NOTE ON SOME FRACTIONAL BI-INHOMOGENEOUS SCHRÖDINGER-CHOQUARD EQUATIONS

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Abstract. In the subcritical energy case, local well-posedness is established in the radial energy space for a class of fractional inhomogeneous Choquard equations. The best constant of a Gagliardo-Nirenberg type inequality is obtained. Moreover, a sharp threshold of global existence versus blow-up dichotomy is obtained for mass super-critical and energy subcritical solutions.

1. Introduction

Our purpose in this paper is to investigate the Cauchy problem for the following fractional inhomogeneous Schrödinger equation of Choquard type:

$$\begin{cases} i\dot{u} - (-\triangle)^s u = \epsilon (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2}u, \\ u(0,\cdot) = u_0, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ for some $N \geq 3, s \in (0,1), p > 1, \epsilon = \pm 1, \beta < 0$ and $0 < \alpha < N$. The fractional Laplacian operator stands for

$$(-\triangle)^s \cdot := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}\cdot),$$

where $\mathcal{F}$ is the Fourier transform. The Riesz-potential is defined as

$$I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha |x|^{N-\alpha}} := \frac{\mathcal{K}}{|x|^{N-\alpha}}, x \in \mathbb{R}^N.$$

Here and hereafter, we assume that

$$\min\{\alpha, -\beta, N-\alpha, N+\beta, N+\alpha+2\beta, 2s+\alpha+2\beta\} > 0 \tag{1.2}$$

In three space dimension, for $s = 1$ and $\beta = 0$, the problem (1.1) corresponds to the homogeneous Schrödinger-Choquard equation which has several physical origins such as quantum mechanics [15], Hartree-Fock theory [17], and non-relativistic quantum theory [11]. If $p = 2, s = \frac{1}{2}, \beta = 0$, then (1.1) models the dynamics of boson stars, where the potential is the Newtonian gravitational potential in the appropriate physical units [7, 14]. When $s = 1$ and $\beta < 0$, some particular cases of the equation (1.1) arise in the study of the mean-field limit of large systems of non-relativistic bosonic atoms and molecules [24, 9]. Before we proceed to the discussion, it is useful to look at the most vital symmetry which is scaling. Indeed, the equation (1.1) enjoys the following scaling invariance

$$u_\lambda(t) = \lambda^{\frac{\alpha+2s+2\beta}{2(p-1)}} u(\lambda^{2s}t, \lambda\cdot), \lambda > 0.$$

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For a real number μ, we have

$$\|u_\lambda(t)\|_{\dot{H}^\mu} = \lambda^{\mu-\frac{N}{2}+\frac{\alpha+2s+2\beta}{2(p-1)}}\|u(\lambda^{2s}t,\lambda\cdot)\|_{\dot{H}^\mu}.$$

So, the critical exponent is

$$s_c := \frac{N}{2} - \frac{\alpha+2s+2\beta}{2(p-1)},$$

for which the $\dot{H}^\mu$ norm is unaffected by scaling. The case $s_c = 0$ corresponds to the mass critical exponent $p_* = 1 + \frac{\alpha+2s+2\beta}{N}$. The energy critical case $s_c = s$ corresponds to $p^* = 1 + \frac{\alpha+2s+2\beta}{N-2s}$. For smaller p, that is $p \in]1, p^*[$ which is called the energy subcritical exponent, contracting time reduces the size of the $\dot{H}^s$ norm. This is the effect that will be exploited to build up solutions.

In the case $s = 1$ and $\beta = 0$, the problem (1.1) was investigated in [8, 23], some particular cases were considered by different authors [4, 10, 2]. In [19], the existence and asymptotic properties of standing waves were investigated. Few paper deal with the general form. Recently, in [21] the above problem was studied in the case $s \in (\frac{1}{2}, 1)$ and $\beta = 0$. Also, the case $s = 1$ and $\beta < 0$ was investigated in [22].

Any solutions to (1.1), in the energy space, satisfies the following conserved quantities:

$$M(u(t)) := \int_{\mathbb{R}^N} |u(t)|^2 dx = M(u_0),$$

$$E(u(t)) := \int_{\mathbb{R}^N} \{|(-\triangle)^{\frac{s}{2}}u(t)|^2 + \frac{\epsilon}{p}(I_\alpha * |\cdot|^\beta |u(t)|^p)|x|^\beta |u(t)|^p\} dx = E(u_0).$$

It is well known that $\epsilon = 1$ corresponds to the defocusing case. Thus, any subcritical energy solution is claimed to be global. When $\epsilon = -1$, which is said the focusing case, any chance to control the $\dot{H}^s$ norm of the solution with the conservation laws. So, maximal solution of (1.1) may blow-up in finite or infinite time.

It is the aim of this paper to investigate the problem (1.1). Indeed, local well-posedness in the radial energy space and global existence in the defocusing case are obtained here. In the focusing sign, the existence of global and non global solutions is discussed with respect to a sharp Gagliardo-Nirenberg inequality related to (1.1).

The manuscript is organized as follows: Section two summarizes the main results. Section three presents some technical tools needed here. Section four is devoted to establish the existence of ground state for (1.1) and to prove a sharp Gagliardo-Nirenberg type inequality. In Section five, a Virial type inequality is established. The two last Sections are devoted, to show the well-posedness of the main problem, to give a sharp dichotomy of global/non global existence of solutions, and to derive blow-up results.

We end this section with some notations. We consider the Lebesgue spaces $L^r := L^r(\mathbb{R}^N)$ equipped with norms $\|f\|_r := \|f\|_{L^r} = (\int_{\mathbb{R}^N} |f(x)|^r dx)^{\frac{1}{r}}$ if $r < \infty$, else $\|f\|_\infty := \|f\|_{L^\infty} = \sup ess_{x\in\mathbb{R}^N}|f(x)|$. For vector valued functions $\|(f_j)\|_\infty := \sup_j \|f_j\|_\infty$. When $r = 2$, let $\|f\| := \|f\|_2$. The usual inhomogeneous Sobolev space is denoted by $W^{s,r} := W^{s,r}(\mathbb{R}^N)$ and endowed with the complete norm $\|f\|_{W^{s,r}} := (\|f\|^r + \|(-\triangle)^{\frac{s}{2}}f\|^r)^{\frac{1}{r}}$, in the case $r = 2$ we denote $H^s := W^{s,2}$ which is equipped with $\|f\|_{H^s} := (\|f\|^2 + \|(-\triangle)^{\frac{s}{2}}f\|^2)^{\frac{1}{2}}$. We need also to introduce some Böchner spaces $L^q(L^r)$, $L^q(H^s)$ and $L^q(W^{s,r})$ equipped with their naturally norms. If X is an abstract space, the set of continuous functions defined on $[0, T[$

and valued in X is denoted by $C_T(X) := C([0,T), X)$, if necessary the interval of time may be closed. Also, we denote $L^q_T(X) := L^q([0,T), X)$. The set X_{rd} stands for the set of radial elements in X. Any constant will be denoted by C which may vary from line to line. For simplicity, let $\int f(x)dx := \int_{\mathbb{R}^N} f(x)dx$ and $\int f(x,y)dxdy := \int\int f(x,y)dxdy$. Finally, if A and B are non-negative quantities, we write $A \lesssim B$ to denote $A \leq CB$. Moreover, we use the notation $A = \mathcal{O}(B)$(respectively $A = o(B)$, $A \sim B$) by which we mean that $A \lesssim B$(respectively $A \lesssim \varepsilon B$, $A = B + o(B)$) holds.

2. Main results

At First, let us introduce the following quantities

$$B := \frac{1}{s}(Np - N - \alpha - 2\beta) \ , \ A := 2p - B;$$

$$J(u) := \frac{\|u\|^A \|u\|^B_{\dot{H}^s}}{\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx} \text{ for } u \in H^s - \{0\}.$$

For $a, b \in \mathbb{R}$, let

$$\underline{\mu} = \min(2a + (N-2s)b, 2a + Nb) \ , \ \overline{\mu} = \max(2a + (N-2s)b, 2a + Nb);$$

and

$$\mathcal{A} = \{(a,b) \in \mathbb{R}^*_+ \times \mathbb{R} \text{ s.t } 2ap \geq \overline{\mu} > 0 \text{ and } \underline{\mu} \geq 0\}.$$

We denote also

$$v^\lambda_{a,b} := \lambda^a v(\lambda^{-b}\cdot) \ , \ \mathcal{L}_{a,b}(v) := (\partial_\lambda v^\lambda_{a,b})|_{\lambda=1};$$

$$K^Q_{a,b}(v) := (2a + (N-2s)b)\|v\|^2_{\dot{H}^s} + (2a + Nb)\|v\|^2;$$

and

$$K^N_{a,b}(v) := -\frac{1}{p}(2ap + b(N + \alpha + 2\beta)) \int (I_\alpha * |\cdot|^\beta |v|^p)|x|^\beta |v|^p dx;$$

$$S := M + E \ , \ K_{a,b} := \mathcal{L}_{a,b} S = K^Q_{a,b} + K^N_{a,b};$$

$$H_{a,b} := (1 - \frac{\mathcal{L}_{a,b}}{\overline{\mu}})S = S - \frac{K_{a,b}}{\overline{\mu}}.$$

Now, we have to define the so called ground state solution of the problem (1.1).

Definition 1. *Any solution to*

$$-(-\triangle)^s \phi - \phi + (I_\alpha * |\cdot|^\beta |\phi|^p)|x|^\beta |\phi|^{p-2}\phi = 0, \phi \in H^s - \{0\}. \tag{2.1}$$

which minimizes the problem

$$m_{a,b} := \inf_{v \in H^s - \{0\}} \{S(v) \ s.t \ K_{a,b}(v) = 0\}, \tag{2.2}$$

is called ground state of the problem (1.1).

Our main results are the following:

Theorem 1. *(Existence of ground state)*
Let $N \geq 3, s \in (0,1), p_* < p < p^*$ and taking $(a,b) \in \mathcal{A}$, then

1. *$m := m_{a,b}$ is nonzero and independent of (a,b);*
2. *there is a ground state solution to* (2.1) *and* (2.2).

Remark 1. *The condition $2ap \geq \overline{\mu}$ is due to the presence of β and it is used only in* (4.1)*.We can omit this condition if we assume that $\alpha + 2\beta \geq 0$. The condition $p_* < p$ is needed only in* (4.2)*. If we relax the set $\mathcal{A}$ to*

$$\mathcal{A}' := \{(a, b) \in \mathbb{R}_+^* \times \mathbb{R} \ s.t \ 2ap + (N + \alpha + 2\beta)b > \overline{\mu} > 0 \ \text{and} \ \underline{\mu} \geq 0\}.$$

Then, the previous Theorem holds for any exponent $1 + \frac{\alpha+2\beta}{N} < p < p^*$.

Theorem 2. *(Sharp Gagliardo-Nirenberg inequality)*
Let $N \geq 3, s \in (0, 1)$, α and β satisfying (1.2) *and $1 + \frac{\alpha+2\beta}{N} < p < p^*$. Then*

(1) *there exists $C(N, p, s, \alpha, \beta) > 0$ such that*

$$\forall u \in H^s, \int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx \leq C(N, p, s, \alpha, \beta)\|u\|^A \|u\|_{\dot{H}^s}^B. \tag{2.3}$$

(2) *the minimization problem*

$$\frac{1}{C(N, p, s, \alpha, \beta)} = \inf_{v \in H^s - \{0\}} J(v)$$

is attained in some $Q \in H^s$ satisfying $C(N, p, s, \alpha, \beta) = \int (I_\alpha * |\cdot|^\beta |Q|^p)|x|^\beta |Q|^p dx$ and

$$B(-\triangle)^s Q + AQ - \frac{2p}{C(N, p, s, \alpha, \beta)}(I_\alpha * |\cdot|^\beta |Q|^p)|x|^\beta |Q|^{p-2}Q = 0. \tag{2.4}$$

(3) *Moreover, there is ϕ a ground state solution to* (2.1) *such that*

$$C(N, p, s, \alpha, \beta) = \frac{2p}{A}\left(\frac{A}{B}\right)^{\frac{B}{2}} \|\phi\|^{-2(p-1)}. \tag{2.5}$$

Remark 2. *In the sequel, it is useful for some time that $B > 2$ in* (2.3)*, this is verified under the assumption $p_* < p$.*

Next, we show that (1.1) is well posed in H^s_{rd} for any exponent $2 \leq p < p^*$. However, the energy is well defined for exponents such that $1 + \frac{\alpha+2\beta}{N} < p \leq p^*$. Such restriction is due to contraction arguments which are used in the proof, since the source term is singular for $1 + \frac{\alpha+2\beta}{N} < p < 2$.

Theorem 3. *(Well-posedness in the radial energy space)*
Let $N \geq 3, s \in [\frac{N}{2N-1}, 1), \alpha, \beta$ satisfying (1.2)*. In addition, we suppose*

$$N - 4s < \alpha + 2\beta, N + \beta > s, N + \alpha + 2\beta - 2s > 0 \ \text{and} \ 2 \leq p < p^*.$$

Then, for all $u_0 \in H^s_{rd}$ there exists $T^* := T^*(\|u_0\|_{H^s_{rd}}) > 0$ such that (1.1) *admits a unique maximal solution*

$$u \in C_{T^*}(H^s_{rd}) \cap L^q_{loc}((0, T^*), W^{s,r}), \forall (q, r) \in \Gamma_s.$$

In addition, such solution satisfies the conservation lows

$$M(u(t)) = M(u_0) \ \text{and} \ E(u(t)) = E(u_0),$$

and it is global if one of the following assertions holds:

(1) *$\epsilon = 1$ and $p < p^*$;*
(2) *$p < p_*$;*
(3) *$p = p_*$ and $M(u_0) < (\frac{p}{C(N,p,s,\alpha,\beta)})^{\frac{2}{A}}$.*

Remark 3. *We remark that*

(1) *the condition* $N-4s<\alpha+2\beta$ *means that* $2<p^*$,
(2) *results obtained in Theorems 1, 2, 3 are valid also for the following inhomogeneous problem*

$$\begin{cases} i\dot{u}-(-\triangle)^s u=\epsilon|x|^{\beta}(I_\alpha*|u|^p)|u|^{p-2}u, \\ u(0,\cdot)=u_0, \end{cases} \tag{2.6}$$

Indeed, replacing 2β *by* β *and slight modifications in their proofs leads to the desired results.*

Now, we are interested on non-global solutions to (1.1). So, we establish first some Virial type inequality. For that, we make use of $\psi\in C_0^\infty(\mathbb{R}^N)$ a radial cutoff function which is introduced in [3] and defined as follows:

$$\psi(x)=\begin{cases} \frac{1}{2}|x|^2, \text{ for } |x|\le 1, \\ C, \text{ for } |x|\ge 10. \end{cases} \quad \text{and } \psi''\le 1. \tag{2.7}$$

For all $R>0$, we denote $\psi_R:=R^2\psi(\frac{\cdot}{R})$. It is well known that ψ_R satisfies some properties [3], mainly

$$\psi_R''\le 1,\ \psi_R'(r)\le r,\ \triangle\psi_R\le N.$$

The localized Virial is denoted

$$M_\psi[u]:=2\Im\Big(\int\overline{u}\nabla\psi.\nabla u dx\Big)=2\Im\Big(\sum_{k=1}^{N}\int\overline{u}\partial_k\psi\partial_k u dx\Big).$$

Theorem 4. *(Virial type identity)*
Let $N\ge 3, s\in(\frac{1}{2},1)$, α,β *satisfying* (1.2) *and* $1+\frac{\alpha+2\beta}{N}<p\le p^*$. *Assume that* $u\in C_T(H^s_{rd})$ *is a solution of* (1.1). *Then*

(1) *For* $R>0$ *and* $\varepsilon>0$ *small enough, we have*

$$\begin{aligned}\frac{d}{dt}M_{\psi_R}[u]&\le 2(N(p-1)-\alpha-2\beta)E(u_0)-2N(p-p_*)\|u\|^2_{\dot{H}^s}\\ &+C\Big(R^{-2s}+\frac{\|u\|_{\dot{H}^s}^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}}{R^{(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}}\Big).\end{aligned}$$

(2) *If* $p=p_*, E(u_0)<0$ *and* $N>1+2s+\alpha+2\beta$, *then for* R *large enough, we have*

$$\frac{d}{dt}M_{\psi_R}[u]<2sE(u_0).$$

Remark 4. *These estimates are established only for solutions of the main problem* (1.1) *and do not concerns* (2.6). *In fact, the nonlinearity in* (1.1) *has a symmetry with respect to the term* $|x|^\beta$, *which is crucial in computations.*

Theorem 5. *(Blow-up vs global well-posedness)*
Let $N\ge 3, s\in(0,1),\alpha,\beta$ *satisfying* (1.2). *In addition we suppose* $N-4s\le\alpha+2\beta$ *and* $0\le s_c<s$. *Let* ϕ *be a ground state solution to* (2.1), $u_0\in H^s_{rd}$, *and* $u\in C_{T^*}(H^s_{rd})$ *be a maximal solution for* (1.1).

(1) *Assume that* $s \in (\frac{1}{2}, 1)$ *and* $1 + \frac{\alpha+2\beta}{N} < p < 1 + 2s + \frac{\alpha+2\beta}{N}$. *Suppose that either* $E(u_0) < 0$ *or* $E(u_0) \geq 0$ *with the two next inequalities*

$$E(u_0)^{s_c} M(u_0)^{s-s_c} < E(\phi)^{s_c} M(\phi)^{s-s_c} \tag{2.8}$$

$$\|u_0\|^{s_c}_{\dot{H}^s} \|u_0\|^{s-s_c} > \|\phi\|^{s_c}_{\dot{H}^s} \|\phi\|^{s-s_c} \tag{2.9}$$

Then,

(a) *if* $0 < s_c < s$, *then* u *blows-up in finite time, in particular* $T^* < \infty$ *and* $\limsup_{t \to T^*} \|u(t)\|_{\dot{H}^s} = +\infty$;

(b) *if* $s_c = 0$, *then* u *either blows-up in finite time or there exist* $C > 0$ *and an instant of time* $t^* > 0$ *such that*

$$\forall t \geq t^*,\ \|u(t)\|_{\dot{H}^s} \geq C t^s.$$

(2) *Suppose that* $E(u_0) \geq 0$ *with* (2.8) *and*

$$\|u_0\|^{s_c}_{\dot{H}^s} \|u_0\|^{s-s_c} < \|\phi\|^{s_c}_{\dot{H}^s} \|\phi\|^{s-s_c}, \tag{2.10}$$

then $T^* = \infty$.

Remark 5. *In the previous Theorem, the assumption* $p - 1 < 2s + \frac{\alpha+2\beta}{N}$ *is a natural extension for the condition "*$\sigma < 2s$*" used in* [3] *to our problem. In the case* $\beta = 0$, *that would give more wide interval for the exponent* p, *when compared with the assumption* $p < \frac{1}{2s+1}(1 + \frac{\alpha}{N} + 4s)$ *from* [21].

3. Tools

The following Lemma summarize some classical results from[1, 5, 18], mainly Sobolev injections and interpolation inequalities.

Lemma 1. *Let* $N \geq 2$ *and* $s \in (0, 1)$. *Then*

- $H^s \hookrightarrow L^q$, *for any* $q \in [2, \frac{2N}{N-2s}]$;
- $H^s_{rd} \hookrightarrow\hookrightarrow L^q$ *for any* $q \in (2, \frac{2N}{N-2s})$;
- *for any* $r \in (1, \frac{N}{s})$ *and* $q \in (r, \frac{Nr}{N-rs}]$, *we have* $W^{s,r} \hookrightarrow L^q$;
- *for any* $q \in [2, \frac{2N}{N-2s}]$, *let* $\theta := \frac{N}{s}(\frac{1}{2} - \frac{1}{q})$, *we have the so-called fractional Gagliardo-Nirenberg inequality*

$$\|u\|_q \lesssim \|u\|^{1-\theta} \|u\|^{\theta}_{\dot{H}^s}; \tag{3.1}$$

- *if* $\frac{1}{2} < \mu < s < \frac{N}{2}$, *then for all* $u \in H^s$, *we have*

$$\|(-\triangle)^{\frac{\mu}{2}} u\| \leq \|u\|^{1-\frac{\mu}{s}} \|(-\triangle)^{\frac{s}{2}} u\|^{\frac{\mu}{s}}; \tag{3.2}$$

- *if* $\mu \in (\frac{1}{2}, \frac{N}{2})$, *then for all* $u \in H^{\mu}_{rd}$, *we have*

$$\sup_{x \in \mathbb{R}^N - \{0\}} |x|^{\frac{N}{2}-\mu} |u(x)| \leq C(N, \mu) \|(-\triangle)^{\frac{\mu}{2}} u\|. \tag{3.3}$$

The next fractional chain rules [6] will be useful.

Lemma 2. *Let* $s \in (0, 1]$ *and* $1 < p, p_i, q_i < \infty$ *satisfying* $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$. *Then*

- *if* $G \in C^1(\mathbb{C})$, *then*

$$\||\nabla|^s G(u)\|_p \lesssim \|G'(u)\|_{p_1}\||\nabla|^s u\|_{q_1};$$

-

$$\||\nabla|^s (uv)\|_p \lesssim \||\nabla|^s u\|_{p_1}\|v\|_{q_1} + \||\nabla|^s v\|_{p_2}\|u\|_{q_2}.$$

Recall the Hardy-Littlewood-Sobolev inequality [16, 21].

Lemma 3. *Let* $N \geq 3, 0 < \lambda < N$ *and* $1 < r, s < \infty$ *and* $f \in L^r, g \in L^s$. *If* $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, *then*

$$\int \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq C(N, s, \lambda)\|f\|_r\|g\|_s.$$

Corollary 1. *Let* $N \geq 3, 0 < \alpha < N$, $1 < q, r, s < \infty$ *and* $f \in L^r, g \in L^s$.

- *if* $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$, *then*

$$\int (I_\alpha * f)(x)g(y)dxdy \leq C(N, s, \alpha)\|f\|_r\|g\|_s;$$

- *if* $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$, *then*

$$\|(I_\alpha * f)g\|_{q'} \leq C(N, s, \alpha)\|f\|_r\|g\|_s.$$

The last estimate is known as the Hardy-Littlwood-Paley inequality. Next, recall also the generalized Pohozaev identity [13, 20].

Proposition 1. *Let* $\phi \in H^s$, *we have*

$$\phi \text{ is solution to } (2.1) \Leftrightarrow S'(\phi) = 0,$$

Moreover, in that case $K_{a,b}(\phi) = 0, \forall (a, b) \in \mathbb{R}^2$.

Finally, let us recall the so-called radial Strichartz estimate [12].

Definition 2. *A couple of real numbers* (q, r) *is said to be admissible, we denote* $(q, r) \in \Gamma_s$, *if*

$$q \geq 2, r \in [2, +\infty), (q, r) \neq (2, \frac{4N-2}{4N-3}) \text{ and } N(\frac{1}{2} - \frac{1}{r}) = \frac{2s}{q}.$$

Proposition 2. *Let* $N \geq 2, \frac{N}{2N-1} \leq s < 1$ *and* $u_0 \in L^2_{rd}$. *Then for any admissible pairs* (q, r) *and* $(\tilde{q}, \tilde{r})$ *from* Γ_s, *we have*

$$\|u\|_{L^q_t(L^r)} \lesssim \|u_0\| + \|i\dot{u} - (-\triangle)^s u\|_{L^{\tilde{q}'}_t(L^{\tilde{r}'})}.$$

Remark 6. *The Strichartz inequality is compatible with truncations. Indeed, if we have* $i\dot{u} - (-\triangle)^s u = h$ *and* $(q, r), (\tilde{q}, \tilde{r}), (\tilde{q}_1, \tilde{r}_1) \in \Gamma_s$ *then*

$$\|u\|_{L^q_t(L^r)} \lesssim \|u_0\| + \|h\|_{L^{\tilde{q}'}_t(L^{\tilde{r}'}(|x|<1))} + \|h\|_{L^{\tilde{q}'_1}_t(L^{\tilde{r}'_1}(|x|>1))}.$$

4. Ground state and Sharp Gagliardo-Nirenberge inequality

4.1. Auxiliary lemmas. In this section, we are going to prove some intermediate results which will be crucial to prove Theorem 1. For easy notations, let

$$u^\lambda := u^\lambda_{a,b}, K := K^\lambda_{a,b}, K^Q := K^Q_{a,b}, K^N := K^N_{a,b}, \mathcal{L} := \mathcal{L}_{a,b};$$

and we keep the notation $H_{a,b}$.

Lemma 4. *Let $(a,b) \in \mathcal{A}$. Then*

(1) $\min(\mathcal{L}H_{a,b}(u), H_{a,b}(u)) > 0$ *for all* $u \in H^s - \{0\}$,
(2) $\lambda \to H_{a,b}(u^\lambda)$ *is increasing.*

Proof. Let $u \in H^s - \{0\}$, we have

$$\begin{aligned} H_{a,b}(u) &= (1 - \frac{\mathcal{L}}{\bar{\mu}})S(u) \\ &= \frac{1}{\bar{\mu}}(\bar{\mu}S(u) - K(u)) \\ &= \frac{1}{\bar{\mu}}\{(\bar{\mu} - (2a + (N-2s)b))\|(-\triangle)^{\frac{s}{2}}u\|^2 + (\bar{\mu} - (2a+Nb))\|u\|^2 \\ &+ \frac{1}{p}(2ap + b(N+\alpha+2\beta) - \bar{\mu}) \int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx\}. \end{aligned}$$

At this step, if we adopt the set $\mathcal{A}'$ instead of $\mathcal{A}$, one gets $2ap + b(N+\alpha+2\beta) - \bar{\mu} > 0$ which implies clearly that $H_{a,b}(u) > 0$. Otherwise, we have

$$\bar{\mu} = \begin{cases} 2a + Nb, \text{ if } b \geq 0 \\ 2a + (N-2s)b, \text{ if } b \leq 0. \end{cases}$$

In the case $b > 0$, we have

$$2ap + b(N+\alpha+2\beta) - \bar{\mu} = 2a(p-1) + b(\alpha+2\beta) > 0.$$

Indeed, from (1.2), one has $\alpha + 2\beta > -N$. So

$$2a(p-1) + b(\alpha+2\beta) > 2a(p-1) - Nb = 2ap - (2a+Nb) = 2ap - \underline{\mu} \geq 0. \tag{4.1}$$

In the case $b \leq 0$, from $\underline{\mu} \geq 0$ we obtain $b \geq -\frac{2a}{N}$. Hence

$$2a(p-1) + b(\alpha+2\beta+2s) \geq 2a(p-1) - \frac{2a}{N}(\alpha+2\beta+2s) \geq 2a(p-p_*).$$

Since $p^* < p$, we obtain

$$2ap + b(N+\alpha+2\beta) - \bar{\mu} = 2a(p-1) + b(\alpha+2\beta+2s) > 0. \tag{4.2}$$

In summary, we obtain

$$2ap + b(N+\alpha+2\beta) - \bar{\mu} > 0, \forall (a,b) \in \mathcal{A}. \tag{4.3}$$

Thus $H_{a,b}(u) > 0$ for any $(a,b) \in \mathcal{A}$.
Furthermore, we have

$$\begin{aligned}\mathcal{L}H_{a,b}(u) &= \mathcal{L}(1-\frac{\mathcal{L}}{\bar{\mu}})S(u) = -\frac{1}{\bar{\mu}}(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})S(u) + \underline{\mu}(1-\frac{\mathcal{L}}{\bar{\mu}})S(u)\\ &= -\frac{1}{\bar{\mu}}(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})S(u) + \underline{\mu}H_{a,b}(u) > -\frac{1}{\bar{\mu}}(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})S(u).\end{aligned}$$

Clearly, we have $(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})\|u\|^2_{H^s} = 0$, then

$$\begin{aligned}\mathcal{L}H_{a,b}(u) &> \frac{1}{\bar{\mu}}(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})(\frac{1}{p}\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx)\\ &= \frac{1}{p\bar{\mu}}(2ap + b(N+\alpha+2\beta) - \bar{\mu})(2ap + b(N+\alpha+2\beta) - \underline{\mu})\\ &\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx\\ &\geq \frac{1}{p\bar{\mu}}(2ap + b(N+\alpha+2\beta) - \bar{\mu})^2 \int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx.\end{aligned}$$

Hence $\mathcal{L}H_{a,b}(u) > 0$, this complete the proof of the first point. The second point follows from the equality $\partial_\lambda H_{a,b}(u^\lambda) = \mathcal{L}H_{a,b}(u^\lambda)$. □

Lemma 5. *Let $(a,b) \in \mathcal{A}$. If $u_n \in H^s - \{0\}$ is a bounded sequence such that $\lim_n K^Q(u_n) = 0$, then there exists $n_0 \in \mathbb{N}$ such that: $K(u_n) > 0, \forall n \geq n_0$.*

Proof. First, let us remark that for any $(a,b) \in \mathcal{A}$, we have

$$K^Q(u_n) > 0 \text{ and } \|(-\triangle)^{\frac{s}{2}} u_n\|^2 \lesssim K^Q(u_n).$$

In fact, the conditions $\overline{\mu} > 0$ and $\underline{\mu} \geq 0$ ensure that $2a + (N-2s)b > 0$ in any cases. Since $p_* < p < p^*$, then (2.3) holds with $B > 2$. Hence

$$\int (I_\alpha * |\cdot|^\beta |u_n|^p)|x|^\beta |u_n|^p dx \lesssim \|u_n\|^A \|(-\triangle)^{\frac{s}{2}} u_n\|^B = o(\|(-\triangle)^{\frac{s}{2}} u_n\|^2) = o(K^Q(u_n)).$$

Thus, when n goes to $+\infty$, one gets

$$\begin{aligned}K(u_n) &= K^Q(u_n) - \frac{1}{p}(2ap + b(N+\alpha+2\beta)) \int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx\\ &= K^Q(u_n) + o(K^Q(u_n)) \sim K^Q(u_n) > 0\end{aligned}$$

□

Remark 7. *Whenever $(a,b) \neq (1, \frac{-2}{N})$ then $2a + Nb > 0$. In that case*

$$\int (I_\alpha * |\cdot|^\beta |u_n|^p)|x|^\beta |u_n|^p dx \lesssim \|u_n\|^A \|(-\triangle)^{\frac{s}{2}} u_n\|^B \lesssim K^Q(u_n)^{2p}.$$

Then $K(u_n) \sim K^Q(u_n)$ holds for any exponent $1 + \frac{\alpha+2\beta}{N} < p < p^*$.

The last intermediate result is the following.

Lemma 6. *Let $(a,b) \in \mathcal{A}$. Then*

$$m_{a,b} = \inf_{u \in H^s - \{0\}} \{H_{a,b}(u) \ s.t \ K(u) \leq 0\}.$$

Proof. We have

$$\|u^\lambda\| = \lambda^{\frac{2a+Nb}{2}}\|u\| \text{ and } \|u^\lambda\|_{\dot H^s} = \lambda^{\frac{2a+(N-2s)b}{2}}\|u\|_{\dot H^s}.$$

Then the set $\{u^\lambda, \lambda\in(0,1]\}$ is bounded in H^s. Now, let

$$r := \inf_{u\in H^s-\{0\}}\{H_{a,b}(u) \text{ s.t } K(u)\le 0\}.$$

It is obvious that

$$\{S(u) \text{ s.t } K(u)=0\} = \{H_{a,b}(u) \text{ s.t } K(u)=0\} \subset \{H_{a,b}(u) \text{ s.t } K(u)\le 0\}.$$

Hence $r\le m_{a,b}$. Conversly, take $u\in H^s-\{0\}$ such that $K(u)<0$. We have $\lim_{\lambda\to 0} K^Q(u^\lambda) = 0$, taking account of Lemma 5, there exists $\lambda\in(0,1)$ such that $K(u^\lambda)>0$. Knowing that $K(u^1)=K(u)<0$, so by a continuity argument there exists $\lambda_0\in(0,1)$ such that $K(u^{\lambda_0})=0$. The function $\lambda\to H_{a,b}(u^\lambda)$ is increasing, then

$$m_{a,b}\le S(u^{\lambda_0}) = H_{a,b}(u^{\lambda_0}) \le H_{a,b}(u^1) = H_{a,b}(u).$$

This valid for any $u\in H^s-\{0\}$ such that $K(u)\le 0$, so we deduce $m_{a,b}\le r$. This finishes the proof. □

4.2. **Proof of Theorem 1.** Let (ϕ_n) be a minimizing sequence, with a rearrangement argument via Lemma 6, we can assume that (ϕ_n) is radial decreasing and satisfies

$$\phi_n\in H^s_{rd}-\{0\}, K(\phi_n)=0 \text{ and } \lim H_{a,b}(\phi_n) = \lim S(\phi_n) = m_{a,b}. \tag{4.4}$$

First step. At first, we will prove that (ϕ_n) is bounded in H^s. In the case $b>0$, let $\lambda := \frac{b}{2a}$. Since $K(\phi_n)=0$ then dividing by $2a>0$, we get

$$\begin{aligned}&(1+(N-2s)\lambda)\|(-\triangle)^{\frac{s}{2}}\phi_n\|^2 + (1+N\lambda)\|\phi_n\|^2\\ &= (1+\lambda\frac{N+\alpha+2\beta}{p})\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx.\end{aligned}$$

Therefore

$$\begin{aligned}\|\phi_n\|^2_{H^s} = &\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx + \lambda\{(2s-N)\|(-\triangle)^{\frac{s}{2}}\phi_n\|^2 - N\|\phi_n\|^2\\ &+\frac{N+\alpha+2\beta}{p}\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx\}.\end{aligned}$$

So

$$\begin{aligned}&\|\phi_n\|^2_{H^s} - 2s\lambda\|(-\triangle)^{\frac{s}{2}}\phi_n\|^2 - (1+\lambda\frac{\alpha+2\beta}{p})\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx\\ &= -\lambda N\{\|\phi_n\|^2_{H^s} - \frac{1}{p}\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx\}\\ &= -\lambda N S(\phi_n).\end{aligned}$$

Knowing that $S(\phi_n)\to m_{a,b}$, then $S(\phi_n)$ defines a bounded real sequence. Hence, for any real number δ, the following sequence is bounded:

$$2s\lambda\|(-\triangle)^{\frac{s}{2}}\phi_n\|^2 + (\delta-1)\|\phi_n\|^2_{H^s} + (1-\frac{\delta}{p}+\frac{\lambda(\alpha+2\beta)}{p})\int(I_\alpha*|\cdot|^\beta|\phi_n|^p)|x|^\beta|\phi_n|^p dx.$$

Taking $\delta \in (1,p)$, it follows that (ϕ_n) is bounded in H^s. In the case $\frac{-2a}{N} \leq b \leq 0$, since $K(\phi_n) = 0$ then

$$\begin{aligned}\bar{\mu}S(\phi_n) &= \bar{\mu}S(\phi_n) - K(\phi_n)\\ &= (2a + (N-2s)b)S(\phi_n) - K(\phi_n)\\ &= -2bs\|\phi_n\|^2 + \frac{1}{p}(2a(p-1) + b(\alpha + 2\beta + 2s))\int (I_\alpha * |\cdot|^\beta |\phi_n|^p)|x|^\beta |\phi_n|^p dx\\ &\geq \frac{1}{p}(2a(p-1) + b(\alpha + 2\beta + 2s))\int (I_\alpha * |\cdot|^\beta |\phi_n|^p)|x|^\beta |\phi_n|^p dx.\end{aligned}$$

Implies

$$\bar{\mu}S(\phi_n) + (2a(p-1) + b(\alpha + 2\beta + 2s))S(\phi_n) \geq (2a(p-1) + b(\alpha + 2\beta + 2s))\|\phi_n\|^2_{H^s}.$$

Since

$$2a(p-1) + b(\alpha + 2\beta + 2s) \geq 2a(p-1) - \frac{2a}{N}(\alpha + 2\beta + 2s) = 2(p - p_*) > 0, \tag{4.5}$$

then ϕ_n is bounded in H^s.

Second step. In this step, we will prove that $m_{a,b} > 0$. Thanks to the compact injections, one obtains

$$\phi_n \rightharpoonup \phi \text{ in } H^s \text{ and } \phi_n \to \phi \text{ in } L^r, \forall r \in (2, \frac{2N}{N-2s}).$$

Assume $\phi = 0$, using the Hardy-Littlewood-Sobolev inequality, one gets

$$\int (I_\alpha * |\cdot|^\beta |\phi_n|^p)|x|^\beta |\phi_n|^p dx \lesssim \||x|^\beta |\phi_n|^p\|^2_{\frac{2N}{\alpha+N}}. \tag{4.6}$$

Let $\mu := (\frac{N}{-\beta})^- > 1$ and $r := \frac{2Np}{\alpha+N+2\beta-\varepsilon}$ for some $\varepsilon = 0^+$. We have $\frac{1}{\mu} + \frac{p}{r} = \frac{\alpha+N}{2N}$, then

$$\||x|^\beta |\phi_n|^p\|_{\frac{2N}{\alpha+N}(|x|<1)} \leq \||x|^\beta\|_{\mu(|x|<1)} \||\phi_n|^p\|_{\frac{r}{p}} \lesssim \|\phi_n\|^p_r.$$

Because $p_* < p < p^*$, one gets $2 < r < \frac{2N}{N-2s}$. Thus, when n goes to $+\infty$, we have

$$\||x|^\beta |\phi_n|^p\|_{\frac{2N}{\alpha+N}(|x|<1)} \lesssim \|\phi_n\|^p_r \to 0.$$

Similarly, by choosing $\mu := (\frac{N}{-\beta})^+ > 1$ and $r := \frac{2Np}{\alpha+N+2\beta+\varepsilon}$ for some $\varepsilon = 0^+$, we obtain

$$\||x|^\beta |\phi_n|^p\|_{\frac{2N}{\alpha+N}(|x|>1)} \to 0 \text{ when } n \to +\infty.$$

Therefore, by using (4.6), one gets

$$\int (I_\alpha * |\cdot|^\beta |\phi_n|^p)|x|^\beta |\phi_n|^p dx \to 0 \text{ when } n \to +\infty.$$

Since $K(\phi_n) = 0$, then $\lim_n K^Q(\phi_n) = 0$. So, by using Lemma 5 we obtain $K(\phi_n) > 0$ for large value of n. Then, $\phi \neq 0$ by contradiction. Next, we have to prove that $m_{a,b} > 0$.

With lower semi-continuity of the H^s norm, we have

$$\begin{aligned}0 &= \liminf_n K(\phi_n)\\ &\geq (2a+(N-2s)b)\liminf_n \|(-\triangle)^{\frac{s}{2}}\phi_n\|^2 + (2a+Nb)\liminf_n \|\phi_n\|^2\\ &- \frac{2ap+b(N+\alpha+2\beta)}{p}\int (I_\alpha * |\cdot|^\beta|\phi|^p)|x|^\beta|\phi|^p dx\\ &\geq K(\phi).\end{aligned}$$

Similarly, we obtain $H_{a,b}(\phi)\leq m_{a,b}$. Furthermore, if $K(\phi)<0$ then there exists $\lambda_0\in(0,1)$ such that $K(\phi^{\lambda_0})=0$. Therefore

$$m_{a,b}\leq S(\phi^{\lambda_0}) = H_{a,b}(\phi^{\lambda_0})\leq H_{a,b}(\phi)\leq m_{a,b}.$$

Then

$$m_{a,b} = S(\phi^{\lambda_0}) = H_{a,b}(\phi^{\lambda_0}) > 0.$$

Let $\phi := \phi^{\lambda_0}$, then ϕ is a minimizer for (2.2) which satisfies (4.4).

Third step. Finally, we are going to prove that ϕ satisfies (2.1). There exists a Lagrange multiplier $\mu\in\mathbb{R}$ such that $S'(\phi)=\mu K'(\phi)$. Hence

$$0 = K(\phi) = \mathcal{L}S(\phi) = \langle S'(\phi),\mathcal{L}(\phi)\rangle = \mu\langle K'(\phi),\mathcal{L}(\phi)\rangle = \mu\mathcal{L}K(\phi) = \mu\mathcal{L}^2 S(\phi).$$

We have

$$\begin{aligned}-\mathcal{L}^2S(\phi) - \bar{\mu}\underline{\mu}S(\phi) &= -(\mathcal{L}-\bar{\mu})(\mathcal{L}-\underline{\mu})S(\phi)\\ &\geq \frac{1}{p}(2ap+b(N+\alpha+2\beta)-\bar{\mu})^2\int (I_\alpha * |\cdot|^\beta|\phi|^p)|x|^\beta|\phi|^p dx.\end{aligned}$$

Taking account of (4.3), one gets $\mathcal{L}^2S(\phi)<0$. Consequently $\mu=0$ and then $S'(\phi)=0$. Therefore, ϕ is a ground state and $m := m_{a,b}$ is independent of (a,b).

4.3. **Proof of Theorem 2.**

4.3.1. *The interpolation inequality* (2.3). By using of the Hardy-Littlewood-Sobolev inequality from Corollary 1, we have

$$\Big(\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\Big)^{\frac{2N}{\alpha+N}} \lesssim \||x|^\beta|u|^p\|_{\frac{2N}{\alpha+N}}^{\frac{4N}{\alpha+N}}.$$

For any Hölder couples (r,r') such that $r:=(\frac{\alpha+N}{-2\beta})^-$ and $r':=(1+\varepsilon)\frac{\alpha+N}{\alpha+2\beta+N}$ where $\varepsilon=0^+$. We have

$$\||x|^\beta|u|^p\|_{\frac{2N}{\alpha+N}(|x|<1)}^{\frac{4N}{\alpha+N}} \lesssim \||x|^{\frac{2N\beta}{\alpha+N}}\|^2_{r(|x|<1)}\|u\|_{\frac{2Npr'}{\alpha+N}}^{\frac{4Np}{\alpha+N}} \lesssim \|u\|_{\frac{2Npr'}{\alpha+N}}^{\frac{4Np}{\alpha+N}}.$$

Since $1+\frac{\alpha+2\beta}{N}<p<p^*$, then $2<\frac{2Npr'}{\alpha+N}=(1+\varepsilon)\frac{2Np}{\alpha+2\beta+N}<\frac{2N}{N-2s}$. Thanks to the fractional Gagliardo-Nirenberg inequality from Lemma 1, one gets

$$\|u\|_{\frac{2Npr'}{\alpha+N}} \lesssim \|u\|^\theta_{\dot{H}^s}\|u\|^{1-\theta},\ \text{where}\ \theta := \frac{N}{s}\Big(\frac{1}{2}-\frac{\alpha+N}{2Npr'}\Big).$$

So

$$\||x|^\beta|u|^p\|_{\frac{2N}{\alpha+N}(|x|<1)}^{\frac{4N}{\alpha+N}} \lesssim \big(\|u\|^\theta_{\dot{H}^s}\|u\|^{1-\theta}\big)^{\frac{4Np}{\alpha+N}}.$$

Similarly, by taking $r := (\frac{\alpha+N}{-2\beta})^+$ and $r' := (1-\varepsilon)\frac{\alpha+N}{\alpha+2\beta+N}$, we estimate integrals on $|x|>1$. So, for some ε small enough, we have

$$\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx \lesssim \|u\|_{\dot H^s}^B \|u\|^A.$$

4.3.2. *The best constant of the Gagliardo-Nirenberg inequality.* First, let us prove the equation (2.4). For that, let

$$\xi := \frac{1}{C(N,p,s,\alpha,\beta)} = \inf_{v\in H^s-\{0\}} J(v).$$

Using Schwartz symmetrization argument, there exists a sequence (v_n) in $H^s_{rd}-\{0\}$ such that $\xi = \lim_n J(v_n)$. For $\lambda,\mu\in\mathbb{R}$, denoting $u^{\lambda,\mu} := \lambda u(\mu\cdot)$. we have

$$\|(-\triangle)^{\frac{s}{2}} u^{\lambda,\mu}\|^2 = \lambda^2\mu^{2s-N}\|(-\triangle)^{\frac{s}{2}}u\|^2;\ \|u^{\lambda,\mu}\|^2 = \lambda^2\mu^{-N}\|u\|^2;$$

$$\int (I_\alpha * |\cdot|^\beta |u^{\lambda,\mu}|^p)|x|^\beta |u^{\lambda,\mu}|^p dx = \lambda^{2p}\mu^{-N-\alpha-2\beta}\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx.$$

Hence, by some elementary computations we get

$$J(u^{\lambda,\mu}) = J(u).$$

Let us define

$$\lambda_n := \frac{\|v_n\|^{\frac{N}{2s}-1}}{\|(-\triangle)^{\frac{s}{2}}v_n\|^{\frac{N}{2s}}} \text{ and } \mu_n := \Big(\frac{\|v_n\|}{\|(-\triangle)^{\frac{s}{2}}v_n\|}\Big)^{\frac{1}{s}}.$$

So, $\psi_n := v_n^{\lambda_n,\mu_n}$ satisfies

$$\|\psi_n\| = \|(-\triangle)^{\frac{s}{2}}\psi_n\| = 1 \text{ and } \xi = \lim_n J(\psi_n).$$

Then, for a subsequence denoted also (ψ_n) there exist $\psi\in H^s_{rd}$ such that $\psi_n \rightharpoonup \psi$. Now, we are going to prove that

$$\int (I_\alpha * |\cdot|^\beta |\psi_n|^p)|x|^\beta |\psi_n|^p dx \to \int (I_\alpha * |\cdot|^\beta |\psi|^p)|x|^\beta |\psi|^p dx.$$

For that, let

$$I_n := \int (I_\alpha * |\cdot|^\beta |\psi_n|^p)|x|^\beta |\psi_n|^p dx - \int (I_\alpha * |\cdot|^\beta |\psi|^p)|x|^\beta |\psi|^p dx.$$

We have

$$\begin{aligned} I_n &= \int (I_\alpha * |\cdot|^\beta (|\psi_n|^p - |\psi|^p))|x|^\beta |\psi_n|^p dx + \int (I_\alpha * |\cdot|^\beta |\psi|^p)|x|^\beta (|\psi_n|^p - |\psi|^p) dx \\ &= \int (I_\alpha * |\cdot|^\beta (|\psi_n|^p - |\psi|^p))|x|^\beta |\psi_n|^p dx + \int (I_\alpha * |\cdot|^\beta (|\psi_n|^p - |\psi|^p))|x|^\beta |\psi|^p dx. \end{aligned}$$

By using the Hardy-Littlewood-Sobolev inequality, we get

$$I_n \lesssim \{\||x|^\beta |\psi_n|^p\|_{\frac{2N}{\alpha+N}} + \||x|^\beta |\psi|^p\|_{\frac{2N}{\alpha+N}}\}\||x|^\beta (|\psi_n|^p - |\psi|^p)\|_{\frac{2N}{\alpha+N}}.$$

Denote

$$I_n^1 := \||x|^\beta (|\psi_n|^p - |\psi|^p)\|_{\frac{2N}{\alpha+N}(|x|<1)} \text{ and } I_n^2 := \||x|^\beta (|\psi_n|^p - |\psi|^p)\|_{\frac{2N}{\alpha+N}(|x|>1)}$$

From (1.2), we have $N > -\beta$, $N+\alpha+2\beta > 0$ and $N > \alpha+2\beta$. So, let $\mu := (\frac{N}{-\beta})^{-}$ and $r := \frac{2N}{\alpha+N-\frac{2N}{\mu}} > 1$. By using Hölder inequality, the Mean Value Theorem and the function $x \to x^r$ is convex, we get

$$\begin{aligned} I_n^1 &\leq \||x|^\beta\|_{\mu(|x|<1)} \||\psi_n|^p - |\psi|^p\|_{r(|x|<1)} \\ &\lesssim \||\psi_n|^p - |\psi|^p\|_r \\ &\lesssim \|\{|\psi_n|^{p-1} + |\psi|^{p-1}\}|\psi_n - \psi|\|_r \\ &\lesssim \|\{|\psi_n|^{r(p-1)} + |\psi|^{r(p-1)}\}|\psi_n - \psi|^r\|_1^{\frac{1}{r}} \\ &\lesssim (\|\psi_n\|_{pr}^{r(p-1)} + \|\psi\|_{pr}^{r(p-1)})^{\frac{1}{r}} \|\psi_n - \psi\|_{pr}. \end{aligned}$$

By using the inequality $(a+b)^\rho \leq 2(a^\rho + b^\rho)$ for a, b nonnegative and $0 \leq \rho \leq 2$, we deduce

$$I_n^1 \lesssim (\|\psi_n\|_{pr}^{p-1} + \|\psi\|_{pr}^{p-1})\|\psi_n - \psi\|_{pr}.$$

Since $1 + \frac{\alpha+2\beta}{N} < p < p^*$, then $2 < rp < \frac{2N}{N-2s}$. So, by compact Sobolev injections one gets $\lim\limits_n I_n^1 = 0$. Similarly, one shows that $\||x|^\beta |\psi_n|^p\|_{\frac{2N}{\alpha+N}}$ defines a bounded sequence and that

$$I_n^2 \lesssim (\|\psi_n\|_{pr}^{p-1} + \|\psi\|_{pr}^{p-1})\|\psi_n - \psi\|_{pr},$$

Consequently, one obtains $\lim\limits_n I_n = 0$. Hence, when n goes to $+\infty$, we have

$$J(\psi_n) = \frac{1}{\int (I_\alpha * |\cdot|^\beta |\psi_n|^p)|x|^\beta |\psi_n|^p dx} \to \frac{1}{\int (I_\alpha * |\cdot|^\beta |\psi|^p)|x|^\beta |\psi|^p dx}.$$

Using lower semi-continuity of the H^s norm, we get $\|\psi\| \leq 1$ and $\|(-\triangle)^{\frac{s}{2}}\psi\| \leq 1$. If $\|\psi\| < 1$ or $\|(-\triangle)^{\frac{s}{2}}\psi\| < 1$, then $\|\psi\|^A \|(-\triangle)^{\frac{s}{2}}\psi\|^B < 1$, implies $J(\psi) < \xi$ which contradicts the definition of ξ. Thus, we have $\|\psi\| = \|(-\triangle)^{\frac{s}{2}}\psi\| = 1$. Therefore $\psi_n \to \psi$ in H_{rd}^s and

$$C(N,p,s,\alpha,\beta) = \frac{1}{\xi} = \frac{1}{J(\psi)} = \int (I_\alpha * |\cdot|^\beta |\psi|^p)|x|^\beta |\psi|^p dx.$$

The minimizer ψ satisfies the Euler equation

$$\partial_\varepsilon J(\psi + \varepsilon\mu)|_{\varepsilon=0} = 0, \forall \mu \in C_0^\infty \cap H^s.$$

Hence, ψ satisfies the equation (2.4). It remains now to prove (2.5). For $\lambda, \mu \in \mathbb{R}$, we introduce the scaling ϕ such that

$$\psi = \phi^{\lambda,\mu} := \lambda\phi(\mu\cdot).$$

In the equation (2.4), replacing ψ by $\phi^{\lambda,\mu}$, we obtain

$$-\frac{B}{A}\mu^{2s}(-\triangle)^s\phi + \phi - 2\frac{\xi}{A}p\lambda^{2p-2}\mu^{-\alpha-2\beta}(I_\alpha * |\cdot|^\beta|\phi|^p)|x|^\beta|\phi|^{p-2}\phi = 0.$$

Taking

$$\mu = (\frac{A}{B})^{\frac{1}{2s}} \text{ and } \lambda = ((\frac{A}{B})^{\frac{\alpha+2\beta}{2s}}\frac{A}{2p\xi})^{\frac{1}{2(p-1)}}.$$

We get

$$(-\triangle)^s\phi - \phi + (I_\alpha * |\cdot|^\beta|\phi|^p)|x|^\beta|\phi|^{p-2}\phi = 0.$$

Since that $1 = \|\psi\| = \lambda\mu^{-\frac{N}{2}}\|\phi\|$, then

$$\lambda\|\phi\| = (\frac{A}{B})^{\frac{N}{4s}}; \lambda^{2(p-1)}\|\phi\|^{2(p-1)} = (\frac{A}{B})^{\frac{N(p-1)}{2s}}; (\frac{A}{B})^{\frac{\alpha+2\beta}{2s}}\frac{A}{2p\xi}\|\phi\|^{2(p-1)} = (\frac{A}{B})^{\frac{N(p-1)}{2s}}.$$

We deduce the following

$$\begin{aligned}
\xi &= \frac{A}{2p}(\frac{A}{B})^{\frac{\alpha+2\beta}{2s}}(\frac{A}{B})^{-\frac{N(p-1)}{2s}}\|\phi\|^{2(p-1)} \\
&= \frac{A}{2p}(\frac{A}{B})^{-\frac{N(p-1)-\alpha-2\beta}{2s}}\|\phi\|^{2(p-1)} \\
&= \frac{A}{2p}(\frac{A}{B})^{-\frac{B}{2}}\|\phi\|^{2(p-1)}.
\end{aligned}$$

Therefore

$$C(N,p,s,\alpha,\beta) = \frac{1}{\xi} = \frac{2p}{A}(\frac{A}{B})^{\frac{B}{2}}\|\phi\|^{-2(p-1)}.$$

5. Well-posedness

In this section, we are going to prove Theorem 3.

5.1. **Local well-posedness.**

We are concerned with the existence and uniqueness of solution to the Cauchy problem (1.1). The sign of ϵ has no local effect, for that we assume here $\epsilon = 1$. For any $T > 0$ and $R > 0$ let us define

$$X_T := \bigcap_{(q,r)\in\Gamma_s} L^q_T(W^{s,r}),$$

and

$$B_T(R) := \{v \in X_T, \text{ s.t} \sup_{(q,r)\in\Gamma_s}\|v\|_{L^q_T(W^{s,r})} \leq R\}.$$

The closed ball $B_T(R)$ is equipped with the complete distance

$$d(u,v) := \|u-v\|_{S_T} := \sup_{(q,r)\in\Gamma_s}\|u-v\|_{L^q_T(L^r)}.$$

Next, we introduce the so called Schrödinger mapping

$$\Phi(u) := \exp(-i\cdot(-\triangle)^s)u_0 - \int_0^{\cdot}\exp(-i(\cdot-\tau)(-\triangle)^s)((I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}u)d\tau.$$

Using the Strichartz estimate from Proposition 2 and Remark 6, one has for all $u, v \in B_T(R)$ and for any $(q, r), (q_1, r_1) \in \Gamma_s$:

$$\begin{aligned}
d(\Phi(u), \Phi(v)) &\lesssim \|(I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2}u - (I_\alpha * |\cdot|^\beta |v|^p)|x|^\beta |v|^{p-2}v\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&+ \|(I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2}u - (I_\alpha * |\cdot|^\beta |v|^p)|x|^\beta |v|^{p-2}v\|_{L_T^{q_1'}(L^{r_1'}(|x|>1))} \\
&\lesssim \|(I_\alpha * [|\cdot|^\beta |u|^p - |\cdot|^\beta |v|^p])|x|^\beta |u|^{p-2}u\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&+ \|(I_\alpha * |\cdot|^\beta |v|^p)[|x|^\beta |u|^{p-2}u - |x|^\beta |v|^{p-2}v]\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&+ \|(I_\alpha * [|\cdot|^\beta |u|^p - |\cdot|^\beta |v|^p])|x|^\beta |u|^{p-2}u\|_{L_T^{q_1'}(L^{r_1'}(|x|>1))} \\
&+ \|(I_\alpha * |\cdot|^\beta |v|^p)[|x|^\beta |u|^{p-2}u - |x|^\beta |v|^{p-2}v]\|_{L_T^{q_1'}(L^{r_1'}(|x|>1))}.
\end{aligned}$$

Denote

$$(I) := \|(I_\alpha * |\cdot|^\beta |v|^p)[|x|^\beta |u|^{p-2}u - |x|^\beta |v|^{p-2}v]\|_{L_T^{q'}(L^{r'}(|x|<1))}.$$

By the Mean Value Theorem, we have

$$(I) \lesssim \|(I_\alpha * |\cdot|^\beta |v|^p)|x|^\beta (|u|^{p-2} + |v|^{p-2})|u - v|\|_{L_T^{q'}(L^{r'}(|x|<1))}.$$

Let $\mu := (\frac{N}{-\beta})^-$ and $r = \frac{2Np}{N+\alpha+2\beta+2(p-1)s-\varepsilon}$ where $\varepsilon = 0^+$. If $k := \frac{Nr}{N-sr}$ then

$$\begin{aligned}
1 + \frac{\alpha}{N} &= \frac{2}{r} + \frac{2}{\mu} + \frac{2p-2}{k} \\
&= \frac{1}{r} + (\frac{1}{\mu} + \frac{p}{k}) + (\frac{1}{\mu} + \frac{p-2}{k} + \frac{1}{r}).
\end{aligned}$$

So, by using the Hardy-Littlwood-Paley inequality, one gets

$$\begin{aligned}
(I) &\lesssim \|\||x|^\beta\|_{L^\mu(|x|<1)} \|v\|_k^p \||x|^\beta\|_{L^\mu(|x|<1)} (\|u\|_k^{p-2} + \|v\|_k^{p-2}) \|u - v\|_r\|_{L_T^{q'}} \\
&\lesssim \|(\|u\|_k^{2(p-1)} + \|v\|_k^{2(p-1)}) \|u - v\|_r\|_{L_T^{q'}}.
\end{aligned}$$

Since $N+\alpha+2\beta-2s > 0$, then $1 < r < \frac{N}{s}$. So, by Sobolev injections, we have $W^{s,r} \hookrightarrow L^k$, then

$$(I) \lesssim \|(\|u\|_{W^{s,r}}^{2(p-1)} + \|v\|_{W^{s,r}}^{2(p-1)}) \|u - v\|_r\|_{L_T^{q'}}.$$

Since $p < p^*$, then we can suppose $\varepsilon = 0^+$ small enough such that $p + \frac{\varepsilon}{N-2s} < p^*$. So

$$p(N - 2s) + \varepsilon < (N - 2s) + \alpha + 2\beta + 2s.$$

Then

$$\frac{p(N - 2s) - ((N - 2s) + \alpha + 2\beta) + \varepsilon}{2s} < 1.$$

Thus

$$\frac{Np}{s}(\frac{1}{2} - \frac{1}{r}) < 1,$$

and then $\frac{2p}{q} = \frac{Np}{s}(\frac{1}{2} - \frac{1}{r}) < 1$ which implies $(2p-1)q' < q$. So, there exists $\theta > 0$ such that $\frac{1}{q'} = \frac{2p-1}{q} + \frac{1}{\theta}$. Using the Hölder inequality, one gets

$$\begin{aligned}(I) &\lesssim T^{\frac{1}{\theta}}(\|u\|^{2(p-1)}_{L^q(W^{s,r})} + \|v\|^{2(p-1)}_{L^q(W^{s,r})})\|u-v\|_{L^q(L^r)} \\ &\lesssim T^{\frac{1}{\theta}}R^{2(p-1)}d(u,v).\end{aligned}$$

In a similar way as previous, by using the Hardy-Littlwood-Paley inequality with respect to the following decomposition

$$1 + \frac{\alpha}{N} = \frac{1}{r} + (\frac{1}{\mu} + \frac{p-1}{k} + \frac{1}{r}) + (\frac{1}{\mu} + \frac{p-1}{k}),$$

we obtain

$$\begin{aligned}(II) &:= \|(I_\alpha * [|\cdot|^\beta|u|^p - |\cdot|^\beta|v|^p])|x|^\beta|u|^{p-2}u\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &\lesssim \|(I_\alpha * |\cdot|^\beta[|u|^{p-1} + |v|^{p-1}]|u-v|)|x|^\beta|u|^{p-2}u\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &\lesssim \|(\|u\|_k^{2(p-1)} + \|v\|_k^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}} \\ &\lesssim \|(\|u\|_{W^{s,r}}^{2(p-1)} + \|v\|_{W^{s,r}}^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}} \\ &\lesssim T^{\frac{1}{\theta}}R^{2(p-1)}d(u,v).\end{aligned}$$

Now, in order to estimate integrals on $|x| > 1$, we make use of $\mu := (\frac{N}{-\beta})^+$ and $r_1 := \frac{2Np}{N+\alpha+2\beta+2(p-1)s+\varepsilon}$ with some $\varepsilon = 0^+$. We obtain

$$\begin{aligned}\|(I_\alpha * [|\cdot|^\beta|u|^p - |\cdot|^\beta|v|^p])|x|^\beta|u|^{p-2}u\|_{L_T^{q_1'}(L^{r_1'}(|x|>1))} &\lesssim T^{\frac{1}{\theta}}R^{2(p-1)}d(u,v) \\ \|(I_\alpha * |\cdot|^\beta|v|^p)[|x|^\beta|u|^{p-2}u - |x|^\beta|v|^{p-2}v]\|_{L_T^{q_1'}(L^{r_1'}(|x|>1))} &\lesssim T^{\frac{1}{\theta}}R^{2(p-1)}d(u,v).\end{aligned}$$

In summary, we obtain

$$d(\Phi(u),\Phi(v)) \lesssim T^{\frac{1}{\theta}}R^{2(p-1)}d(u,v).$$

When $v = 0$, it becomes

$$\|\Phi(u) - \exp(-i\cdot(-\triangle)^s)u_0\|_{S_T} \le CT^{\frac{1}{\theta}}R^{2p-1}.$$

Then

$$\begin{aligned}\|\Phi(u)\|_{S_T} &\le \|\exp(-i\cdot(-\triangle)^s)u_0\|_{S_T} + CT^{\frac{1}{\theta}}R^{2p-1} \\ &\le \|u_0\| + CT^{\frac{1}{\theta}}R^{2p-1}.\end{aligned}$$

Now, it remains to estimate

$$\sup_{(q,r)\in\Gamma_s}\|\Phi(u)\|_{L_T^q(\dot W^{s,r})} = \|(-\triangle)^{\frac{s}{2}}\Phi(u)\|_{S_T}.$$

Let $(III) := \|(-\triangle)^{\frac{s}{2}}\Phi(u)\|_{S_T} - \|(-\triangle)^{\frac{s}{2}}u_0\|$. Thanks to the Strichartz estimate, one has

$$\begin{aligned}(III) &\lesssim \|(-\triangle)^{\frac{s}{2}}(I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2}u\|_{L_T^{q'}(L^{r'})}\\ &\lesssim \|(I_\alpha * |\cdot|^{\beta-s}|u|^p)|x|^\beta |u|^{p-1} + (I_\alpha * |\cdot|^\beta |u|^{p-1}|(-\triangle)^{\frac{s}{2}}u|)|x|^\beta|u|^{p-1}\\ &+ (I_\alpha * |\cdot|^\beta |u|^p)|x|^{\beta-s}|u|^{p-1} + (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2}|(-\triangle)^{\frac{s}{2}}u|\|_{L_T^{q'}(L^{r'})}.\end{aligned}$$

Taking in account the chain rules from Lemma 2 and the Hardy-Littlwood-Paley inequality, one controls the second and the fourth terms in the last inequality as previous. Therefore, we obtain

$$\begin{aligned}(III) &\lesssim T^{\frac{1}{\theta}}R^{2p-1} + \|(I_\alpha * |\cdot|^{\beta-s}|u|^p)|x|^\beta |u|^{p-1}\|_{L_T^{q'}(L^{r'})}\\ &+ \|(I_\alpha * |\cdot|^\beta |u|^p)|x|^{\beta-s}|u|^{p-1}\|_{L_T^{q'}(L^{r'})}.\end{aligned}$$

Now, we introduce $\rho := (\frac{N}{s-\beta})^-$. By the assumption $N+\beta-s>0$, we have $\rho>1$. Once again, we make use of $\mu := (\frac{N}{-\beta})^-$, $r = \frac{2Np}{N+\alpha+2\beta+2(p-1)s-\varepsilon}$ and $k = \frac{Nr}{N-sr}$. We have

$$1+\frac{\alpha}{N} = \frac{1}{r} + (\frac{1}{\rho}+\frac{p}{k}) + (\frac{1}{\mu}+\frac{p-1}{k}).$$

So, by using the Hardy-Littlwood-paley and Hölder inequalities, taking account of Sobolev injections, we obtain

$$\begin{aligned}\|(I_\alpha * |\cdot|^{\beta-s}|u|^p)|x|^\beta |u|^{p-1}\|_{L_T^{q'}(L^{r'}(|x|<1))} &\lesssim \|\||x|^{\beta-s}\|_{\rho(|x|<1)}\|u\|_k^p \||x|^\beta\|_{\mu(|x|<1)}\|u\|_k^{p-1}\|_{L_T^{q'}}\\ &\lesssim \|\|u\|_{W^{s,r}}^{2p-1}\|_{L_T^{q'}}\\ &\lesssim T^{\frac{1}{\theta}}R^{2p-1}.\end{aligned}$$

Similarly

$$\|(I_\alpha * |\cdot|^\beta |u|^p)|x|^{\beta-s}|u|^{p-1}\|_{L_T^{q'}(L^{r'}(|x|<1))} \lesssim T^{\frac{1}{\theta}}R^{2p-1}.$$

Also, to control integrals on $|x|>1$, we use $\rho := (\frac{N}{s-\beta})^+$, $\mu := (\frac{N}{-\beta})^+$ and $r := \frac{2Np}{N+\alpha+2\beta+2(p-1)s+\varepsilon}$. Thus, one gets

$$\|(I_\alpha * |\cdot|^{\beta-s}|u|^p)|x|^\beta |u|^{p-1}\|_{L_T^{q'}(L^{r'}(|x|>1))} \lesssim T^{\frac{1}{\theta}}R^{2p-1}.$$

and

$$\|(I_\alpha * |\cdot|^\beta |u|^p)|x|^{\beta-s}|u|^{p-1}\|_{L_T^{q'}(L^{r'}(|x|>1))} \lesssim T^{\frac{1}{\theta}}R^{2p-1}.$$

In summary, we obtain

$$\|(-\triangle)^{\frac{s}{2}}\Phi(u)\|_{S_T} \le \|u_0\|_{\dot H^s} + CT^{\frac{1}{\theta}}R^{2p-1}.$$

Then, by taking $R > \|u_0\|_{H^s}$ and T small enough, it follows that Φ is a contraction of $B_T(R)$. So, it's fix point is the unique solution to (1.1) in $B_T(R)$. Uniqueness of the maximal solution follows from previous computations and standard translation argument.

5.2. **Global well-posedness.**

5.2.1. *The defocusing* ($\epsilon = 1$) *energy subcritical case.* Let $u \in C([0, T^*[, H^s)$ be the unique maximal solution of (1.1). Suppose that $T^* < \infty$ and take $0 < \tau < T^*$, we consider the following Cauchy problem

$$\begin{cases} i\dot{v} - (-\triangle)^s v = (I_\alpha * |\cdot|^\beta |v|^p)|x|^\beta |v|^{p-2} v, t \geq \tau, \\ v(\tau, \cdot) = u(\tau, \cdot). \end{cases} \tag{5.1}$$

Using contraction arguments as for the main problem (1.1), we prove the existence of $T > 0$ and $v \in C([\tau, \tau + T], H^s)$ solution to (5.1). Thanks to the conservation laws, the instant of time T does not depend on τ. Hence, taking in account that $\|(-\triangle)^{\frac{s}{2}} u\|$ remains bounded, let τ be close to T^* such that $T^* < \tau + T$, this contradicts the maximality of T^*. Then $T^* = \infty$ and $u \in C(\mathbb{R}_+, H^s)$.

5.2.2. *The focusing* ($\epsilon = -1$) *mass subcritical case.* By using of the Gagliardo-Nirenberg inequality (2.3), we have

$$\begin{aligned} E(u_0) &:= \|u(t)\|^2_{\dot{H}^s} - \frac{1}{p} \int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^{p-2} u dx \\ &\geq \|u(t)\|^2_{\dot{H}^s} - \frac{C(N,p,s,\alpha,\beta)}{p} \|u(t)\|^A \|u(t)\|^B_{\dot{H}^s} \\ &\geq \|u(t)\|^2_{\dot{H}^s} (1 - \frac{C(N,p,s,\alpha,\beta)}{p} M(u_0)^{\frac{A}{2}} \|u(t)\|^{B-2}_{\dot{H}^s}) \\ &\geq \|u(t)\|^2_{\dot{H}^s} (1 - \frac{C(N,p,s,\alpha,\beta)}{p} M(u_0)^{\frac{A}{2}} \|u(t)\|^{\frac{N}{s}(p-p_*)}_{\dot{H}^s}). \end{aligned}$$

Consequently, if $p < p_*$ or $p = p_*$ and $M(u_0) < (\frac{p}{C(N,p,s,\alpha,\beta)})^{\frac{2}{A}}$, then $\sup_{[0,T^*[} \|u(t)\|_{\dot{H}^s} < \infty$ which implies that $T^* = \infty$.

6. Virial type identity

In this section we have to prove Theorem 4.

6.1. **Preliminary.** Let us define the self-adjoint differential operator

$$\Gamma_\psi := -i(\nabla.\nabla\psi + \nabla\psi.\nabla);$$

which acts on functions as follows

$$\Gamma_\psi f = -i(\nabla.((\nabla\psi)f) + (\nabla\psi).(\nabla f)).$$

Then

$$M_{\psi_R}[u(t)] = \langle u(t), \Gamma_\psi u(t)\rangle.$$

For $m > 0$, it is useful to define the function

$$u_m := \sqrt{\frac{\sin(\pi s)}{\pi}} \frac{1}{m - \triangle} u = \sqrt{\frac{\sin(\pi s)}{\pi}} \mathcal{F}^{-1}(\frac{\mathcal{F}u}{|\cdot|^2 + m}).$$

Let $[X, Y] := XY - YX$ denotes the commutator of X and Y. We have the following general result.

Lemma 7. *For sufficiently smooth function g, we have*

$$\langle u, [-g, i\Gamma_\psi]u\rangle = 2\int |u|^2\nabla\psi.\nabla g dx.$$

Proof.

$$\begin{aligned}\langle u, [-g, i\Gamma_\psi]u\rangle &= -\langle u, [g, \nabla\psi.\nabla + \nabla.\nabla\psi]u\rangle \\ &= -\langle u, g\nabla\psi.\nabla u + g\nabla((\nabla\psi)u)\rangle \\ &+ \langle u, \nabla\psi.\nabla(gu) + \nabla((\nabla\psi)gu)\rangle \\ &= -2\langle u, g\nabla\psi.\nabla u\rangle + 2\langle u, \nabla\psi.\nabla(gu)\rangle \\ &= 2\langle u, u\nabla\psi.\nabla g\rangle = 2\int \bar{u}u\nabla\psi.\nabla g dx = 2\int |u|^2\nabla\psi.\nabla g dx.\end{aligned}$$

□

6.2. **The energy subcritical case.** Using the equation (1.1), it follows that

$$\frac{d}{dt}M_{\psi_R}[u(t)] = \langle u(t), [(-\triangle)^s, i\Gamma_{\psi_R}]u(t)\rangle + \langle u(t), [-(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}, i\Gamma_{\psi_R}]u(t)\rangle.$$

According to computations in [3], one controls the dispersive term as follows

$$\langle u(t), [(-\triangle)^s, i\Gamma_{\psi_R}]u(t)\rangle \leq 4s\|u(t)\|^2_{\dot{H}^s} + CR^{-2s}.$$

Our aim now is to estimate the nonlinear term

$$\mathcal{N} := \langle u(t), [-(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}, i\Gamma_{\psi_R}]u(t)\rangle.$$

Let $g(x) := (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}$, we obtain

$$\begin{aligned}\mathcal{N} &= 2\int |u|^2\nabla\psi_R.\nabla((I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2})dx. \\ &= -2\int \{\nabla(|u|^2).\nabla\psi_R + |u|^2\triangle\psi_R\}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}dx. \\ &= -\frac{4}{p}\int \nabla\psi_R.\nabla(|u|^p)(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta dx - 2\int \triangle\psi_R(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\ &= (\frac{4}{p} - 2)\int \triangle\psi_R(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \frac{4}{p}\int \nabla\psi_R.(\nabla I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\ &+ \frac{4}{p}\int \nabla\psi_R.\nabla(|x|^\beta)(I_\alpha * |\cdot|^\beta|u|^p)|u|^p dx.\end{aligned}$$

Denote

$$M_1 := \frac{4}{p}\int \nabla\psi_R.(\nabla I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx;$$

and

$$M_2 := \frac{4}{p}\int \nabla\psi_R.\nabla(|x|^\beta)(I_\alpha * |\cdot|^\beta|u|^p)|u|^p dx.$$

Let $D := \{x \in \mathbb{R}^N \text{ s.t } |x| \leq R\} \times \{y \in \mathbb{R}^N \text{ s.t } |y| \leq R\} \subset \mathbb{R}^{2N}$ and $D^c = \mathbb{R}^{2N} - D$. We have $\nabla I_\alpha(x) = -\mathcal{K}(N-\alpha)\frac{x}{|x|^{N-\alpha+2}}$, then

$$\begin{aligned}
M_1 &= -\frac{4\mathcal{K}}{p}(N-\alpha)\int \nabla\psi_R(x).\frac{x-y}{|x-y|^{N-\alpha+2}}|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p dydx \\
&= -\frac{4\mathcal{K}}{p}(N-\alpha)\int (\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^{N-\alpha+2}}|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p dydx \\
&+ \frac{4\mathcal{K}}{p}(N-\alpha)\int \nabla\psi_R(y).\frac{y-x}{|y-x|^{N-\alpha+2}}|x|^\beta|u(x)|^p|y|^\beta|u(y)|^p dxdy.
\end{aligned}$$

The last term is $-M_1$, so

$$M_1 = -\frac{2\mathcal{K}}{p}(N-\alpha)\int (\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^{N-\alpha+2}}|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p dydx.$$

Since $\nabla\psi_R(x)-\nabla\psi_R(y) = x-y$ on D, then

$$\begin{aligned}
-\frac{p}{2\mathcal{K}(N-\alpha)}M_1 &= \int_D \frac{|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p}{|x-y|^{N-\alpha}}dydx \\
&+ \int_{D^c}(\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^{N-\alpha+2}}|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p dydx \\
&= \int \frac{|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p}{|x-y|^{N-\alpha}}dydx \\
&+ \int_{D^c}\{(\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^2}-1\}\frac{|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p}{|x-y|^{N-\alpha}}dydx.
\end{aligned}$$

For the last integral on D^c, we have the following two cases: if $|x-y| < R$, then using the Mean Value Theorem and the property $\|\nabla^j\psi_R\|_\infty \lesssim R^{2-j}(0 \leq j \leq 4)$, one has

$$|(\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^2}-1| \lesssim \|\nabla^2\psi_R\|_\infty + 1 \lesssim 1.$$

Else, if $|x-y| \geq R$, then

$$|(\nabla\psi_R(x)-\nabla\psi_R(y)).\frac{x-y}{|x-y|^2}-1| \lesssim \frac{2\|\nabla\psi_R\|_\infty}{R} + 1 \lesssim 1.$$

Moreover, by symmetry

$$\begin{aligned}
\int_{D^c}\frac{|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p}{|x-y|^{N-\alpha}}dydx &\leq 2\int_{(|x|>R)\times\mathbb{R}^N}\frac{|y|^\beta|u(y)|^p|x|^\beta|u(x)|^p}{|x-y|^{N-\alpha}}dydx \\
&= \frac{2}{\mathcal{K}}\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx.
\end{aligned}$$

Therefore

$$M_1 = -\frac{2(N-\alpha)}{p}\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).$$

Next, we have to control M_2. We have

$$\begin{aligned}\frac{p}{4\beta}M_2 &= \frac{1}{\beta}\int \nabla_{\psi_R}(x).\nabla(|x|^\beta)(I_\alpha*|\cdot|^\beta|u|^p)|u|^p dx\\ &= \int \nabla_{\psi_R}(x).\frac{x}{|x|^2}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\\ &= \int_{|x|<R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \int_{|x|>R}\nabla_{\psi_R}(x).\frac{x}{|x|^2}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\\ &= \int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \int_{|x|>R}\{\nabla_{\psi_R}(x).\frac{x}{|x|^2}-1\}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx.\end{aligned}$$

For all $|x|>R$, we have

$$|\nabla_{\psi_R}(x).\frac{x}{|x|^2}-1| \lesssim \frac{\|\nabla\psi_R\|_\infty}{R}+1 \lesssim 1.$$

Then

$$M_2 = \frac{4\beta}{p}\int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).$$

Replacing in $\mathcal{N}$ and taking in account that $\triangle\psi_R(x)=N$ when $|x|<R$, we find

$$\begin{aligned}\mathcal{N} &= 2(\frac{2}{p}-1)\int \triangle\psi_R(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx - \frac{2(N-\alpha)}{p}\int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\\ &+ \frac{4\beta}{p}\int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).\\ &= \{2N(\frac{2}{p}-1)-\frac{2(N-\alpha)}{p}+\frac{4\beta}{p}\}\int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\\ &+ \mathcal{O}(\int_{|x|>R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).\\ &= -2\frac{Np-N-\alpha-2\beta}{p}\int (I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx\\ &+ \mathcal{O}(\int_{|x|>R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).\end{aligned}$$

Let

$$M_3 := \int_{|x|>R}(I_\alpha*|\cdot|^\beta|u|^p)|x|^\beta|u|^p dx.$$

Taking $\mu := (\frac{N}{-\beta})^{+}$ and $r := \frac{2N}{N+\alpha+2\beta+\varepsilon}$ where $\varepsilon = 0^{+}$. Using Hardy-Littlwood-Sobolev inequality, we obtain

$$\begin{aligned}
M_3 &\lesssim \||x|^{\beta}\|^2_{\mu(|x|>R)}\||u|^p\|^2_{r(|x|>R)} \\
&\lesssim (\int_{|x|>R} |u|^{\frac{2Np}{N+\alpha+2\beta}}dx)^{\frac{N+\alpha+2\beta}{N}} \\
&\lesssim (\int_{|x|>R} |u|^{\frac{2Np}{N+\alpha+2\beta}-2}|u|^2dx)^{\frac{N+\alpha+2\beta}{N}} \\
&\lesssim \|u\|^{2(p-1-\frac{\alpha+2\beta}{N})}_{\infty(|x|\geq R)}\|u\|^{2(\frac{N+\alpha+2\beta}{N})} \\
&\lesssim \|u\|^{2(p-1-\frac{\alpha+2\beta}{N})}_{\infty(|x|\geq R)}.
\end{aligned}$$

Let $\delta := \frac{1+\varepsilon}{2}$ such that $\frac{1}{2} < \delta < s < \frac{N}{2}$. By using (3.2) and (3.3), yields

$$\begin{aligned}
M_3 &\lesssim \|u\|^{2(p-1-\frac{\alpha+2\beta}{N})}_{\infty(|x|\geq R)} \\
&\lesssim (R^{-(\frac{N}{2}-\delta)}\|(-\triangle)^{\frac{\delta}{2}}u\|)^{2(p-1-\frac{\alpha+2\beta}{N})} \\
&\lesssim R^{-(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}(\|u\|^{1-\frac{\delta}{s}}\|(-\triangle)^{\frac{s}{2}}u\|^{\frac{\delta}{s}})^{2(p-1-\frac{\alpha+2\beta}{N})} \\
&\lesssim \frac{1}{R^{(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}}\|(-\triangle)^{\frac{s}{2}}u\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}.
\end{aligned}$$

From previous computations, we obtain

$$\begin{aligned}
\frac{d}{dt}M_{\psi_R}[u] &= \langle u(t), [(-\triangle)^s, i\Gamma_{\psi_R}]u(t)\rangle + \langle u(t), [-(I_\alpha * |\cdot|^{\beta}|u|^p)|x|^{\beta}|u|^{p-2}, i\Gamma_{\psi_R}]u(t)\rangle \\
&\leq 4s\|u\|^2_{\dot{H}^s} + CR^{-2s} - 2\frac{Np-N-\alpha-2\beta}{p}\int(I_\alpha * |\cdot|^{\beta}|u|^p)|x|^{\beta}|u|^p dx \\
&+ C\frac{1}{R^{(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}}\|u\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}_{\dot{H}^s}.
\end{aligned}$$

Thus, we deduce the following desired estimate

$$\begin{aligned}
\frac{d}{dt}M_{\psi_R}[u] &\lesssim 2(N(p-1)-\alpha-2\beta)E(u_0) - 2N(p-p_*)\|u\|^2_{\dot{H}^s} \\
&\quad + \mathcal{O}(R^{-2s} + \frac{1}{R^{(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}}\|u\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}_{\dot{H}^s}).
\end{aligned}$$

6.3. **The mass critical case.** Let us define $\psi_1 := 1 - \psi_R''$ and $\psi_2 := N - \triangle\psi_R$. From previous, we have

$$\begin{aligned}
\mathcal{N} &= -2(1-\frac{2}{p_*})\int \triangle\psi_R(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + M_1 + M_2 \\
&= -2N(1-\frac{2}{p_*})\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + 2(1-\frac{2}{p_*})\int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\
&- 2\frac{N-\alpha-2\beta}{p_*}\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx) \\
&= -2\{N(1-\frac{2}{p_*}) + \frac{N-\alpha-2\beta}{p_*}\}\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\
&+ 2(1-\frac{2}{p_*})\int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx) \\
&= -\frac{4s}{p_*}\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + 2(1-\frac{2}{p_*})\int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\
&+ \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).
\end{aligned}$$

According to computations in [3], we get

$$\begin{aligned}
\frac{d}{dt}M_{\psi_R}[u] &= 4s\|u\|_{\dot H^s}^2 - 4\int_0^{+\infty} m^s \int \psi_1|\nabla u_m|^2 dxdm + \mathcal{O}(R^{-2s}) \\
&- \frac{4s}{p_*}\int (I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + 2(1-\frac{2}{p_*})\int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx \\
&+ \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx) \\
&= 4sE(u_0) - 4\int_0^{+\infty} m^s \int \psi_1|\nabla u_m|^2 dxdm + \mathcal{O}(R^{-2s}) \\
&+ 2(1-\frac{2}{p_*})\int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx + \mathcal{O}(\int_{|x|>R}(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx).
\end{aligned}$$

Let $\rho := \frac{N+\alpha+2\beta}{\alpha+2\beta+2s} > 1$ and

$$M_4 := \int \psi_2(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^p dx.$$

Taking $\mu := (\frac{N}{-\beta})^+, r = \frac{2N}{N+\alpha+2\beta+\varepsilon}$ where $\varepsilon = 0^+$. By Hardy-Littlwood-Sobolev and Hölder inequalities, we have

$$\begin{aligned}
M_4 &\lesssim \||x|^\beta\|_\mu^2 \||u|^{p_*}\|_r \|\psi_2 |u|^{p_*}\|_r \\
&\lesssim \|u\|^{p_*}_{\frac{2Np_*}{N+\alpha+2\beta}} (\int_{|x|>R} (|\psi_2||u|^{p_*})^{\frac{2N}{N+\alpha+2\beta}} dx)^{\frac{N+\alpha+2\beta}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} (\int_{|x|>R} (|\psi_2 u|)^{\frac{2N}{N+\alpha+2\beta}} |u|^{(p_*-1)\frac{2N}{N+\alpha+2\beta}} dx)^{\frac{N+\alpha+2\beta}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} \{(\int_{|x|>R} |\psi_2 u|^{\frac{2N}{N+\alpha+2\beta}\rho'} dx)^{\frac{1}{\rho'}} (\int |u|^2 dx)^{\frac{1}{\rho}}\}^{\frac{N+\alpha+2\beta}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} (\int_{|x|>R} |\psi_2 u|^{\frac{2N}{N+\alpha+2\beta}\rho'} dx)^{\frac{1}{\rho'}\frac{N+\alpha+2\beta}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} (\int_{|x|>R} |\psi_2 u|^{\frac{2N}{N-2s}} dx)^{\frac{N-2s}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} (\int_{|x|>R} |\psi_2 u|^{\frac{4s}{N-2s}} |\psi_2 u|^2 dx)^{\frac{N-2s}{2N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} \|\psi_2 u\|^{\frac{2s}{N}}_{\infty(|x|\geq R)}.
\end{aligned}$$

Using (3.2) and (3.3), for any δ such that $0 < \delta := \frac{1+\varepsilon}{2} < s < \frac{N}{2}$, we get the following

$$\begin{aligned}
M_4 &\lesssim \|u\|^{p_*}_{\dot H^s} \|\psi_2 u\|^{\frac{2s}{N}}_{\infty(|x|\geq R)} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} (R^{-\frac{N}{2}+\delta} \|(-\triangle)^{\frac{\delta}{2}} \psi_2 u\|)^{\frac{2s}{N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}} (\|\psi_2 u\|^{1-\frac{\delta}{s}} \|(-\triangle)^{\frac{s}{2}} \psi_2 u\|^{\frac{\delta}{s}})^{\frac{2s}{N}} \\
&\lesssim \|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}} \|\psi_2 u\|^{\frac{1+\varepsilon}{N}}_{\dot H^s}.
\end{aligned}$$

Hence, for any $\eta > 0$ and $q := \frac{2N}{1+\varepsilon}$, by using the Young inequality, we have

$$M_4 \lesssim \eta \|\psi_2 u\|^2_{\dot H^s} + \eta^{1-q'} (\|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'}.$$

Estimating similarly as for M_4, we get

$$\begin{aligned}
M_3 &\lesssim \|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}} \|u\|^{\frac{1+\varepsilon}{N}}_{\dot H^s} \\
&\lesssim \eta \|u\|^2_{\dot H^s} + \eta^{1-q'} (\|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'}.
\end{aligned}$$

In summary, from previous computations there exists C a positive constant such that

$$\begin{aligned}
\frac{d}{dt} M_{\psi_R}[u] &\leq 4sE(u_0) - 4\int_0^{+\infty} m^s \int \psi_1 |\nabla u_m|^2 dx dm + CR^{-2s} \\
&+ \eta C \|\psi_2 u\|^2_{\dot H^s} + \eta C \|u\|^2_{\dot H^s} + 2C\eta^{1-q'} (\|u\|^{p_*}_{\dot H^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'}.
\end{aligned}$$

According to computations in [3], we have

$$\begin{aligned} s\|\psi_2 u\|^2_{\dot{H}^s} &= \int_0^{+\infty} m^s \int |\nabla(\psi_2 u)_m|^2 dxdm \\ &= \int_0^{+\infty} m^s \int |\psi_2|^2 |\nabla u_m|^2 dxdm + \mathcal{O}(1 + \|\nabla\psi_2\|^2_\infty + \|\triangle\psi_2\|^2_\infty) \\ &= \int_0^{+\infty} m^s \int |\psi_2|^2 |\nabla u_m|^2 dxdm + \mathcal{O}(1 + R^{-2} + R^{-4}). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} M_{\psi_R}[u] \leq 4sE(u_0) &- 4\int_0^{+\infty} m^s \int (\psi_1 - \frac{\eta}{4s} C|\psi_2|^2 - \frac{\eta}{4s} C)|\nabla u_m|^2 dxdm \\ &+ 2C\eta^{1-q'} (\|u\|^{p_*}_{\dot{H}^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'} + \mathcal{O}(\eta + R^{-2s} + R^{-2} + R^{-4}). \end{aligned}$$

Let $\gamma := \frac{2}{q' p_*}$, with the assumption $N > 1 + 2s + \alpha + 2\beta$, we have $\gamma > 1$. Moreover, by using the Young inequality once again, we obtain

$$\begin{aligned} \eta^{1-q'} (\|u\|^{p_*}_{\dot{H}^s} \frac{1}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'} &= ((\eta\|u\|_{\dot{H}^s})^{p_*} \frac{\eta^{-\frac{1}{q} - p_*}}{R^{\frac{s}{N}(N-1-\varepsilon)}})^{q'} \\ &\lesssim \eta^2 \|u\|^2_{\dot{H}^s} + \frac{\eta^{-\gamma' q'(\frac{1}{q} + p_*)}}{R^{\frac{\gamma' q' s}{N}(N-1-\varepsilon)}}. \end{aligned}$$

Consequently

$$\begin{aligned} \frac{d}{dt} M_{\psi_R}[u] \leq 4sE(u_0) &- 4\int_0^{+\infty} m^s \int (\psi_1 - \frac{\eta}{4s} C|\psi_2|^2 - \frac{\eta + \eta^2}{4s} C)|\nabla u_m|^2 dxdm \\ &+ \mathcal{O}(\eta + R^{-2s} + R^{-2} + R^{-4} + \frac{\eta^{-\gamma' q'(\frac{1}{q} + p_*)}}{R^{\frac{\gamma' q' s}{N}(N-1-\varepsilon)}}). \end{aligned}$$

Taking $\eta = R^{-\sigma}$, where $0 < \sigma < \frac{2s(N-1-\varepsilon)}{2Np_* + 1 + \varepsilon}$. We get

$$\frac{\eta^{-\gamma' q'(\frac{1}{q} + p_*)}}{R^{\frac{\gamma' q' s}{N}(N-1-\varepsilon)}} = \frac{1}{R^{\gamma' q' \{-\sigma(\frac{1}{q} + p_*) + \frac{s}{N}(N-1-\varepsilon)\}}}.$$

Since $-\sigma(\frac{1}{q} + p_*) + \frac{s}{N}(N-1-\varepsilon) > 0$, then by taking R large enough and using properties of ψ_i, one gets

$$\frac{d}{dt} M_{\psi_R}[u] \leq 2sE(u_0).$$

7. Sharp criteria for global/non global solutions

7.1. **Intermediate results.** In order to prove Theorem 5, we need the following auxiliary results.

Lemma 8. *Let $s \in (\frac{1}{2}, 1)$, $u_0 \in H^s_{rd}$ such that $E(u_0) \neq 0$ and $u \in C_{T^*}(H^s_{rd})$ be the maximal solution of* (1.1). *If there exist $R > 0, t_0 > 0$ and $C > 0$ such that*

$$M_{\psi_R}[u(t)] \leq -C \int_{t_0}^t \|u(\tau)\|_{\dot{H}^s} d\tau, \forall t \geq t_0,$$

then $T^* < \infty$.

Proof. See [3, 21]. $\square$

Lemma 9. *Under the flow of* (1.1)*, the following conditions are invariant:*
1/ (2.8) *and* (2.10)*;*
2/ (2.8) *and* (2.9)*.*

Proof. Together conservation laws and the Gagliardo-Nirenberg inequality (2.3), give

$$\begin{aligned} E(u_0) &= \|u(t)\|^2_{\dot H^s} - \frac{1}{p}\int (I_\alpha * |\cdot|^\beta |u|^p)|x|^\beta |u|^p dx \\ &\geq \|u(t)\|^2_{\dot H^s} - \frac{C(N,p,s,\alpha,\beta)}{p} M(u_0)^{\frac{A}{2}} \|u(t)\|^B_{\dot H^s}. \end{aligned}$$

Denoting

$$X(t) := \|u(t)\|^2_{\dot H^s} \text{ and } D := \frac{C(N,p,s,\alpha,\beta)}{p} M(u_0)^{\frac{A}{2}}.$$

So

$$X(t) - DX(t)^{\frac{B}{2}} \leq E(u_0), \forall t \in [0, T^*). \tag{7.1}$$

The function $f(x) := x - Dx^{\frac{B}{2}}$ has a global maximum on $\mathbb{R}_+$, at the point $x_m := (\frac{2}{BD})^{\frac{2}{B-2}}$ with the maximum value $f(x_m) := \frac{B-2}{B}(\frac{2}{BD})^{\frac{2}{B-2}} = \frac{B-2}{B} x_m$, equivalently $x_m = \frac{B}{B-2} f(x_m)$. Using the Pohozaev identities, we have $K_{N+\alpha+2\beta,-2p}(\phi) = 0$ and $K_{N,-2}(\phi) = 0$, then

$$\|\phi\|^2_{\dot H^s} = \frac{B}{A}\|\phi\|^2 \text{ and } \int (I_\alpha * |\cdot|^\beta |\phi|^p)|x|^\beta |\phi|^p dx = \frac{2p}{B}\|\phi\|^2_{\dot H^s}.$$

Thus

$$E(\phi) = \frac{B-2}{B}\|\phi\|^2_{\dot H^s} = \frac{B-2}{A}\|\phi\|^2.$$

Using previous relation, the condition (2.8) becomes

$$E(u_0) < \frac{B-2}{A} M(\phi)^{\frac{s}{s_c}} M(u_0)^{\frac{s_c - s}{s_c}}.$$

By using (2.5), we have

$$\begin{aligned} f(x_m) &= \frac{B-2}{B}\Big(\frac{2p}{C(N,p,s,\alpha,\beta) M(u_0)^{\frac{A}{2}} B}\Big)^{\frac{2}{B-2}} \\ &= \frac{B-2}{B}\Big(\Big(\frac{A}{B}\Big)^{\frac{2-B}{2}} M(\phi)^{p-1} M(u_0)^{-\frac{A}{2}}\Big)^{\frac{2}{B-2}} \\ &= \frac{B-2}{A} M(\phi)^{(p-1)\frac{2}{B-2}} M(u_0)^{-\frac{A}{B-2}} \\ &= \frac{B-2}{A} M(\phi)^{\frac{s}{s_c}} M(u_0)^{\frac{s_c-s}{s_c}} > E(u_0). \end{aligned}$$

Therefore

$$x_m = \frac{B}{B-2} f(x_m) = \frac{B}{A} M(\phi)^{\frac{s}{s_c}} M(u_0)^{\frac{s_c-s}{s_c}}.$$

From the inequality (7.1), we deduce

$$f(\|u(t)\|^2_{\dot H^s}) \leq E(u_0) < f(x_m). \tag{7.2}$$

1/ The condition (2.10) implies

$$(\|u_0\|^2_{\dot{H}^s})^{s_c} M(u_0)^{s-s_c} < (\|\phi\|^2_{\dot{H}^s})^{s_c} M(\phi)^{s-s_c}.$$

Equivalently

$$\begin{aligned}\|u_0\|^2_{\dot{H}^s} &< \|\phi\|^2_{\dot{H}^s}\left(\frac{M(\phi)}{M(u_0)}\right)^{\frac{s-s_c}{s_c}}\\ &< \frac{B}{A} M(\phi)^{\frac{s}{s_c}} M(u_0)^{\frac{s-s_c}{s_c}} = x_m.\end{aligned}$$

Knowing that $u \in C_{T^*}(\dot{H}^s)$, then by (7.2) and a continuity argument, one gets

$$\|u(t)\|^2_{\dot{H}^s} < x_m, \forall t \in [0, T^*).$$

2/ In a similar way as previous, the condition (2.9) is equivalent to $\|u_0\|^2_{\dot{H}^s} > x_m$. So, by a continuity argument, we obtain

$$\|u(t)\|^2_{\dot{H}^s} > x_m, \forall t \in [0, T^*).$$

Therefore, conditions (2.8) and (2.10), as well as (2.8) and (2.9) are invariant under the flow of (1.1). □

7.2. **Proof of Theorem 5.**

7.2.1. *The case* $0 < s_c < s$ *and* $E(u_0) < 0$*:* Let $u \in C_{T^*}(H^s_{rd})$ a solution of (1.1) and $\delta := N(p - p_*)$, by the Virial type identity, we have

$$\begin{aligned}\frac{d}{dt} M_{\psi_R}[u(t)] &\le 2(N(p-1) - \alpha - 2\beta)E(u_0) - 2\delta\|u\|^2_{\dot{H}^s}\\ &\quad + C\left(R^{-2s} + \frac{1}{R^{(N-1-\varepsilon)(p-1-\frac{\alpha+2\beta}{N})}}\|u\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}_{\dot{H}^s}\right).\end{aligned}$$

The assumption $p < 1 + 2s + \frac{\alpha+2\beta}{N}$ makes possible the choose of ε small enough such that $\frac{1+\varepsilon}{s}(p - 1 - \frac{\alpha+2\beta}{N}) < 2$. Using the lower bound of $\|u\|_{\dot{H}^s}$, for R large enough, uniformly on time, we have

$$\frac{d}{dt} M_{\psi_R}[u(t)] \le (N(p-1) - \alpha - 2\beta)E(u_0) - \delta\|u\|^2_{\dot{H}^s}.$$

By integrating on time, there exists t_0 sufficiently large such that

$$M_{\psi_R}[u(t)] < 0, \forall t \ge t_0.$$

Integrating once again on $[t_0, t]$, we obtain

$$M_{\psi_R}[u(t)] \le -\delta \int_{t_0}^{t} \|u(\tau)\|^2_{\dot{H}^s} d\tau, \forall t \ge t_0.$$

Thanks to Lemma 8, one gets $T^* = +\infty$.

7.2.2. *The case* $0 < s_c < s$ *and* $E(u_0) > 0$ *under* (2.8) *and* (2.9)*:* Taking $0 < \mu < 1$ such that

$$E(u_0)^{s_c} M(u_0)^{s-s_c} < ((1-\mu)E(\phi))^{s_c} M(\phi)^{s-s_c}.$$

So

$$(\frac{E(u_0)}{1-\mu})^{s_c} M(u_0)^{s-s_c} < E(\phi)^{s_c} M(\phi)^{s-s_c}. \tag{7.3}$$

Then

$$\frac{E(u_0)}{1-\mu} < f(x_m).$$

We deduce that

$$\frac{f(\|u(t)\|^2_{\dot{H}^s})}{1-\mu} \le \frac{E(u_0)}{1-\mu} < f(x_m).$$

Thus

$$f(\|u(t)\|^2_{\dot{H}^s}) \le E(u_0) < (1-\mu) f(x_m). \tag{7.4}$$

For μ small enough, we have

$$(1-\mu) f(x_m) < f(\frac{x_m}{1-\mu}). \tag{7.5}$$

Indeed, we have

$$\begin{aligned} f(\frac{x_m}{1-\mu}) &= \frac{x_m}{1-\mu} - D\frac{x_m^{\frac{B}{2}}}{(1-\mu)^{\frac{B}{2}}} \\ &= \frac{x_m}{1-\mu}(1 - D\frac{1}{(1-\mu)^{\frac{B-2}{2}}} x_m^{\frac{B-2}{2}}) \\ &= \frac{x_m}{1-\mu}(1 - \frac{2}{B}\frac{1}{(1-\mu)^{\frac{B-2}{2}}}). \end{aligned}$$

Since

$$\lim_{\mu\to 0} f(\frac{x_m}{1-\mu}) = \lim_{\mu\to 0} \frac{x_m}{1-\mu}(1 - \frac{2}{B}\frac{1}{(1-\mu)^{\frac{B-2}{2}}}) = \frac{B-2}{B} x_m = f(x_m).$$

Then there exists μ small enough such that (7.5) is satisfied. Together with (7.4), give

$$f(\|u(t)\|^2_{\dot{H}^s}) < f(\frac{x_m}{1-\mu}), \forall t \in [0, T^*). \tag{7.6}$$

In the other hand, since $\|\phi\|^2_{\dot{H}^s} = \frac{B}{B-2} E(\phi)$ then by using (2.9) and (7.3), one gets

$$\begin{aligned} \|u_0\|^{2s_c}_{\dot{H}^s} &> (\frac{B}{B-2} E(\phi))^{s_c} M(\phi)^{s-s_c} M(u_0)^{s_c-s} \\ &> (\frac{B}{B-2})^{s_c} \{E(\phi)^{s_c} M(\phi)^{s_c}\} M(u_0)^{s_c-s} \\ &> (\frac{B}{B-2})^{s_c} \{(\frac{E(u_0)}{1-\mu})^{s_c} M(u_0)^{s-s_c}\} M(u_0)^{s_c-s} \\ &= (\frac{B}{B-2})^{s_c} (\frac{E(u_0)}{1-\mu})^{s_c}. \end{aligned}$$

Hence

$$\|u_0\|^2_{\dot{H}^s} > \frac{B}{B-2}\frac{E(u_0)}{1-\mu} \geq \frac{B}{B-2}\frac{f(x_m)}{1-\mu} = \frac{x_m}{1-\mu}. \tag{7.7}$$

In summary, by using (7.6), (7.7) and a continuity argument, we deduce

$$\|u(t)\|^2_{\dot{H}^s} > \frac{B}{B-2}\frac{E(u_0)}{1-\mu}, \text{ equivalently } (1-\mu)(B-2)\|u(t)\|^2_{\dot{H}^s} > BE(u_0), \forall t \in [0, T^*).$$

Inserting this bound in to the Virial type identity, we obtain

$$\begin{aligned}
\frac{d}{dt}M_{\psi_R}[u(t)] &\leq 2sBE(u_0) - 2(N-p_*)\|u(t)\|^2_{\dot{H}^s} \\
&\quad + \mathcal{O}_R(1)(1 + \|u(t)\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}_{\dot{H}^s}) \\
&\leq -2s\mu(B-2)\|u(t)\|^2_{\dot{H}^s} + \mathcal{O}_R(1)(1 + \|u(t)\|^{\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N})}_{\dot{H}^s}) \\
&\leq -s\mu(B-2)\|u(t)\|^2_{\dot{H}^s}.
\end{aligned}$$

In the last inequalities $\mathcal{O}_R(1) \to 0$ when $R \to +\infty$ uniformly on time and $\varepsilon > 0$ small enough such that $\frac{1+\varepsilon}{s}(p-1-\frac{\alpha+2\beta}{N}) < 2$, which is possible since $p < 1 + 2s + \frac{\alpha+2\beta}{N}$ by assumption. Therefore, the proof follows once again by Lemma 8.

7.2.3. *The case* $0 \leq s_c < s$ *and* $E(u_0) \geq 0$ *under* (2.8) *and* (2.10)*:* Thanks to Lemma 9, in this case we have $\sup_{t\in[0,T^*)} \|u(t)\|_{H^s} < \infty$. Then $T^* = +\infty$, and u is a global solution for (1.1).

7.2.4. *The case* $s_c = 0$*:* In the mass critical case $s_c = 0$, the blow up condition stated above leads us to a contradiction. Indeed, conditions (2.8) and (2.9) becomes $M(u_0) < M(\phi)$ and $M(u_0) > M(\phi)$. Thus, for $s_c = 0$ the only admissible condition is $E(u_0) < 0$. From Theorem 4, we have

$$\frac{d}{dt}M_{\psi_R}[u(t)] \leq 2sE(u_0), \forall t \in [0, T^*).$$

So, in a similar way as in [3, 21], the solution u either blows up in finite time or blows up in infinite time such that

$$\|u(t)\|_{\dot{H}^s} \geq Ct^s, \forall t \geq t_*.$$

with some constants $C > 0$ and $t_* > 0$.

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