{R}^3). \tag{2.1}$$

That is, ϕ_u is the weak solution of $-\Delta\phi = K(x)u^2$. Moreover, ϕ_u can be represented by

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}dy. \tag{2.2}$$

Therefore, we can transform system $(SP_{\mu,\lambda})$ into a nonlinear Schrödinger equation with a non-local term as follows:

$$-\Delta u + \mu V(x)u + K(x)\phi_u u = \lambda f(x)u + g(x)|u|^{p-2}u \tag*{$(S_{\mu,\lambda})$}$$

with ϕ_u as in (2.2). Next, we set the space

$$X = \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} Vu^2 dx < \infty \right\}$$

with the following inner product and norm

$$\langle u, v\rangle_\mu = \int_{\mathbb{R}^3} (\nabla u \nabla v + \mu V uv)dx, \quad \|u\|_\mu = \langle u, u\rangle_\mu^{1/2},$$

for $\mu > 0$. By condition $(V2)$, the Hölder and Sobolev inequalities, we deduce that

$$\int_{\mathbb{R}^3} u^2 dx = \int_{\{V \ge c_0\}} u^2 dx + \int_{\{V<c_0\}} u^2 dx \le \frac{1}{\mu c_0}\int_{\mathbb{R}^3} \mu V u^2 dx + \frac{|\{V<c_0\}|^{\frac{2}{3}}}{S^2}\|u\|^2_{D^{1,2}},$$

which implies that the embedding $X \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Furthermore, for $2 \le r \le 6$ and $\mu \ge \mu_0 := S^2\left(c_0|\{V<c_0\}|^{\frac{2}{3}}\right)^{-1}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |u|^r dx &\le \left(\int_{\mathbb{R}^3} u^2 dx\right)^{\frac{6-r}{4}} \left(\int_{\mathbb{R}^3} u^6 dx\right)^{\frac{r-2}{4}} \\
&\le \left(\frac{1}{\mu c_0}\int_{\mathbb{R}^3} \mu V u^2 dx + \frac{|\{V<c_0\}|^{\frac{2}{3}}}{S^2}\|u\|^2_{D^{1,2}}\right)^{\frac{6-r}{4}} \left(\frac{\|u\|^6_{D^{1,2}}}{S^6}\right)^{\frac{r-2}{4}} \\
&\le |\{V<c_0\}|^{\frac{6-r}{6}} S^{-r}\|u\|^r_\mu.
\end{aligned} \tag{2.3}$$

Now, for Eq. $(S_{\mu,\lambda})$, we set the energy functional $J_{\mu,\lambda} : X \to \mathbb{R}$ which is defined by

$$J_{\mu,\lambda}(u) = \frac{1}{2}\|u\|^2_\mu + \frac{1}{4}\int_{\mathbb{R}^3} K\phi_u u^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} fu^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} g|u|^p dx.$$

The functional $J_{\mu,\lambda}$ is a C^1 functional with the derivative given by

$$\langle J'_{\mu,\lambda}(u),\varphi\rangle=\int_{\mathbb{R}^3}(\nabla u\nabla\varphi+\mu Vu\varphi)dx+\int_{\mathbb{R}^3}K\phi_u u\varphi dx-\lambda\int_{\mathbb{R}^3}fu\varphi dx-\int_{\mathbb{R}^3}g|u|^{p-2}u\varphi dx$$

for all $\varphi\in X$, where $J'_{\mu,\lambda}$ denotes the Fréchet derivative of $J_{\mu,\lambda}$. One can see that the critical points of $J_{\mu,\lambda}$ are corresponding to the solutions of Eq. $(S_{\mu,\lambda})$. Therefore, we conclude that u is a critical point of $J_{\mu,\lambda}$ if and only if (u,ϕ_u) solves system $(SP_{\mu,\lambda})$.

Let us define the operator $\Phi: H^1(\mathbb{R}^3)\to D^{1,2}(\mathbb{R}^3)$ and the functional $N: H^1(\mathbb{R}^3)\to\mathbb{R}$ by $\Phi(u):=\phi_u$ and

$$N(u):=\int_{\mathbb{R}^3}K\phi_u u^2dx=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|}dydx \tag{2.4}$$

respectively. By (2.1) and (2.3), we have the following estimates for $\mu\geq\mu_0$:

$$\|\phi_u\|^2_{D^{1,2}}=N(u)\leq\begin{cases} S^{-2}\|K\|^2_{L^2}\|u\|^4_{L^6}\leq S^{-6}\|K\|^2_{L^2}\|u\|^4_{\mu}, & \text{if } K\in L^2(\mathbb{R}^3),\\ S^{-2}\|K\|^2_{L^\infty}\|u\|^4_{L^{\frac{12}{5}}}\leq S^{-6}|\{V<c_0\}|\|K\|^2_{L^\infty}\|u\|^4_{\mu}, & \text{if } K\in L^\infty(\mathbb{R}^3).\end{cases} \tag{2.5}$$

Next, we state some useful properties.

Proposition 2.1 ([9], Lemma 2.1)**.** *(i) Φ is continuous and $\Phi(tu)=t^2\Phi(u)$ for all $t\in\mathbb{R}$.*
(ii) Φ maps bounded sets into bounded sets.

Proposition 2.2. *Assume that the sequence $\{u_n\}\subset H^1(\mathbb{R}^3)$ is bounded. Then there exist a subsequence $\{u_n\}$ (still denote by $\{u_n\}$) and $u\in H^1(\mathbb{R}^3)$ such that*
(i) for $K\in L^2(\mathbb{R}^3)$, we have $N(u_n)=N(u)+o(1)$ and $N'(u_n)=N'(u)+o(1)$ in $H^{-1}(\mathbb{R}^3)$;
(ii) for $K\in L^\infty(\mathbb{R}^3)$, we have $N(u_n-u)=N(u_n)-N(u)+o(1)$ and $N'(u_n)=N'(u)+o(1)$ in $H^{-1}(\mathbb{R}^3)$.

Proposition 2.3 ([34], Lemma 2.13)**.** *If $f\in L^{\frac{3}{2}}(\mathbb{R}^3)$, the functional $u\mapsto\int_{\mathbb{R}^3}fu^2dx$ is weakly continuous on $H^1(\mathbb{R}^3)$.*

Remark 2.4. *According to the fact that the embedding $X\hookrightarrow H^1(\mathbb{R}^3)$ is continuous, the above all properties are still valid as the space $H^1(\mathbb{R}^3)$ replaced by X.*

We omit the proofs of Propositions 2.1-2.3 and the readers are referred to [9, 20, 26, 34].

3 Eigenvalue problems

In this section, we study the following eigenvalue problem

$$-\Delta u+\mu V(x)u=\lambda f(x)u \quad \text{in} \quad X. \tag{3.1}$$

In order to find the positive principal eigenvalue of (3.1) we solve the following minimization problem:

$$\min\left\{\int_{\mathbb{R}^3}(|\nabla u|^2+\mu Vu^2)dx : u\in X, \int_{\mathbb{R}^3}fu^2dx=1\right\}. \tag{3.2}$$

Let

$$\lambda_{1,\mu}(f):=\inf\left\{\int_{\mathbb{R}^3}(|\nabla u|^2+\mu Vu^2)dx : u\in X, \int_{\mathbb{R}^3}fu^2dx=1\right\}.$$

By condition (F), we deduce that

$$\frac{\int_{\mathbb{R}^3}(|\nabla u|^2+\mu V u^2)dx}{\int_{\mathbb{R}^3} f u^2 dx} \geq \frac{\|u\|^2_{D^{1,2}}}{\|f\|_{L^{\frac{3}{2}}} S^{-2}\|u\|^2_{D^{1,2}}} = \frac{S^2}{\|f\|_{L^{\frac{3}{2}}}} > 0,$$

which implies that $\lambda_{1,\mu}(f) \geq S^2\|f\|^{-1}_{L^{\frac{3}{2}}} > 0$ for all $\mu > 0$. Moreover, by condition (V3),

$$\inf_{u\in X\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2+\mu V u^2 dx}{\int_{\mathbb{R}^3} f u^2 dx} \leq \inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2+\mu V u^2 dx}{\int_{\mathbb{R}^3} f u^2 dx} = \inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} f_{\overline{\Omega}} u^2 dx},$$

which indicates that $\lambda_{1,\mu}(f) \leq \lambda_1(f_{\overline{\Omega}})$ for all $\mu > 0$, where $\lambda_1(f_{\overline{\Omega}})$ is the positive principal eigenvalue of (1.8). Next, we state the useful lemma to solve problem (3.2).

Lemma 3.1 ([35], Lemma 2.4)**.** *Let $\mu_n \to \infty$ as $n \to \infty$ and $\{v_n\} \subset X$ with $\|v_n\|_{\mu_n} \leq C_0$ for some $C_0 > 0$. Then there exists a subsequence $\{v_n\}$ and $v_0 \in H^1_0(\Omega)$ such that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^r(\mathbb{R}^3)$ for all $2 \leq r < 6$. Furthermore, we have $N(v_n) \to N(v_0)$ as $n \to \infty$.*

Lemma 3.2. *For each $\mu > 0$, problem (3.2) has a positive solution $e_{1,\mu} \in X$ such that*

$$\lambda_{1,\mu}(f) = \int_{\mathbb{R}^3} |\nabla e_{1,\mu}|^2 + \mu V e^2_{1,\mu} dx.$$

Furthermore, we have
(i) $\lambda_{1,\mu}(f)$ is simple and principal eigenvalue of Eq. (3.1) and $e_{1,\mu}$ is a corresponding eigenfunction.
(ii) $\lambda_{1,\mu}(f) < \lambda_1(f_{\overline{\Omega}})$, $\lambda_{1,\mu}(f) \to \lambda_1^-(f_{\overline{\Omega}})$ and $e_{1,\mu} \to e_1$ in X as $\mu \to \infty$, where e_1 is the positive principal eigenfunction corresponding to $\lambda_1(f_{\overline{\Omega}})$.

Proof. Some results has been proved in [35]. For the reader's convenience, we give the proof in detail here. Let $\{u_n\} \subset X$ be a minimizing sequence of problem (3.2). Clearly, $\{u_n\}$ is bounded and thus there exist a subsequence $\{u_n\}$ and $e_{1,\mu} \in X$ such that

$$u_n \rightharpoonup e_{1,\mu} \quad \text{in } X; \tag{3.3}$$

$$u_n \to e_{1,\mu} \quad \text{in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 \leq r < 6. \tag{3.4}$$

Then by Proposition 2.3 and (3.3), we have

$$\int_{\mathbb{R}^3} f e^2_{1,\mu} dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} f u_n^2 dx = 1. \tag{3.5}$$

We next show that $u_n \to e_{1,\mu}$ in X. If it is false, then we have

$$\int_{\mathbb{R}^3} |\nabla e_{1,\mu}|^2 + \mu V e^2_{1,\mu} dx < \liminf_{n\to\infty} \|u_n\|^2_\mu = \lambda_{1,\mu}(f),$$

which is impossible accroding to the definition of $\lambda_{1,\mu}(f)$ and (3.5). Hence $u_n \to e_{1,\mu}$ in X and $\lambda_{1,\mu}(f) = \|e_{1,\mu}\|^2_\mu$. Since $|e_{1,\mu}| \in X$ and $\||e_{1,\mu}|\|^2_\mu = \|e_{1,\mu}\|^2_\mu = \lambda_{1,\mu}(f)$, we may assume that $e_{1,\mu} \geq 0$.

Let

$$F(t) := \frac{\|e_{1,\mu} + t\varphi\|^2_\mu}{\int_{\mathbb{R}^3} f(e_{1,\mu}+t\varphi)^2 dx}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Then we have $F'(0) = 0$ since F has a minimizer at $t = 0$. By $F'(0) = 0$, we deduce that

$$\int_{\mathbb{R}^3} (\nabla e_{1,\mu} \nabla \varphi + \mu V e_{1,\mu}\varphi) dx = \lambda_{1,\mu}(f) \int_{\mathbb{R}^3} f e_{1,\mu} \varphi dx,$$

which indicates that $e_{1,\mu}$ is an eigenfunction of (3.1) corresponding to the eigenvalue $\lambda_{1,\mu}(f)$. According to the strong maximum principle, we may assume that $e_{1,\mu} > 0$.

Here we show that $\lambda_{1,\mu}(f)$ is simple. By Sobolev embedding and conditions (V1) and (F), we conclude that any corresponding eigenfunctions belong to $W^{2,q}_{loc}(\mathbb{R}^3)\cap C_{loc}(\mathbb{R}^3)$ for $2 \leq q \leq 6$. Suppose that $\lambda_{1,\mu}(f)$ is not simple. Then there exist an eigenfunction $v \in X$ and $d > 0$ such that $u = v - de_{1,\mu}$ both takes positive and negative values. Thus, there exists $x_0 \in \mathbb{R}^3$ such that $u(x_0) = 0$ since $v, e_{1,\mu} \in C_{loc}(\mathbb{R}^3)$. Moreover, we may obtain

$$-\Delta|u| + \mu V|u| = \lambda_{1,\mu}(f)f|u|.$$

By the strong maximum principle, we have $|u| > 0$, which contradicts $u(x_0) = 0$. Hence $\lambda_{1,\mu}(f)$ is simple.

Next, we show that $\lambda_{1,\mu}(f) < \lambda_1(f_{\overline{\Omega}})$. Arguing by contradiction, we assume that $\lambda_{1,\mu}(f) = \lambda_1(f_{\overline{\Omega}})$. Then one can obtain e_1 is also an eigenfunction of (3.1), which is a contradiction due to Harnack inequality. Thus, we conclude that $\lambda_{1,\mu}(f) < \lambda_1(f_{\overline{\Omega}})$.

For any sequence $\mu_n \to \infty$, let $v_n = e_{1,\mu_n}$ be the minimizer of $\lambda_{1,\mu_n}(f)$, then we have

$$\int_{\mathbb{R}^3} f v_n^2 dx = 1 \quad \text{and} \quad \lambda_{1,\mu_n}(f) = \|v_n\|^2_{\mu_n} < \lambda_1(f_{\overline{\Omega}}).$$

By Lemma 3.1, there exist a subsequence $\{v_n\}$ and $v_0 \in H^1_0(\Omega)$ such that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 6$. Then by Proposition 2.3, we have

$$\int_{\Omega} f v_0^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} f v_n^2 dx = 1.$$

Moreover,

$$\int_{\Omega} |\nabla v_0|^2 dx = \int_{\mathbb{R}^3} |\nabla v_0|^2 + V v_0^2 dx \leq \liminf_{n\to\infty} \|v_n\|^2_{\mu_n} \leq \lambda_1(f_{\overline{\Omega}}).$$

By Remark 1.1, we conclude that $v_0 = e_1$ and $v_n \to e_1$ in X. It is easy to deduce that $\lambda_{1,\mu_1}(f) \leq \lambda_{1,\mu_2}(f)$ for $\mu_1 < \mu_2$, and thus we can conclude that $e_{1,\mu} \to e_1$ in X. This completes the proof. $\square$

In order to find the other positive eigenvalues of (3.1) we solve the following problem

$$\min\left\{\int_{\mathbb{R}^3} |\nabla u|^2 + \mu V u^2 dx : u \in X, \int_{\mathbb{R}^3} f u^2 dx = 1, \langle u, e_{i,\mu}\rangle_\mu = 0 \ \text{ for all } \ 1 \leq i \leq n-1\right\}. \tag{3.6}$$

Lemma 3.3. *For each $\mu > 0$ and $n \geq 2$, problem* (3.6) *has a solution $e_{n,\mu} \in X$. Furthermore, $e_{n,\mu}$ is an eigenfunction of* (3.1) *corresponding to the eigenvalue $\lambda_{n,\mu}(f) := \|e_{n,\mu}\|^2_\mu$ and*

$$\lambda_{2,\mu}(f) > \frac{\lambda_1(f_{\overline{\Omega}}) + \lambda_2(f_{\overline{\Omega}})}{2} \quad \textit{for all } \mu \textit{ large enough.}$$

Proof. The existence of $e_{n,\mu}$ is proved as in Lemma 3.2. However, we cannot assume that $e_{n,\mu} \geq 0$ because of the condition $\langle e_{n,\mu}, e_{i,\mu}\rangle_\mu = 0$. Let $v \in X$ satisfying $\langle v, e_{i,\mu}\rangle_\mu = 0$ for all $1 \leq i \leq n-1$ and

$$F(t) = \frac{\|e_{n,\mu} + tv\|^2_\mu}{\int_{\mathbb{R}^3} f(e_{n,\mu} + tv)^2 dx}.$$

Then $F'(0) = 0$ implies that

$$\int_{\mathbb{R}^3} (\nabla e_{n,\mu} \nabla v + \mu V e_{n,\mu} v) dx = \lambda_{n,\mu}(f) \int_{\mathbb{R}^3} f e_{n,\mu} v dx. \tag{3.7}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ and $w = \varphi - \sum_{i=1}^{n-1} c_i e_{i,\mu}$, where $c_i = \frac{\langle \varphi, e_{i,\mu}\rangle_\mu}{\langle e_{i,\mu}, e_{i,\mu}\rangle_\mu}$. Then $\langle w, e_{i,\mu}\rangle_\mu = 0$ for all $1 \le i \le n-1$, and thus by (3.7),

$$\int_{\mathbb{R}^3} \nabla e_{n,\mu}\nabla\varphi + \mu V e_{n,\mu}\varphi dx = \int_{\mathbb{R}^3} \nabla e_{n,\mu}\nabla w + \mu V e_{n,\mu} w dx = \lambda_{n,\mu}(f)\int_{\mathbb{R}^3} f e_{n,\mu} w dx = \lambda_{n,\mu}(f)\int_{\mathbb{R}^3} f e_{n,\mu}\varphi dx.$$

This indicates that $e_{n,\mu}$ is an eigenfunction of (3.1) corresponding to the eigenvalue $\lambda_{n,\mu}(f)$.

We now show that $\lambda_{2,\mu}(f) > \frac{1}{2}\left(\lambda_1(f_{\overline{\Omega}}) + \lambda_2(f_{\overline{\Omega}})\right)$ for all μ large enough. Suppose on the contrary. Then there exists a sequence $\{\mu_n\}$ with $\mu_n \to \infty$ such that

$$\lambda_{2,\mu_n}(f) \le \frac{\lambda_1(f_{\overline{\Omega}}) + \lambda_2(f_{\overline{\Omega}})}{2}.$$

Let $v_n = e_{2,\mu_n}$ be the minimizer of $\lambda_{2,\mu_n}(f)$, then $\int_{\mathbb{R}^3} f v_n^2 dx = 1, \;\; \langle v_n, e_{1,\mu_n}\rangle_{\mu_n} = 0$ and

$$\lambda_{2,\mu_n}(f) = \|v_n\|_{\mu_n}^2 \le \frac{\lambda_1(f_{\overline{\Omega}}) + \lambda_2(f_{\overline{\Omega}})}{2}. \tag{3.8}$$

By Lemma 3.1, there exist a subsequence $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^r(\mathbb{R}^3)$ for $2 \le r < 6$. Then by Proposition 2.3, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f v_n^2 dx = \int_\Omega f v_0^2 dx = 1. \tag{3.9}$$

Moreover,

$$\int_\Omega |\nabla v_0|^2 dx = \int_{\mathbb{R}^3} |\nabla v_0|^2 + V v_0^2 dx \le \liminf_{n\to\infty}\|v_n\|_{\mu_n}^2 \le \frac{\lambda_1(f_{\overline{\Omega}}) + \lambda_2(f_{\overline{\Omega}})}{2}.$$

According to the fact in Lemma 3.2 that $e_{1,\mu} \to e_1$ in X as $\mu \to \infty$, we deduce that

$$\|e_{1,\mu_n} - e_1\|_{\mu_n} \to 0 \quad \text{as} \;\; n \to \infty. \tag{3.10}$$

It follows from (3.8), (3.10) and $v_n \rightharpoonup v_0$ in X that

$$\lim_{n\to\infty}\langle v_n, e_{1,\mu_n}\rangle_{\mu_n} = \int_\Omega \nabla v_0 \nabla e_1 dx = 0. \tag{3.11}$$

From (3.9), (3.11) and Remark 1.1, we conclude that $\int_\Omega |\nabla v_0|^2 dx \ge \lambda_2(f_{\overline{\Omega}})$ which is a contradiction. This completes the proof. □

4 Palais-Smale sequence

In this section, we study the mountain pass geometry and the compactness condition for the functional $J_{\mu,\lambda}$. Let us recall the well known mountain pass theorem.

Theorem 4.1 ([36], Mountain pass theorem)**.** *Let E be a Banach space, $J \in C^1(E,\mathbb{R}), v \in E$ and $\rho > 0$ be such that $\|v\| > \rho$ and*

$$b := \inf_{\|u\|=\rho} J(u) > J(0) \ge J(v).$$

If J satisfies the Palais-Smale condition at level α with

$$\alpha := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \quad \textit{and} \quad \Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = v\},$$

then α is a critical value of J and $\alpha \ge b$.

Definition 4.2. *(i) A sequence $\{u_n\} \subset X$ is called a Palais-Smale sequence at level $\alpha \in \mathbb{R}$ ($(PS)_\alpha$-sequence for short) for the functional $J_{\mu,\lambda}$ if $J_{\mu,\lambda}(u_n) \to \alpha$ and $J'_{\mu,\lambda}(u_n) \to 0$.*
(ii) We say that the functional $J_{\mu,\lambda}$ satisfies the Palais-Smale condition at level α ($(PS)_\alpha$-condition) if every $(PS)_\alpha$-sequence has a convergent subsequence.

Next, we show that the functional $J_{\mu,\lambda}$ has the mountain pass geometry in $\lambda > \lambda_1(f_{\overline{\Omega}})$ and near $\lambda_1(f_{\overline{\Omega}})$ for $\mu > 0$ large enough. To prove the mountain pass geometry of the functional $J_{\mu,\lambda}$, we decompose each $u \in X$ as $u = te_{1,\mu} + w$, where $t \in \mathbb{R}$, $w \in X$ and $\langle w, e_{1,\mu}\rangle_\mu = 0$. Then

$$\|u\|_\mu^2 = \|e_{1,\mu}\|_\mu^2 t^2 + \|w\|_\mu^2 = \lambda_{1,\mu}(f)t^2 + \|w\|_\mu^2. \tag{4.1}$$

Moreover, by Lemmas 3.2 and 3.3, we have

$$\lambda_{1,\mu}(f)\int_{\mathbb{R}^3} fe_{1,\mu}wdx = \int_{\mathbb{R}^N} \nabla e_{1,\mu}\nabla w + \mu V e_{1,\mu}wdx = 0 \tag{4.2}$$

and

$$\lambda_{2,\mu}(f)\int_{\mathbb{R}^3} fw^2dx \leq \|w\|_\mu^2. \tag{4.3}$$

Thus, by $(4.1) - (4.3)$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\left(\|u\|_\mu^2 - \lambda\int_{\mathbb{R}^3} fu^2dx\right)\\
&= \frac{1}{2}\left(\lambda_{1,\mu}(f)t^2 + \|w\|_\mu^2 - \lambda\int_{\mathbb{R}^3}(t^2fe_{1,\mu}^2 + 2tfe_{1,\mu}w + fw^2)dx\right)\\
&\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,\mu}(f)}\right)\lambda_{1,\mu}(f)t^2 + \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{2,\mu}(f)}\right)\|w\|_\mu^2\\
&= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,\mu}(f)}\right)\|u\|_\mu^2 + \frac{\lambda}{2}\left(\frac{1}{\lambda_{1,\mu}(f)} - \frac{1}{\lambda_{2,\mu}(f)}\right)\|w\|_\mu^2.
\end{aligned} \tag{4.4}$$

Lemma 4.3. *Suppose that $4 \leq p < 6$ and conditions (V1)-(V3), (F), (G) and (K1) hold. In addition, for $p = 4$, assume (K3) and (D1). Then there exists $\delta_0 > 0$ such that for every $0 < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, there exist $\rho_\lambda, \eta_\lambda > 0$ and $\varphi_0 \in H_0^1(\Omega)$ such that $\|\varphi_0\|_\mu > \rho_\lambda$ and*

$$\inf_{\|u\|_\mu = \rho_\lambda} J_{\mu,\lambda}(u) \geq \eta_\lambda > 0 > J_{\mu,\lambda}(\varphi_0)$$

for $\mu > 0$ large enough.

Proof. We first show that there exists $\delta_0 > 0$ such that for every $0 < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, there exist $\rho_\lambda, \eta_\lambda > 0$ such that

$$\inf_{\|u\|_\mu = \rho_\lambda} J_{\mu,\lambda}(u) \geq \eta_\lambda > 0$$

as $\mu > 0$ large enough. We separate this part into two cases.
Case (1): $0 < \lambda < \lambda_1(f_{\overline{\Omega}})$. By Lemma 3.2$(ii)$, we have

$$\lambda_{1,\mu}(f) \geq \frac{\lambda + \lambda_1(f_{\overline{\Omega}})}{2} \quad \text{for } \mu > 0 \text{ large enough}. \tag{4.5}$$

By (2.3), (4.4), (4.5) and condition (G), we deduce that for $\mu > 0$ large enough,

$$\begin{aligned} J_{\mu,\lambda}(u) &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,\mu}(f)}\right)\|u\|_\mu^2 + \frac{1}{4}N(u) - \frac{\|g\|_{L^\infty}|\{V < c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \\ &\geq \|u\|_\mu^2\left(\frac{1}{2}\left(\frac{\lambda_1(f_{\overline{\Omega}}) - \lambda}{\lambda_1(f_{\overline{\Omega}}) + \lambda}\right) - \frac{\|g\|_{L^\infty}|\{V < c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^{p-2}\right), \end{aligned}$$

where $N(u)$ is as in (2.4). Let

$$\rho_\lambda = \left(\frac{1}{4}\left(\frac{\lambda_1(f_{\overline{\Omega}}) - \lambda}{\lambda_1(f_{\overline{\Omega}}) + \lambda}\right)\frac{pS^p}{\|g\|_{L^\infty}|\{V < c_0\}|^{\frac{6-p}{6}}}\right)^{\frac{1}{p-2}}.$$

Then for $\mu > 0$ large enough and $\|u\|_\mu = \rho_\lambda$, we have

$$J_{\mu,\lambda}(u) \geq \frac{1}{4}\left(\frac{\lambda_1(f_{\overline{\Omega}}) - \lambda}{\lambda_1(f_{\overline{\Omega}}) + \lambda}\right)\rho_\lambda^2 =: \eta_\lambda > 0.$$

Case (2): $\lambda \geq \lambda_1(f_{\overline{\Omega}})$. Using (2.3) and (4.4), for $\mu > 0$ large enough, we have

$$J_{\mu,\lambda}(u) \geq \Lambda_{1,\mu}\|u\|_\mu^2 + \Lambda_{2,\mu}\|w\|_\mu^2 + \frac{N(te_{1,\mu})}{4} + \frac{N(u) - N(te_{1,\mu})}{4} - \frac{\|g\|_{L^\infty}|\{V < c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p, \tag{4.6}$$

where $u = te_{1,\mu} + w$ with $t \in \mathbb{R}$ and $w \in \{v \in X : \langle v, e_{1,\mu}\rangle_\mu = 0\}$,

$$\Lambda_{1,\mu} = \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{1,\mu}(f)}\right) \quad \text{and} \quad \Lambda_{2,\mu} = \frac{\lambda}{2}\left(\frac{1}{\lambda_{1,\mu}(f)} - \frac{1}{\lambda_{2,\mu}(f)}\right).$$

By Lemma 3.3, we may deduce that for $\mu > 0$ large enough and $\lambda \geq \lambda_1(f_{\overline{\Omega}})$,

$$\Lambda_{2,\mu} \geq \frac{1}{2}\left(1 - \frac{\lambda_1(f_{\overline{\Omega}})}{\lambda_{2,\mu}(f)}\right) \geq \frac{\lambda_2(f_\Omega) - \lambda_1(f_{\overline{\Omega}})}{2(\lambda_2(f_{\overline{\Omega}}) + \lambda_1(f_{\overline{\Omega}}))} =: \Lambda_0. \tag{4.7}$$

By Proposition 2.1(i) and Lemma 3.2(ii), we deduce that $N(e_{1,\mu}) \to N(e_1)$ as $\mu \to \infty$, which imples that

$$N(e_{1,\mu}) \geq \frac{1}{2}N(e_1) \quad \text{for } \mu > 0 \text{ large enough.} \tag{4.8}$$

Moreover, by condition (K1), we have $N(e_1) > 0$. By the mean value theorem, there exists θ with $0 < \theta < 1$ such that

$$\frac{1}{4}|N(te_{1,\mu} + w) - N(te_{1,\mu})| = \left|\int_{\mathbb{R}^3} K\phi_{te_{1,\mu}+\theta w}(te_{1,\mu} + \theta w)wdx\right|. \tag{4.9}$$

When $K \in L^2(\mathbb{R}^3)$, by (2.5), Hölder, Sobolev and Young's inequalities, we deduce that

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} K\phi_{te_{1,\mu}+\theta w}(te_{1,\mu} + \theta w)wdx\right| \\ &\leq S^{-1}\|K\|_{L^2}\|\phi_{te_{1,\mu}+\theta w}\|_{D^{1,2}}\|te_{1,\mu} + \theta w\|_{L^6}\|w\|_{L^6} \\ &\leq S^{-6}\|K\|_{L^2}^2\|te_{1,\mu} + \theta w\|_\mu^3\|w\|_\mu \\ &= S^{-6}\|K\|_{L^2}^2\left(t^2\|e_{1,\mu}\|_\mu^2 + \theta^2\|w\|_\mu^2\right)^{\frac{3}{2}}\|w\|_\mu \\ &\leq \sqrt{2}S^{-6}\|K\|_{L^2}^2|t|^3\|e_{1,\mu}\|_\mu^3\|w\|_\mu + \sqrt{2}S^{-6}\|K\|_{L^2}^2\|w\|_\mu^4 \\ &< \frac{N(e_1)}{16}t^4 + \frac{12^3\|K\|_{L^2}^8\lambda_1(f_{\overline{\Omega}})^6}{S^{24}N(e_1)^3}\|w\|_\mu^4 + \sqrt{2}S^{-6}\|K\|_{L^2}^2\|w\|_\mu^4. \end{aligned} \tag{4.10}$$

Analogously, for $K \in L^\infty(\mathbb{R}^3)$, by (2.3), (2.5) and Young's inequality, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} K\phi_{te_{1,\mu}+\theta w}(te_{1,\mu}+\theta w)wdx\right| \\
&\leq \|K\|_{L^\infty}\|\phi_{te_{1,\mu}+\theta w}\|_{L^6}\|te_{1,\mu}+\theta w\|_{L^{\frac{12}{5}}}\|w\|_{L^{\frac{12}{5}}} \\
&\leq \sqrt{2}S^{-6}\left|\{V<c_0\}\right|\|K\|_{L^\infty}^2|t|^3\|e_{1,\mu}\|_\mu^3\|w\|_\mu + \sqrt{2}S^{-6}\left|\{V<c_0\}\right|\|K\|_{L^\infty}^2\|w\|_\mu^4 \\
&< \frac{N(e_1)}{16}t^4 + \frac{12^3\left|\{V<c_0\}\right|^4\|K\|_{L^\infty}^8\lambda_1(f_{\overline{\Omega}})^6}{S^{24}N(e_1)^3}\|w\|_\mu^4 + \sqrt{2}S^{-6}\left|\{V<c_0\}\right|\|K\|_{L^\infty}^2\|w\|_\mu^4. \qquad (4.11)
\end{aligned}
$$

Subsequently, combining (4.6) – (4.11), for $\mu > 0$ large enough, we have

$$
\begin{aligned}
&J_{\mu,\lambda}(u) \\
\geq& -|\Lambda_{1,\mu}|\|u\|_\mu^2 + \Lambda_0\|w\|_\mu^2 + \frac{N(e_1)}{8}t^4 + \frac{N(te_{1,\mu}+w)-N(te_{1,\mu})}{4} - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \\
\geq& -|\Lambda_{1,\mu}|\|u\|_\mu^2 + \Lambda_0\|w\|_\mu^2 + \frac{N(e_1)}{16}t^4 - C_K\|w\|_\mu^4 - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \\
=& -|\Lambda_{1,\mu}|\|u\|_\mu^2 + \Lambda_0\|w\|_\mu^2 + \frac{N(e_1)}{16\lambda_{1,\mu}(f)^2}\left(\|u\|_\mu^2 - \|w\|_\mu^2\right)^2 - C_K\|w\|_\mu^4 - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \\
\geq& -|\Lambda_{1,\mu}|\|u\|_\mu^2 + \Lambda_0\|w\|_\mu^2 + \frac{N(e_1)}{16\lambda_1(f_{\overline{\Omega}})^2}\left(\frac{\|u\|_\mu^4}{2} - \|w\|_\mu^4\right) - C_K\|w\|_\mu^4 - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \\
=& -|\Lambda_{1,\mu}|\|u\|_\mu^2 + \|u\|_\mu^4\left(\frac{N(e_1)}{32\lambda_1(f_{\overline{\Omega}})^2} - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^{p-4}\right) \\
&+ \|w\|_\mu^2\left(\Lambda_0 - \left(\frac{N(e_1)}{16\lambda_1(f_{\overline{\Omega}})^2} + C_K\right)\|w\|_\mu^2\right), \qquad (4.12)
\end{aligned}
$$

where

$$
C_K = \begin{cases} \dfrac{12^3\|K\|_{L^2}^8\lambda_1(f_{\overline{\Omega}})^6}{S^{24}N(e_1)^3} + \dfrac{\sqrt{2}\|K\|_{L^2}^2}{S^6}, & \text{for } K \in L^2(\mathbb{R}^3), \\ \dfrac{12^3\left|\{V<c_0\}\right|^4\|K\|_{L^\infty}^8\lambda_1(f_{\overline{\Omega}})^6}{S^{24}N(e_1)^3} + \dfrac{\sqrt{2}\left|\{V<c_0\}\right|\|K\|_{L^\infty}^2}{S^6}, & \text{for } K \in L^\infty(\mathbb{R}^3). \end{cases}
$$

Let

$$
\rho_\lambda = \begin{cases} \min\left\{\left(\frac{N(e_1)pS^p}{64\lambda_1(f_{\overline{\Omega}})^2\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}\right)^{\frac{1}{p-4}}, \left(\Lambda_0\left(\frac{N(e_1)}{16\lambda_1(f_{\overline{\Omega}})^2} + C_K\right)^{-1}\right)^{\frac{1}{2}}\right\}, & \text{for } 4<p<6, \\ \left(\Lambda_0\left(\frac{N(e_1)}{16\lambda_1(f_{\overline{\Omega}})^2} + C_K\right)^{-1}\right)^{\frac{1}{2}}, & \text{for } p=4. \end{cases} \qquad (4.13)
$$

Then for $\|u\|_\mu = \rho_\lambda$, we have

$$
\frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^{p-4} \leq \frac{N(e_1)}{64\lambda_1(f_{\overline{\Omega}})^2} \quad \text{and} \quad \Lambda_0 - \left(\frac{N(e_1)}{16\lambda_1(f_{\overline{\Omega}})^2} + C_K\right)\|w\|_\mu^2 \geq 0. \qquad (4.14)
$$

Note that under condition (D1), the first inequality in (4.14) holds for $p = 4$. Let

$$
\delta_0 = \frac{N(e_1)}{256\lambda_1(f_{\overline{\Omega}})}\rho_\lambda^2 \qquad (4.15)
$$

and

$$\delta_\mu = \frac{N(e_1)\lambda_{1,\mu}(f)}{64\lambda_1(f_{\overline{\Omega}})^2}\rho_\lambda^2.$$

By Lemma 3.2(ii), we deduce that for $\mu > 0$ large enough,

$$\lambda_1(f_{\overline{\Omega}}) + \delta_0 \le \lambda_{1,\mu}(f) + 2\delta_0 \le \lambda_{1,\mu}(f) + \delta_\mu.$$

This implies that for every $\lambda_1(f_{\overline{\Omega}}) \le \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$ and $\mu > 0$ large enough,

$$|\Lambda_{1,\mu}| = \frac{1}{2}\left(\frac{\lambda}{\lambda_{1,\mu}(f)} - 1\right) \le \frac{1}{2}\left(\frac{\lambda_{1,\mu}(f) + \delta_\mu}{\lambda_{1,\mu}(f)} - 1\right) = \frac{N(e_1)}{128\lambda_1(f_{\overline{\Omega}})^2}\rho_\lambda^2. \tag{4.16}$$

It follows from (4.12), (4.14) and (4.16) that for every $\lambda_1(f_{\overline{\Omega}}) \le \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, $\|u\|_\mu = \rho_\lambda$ and $\mu > 0$ large enough,

$$J_{\mu,\lambda}(u) \ge \frac{N(e_1)}{128\lambda_1(f_{\overline{\Omega}})^2}\rho_\lambda^4 =: \eta_\lambda > 0.$$

Consequently, for every $0 < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, we have $\inf_{\|u\|_\mu = \rho_\lambda} J_{\mu,\lambda}(u) \ge \eta_\lambda > 0$ as $\mu > 0$ large enough.

Next, we show that there exists $\varphi_0 \in H_0^1(\Omega)$ such that $\|\varphi_0\|_\mu > \rho_\lambda$ and $J_{\mu,\lambda}(\varphi_0) < 0$. We divide the part into two cases:
Case (I): $4 < p < 6$. By condition (G), we may choose a function $\varphi \in H_0^1(\Omega)$ such that $\int_{\mathbb{R}^3} g|\varphi|^p dx > 0$. Let $s > 0$, then

$$J_{\mu,\lambda}(s\varphi) = \frac{\|\varphi\|_\mu^2 - \lambda\int_{\mathbb{R}^3} f\varphi^2 dx}{2}s^2 + \frac{N(\varphi)}{4}s^4 - \frac{\int_{\mathbb{R}^3} g|\varphi|^p dx}{p}s^p.$$

This implies that there exists $s_0 > 0$ such that $\|s_0\varphi\|_\mu > \rho_\lambda$ and $J_{\mu,\lambda}(s_0\varphi) < 0$.
Case (II): $p = 4$. By conditions (G) and (K3), we can choose a function $\varphi \in H_0^1(\Omega)$ such that $\int_{\mathbb{R}^3} g\varphi^4 dx > 0$ and $N(\varphi) = 0$. Let $s > 0$, then

$$J_{\mu,\lambda}(s\varphi) = \frac{\|\varphi\|_\mu^2 - \lambda\int_{\mathbb{R}^3} f\varphi^2 dx}{2}s^2 - \frac{\int_{\mathbb{R}^3} g\varphi^4 dx}{4}s^4,$$

Thus, there exists $s_0 > 0$ such that $\|s_0\varphi\|_\mu > \rho_\lambda$ and $J_{\mu,\lambda}(s_0\varphi) < 0$. We complete the proof. □

Next, we study the $(PS)_{\beta_\lambda(\mu)}$-condition for the functional $J_{\mu,\lambda}$, where $\beta_\lambda(\mu)$ is a real value function defined in $\mu > 0$ and $\beta_\lambda(\mu) > 0$ for all $\lambda, \mu > 0$. Then we have the following results.

Lemma 4.4. *Suppose that $4 \le p < 6$ and conditions (V1)-(V3), (F), (G), (K1) and (K2) hold. In addition, for $p = 4$, assume (K3). If there exists $d_\lambda > 0$ such that*

$$0 < \beta_\lambda(\mu) < d_\lambda$$

for all $\mu > 0$ large enough, then there exists $D_0 > 0$ such that the $(PS)_{\beta_\lambda(\mu)}$-sequence $\{u_n\}$ for $J_{\mu,\lambda}$ satisfies $\|u_n\|_\mu < D_0$ for all $\mu > 0$ large enough.

Proof. Suppose on the contrary. Then there exist two sequences $\{\mu_n\}, \{D_n\} \subset \mathbb{R}^+$ with $\mu_n \to \infty, D_n \to \infty$ as $n \to \infty$ such that for every $n \in \mathbb{N}$, there exists a $(PS)_{\beta_\lambda(\mu_n)}$-sequence $\{u_{n,m}\}_{m\in\mathbb{N}}$ with $\|u_{n,m}\|_{\mu_n} > D_n$ for all $m \in \mathbb{N}$,

$$\frac{1}{2}\|u_{n,m}\|_{\mu_n}^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3} fu_{n,m}^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} g|u_{n,m}|^p dx + \frac{1}{4}N(u_{n,m}) \to \beta_\lambda(\mu_n) \tag{4.17}$$

and

$$\int_{\mathbb{R}^3}(\nabla u_{n,m}\nabla\varphi+\mu_n V u_{n,m}\varphi-\lambda f u_{n,m}\varphi)dx-\int_{\mathbb{R}^3} g|u_{n,m}|^{p-2}u_{n,m}\varphi dx+\int_{\mathbb{R}^3} K\phi_{u_{n,m}}u_{n,m}\varphi dx\to 0 \quad (4.18)$$

as $m\to\infty$.

Let $w_n=u_{n,n}$, then $\|w_n\|_{\mu_n}\to\infty$ as $n\to\infty$. Let $v_n=\frac{w_n}{\|w_n\|_{\mu_n}}$, then $\|v_n\|_{\mu_n}=1$. By Lemma 3.1, there exist a subsequence $\{v_n\}$ and $v_0\in H_0^1(\Omega)$ such that $v_n\rightharpoonup v_0$ in X, $v_n\to v_0$ in $L^r(\mathbb{R}^3)$ for $2\le r<6$, and $\lim_{n\to\infty}N(v_n)=N(v_0)$. Then by conditions (F) and (G), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} fv_n^2dx=\int_\Omega fv_0^2dx \quad\text{and}\quad \lim_{n\to\infty}\int_{\mathbb{R}^3} g|v_n|^pdx=\int_\Omega g|v_0|^pdx.$$

Dividing (4.17) by $\|w_n\|_{\mu_n}^2$ and the boundedness of $\beta_\lambda(\mu_n)$, we obtain

$$\frac{1}{2}-\frac{\lambda}{2}\int_{\mathbb{R}^3} fv_n^2dx-\frac{1}{p}\|w_n\|_{\mu_n}^{p-2}\int_{\mathbb{R}^3} g|v_n|^pdx+\frac{1}{4}\|w_n\|_{\mu_n}^2N(v_n)\to 0. \quad (4.19)$$

Dividing (4.18) by $\|w_n\|_{\mu_n}$, we have

$$\int_{\mathbb{R}^3}(\nabla v_n\nabla\varphi+\mu_n V v_n\varphi-\lambda f v_n\varphi)dx-\|w_n\|_{\mu_n}^{p-2}\int_{\mathbb{R}^3} g|v_n|^{p-2}v_n\varphi dx+\|w_n\|_{\mu_n}^2\int_{\mathbb{R}^3} K\phi_{v_n}v_n\varphi dx\to 0. \quad (4.20)$$

If the assumption that $v_0=0$ a.e. in Ω holds, then we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} fv_n^2dx=\int_\Omega fv_0^2dx=0. \quad (4.21)$$

Choosing $\varphi=v_n$ in (4.20), we obtain

$$1-\lambda\int_{\mathbb{R}^3} fv_n^2dx-\|w_n\|_{\mu_n}^{p-2}\int_{\mathbb{R}^3} g|v_n|^pdx+\|w_n\|_{\mu_n}^2N(v_n)\to 0. \quad (4.22)$$

For $4<p<6$, combining (4.19), (4.21) and (4.22), we deduce that

$$\lim_{n\to\infty}\|w_n\|_{\mu_n}^{p-2}\int_{\mathbb{R}^3} g|v_n|^pdx=\frac{p}{4-p}<0$$

However, by (4.21), (4.22) implies that $\lim_{n\to\infty}\|w_n\|_{\mu_n}^{p-2}\int_{\mathbb{R}^3} g|v_n|^pdx>0$, which is a contradiction. Similarly, we also have a contradiction for $p=4$ by (4.19), (4.21) and (4.22).

Now, we prove the assumption that $v_0=0$ a.e. in Ω. For $4<p<6$, dividing (4.20) by $\|w_n\|_{\mu_n}^{p-2}$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} g|v_n|^{p-2}v_n\varphi dx=\int_\Omega g|v_0|^{p-2}v_0\varphi dx=0. \quad (4.23)$$

Clearly, $v_0\chi_{\{x\in\Omega:g(x)>0\}}\in L^r(\Omega)$ for $2\le r\le 6$. Then there exists a sequence $\{\varphi_n\}\subset C_0^\infty(\Omega)$ such that $\varphi_n\to v_0\chi_{\{x\in\Omega:g(x)>0\}}$ in $L^r(\Omega)$ for $2\le r\le 6$. Therefore, choosing $\varphi=\varphi_n$ in (4.23) and taking $n\to\infty$, we have

$$0=\lim_{n\to\infty}\int_\Omega g|v_0|^{p-2}v_0\varphi_ndx=\int_{\{x\in\Omega:g(x)>0\}} g|v_0|^pdx,$$

which implies that $v_0=0$ a.e. in $\{x\in\Omega:g(x)>0\}$. In a same way, we obtain that $v_0=0$ a.e. in $\{x\in\Omega:g(x)<0\}$. For the remaining part, by combining (4.19) and (4.22), we have

$$0=\lim_{n\to\infty}\int_{\mathbb{R}^3} K\phi_{v_n}v_n^2dx=\int_{\{x\in\Omega:g(x)=0\}} K\phi_{v_0}v_0^2dx.$$

By condition (K2), we obtain that $v_0 = 0$ a.e. in $\{x \in \Omega : g(x) = 0\}$.

For $p = 4$, dividing (4.20) by $\|w_n\|_{\mu_n}^2$, we obtain

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} g v_n^3 \varphi dx - \int_{\mathbb{R}^3} K\phi_{v_n} v_n \varphi dx \right) = \int_\Omega g v_0^3 \varphi dx - \int_\Omega K\phi_{v_0} v_0 \varphi dx = 0. \tag{4.24}$$

In a same way as the argument of the case $4 < p < 6$ and by condition (K3), we have

$$0 = \int_{\{x\in\Omega:g(x)>0\}} g v_0^4 dx - \int_{\{x\in\Omega:g(x)>0\}} K\phi_{v_0} v_0^2 dx = \int_{\{x\in\Omega:g(x)>0\}} g v_0^4 dx,$$

which implies that $v_0 = 0$ a.e. in $\{x \in \Omega : g(x) > 0\}$. Similarly, we have

$$\int_{\{x\in\Omega:g(x)<0\}} g v_0^4 dx = \int_{\{x\in\Omega:g(x)<0\}} K\phi_{v_0} v_0^2 dx$$

and

$$\int_{\{x\in\Omega:g(x)=0\}} K\phi_{v_0} v_0^2 dx = 0.$$

By condition (K2), we obtain that $v_0 = 0$ a.e. in $\{x \in \Omega : g(x) \le 0\}$. This completes the proof. □

Lemma 4.5. *Suppose that $2 < p < 6$ and conditions (V1)-(V2), (F), (G) and (K1) hold. Let $\{u_n\}$ be a $(PS)_\beta$-sequence for $J_{\mu,\lambda}$. If there exists $D_0 > 0$ such that*

$$\|u_n\|_\mu < D_0 \tag{4.25}$$

for all $\mu > 0$ large enough, then the sequence $\{u_n\}$ has a convergent subsequence.

Proof. By (4.25), there exist a subsequence $\{u_n\}$ and $u_0 \in X$ such that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^r_{loc}(\mathbb{R}^3)$ for $2 \le r < 6$. It follows from Proposition 2.3 and $X \hookrightarrow H^1(\mathbb{R}^3)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f u_n^2 dx = \int_{\mathbb{R}^3} f u_0^2 dx. \tag{4.26}$$

Next, we show that $u_n \to u_0$ in X. Let $v_n = u_n - u_0$. By (4.25), we deduce that for $\mu > 0$ large enough,

$$\|v_n\|_\mu^2 \le \|u_n\|_\mu + \|u_0\|_\mu < D_1. \tag{4.27}$$

From Brézis-Lieb lemma [37] and Proposition 2.2, we have

$$\|v_n\|_\mu^2 = \|u_n\|_\mu^2 - \|u_0\|_\mu^2 + o(1),$$

$$N(v_n) = N(u_n) - N(u_0) + o(1)$$

and

$$\int_{\mathbb{R}^3} g|v_n|^p dx = \int_{\mathbb{R}^3} g|u_n|^p dx - \int_{\mathbb{R}^3} g|u_0|^p dx + o(1).$$

Moreover, we have

$$\int_{\mathbb{R}^3} v_n^2 dx \le \frac{1}{\mu c_0} \int_{\{V\ge c_0\}} \mu V v_n^2 dx + \int_{\{V<c_0\}} v_n^2 dx \le \frac{1}{\mu c_0} \int_{\mathbb{R}^3} \mu V v_n^2 dx + o(1). \tag{4.28}$$

By (2.3), (4.27) and (4.28), we deduce

$$\begin{aligned}
\int_{\mathbb{R}^3} g|v_n|^p dx &\leq \|g\|_{L^\infty} \left(\int_{\mathbb{R}^3} |v_n|^p dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^3} |v_n|^p dx\right)^{\frac{2}{p}} \\
&\leq \|g\|_{L^\infty} \left(|\{V < c_0\}|^{\frac{6-p}{6}} S^{-p} \|v_n\|_\mu^p\right)^{\frac{p-2}{p}} \left(\|v_n\|_{L^2}^{\frac{6-p}{2}} \|v_n\|_{L^6}^{\frac{3p-6}{2}}\right)^{\frac{2}{p}} \\
&\leq \|g\|_{L^\infty} \left(|\{V < c_0\}|^{\frac{6-p}{6}} S^{-p} D_1^{\frac{p}{2}}\right)^{\frac{p-2}{p}} \left(\left(\frac{1}{\mu c_0}\int_{\mathbb{R}^3} \mu V v_n^2 dx + o(1)\right)^{\frac{6-p}{4}} (S^{-1}\|v_n\|_{D^{1,2}})^{\frac{3p-6}{2}}\right)^{\frac{2}{p}} \\
&\leq \|g\|_{L^\infty} \left(|\{V < c_0\}|^{\frac{6-p}{6}} S^{-p} D_1^{\frac{p}{2}}\right)^{\frac{p-2}{p}} \left((\mu c_0)^{\frac{p-6}{4}} \|v_n\|_\mu^{\frac{6-p}{2}} S^{\frac{6-3p}{2}} \|v_n\|_\mu^{\frac{3p-6}{2}}\right)^{\frac{2}{p}} + o(1) \\
&\leq D_2 \mu^{\frac{p-6}{2p}} \|v_n\|_\mu^2 + o(1),
\end{aligned} \tag{4.29}$$

where $D_2 = \|g\|_{L^\infty} |\{V < c_0\}|^{\frac{(6-p)(p-2)}{6p}} S^{-1-p+\frac{6}{p}} D_1^{\frac{p-2}{2}} c_0^{\frac{p-6}{2p}}$.

Since $J'_{\mu,\lambda}(u_n) \to 0$, we have

$$o(1) = \int_{\mathbb{R}^3} \nabla u_n \nabla \varphi + \mu V u_n \varphi dx - \lambda \int_{\mathbb{R}^3} f u_n \varphi dx - \int_{\mathbb{R}^3} g|u_n|^{p-2} u_n \varphi dx + \int_{\mathbb{R}^3} K \phi_{u_n} u_n \varphi dx, \tag{4.30}$$

which implies that

$$0 = \int_{\mathbb{R}^3} \nabla u_0 \nabla \varphi + \mu V u_0 \varphi dx - \lambda \int_{\mathbb{R}^3} f u_0 \varphi dx - \int_{\mathbb{R}^3} g|u_0|^{p-2} u_0 \varphi dx + \int_{\mathbb{R}^3} K \phi_{u_0} u_0 \varphi dx. \tag{4.31}$$

Choosing $\varphi = u_n$ in (4.30) and $\varphi = u_0$ in (4.31), we obtain

$$o(1) = \|u_n\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f u_n^2 dx - \int_{\mathbb{R}^3} g|u_n|^p dx + N(u_n) \tag{4.32}$$

and

$$0 = \|u_0\|_\mu^2 - \lambda \int_{\mathbb{R}^3} f u_0^2 dx - \int_{\mathbb{R}^3} g|u_0|^p dx + N(u_0). \tag{4.33}$$

By (4.26), (4.32) and (4.33), we have

$$o(1) = \|v_n\|_\mu^2 - \int_{\mathbb{R}^3} g|v_n|^p dx + N(v_n). \tag{4.34}$$

It follows from (4.29), (4.34) and $N(v_n) \geq 0$ that

$$o(1) \geq \|v_n\|_\mu^2 - D_2 \mu^{\frac{p-6}{2p}} \|v_n\|_\mu^2.$$

Therefore, we have $o(1) \geq \frac{1}{2}\|v_n\|_\mu^2$ for μ large enough, which implies that $v_n \to 0$ in X. Hence we have $u_n \to u_0$ in X,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g|u_n|^p dx = \int_{\mathbb{R}^3} g|u_0|^p dx \quad \text{and} \quad \lim_{n\to\infty} N(u_n) = N(u_0),$$

which implies that $J_{\mu,\lambda}(u_0) = \beta_\lambda(\mu)$. This completes the proof. □

5 The proof of Theorems 1.2

In this section, we prove Theorem 1.2. The results will hold by the following two theorems.

Theorem 5.1. *Under the assumptions of Theorems 1.2. There exists $\delta_0 > 0$ such that for every $0 < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, Eq. $(S_{\mu,\lambda})$ has a positive solution u^+ with $J_{\mu,\lambda}(u^+) > 0$ for $\mu > 0$ large enough.*

Proof. By Lemma 4.3, there exists $\delta_0 > 0$ such that for every $0 < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, there exist $\rho_\lambda, \eta_\lambda > 0$ and $\varphi_0 \in H_0^1(\Omega)$ such that $\|\varphi_0\|_\mu > \rho_\lambda$ and

$$\inf_{\|u\|_\mu = \rho_\lambda} J_{\mu,\lambda}(u) \geq \eta_\lambda > 0 > J_{\mu,\lambda}(\varphi_0)$$

for all $\mu > 0$ large enough. Let

$$\alpha_\lambda(\mu) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\mu,\lambda}(\gamma(t)) \quad \text{with} \quad \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = \varphi_0\}.$$

Since $\varphi_0 \in H_0^1(\Omega)$, there exists a constant $d_\lambda > 0$, independent of μ, such that

$$0 < \eta_\lambda \leq \alpha_\lambda(\mu) \leq \max_{0\leq t\leq 1} J_{\mu,\lambda}(t\varphi_0) \leq d_\lambda \quad \text{for all } \mu > 0 \text{ large enough.} \tag{5.1}$$

In order to prove the positively of solutions, we follow from the argument in [21]. Since $J_{\mu,\lambda}(u) = J_{\mu,\lambda}(|u|)$, we may assume that for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ with $\gamma_n(t) \geq 0$ for all $t \in [0,1]$ such that

$$\alpha_\lambda(\mu) \leq \max_{t\in[0,1]} J_{\mu,\lambda}(\gamma_n(t)) < \alpha_\lambda(\mu) + \frac{1}{n}.$$

Denote $J_{\mu,\lambda}(v_n) = \max_{t\in[0,1]} J_{\mu,\lambda}(\gamma_n(t))$. By Ekeland's variational principle [38], we have a $(PS)_{\alpha_\lambda(\mu)}$-sequence $\{u_n\}$ for $J_{\mu,\lambda}$ satisfying

$$\alpha_\lambda(\mu) \leq J_{\mu,\lambda}(u_n) \leq J_{\mu,\lambda}(v_n) < \alpha_\lambda(\mu) + \frac{1}{n}, \quad \|J'_{\mu,\lambda}(u_n)\|_\mu < \frac{1}{\sqrt{n}}$$

and

$$\|v_n - u_n\|_\mu < \frac{1}{\sqrt{n}}. \tag{5.2}$$

By (5.1) and Lemmas 4.4 and 4.5, there exist a subsequence $\{u_n\}$ and $u^+ \in X$ such that $u_n \to u^+$ in X as $n \to \infty$,

$$J_{\mu,\lambda}(u^+) = \alpha_\lambda(\mu) \quad \text{and} \quad J'_{\mu,\lambda}(u^+) = 0.$$

By (5.2) and the fact that $\gamma_n(t) \geq 0$ for all $t \in [0,1]$, we conclude that $u^+ \geq 0$ a.e. in $\mathbb{R}^3$. It follows from $J_{\mu,\lambda}(u^+) > 0$ and the strong maximum principle that $u^+ > 0$ in $\mathbb{R}^3$. This proof is complete. □

Theorem 5.2. *Under the assumptions of Theorems 1.2. There exists $\delta_0 > 0$ such that for every $\lambda_1(f_{\overline{\Omega}}) < \lambda < \lambda_1(f_{\overline{\Omega}}) + \delta_0$, Eq. $(S_{\mu,\lambda})$ has a positive solution u^- with $J_{\mu,\lambda}(u^-) < 0$ for μ large enough.*

Proof. Let $B_{\rho_\lambda} := \{u \in X : \|u\|_\mu \leq \rho_\lambda\}$ with ρ_λ as in Lemma 4.3. We consider the infimum of $J_{\mu,\lambda}$ on B_{ρ_λ} and set

$$\overline{\alpha}_\lambda(\mu) := \inf_{\|u\|_\mu \leq \rho_\lambda} J_{\mu,\lambda}(u).$$

By conditions (F) and (G), we deduce

$$J_{\mu,\lambda}(u) \geq -\frac{\|f\|_{L^{\frac{3}{2}}}}{2S^2}\|u\|_\mu^2 - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\|u\|_\mu^p \geq -\frac{\|f\|_{L^{\frac{3}{2}}}}{2S^2}\rho_\lambda^2 - \frac{\|g\|_{L^\infty}|\{V<c_0\}|^{\frac{6-p}{6}}}{pS^p}\rho_\lambda^p.$$

Moreover, for $t>0$, we have

$$J_{\mu,\lambda}(te_1) = -\frac{\lambda-\lambda_1(f_{\overline{\Omega}})}{2}t^2 + \frac{N(e_1)}{4}t^4 + \frac{\int_{\mathbb{R}^3} ge_1^p dx}{p}t^p.$$

This implies there exists $t_0>0$ such that $\|t_0e_1\|_\mu \leq \rho_\lambda$ and $J_{\mu,\lambda}(t_0e_1)<0$. Hence we conclude that there exist two positive constants $\overline{\eta}_\lambda, \overline{d}_\lambda > 0$ such that

$$-\overline{d}_\lambda \leq \overline{\alpha}_\lambda(\mu) \leq -\overline{\eta}_\lambda \quad \text{for all } \mu>0 \text{ large enough.} \tag{5.3}$$

Since $J_{\mu,\lambda}(u)=J_{\mu,\lambda}(|u|)$, we may assume that there exists $v_n \geq 0$ with $\|v_n\|_\mu \leq \rho_\lambda$ such that

$$\overline{\alpha}_\lambda(\mu) \leq J_{\mu,\lambda}(v_n) < \overline{\alpha}_\lambda(\mu)+\frac{1}{n}.$$

Then by Ekeland's variational principle [38], we have a $(PS)_{\overline{\alpha}_\lambda(\mu)}$-sequence $\{u_n\}\subset B_{\rho_\lambda}$ satisfying

$$\overline{\alpha}_\lambda(\mu) \leq J_{\mu,\lambda}(u_n) \leq J_{\mu,\lambda}(v_n) < \overline{\alpha}_\lambda(\mu)+\frac{1}{n}, \quad \|J'_{\mu,\lambda}(u_n)\|_\mu < \frac{1}{\sqrt{n}}$$

and

$$\|u_n-v_n\|_\mu < \frac{1}{\sqrt{n}}. \tag{5.4}$$

Note that ρ_λ is independent of μ. Thus, by Lemma 4.5, there exist a subsequence $\{u_n\}$ and $u^- \in X$ such that $u_n \to u^-$ in X as $n\to\infty$, $J_{\mu,\lambda}(u^-)=\overline{\alpha}_\lambda(\mu)<0$ and $J'_{\mu,\lambda}(u^-)=0$. By (5.4) and the fact that $v_n\geq 0$, we conclude that $u^-\geq 0$ a.e. in $\mathbb{R}^3$. It follows from $J_{\mu,\lambda}(u^-)<0$ and the strong maximum principle that $u^->0$ in $\mathbb{R}^3$. This proof is complete. □

6 Concentration for Solutions

In this section, we follow the argument in [39] to study the asymptotic behavior of positive solutions of system $(SP_{\mu,\lambda})$. The results of Theorem 1.3 will hold by the following two theorems.

Theorem 6.1. *Let u_μ be the solutions obtained in Theorem 5.1. Then there exists $u_\infty \in H_0^1(\Omega)$ such that $u_\mu \to u_\infty^+$ in X as $\mu\to\infty$ and it is a positive solution of Eq. $(S_{\infty,\lambda})$ with $J_{\infty,\lambda}(u_\infty^+)>0$, where*

$$J_{\infty,\lambda}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega fu^2 dx - \frac{1}{p}\int_\Omega g|u|^p dx + \frac{1}{4}\int_\Omega K\phi_u u^2 dx.$$

Proof. For any sequence $\mu_n\to\infty$, let $u_n := u_{\mu_n}$ be the positive solutions of Eq. $(S_{\mu,\lambda})$ obtained in Theorem 5.1 for λ. From the proof of Theorem 5.1, there exists a constant $D_0>0$ such that $\|u_n\|_{\mu_n} < D_0$ for all n. Then by Lemma 3.1, there exist a subsequence $\{u_n\}$ and $u_\infty^+ \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_\infty^+$ in X and $u_n \to u_\infty^+$ in $L^r(\mathbb{R}^3)$ for $2\leq r<6$. Since $\langle J'_{\mu_n,\lambda}(u_n),\varphi\rangle = 0$ for any $\varphi\in C_0^\infty(\Omega)$, we have

$$\int_\Omega \nabla u_n \nabla\varphi dx - \lambda\int_\Omega fu_n\varphi dx - \int_\Omega g|u_n|^{p-2}u_n\varphi dx + \int_\Omega K\phi_{u_n}u_n\varphi dx = 0 \tag{6.1}$$

which implies that

$$\int_{\Omega} \nabla u_{\infty}^{+} \nabla \varphi dx - \lambda \int_{\Omega} f u_{\infty}^{+} \varphi dx - \int_{\Omega} g |u_{\infty}^{+}|^{p-2} u_{\infty}^{+} \varphi dx + \int_{\Omega} K \phi_{u_{\infty}^{+}} u_{\infty}^{+} \varphi dx = 0. \tag{6.2}$$

That is, u_{∞}^{+} is a weak solution of Eq. $(S_{\infty,\lambda})$.

Next, we show that $u_n \to u_{\infty}^{+}$ in X. Since $\langle J'_{\mu_n,\lambda}(u_n), u_n \rangle = \langle J'_{\mu_n,\lambda}(u_n), u_{\infty}^{+} \rangle = 0$, we have

$$\|u_n\|_{\mu_n}^2 - \lambda \int_{\mathbb{R}^3} f u_n^2 dx - \int_{\mathbb{R}^3} g |u_n|^p dx + \int_{\mathbb{R}^3} K \phi_{u_n} u_n^2 dx = 0 \tag{6.3}$$

and

$$\int_{\Omega} \nabla u_n \nabla u_{\infty}^{+} dx - \lambda \int_{\Omega} f u_n u_{\infty}^{+} dx - \int_{\Omega} g |u_n|^{p-2} u_n u_{\infty}^{+} dx + \int_{\Omega} K \phi_{u_n} u_n u_{\infty}^{+} dx = 0. \tag{6.4}$$

By the fact that $u_n \to u_{\infty}^{+}$ in $L^r(\mathbb{R}^3)$ for $2 \leq r < 6$, we have

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} f u_n^2 dx - \int_{\Omega} f u_n u_{\infty}^{+} dx \right) = 0, \quad \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} g |u_n|^p dx - \int_{\Omega} g |u_n|^{p-2} u_n u_{\infty}^{+} dx \right) = 0$$

and

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} K \phi_{u_n} u_n^2 dx - \int_{\Omega} K \phi_{u_n} u_n u_{\infty}^{+} dx \right) = 0.$$

Thus,

$$\lim_{n\to\infty} \|u_n\|_{\mu_n}^2 = \lim_{n\to\infty} \int_{\Omega} \nabla u_n \nabla u_{\infty}^{+} dx = \int_{\Omega} |\nabla u_{\infty}^{+}|^2 dx \geq \lim_{n\to\infty} \|u_n\|^2$$

which implies $u_n \to u_{\infty}^{+}$ in X, and thus we have $J_{\mu_n,\lambda}(u_n) \to J_{\infty,\lambda}(u_{\infty}^{+})$. Moreover, by (5.1), we can deduce that $\eta_{\lambda} \leq J_{\mu_n,\lambda}(u_n) \leq d_{\lambda}$ for all n, which implies that $\eta_{\lambda} \leq J_{\infty,\lambda}(u_{\infty}^{+}) \leq d_{\lambda}$. Hence we conclude that $u_{\infty}^{+} > 0$. This proof is complete. □

Theorem 6.2. *Let u_{μ} be the solutions obtained in Theorem 5.2. Then there exists $u_{\infty}^{-} \in H_0^1(\Omega)$ such that $u_{\mu} \to u_{\infty}^{-}$ in X as $\mu \to \infty$ and it is a positive solution of Eq. $(S_{\infty,\lambda})$ with $J_{\infty,\lambda}(u_{\infty}^{-}) < 0$.*

Proof. This proof is essential same as that of Theorem 6.1. Hence we omit it here. □

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