Nonlinear polyharmonic problems with the parameter near resonance

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Abstract

This paper is concerned with sublinear perturbations of resonant linear polyharmonic problems. We establish some *a priori* bounds and use these together with Leray-Schauder continuation and bifurcation arguments to obtain extensions of some known results of Mawhin and Schmitt on the multiplicity of solutions of nonlinear elliptic eigenvalue problems with the parameter near resonance.

Keywords: Polyharmonic operator, positive solutions, eigenvalue, bifurcation.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. We study the existence and multiplicity of solutions to the nonlinear polyharmonic problem

$$\begin{aligned} &(-\Delta)^m u - (\lambda_1 + \lambda)u + f(x,u) = h \qquad x \in \Omega, \\ &u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \qquad x \in \partial\Omega, \end{aligned} \tag{1.1$_\lambda$}$$

where λ_1 is principal eigenvalue of $(-\Delta)^m$ with Dirichlet boundary condition. Let ψ be the corresponding eigenfunction of λ_1 with

$$\psi(x) > 0 \quad x \in \Omega; \qquad \int_\Omega \psi^2(x)dx = 1.$$

The case $m = 1$ has been extensively studied, see J. Mawhin, K. Schmitt [13, 14], R. Chiappinelli, J. Mawhin, R. Nugari [1], Ma [12] and the references therein. The linear problem associated with $(1.1)_\lambda$ reads

$$\begin{aligned} -\Delta u - (\lambda_1 + \lambda)u = h \qquad x \in \Omega, \\ u = 0 \qquad x \in \partial\Omega, \end{aligned} \tag{1.2}$$

and the corresponding existence results are well known from the linear theory; namely, if $\lambda \neq 0$, then (1.2) has a unique solution for each given h (provided of course $\lambda_1 + \lambda$ does not touch other eigenvalues) while for $\lambda = 0$, a solution exists if, and only if

$$\int_\Omega h(x)\psi(x)dx = 0. \tag{1.3}$$

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A similar situation arises when introducing a sufficiently "small" nonlinearity f. Assume

$$|f(x,s)| \le c \quad (x,s) \in \bar{\Omega} \times \mathbb{R} \tag{1.4}$$

for some $c \ge 0$, it is easy to see (e.g. by the Schauder fixed point theorem) that if $\lambda \ne 0$, $(1.1)_\lambda$ again has a solution for each given h. If $\lambda = 0$, a by now classical result due to Landesman and Lazer [10] states that $(1.1)_\lambda$ is solvable if h satisfies the condition

$$\int_\Omega f^+(x)\psi(x)dx < \int_\Omega h(x)\psi(x)dx < \int_\Omega f_-(x)\psi(x)dx, \tag{1.5}$$

or

$$\int_\Omega f^-(x)\psi(x)dx < \int_\Omega h(x)\psi(x)dx < \int_\Omega f_+(x)\psi(x)dx, \tag{1.5'}$$

where

$$f^\pm(x) = \limsup_{s\to\pm\infty} f(x,s), \quad f_\pm(x) = \liminf_{s\to\pm\infty} f(x,s),$$

and ψ is the eigenfunction corresponding to λ_1.

In [13], J. Mawhin, K. Schmitt used that f is subject to the growth restriction

$$|f(x,s)| \le a|s|^\alpha + b \qquad x \in \Omega, \ s \in \mathbb{R} \tag{1.6}$$

where $0 \le \alpha < 1$ and a, b are positive constants. They considered the problem of multiplicity of solutions to $(1.1)_\lambda$ with $m = 1$ for λ near 0. Using degree theory together with results on bifurcation from infinity at the simple eigenvalue λ_1, they showed that under condition (1.5), $(1.1)_\lambda$ with $m = 1$ has near λ_1, at least one solution for $\lambda \ge 0$, and at least two solutions for $\lambda < 0$; a similar result holds under $(1.5')$ if we "reverse the sides" of λ with respect to 0. This fact follows in [13] from a more general result which involves nonlinear perturbations of linear operators having an isolated eigenvalue of odd multiplicity. Further applications to boundary-value problems for ordinary differential equations can be found in J. Mawhin, K. Schmitt [14].

One crucial step in [13] consists in obtaining *a priori* bounds for the solutions of $(1.1)_\lambda$, which are to be uniform in λ for λ on one side of 0. More precisely, one shows that if, e.g. (1.3) holds, then there exist $R > 0$ and $\delta > 0$ so that

$$||u|| < R \quad \text{for all possible solutions of } (1.1)_\lambda \text{ with } 0 \le \lambda \le \delta,$$

where the norm of u is taken in a suitable function space. This first gives, by degree and continuation arguments, the existence of one solution in $B_R = \{u : ||u|| < R\}$ for all λ near 0. Furthermore, it implies that the bifurcation branch of solutions of $(1.1)_\lambda$ arising at λ_1 and containing solutions of large norm has to lie (at least locally, i.e. "near (λ_1, ∞)") in the region $\lambda < 0$, it is then easy to deduce the existence, for $\lambda < 0$, of a second solution which lies outside B_R.

It is the purpose of this paper to study the existence and multiplicity of solutions of the nonlinear polyharmonic problem $(1.1)_\lambda$ with $m \geq 1$ under the Landesman-Lazer conditions (1.5) or (1.5′) and the sublinear condition

$$\lim_{|s|\to\infty} \frac{f(x,s)}{s} = 0. \tag{1.7}$$

Notice that (1.7) above is precisely the condition guaranteeing that bifurcation from infinity occurs [17].

Remark 1.1. it is worth remarking that (1.7) is weaker that (1.6).

In fact. Let us consider

$$\psi(s) := s\ln(1+s) \qquad s \in [0,\infty).$$

Obviously, we have for any $p > 1$,

$$\begin{aligned}
\lim_{s\to\infty} \frac{s\ln(1+s)}{s^p} &= \lim_{s\to\infty} \frac{\ln(1+s) + \frac{s}{1+s}}{ps^{p-1}} \\
&= \lim_{s\to\infty} \frac{\ln(1+s)}{ps^{p-1}} + \lim_{s\to\infty} \frac{\frac{s}{1+s}}{ps^{p-1}} \\
&= \lim_{s\to\infty} \frac{\frac{1}{1+s}}{p(p-1)s^{p-2}} = 0.
\end{aligned}$$

Let $q = 1/p$. Let $g_* : [0,\infty) \to [0,\infty)$ be the inverse function of ψ. Then g_* is increasing in $[0,\infty)$ and

$$\lim_{x\to\infty} \frac{g_*(x)}{x^q} = \infty$$

for any $q \in (0,1)$.

Therefore, g_* satisfies (1.7), but g_* does not satisfy (1.6) for any $q \in (0,1)$. $\square$

2 Preliminaries

2.1 Principal eigenvalue

The biharmonic eigenvalue problem with Dirichlet boundary conditions is the following:

$$\begin{aligned}
&\Delta^2\varphi = \lambda\varphi \qquad \text{in } \Omega, \\
&\varphi = \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega.
\end{aligned} \tag{2.1}$$

The famous conjecture for this problem was as follows; by now it has numerous counterexamples.

Conjecture 1 *(Szegö, 1950) If Ω is a nice domain (convex), then the first eigenfunction for (2.1) is of fixed sign.*

Szegö conjecture proved to be wrong, see Duffin and others ([3, 4, 11, 2] and [18]). Coffman in [2] proved that the first eigenfunction on a square changes sign. Sign changing first eigenfunctions are also found in [16].

For the domains

$$A_\epsilon = \{(x,y) \in \mathbb{R}^2 : \epsilon^2 < x^2 + y^2 < 1\} \quad \text{with } 0 < \epsilon < 1.$$

For these domains, Coffman-Duffin-Shaffer proved the following somehow surprising statement.

Lemma 2.1. ([5, Theorem 3.9.]) Let $\Omega = A_\epsilon$ for some $\epsilon \in (0,1)$ and consider problem (2.1). There exists $\epsilon_0 > 0$ such that the following holds.

1. If $\epsilon < \epsilon_0$, then the first eigenvalue has multiplicity two. There exist two independent eigenfunctions for this first eigenvalue with diametral nodal lines.

2. If $\epsilon = \epsilon_0$, then the first eigenvalue has multiplicity three. There exists a positive eigenfunction for this eigenvalue and there are two independent eigenfunctions with diametral nodal lines.

3. If $\epsilon > \epsilon_0$, then the first eigenvalue has multiplicity one and the corresponding eigenfunction is of fixed sign.

In the following, we consider the eigenvalue problem

$$\begin{aligned} &(-\Delta)^m u = \lambda \hat{a}(x) u \qquad && \text{in } \Omega, \\ &D^\alpha u|_{\partial\Omega} = 0 \qquad \text{for } |\alpha| \le m-1, && \end{aligned} \tag{2.2}$$

where $\hat{a} \in C(\bar{\Omega}, (0,\infty))$. The first eigenvalue of (2.2) is defined as

$$\lambda_1 = \begin{cases} \min\limits_{u \in H_0^m(\Omega)\setminus\{0\}} \dfrac{||\Delta^{\frac{m}{2}} u||_{L^2}^2}{||\hat{a}^{\frac{1}{2}} u||_{L^2}^2}, & \text{for } m \text{ even}, \\ \min\limits_{u \in H_0^m(\Omega)\setminus\{0\}} \dfrac{||\nabla\Delta^{\frac{m-1}{2}} u||_{L^2}^2}{||\hat{a}^{\frac{1}{2}} u||_{L^2}^2}, & \text{for } m \text{ odd}, \end{cases} \tag{2.3}$$

where $H_0^m(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the normal $||\cdot||_{W^{m,2}}$. We will show that if Ω is *good* enough, then λ_1 is simple and its corresponding eigenfunction is of one-sign.

Lemma 2.2. There exists $\epsilon_0 > 0$ such that if the following holds:

(i) $\Omega = B$, or

(ii) $\Omega \subset \mathbb{R}^2$ is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$.

Then

(1) the first eigenvalue λ_1 of (2.2) is simple;

(2) the corresponding eigenfunction ψ is of one sign;

(3) $(-1)^m \frac{\partial^m \psi}{\partial \nu^m} > 0 \quad x \in \partial\Omega$.

Proof. (1) can be deduced from [7].

(2) From [6] and [8], there exists $\epsilon_0 > 0$, such that if Ω of class $C^{2m,\alpha}$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$, then the Green function $G(x,y)$ of $(-\Delta)^m$ for Dirichlet problem in Ω is positive, and subsequently,

$$\begin{aligned} &(-\Delta)^m u = f \qquad && \text{in } \Omega, \\ &D^\alpha u|_{\partial\Omega} = 0 \qquad && \text{for } |\alpha| \le m-1, \end{aligned}$$

is sign-preserving, that is,

$$f \geq 0 \implies u \geq 0.$$

Applying this and the standard Krein-Rutman type argument, we may get (1) and (2). (3) is an immediate consequence of F. Gazzola, H.-Ch. Grunau, G. Sweers [8, Theorem 3.2], [6] and [7]. □

Remark 2.3. If $\Omega = B$, then Lemma 2.2 reduces to Gazzola, Grunau and Sweers [5, Theorem 3.7].

Remark 2.4. The similar result for Lemma 2.2 with $N = 1$ and $m = 2$ has been investigated by Ma et al. [15].

2.2 A priori bounds

Let $X = C(\bar{\Omega})$ with its usual norm $||\cdot||$. We first shall establish some boundedness criteria for solutions of $(1.1)_\lambda$.

Let L be the closed Fredholm operator

$$Lu = (-\Delta)^m u - \lambda_1 u \qquad u \in D(L), \tag{2.4}$$

where

$$D(L) = \{u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) : u \text{ satisfies boundary conditions in } (1.1)_\lambda\}.$$

Then 0 is an isolated and simple eigenvalue of L and

$$\int_\Omega z(x)\psi(x)dx = 0 \qquad \forall\, z \in R(L),$$

where $R(L)$ and $N(L)$ are the range and kernel (null space) of L, respectively.

Let $P : X \to X$ be a continuous linear projection onto $N(L)$. We observe that

$$L_P : D(L) \cap N(P) \to R(L)$$

defined by

$$L_P = L|_{D(L)\cap N(P)}$$

is an invertible linear operator with continuous inverse, L_P^{-1}. Let $Q : X \to X$ is a linear projection with $N(Q) = R(L)$. Let

$$(F(u))(x) = f(x, u(x)) \qquad x \in \Omega.$$

Let $h \in X$ be such that for each $y \in N(L)$, $||y|| = 1$, each sequence $\{t_n\}$ with $t_n \to \infty$, each sequence $\{y_n\} \subset N(L)$, $||y_n|| = 1$ with $y_n \to y$ as $n \to \infty$ and each bounded sequence $\{z_n\} \subset N(P)$ one has

$$\langle h, y\rangle > \liminf_{n\to\infty} \langle F(t_n y_n + t_n \rho(t_n) z_n), y\rangle \tag{2.5}$$

or

$$\langle h, y\rangle < \limsup_{n\to\infty}\langle F(t_n y_n + t_n\rho(t_n)z_n), y\rangle \tag{2.6}$$

where $\rho : [0,\infty) \to [0,\infty)$ is a decreasing function, such that

$$\frac{f^*(x,s)}{s} \le \rho(s) \quad s > 0; \qquad \lim_{s\to\infty}\rho(s) = 0, \tag{2.7}$$

$$f^*(x,s) = \max\{f(x,t) : \ 0 \le t \le s\} \qquad s \in [0,\infty). \tag{2.8}$$

Lemma 2.5. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0,\epsilon_0]$. Assume that (2.5) holds, then there exist $R_0 > 0$ and $\delta_0 > 0$ such that any solution u of $(1.1)_\lambda$ satisfies

$$||u|| < R_0 \tag{2.9}$$

as long as

$$0 \le \lambda \le \delta_0 = \frac{1}{2||L_P^{-1}||}. \tag{2.10}$$

Proof. Let u be a solution of $(1.1)_\lambda$ and write

$$u = s\psi + w, \quad s\psi \in N(L), \quad w \in N(P).$$

Then

$$\int_\Omega \big(-\lambda s\psi + f(x, s\psi + w) - h\big)\psi dx = 0 \tag{2.11}$$

and

$$Lw - \lambda w + (I-Q)f(x, s\psi + w) = (I-Q)h. \tag{2.12}$$

Since $L - \lambda I : D(L)\cap N(P) \to R(Q)$ is invertible for $|\lambda| \le \delta_0$, we have, if $||s\psi|| \ne 0$

$$\begin{aligned}
||w|| &\le ||(L-\lambda I)^{-1}(I-Q)(f(\cdot, a\psi + w) - h)|| \\
&\le 2||L_P^{-1}||\,||I-Q||\,||f(\cdot, s\psi + w) - h|| \\
&\le 2||L_P^{-1}||\,||I-Q||\,||f^*(\cdot, ||s\psi|| + ||w||) - h|| \\
&\le 2||L_P^{-1}||\,||I-Q||\,\big(\rho(||s\psi|| + ||w||)\,(||s\psi|| + ||w||) + ||h||\big) \\
&\le 2||L_P^{-1}||\,||I-Q||\,(\rho(||s\psi||)\,(||s\psi|| + ||w||) + ||h||)
\end{aligned}$$

$$\frac{||w||}{||s\psi||} \le 2||L_P^{-1}||\,||I-Q||\,(\rho(||s\psi||)\,(1 + \frac{||w||}{||s\psi||}) + \frac{||h||}{||s\psi||}). \tag{2.13}$$

Obviously, this implies

$$\lim_{s\to\infty}\frac{||w||}{||s\psi||} = 0, \tag{2.14}$$

and there exist two constants C_1, $C_2 \in (0,\infty)$, such that

$$||w|| \le C_1\rho(||s\psi||)||s\psi|| \qquad \text{as long as } ||s\psi|| > C_2. \tag{2.15}$$

If we now assume that the conclusion of the lemma is false, we obtain a sequence $\{\lambda_n\}$, $0 \le \lambda_n \le \delta_0$, $\lambda_n \to 0^+$, and a sequence of corresponding solutions $\{u_n\}$ of $(1.1)_\lambda$ such that

$$||u_n|| \to \infty, \qquad u_n = s_n\psi + w_n.$$

It follows from the calculations leading to (2.15) that necessarily $||s_n\psi|| \ge C_2$ and we may hence assume that

$$||w_n|| \le C_1\rho(||s_n\psi||)||s_n\psi||.$$

Letting $s_n\psi = t_n y_n$, $t_n = ||s_n\psi||$, $y_n = \frac{s_n\psi}{||s_n\psi||}$, $w_n = \rho(t_n)t_n z_n$, we have from (2.15) that

$$||z_n|| \le C_3.$$

By passing to a subsequence, we may assume that $y_n \to y \in N(L)$, $||y|| = 1$. Hence from (2.11) we obtain

$$\int_\Omega (-\lambda_n t_n y_n y)dx + \int_\Omega F(t_n y_n + t_n\rho(t_n)z_n)\, y dx = \int_\Omega h y dx$$

and since

$$\int_\Omega y_n y dx = \int_\Omega y y dx + \int_\Omega (y_n - y) y dx > 0$$

for n large, we get

$$\int_\Omega h y dx \le \int_\Omega F(t_n y_n + t_n\rho(t_n)z_n)\, y dx$$

and

$$\int_\Omega h y dx \le \liminf_{n\to\infty} \int_\Omega f^*(x, t_n y_n + t_n\rho(t_n)z_n) y dx.$$

contradicting (2.5). $\square$

Using a similar argument we may establish the next lemma.

Lemma 2.6. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$. Assume that (2.6) holds, then there exist $R_0 > 0$ and $\delta_0 > 0$ such that any solution u of $(1.1)_\lambda$ satisfies

$$||u||_\infty < R_0$$

as long as

$$-\delta_0 \le \lambda \le 0.$$

By the same method used in J. Mawhin and K. Schmitt [13], with obvious changes, we may get the following results.

Lemma 2.7. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$. Assume that (2.5) holds, then there exists $R_1 \ge R_0$ such that for $0 \le \lambda \le \delta_0$ and $R > R_1$, one has

$$\deg(L - \lambda I + F - h, B(R), 0) = \deg(L - \delta_0 I, B(R), 0) = \pm 1,$$

where $B(R) = \{u \in X : ||u|| < R\}$, and "deg" denotes Leray-Schauder degree when $\lambda \neq 0$ and coincidence degree when $\lambda = 0$ (see [9]). Therefore $(1.1)_\lambda$ has a solution in $B(R)$ for $0 \leq \lambda \leq \delta_0$.

Lemma 2.8. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$. Assume that (2.6) holds, then there exists $R_1 \geq R_0$ such that for $-\delta_0 \leq \lambda \leq 0$ and $R > R_1$, one has

$$\deg(L - \lambda I + F - h, B(R), 0) = \deg(L + \delta_0 I, B(R), 0) = \pm 1.$$

Therefore $(1.1)_\lambda$ has a solution in $B(R)$ for $-\delta_0 \leq \lambda \leq 0$.

Lemma 2.9. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$. Assume that (2.5) holds, then there exists $\delta_1 > 0$ such that for $-\delta_1 < \lambda \leq 0$, one has

$$\deg(L - \lambda I + F - h, B(R), 0) = \deg(L - \delta_0 I, B(R), 0) = \pm 1.$$

Therefore $(1.1)_\lambda$ has a solution in $B(R)$ for $-\delta_1 \leq \lambda \leq \delta_0$.

Lemma 2.10. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$. Assume that (2.6) holds, then there exists $\delta_1 > 0$ such that for $0 \leq \lambda < \delta_1$, one has

$$\deg(L - \lambda I + F - h, B(R), 0) = \deg(L - \delta_0 I, B(R), 0) = \pm 1.$$

Therefore $(1.1)_\lambda$ has a solution in $B(R)$ for $-\delta_0 \leq \lambda \leq \delta_1$.

Remark 2.11. Using "Whyburn's lemma" (see [19]), one can in fact deduce that (1.1_λ) has a continuum $C = \{(\lambda, u_\lambda)\}$ of solutions with $||u_\lambda|| < R_1$ and $|\lambda| < \delta_1$ for δ_1 sufficiently small.

Remark 2.12. Since F is L-completely continuous and satisfies (2.7) and since $\lambda = 0$ is an eigenvalue of odd multiplicity, it follows from bifurcation results of [17] that $\lambda = 0$ is a bifurcation point from infinity for $(1.1)_\lambda$, i.e., there exists a continuum $\mathcal{C}_\infty \subset \mathbb{R} \times X$ of solutions of $(1.1)_\lambda$, bifurcating from infinity at $\lambda = 0$, i.e., there exists $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0]$, there exists a subcontinuum $\mathcal{C}_\sigma \subset \mathcal{C}_\infty$

$$\mathcal{C}_\sigma \subset \{(\lambda, u) \in C : \ |\lambda| < \sigma, \ ||u|| > \frac{1}{\sigma}\} =: U_\sigma.$$

and $\mathcal{C}_\sigma$ connects $(0, \infty)$ to ∂U_σ.

3 The main results

Theorem 3.1. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$.
Assume that (2.5) and (1.8) hold. Then there exist $\lambda_- < 0 < \lambda_+$ such that $(1.1)_\lambda$ has

(1) at least one solution if $\lambda \in [0, \lambda_+]$;

(2) at least two solutions if $\lambda \in [\lambda_-, 0)$.

Assume that (2.6) and (1.8) hold. Then there exist $\lambda_- < 0 < \lambda_+$ such that $(1.1)_\lambda$ has

(3) at least one solution if $\lambda \in [\lambda_-, 0]$;

(4) at least two solutions if $\lambda \in (0, \lambda_+]$.

Proof. It follows from lemmas 2.7 and 2.9 that there exist $-\delta_1 < 0 < \lambda_+$ such that $(1.1)_\lambda$ has at least one solution in $B(R_1)$ for $\lambda \in [-\delta_1, \lambda_+)$; on the other hand, Remark 2.12 shows that there exists a continuum C_σ of solutions of $(1.1)_\lambda$ bifurcating from infinity at $\lambda = 0$. Hence the subcontinua C_σ of Remark 2.12 (see Lemma 2.7) must satisfy

$$C_\sigma \subset \{(\lambda, u) : \, ||u|| > \frac{1}{\sigma}, \, -\sigma < \lambda < 0\},$$

and hence, if $\frac{1}{\sigma}$, we obtain a second solution u, with $||u|| > R_1$. Hence letting $\lambda_- := \max\{-\delta_1, -\frac{1}{R_1}\}$, we obtain (1) and (2). The remaining part is proved using lemmas 2.8 and 2.10 and Remark 2.12. □

Corollary 3.2. Let ϵ_0 be as in Lemma 2.2. Let $\Omega = B$, or $N = 2$ and Ω is a bounded domain of class $C^{2m,\alpha}(\bar{\Omega})$ which is ϵ-close in $C^{2m,\alpha}$ sense to B for any $\epsilon \in (0, \epsilon_0]$.

Assume that (1.5) and (1.8) hold. Then there exist $\lambda_- < 0 < \lambda_+$ such that $(1.1)_\lambda$ has

(1) at least one solution if $\lambda \in [0, \lambda_+]$;

(2) at least two solutions if $\lambda \in [\lambda_-, 0)$.

Assume that $(1.5')$ and (1.8) hold. Then there exist $\lambda_- < 0 < \lambda_+$ such that $(1.1)_\lambda$ has

(3) at least one solution if $\lambda \in [\lambda_-, 0]$;

(4) at least two solutions if $\lambda \in (0, \lambda_+]$.

Proof. Taking $y = y_n = \pm\psi$ in (2.5) and (2.6), it is easy to check that (1.5) implies (2.5), and $(1.5')$ implies (2.6). □

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The authors claim that the research was realized in collaboration with the same responsibility. All authors read and approved the last of the manuscript.

Endnotes

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