The Exponential Decay Functions for Prey and Predator Species in a Fractional-Order Ecosystem Model with Considering Global Warming Phenomena

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Abstract: This study examines the memory effect on predator-prey interactions in the face of global warming, employing a fear function with a Holling type II function to characterize the consumer prey and predator species. The effect of global warming on both prey and predators was studied using the exponential decay function. The system's equilibria are determined, and the system's stability is established around the equilibrium points. A series of numerical simulations are done to evaluate the theoretical component of the work and show the impacts of global warming, anxiety, and fractional order on our model's behavior.
Keywords: Fear effect, global warming, the exponential decay function, Caputo fractional order, predator-prey model, stability, Hopf-bifurcation

1 Introduction

Climate change has a considerable impact on ecosystems, according to the Intergovernmental Panel on Climate Change (IPCC). Climate change has a number of effects on ecosystems. Warming may force species to migrate to higher latitudes or elevations, where temperatures are more suitable for survival. As sea levels rise, saltwater intrusion into freshwater systems may force certain important species to relocate or die, removing predators and prey that are

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essential to the current food chain [1, 2, 3, 4, 5, 6].
Climate change affects ecosystems and species in an indirect manner, and it interacts with other human pressures such as development. Although some stressors have minor effects when employed alone, when they are coupled, they can cause large environmental changes. Climate change, for example, may exacerbate the demands placed on vulnerable coastal areas by land expansion. Furthermore, if climate change results in more severe rainfall events, already devastated forest areas may become prone to erosion [1].
The consequences of climate change on one species can spread across the food chain, affecting a variety of different organisms. As shown in the picture below, the food web of polar bears is exceedingly intricate. Sea ice loss not only affects the number of polar bears by reducing the area of their primary habitat, but it also has a significant impact on their food web. As the length and width of Arctic sea ice shrinks, the amount of ice algae, which thrive in nutrient-rich regions of the ice, drops. These algae are eaten by zooplankton, which is eaten by Arctic cod, which is a key food source for many marine mammals, including seals. Seals are known to be eaten by polar bears. Cows and farm animals, sometimes known as cattle, are responsible for roughly 14% of human-caused climate emissions, with methane from their burps and manure being the most major concern. As a result, living species account for a portion of the global warming increase. Human waste from manufacturing and combustion activities also contributes significantly to global warming [2, 4, 5, 7, 8]. Predator-prey interactions are an important component of biological communities. There are numerous publications [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] have been published in order to gain a better understanding of the predator-prey system's dynamics. Fear can be explored using a mathematical model in predator-prey interactions, which is becoming a hot topic in ecology and theoretical biology. Several studies have been carried out to see how fear affects prey population density [18, 19, 20]. Mondal et al. [17] investigated the dynamical behavior of a predator-prey model with fear and greater food.
Modeling prey-predator systems with fractional-order differential equations has various advantages. For starters, it can capture a biological activity's whole temporal state. Second, the utility and true significance of fractional derivatives have been established in a variety of fields based on that biological process. In a number of papers, researchers have proved the utility of this derivative in modeling biological processes, taking into consideration elements such as the freedom to arrange the derivative, dealing with species memory acquired during their cycle life, hereditary features, and so on. See [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] for further information.

2 Mathematical Model

Our model divides in to two parts: the first is ecological community which contains two classes; prey $x(t)$ and predator $y(t)$ spcies, where the second part contains the Global Warming class $w(t)$. the density of prey class described in the following fractional equation;

$$D^{\alpha}x(t) = \frac{rx}{1+fy} - gx^2 - cF(x)y - x_0we^{-k_1x} - d_1x \tag{1}$$

where r is the internist growth rate of prey, f and $\frac{1}{1+fy}$ are the fear and fear level function respectively. g is conflict between prey species. In the second term the parameter c describe that the Conversion rate of prey to predator and the functional response of Holling Type II is $F(x)$ as:

$$F(X) = \frac{ax}{b+x}$$

where a parametrize the saturation functional response and b; the prey population level where the predation rate per unit prey is half its maximum value. We used the function $x_0we{-}k_1x$ to represent the rate of global warming damage to the prey community. The death rate of prey due to global warming is influenced by a number of factors, including the type of prey, its strength, and the magnitude of global warming's impact on the environment surrounding the prey or its direct impact on the prey.where x_0 is initial amount of prey and k_1 is constant of proportionality for x. finely, d_1 refer to death rates of prey.
The describe of the densety of predator species $y(t)$ as:

$$D^{\alpha}y(t) = F(x)y - y_0we^{-k_2y} - d_2y \tag{2}$$

The function $y_0e^{-k_2y}$, which represents the rate of global warming damage to the prey community. The death rate of prey due to global warming is influenced by a number of factors, including the type of prey, its strength, and the magnitude of global warming's impact on the environment surrounding the prey or its direct impact on the prey.where y_0 is initial amount of prey and k_2 is constant of proportionality of y. In a nutshell, d_2 refers to predator death rates.
In the second part, $w(t)$ is the density of global warming which has the following equation;

$$D^{\alpha}w(t) = \alpha_1x + \alpha_2y + \alpha_3w - d_3w \tag{3}$$

The parameters α_1 and α_2 are the contributions of prey and predator to increasing global warming respectively. Furthermore, human-caused gaseous remnants of combustion from

factories and other pollutants contributed to global warming by α_3. while, the decay rate of global warming represented by d_3.
Now, by collecting all the previus equations , the ecosystem will be in the form:

$$\begin{cases} D^\alpha x(t) &= \frac{rx}{1+fy} - gx^2 - cF(x)y - x_0 we^{-k_1 x} - d_1 x \\ D^\alpha y(t) &= F(x)y - y_0 we^{-k_2 y} - d_2 y \\ D^\alpha w(t) &= \alpha_1 x + \alpha_2 y + \alpha_3 w - d_3 w. \end{cases} \tag{4}$$

We equipped the system (4) with the initial condition

$$x(0) > 0, \quad y(0) > 0 \text{ and } w(0) > 0.$$

All the biological meanings of the symbols are found in the following table:

The Parameter	Environmental Interpretation
r	The internist growth rate of prey
f	The level of fear
g	The conflict between prey species
k_1	The constant of proportionality for x
k_2	is constant of proportionality for y
c	Conversion rate of prey to predator
a	Parametrize the saturation functional response
b	The prey population level where the predation rate per unit prey is half its maximum value
x_0	The rate function of global warming damage to the prey community
y_0	The rate function of global warming damage to the predater community
α_1	Contributions of prey to increasing global warming
α_2	Contributions of predator to increasing global warming
α_3	Contributions of various industrial combustion and avalanches
d_1	Death rate of prey
d_2	Death rate of predator
d_3	The decay rate of global warming

The use of fractional-order differential equations to model prey-predator systems has several advantages. For starters, it can capture the entire temporal state of a biological activity. Second, the utility and genuine relevance of fractional derivatives have previously been determined in numerous disciplines of study based on that biological process. Researchers have demonstrated the utility of this derivative in modeling biological phenomena in a number of articles, taking into account a variety of factors such as the freedom to arrange the derivative,

dealing with species memory acquired during their cycle life, genetic traits, and so on. See [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] for further information.
First, we'll go over some definitions and results related to the Caputo fractional derivative [33, 34].

Definition 1. *For a continius function $h \in C^n$ and $T, \alpha \in R$, In the Caputo interpretation, the fractional order derivative with order α is given by*

$$D^\alpha(h(T)) = \frac{1}{\Gamma(n-\alpha)} \int_0^T \frac{h^n(\tau)}{(T-\tau)^{\alpha+1-n}} d\tau$$

where $n-1 < \alpha < n \in N : \Gamma$ is called the gamma function.

Definition 2. *For a function $h : R^n \to R$; The fractional integral of order α is calculated as follows:*

$$I^\alpha(h(T)) = \frac{1}{\Gamma(\alpha)} \int_0^T h(\tau)(T-\tau)^{\alpha-1} d\tau$$

Definition 3. *Given a dynamical system with the Caputo fractional operator,*

$$D^\alpha h(T) = u(T; h(T)); h(0) = h_0; \alpha \in (0,1);$$

The point h^* if satisfies $u(T;h) = 0$, then it is called the system's equilibrium point. when all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial u}{\partial h}$ evaluated at h^* satisfy $|arg(\lambda_j)| > \frac{\alpha\pi}{2}$, this equilibrium point is locally asymptotically stable.

3 The Fundamental Outcomes

Now we'll talk about the system's existence and uniqueness.

3.1 Solution availability

Theorem 1. *Assume that the initial state is non-negative, then system (4) always has unique solutions.*

Proof. The method utilized by Moustafa et al. [35] is applied. Take into account the area.

$$G = \left\{(x, y, w) \in R^3; \max |x|, |y|, |w| \leq K\right\}$$

We use the term $Q = (x, y, w)$ and $\tilde{Q} = (\tilde{x}, \tilde{y}, \tilde{w})$, Create a mapping

$$E(Q) = (E_1(Q), E_2(Q), E_3(Q))$$

$$\begin{cases} D^{\alpha}x(t) &= \frac{rx}{1+fy} - gx^2 - cF(x)y - x_0 w e^{-k_1 x} - d_1 x \\ D^{\alpha}y(t) &= F(x)y - y_0 w e^{-k_2 y} - d_2 y \\ D^{\alpha}w(t) &= \alpha_1 x + \alpha_2 y + \alpha_3 w - d_3 w. \end{cases} \tag{5}$$

where; $E_1(Q) = \frac{rx}{1+fy} - gx^2 - \frac{caxy}{b+x} - x_0 w e^{-k_1 x} - d_1 x$, $E_2(Q) = \frac{axy}{b+x} - y_0 w e^{-k_2 y} - d_2 y$; $E_3(Q) = \alpha_1 x + \alpha_2 y + \alpha_3 w - d_3 w$

for $Q, \tilde{Q} \in G$, depending on $E(Q)$ we obtain;

$$\begin{aligned} ||E(Q) - E(\tilde{Q})|| &= \Big| E_1(Q) - E_1(\tilde{Q})| + |E_2(Q) - E_2(\tilde{Q})| + |E_3(Q) - E_3(\tilde{Q})\Big| \\ &= \left| \frac{rx}{1+fy} - gx^2 - \frac{caxy}{b+x} - x_0 w e^{-k_1 x} - d_1 x - \frac{r\tilde{x}}{1+f\tilde{y}} + g\tilde{x}^2 + \frac{ca\tilde{x}\tilde{y}}{b+\tilde{x}} \right. \\ &\quad + x_0 \tilde{w} e^{-k_1 \tilde{x}} + d_1 \tilde{x} + \left| \frac{axy}{b+x} - y_0 w e^{-k_2 y} - d_2 y - \frac{a\tilde{x}\tilde{y}}{b+\tilde{x}} + y_0 \tilde{w} e^{-k_2 \tilde{y}} + d_2 \tilde{y} \right| \quad Moustafa \\ &\quad + |\alpha_1 x + \alpha_2 y + \alpha_3 w - d_3 w - \alpha_1 \tilde{x} - \alpha_2 \tilde{y} - \alpha_3 \tilde{w} + d_3 \tilde{w}| \end{aligned}$$

from the equation $E_1(Q), E_2(Q)$ and $E_3(Q)$ we have

$$\begin{aligned} \Big| E_1(Q) - E_1(\tilde{Q})| &\leq (rK + rfK + gK^2 + cabK + d_1 K)\,|x - \tilde{x}| + (rfK + cabK + caK^2)\,|y - \tilde{y}| \\ &\quad + (x_0 e^{k_1 K})\,|w - \tilde{w}| \\ \Big| E_2(Q) - E_2(\tilde{Q})| &\leq abK\,|x - \tilde{x}| + (abK + aK^2 + d_2 K)\,|y - \tilde{y}| + (x_0 e^{k_2 K})\,|w - \tilde{w}| \\ \Big| E_3(Q) - E_3(\tilde{Q})| &\leq \alpha_1 K\,|x - \tilde{x}| + \alpha_2 K\,|y - \tilde{y}| + (\alpha_3 K + d_3 K)\,|w - \tilde{w}| \end{aligned}$$

As a collect of all $\Big| E_1(Q) - E_1(\tilde{Q})|$, $\Big| E_2(Q) - E_2(\tilde{Q})|$ and $\Big| E_3(Q) - E_3(\tilde{Q})|$ we can obtain that;

$$\begin{aligned} ||E(Q) - E(\tilde{Q})|| &\leq (rK + rfK + gK^2 + cabK + d_1 K + abK + \alpha_1)\,|x - \tilde{x}| \\ &\quad + (rfK + cabK + caK^2 + abK + aK^2 + d_2 K + \alpha_2)\,|y - \tilde{y}| \\ &\quad + (x_0 e^{k_1 K} + y_0 e^{k_2 K} + \alpha_3 K + d_3 K)\,|w - \tilde{w}| \\ &\leq K_{max} |Q - \tilde{Q}| \end{aligned}$$

where; $K_{max} = \{K_x, K_y, K_w\}$ and $K_x = rK + rfK + gK^2 + cabK + d_1 K + abK + \alpha_1$, $K_y = rfK + cabK + caK^2 + abK + aK^2 + d_2 K + \alpha_2$ and $K_w = x_0 e^{k_1 K} + y_0 e^{k_2 K} + \alpha_3 K + d_3 K$
. It should be noted that the exponential function's expansion was utilized to simplify computations and achieve results. As a result, $E(Q)$ fulfills the Lipschitz requirement. As a result, the existence and uniqueness of the fractional order system (4) are established. $\square$

3.2 Boundedness

We must identify solutions that are non-negative and constrained in order to be biologically valid. The non-negativity and boundedness of system (4) solutions are established by the following result.

Theorem 2. *The sub region of* R_+^3

$$S = \left\{ (x, y, w) \in R_+^3 ; x + y \leq \frac{r - d_1}{g} ; w \leq \frac{r - d_1}{g}(\alpha_1 + \alpha_2) \right\}$$

is positively invariant region and attracting for system (4)

Proof. we Consider the two purpose functions for species population $V_1(T)$ and global warming population $F(t)$ as;

$$F(t) = x(t) + y(t), G(t) = w(t)$$

from the first function,

$$\begin{aligned} D^\alpha F(t) &= D^\alpha x(t) + D^\alpha y(t) \\ &\leq \frac{rx}{1 + fy} - gx^2 - d^* F \\ &\leq \frac{r(r - d_1)}{g} \end{aligned}$$

where $d^* = \max d_1, d_2$ and $\sup x(t) = \frac{r-d_1}{g}$ with $c - 1$ is positive. Hence;

$$F(t) \leq (F(t_0 - \frac{r - d_1}{g}))E_\alpha(-dt) + \frac{r - d_1}{g}$$

where E_α the Mittag-Leffler function is represented by Choi et al.([37]) provided Lemma 5 and Corollary 6 which can be used to derive

$$F(t) \leq \frac{r - d_1}{g} \quad as \quad t \to \infty$$

In the other hand, from $G(t) = w(t)$ we get

$$G(t) \leq \frac{r - d_1}{g}(\alpha_1 + \alpha_2) + (\alpha_3 - d_3)w$$

In natural case, $\alpha_3 < d_3$, So,

$$G(t) \leq (G(t_0 - \frac{r - d_1}{g}(\alpha_1 + \alpha_2)))E_\alpha(-dt) + \frac{r - d_1}{g}(\alpha_1 + \alpha_2)$$

where E_α the Mittag-Leffler function is represented by Choi et al.([37]) provided Lemma 5 and Corollary 6 which can be used to derive

$$G(t) \leq \frac{r - d_1}{g}(\alpha_1 + \alpha_2) \quad as \quad t \to \infty$$

□

3.3 Non-negativity

We are solely interested in nonnegative solutions because of their biological significance. The non-negativity of system 4 solutions are guaranteed by the following result. To begin, we show that the solutions $x(t)$ that begin with Ω are non-negative. that is, $x(t) \geq 0, \forall t \geq t_0$. If this isn't the case, then there's a constant $t_1 > t_0$ such that

$$\begin{cases} x(t) & > & 0, t_0 \leq t < t_1 \\ x(t_1) & = & 0, \\ x(t_1^+) & < & 0. \end{cases} \tag{6}$$

We can deduce 6 and the first equation of system 4.

$${}_{t_0}^{\alpha}D_{t_1}^{p}|_{x(t_1)=0} = 0$$

We have $x(t_1^+) = 0$ according to Lemma 1 in [38], which contradicts the fact that $x(t_1^+) < 0$. As a result, for all $t \geq t_0$, we have $x(t) \geq 0$. We have $y(t) \geq 0$ and $w(t) \geq 0$ for every $t \geq t_0$ using the same method as before. Hence, the following theorem summrized the non-negativity.

Theorem 3. *All of the system 4 solutions that begin with Ω are non-negative.*

3.4 Steady States Points

The derivative of a constant function is 0 in the Caputo meaning, as is well known. As a result, the equilibrium points for 4 in integer order version and fractional order version are the same. As a result, the system 4 equilibria can we find as .

$$D^{\alpha}x(t) = D^{\alpha}y(t) = D^{\alpha}w(t) = 0$$

Thus, the system 4 admit the following equilibria

- The free equilibrium point $c_0 = (0, 0, 0)$ which alwys exists.
- The prey equilibrium point $c_1 = (\breve{x}, 0, 0)$ which exists when $r > d_1$. where $\breve{x} = \frac{r-d_1}{g}$
- The prey-predator equilibrium point $c_2 = (\breve{x}, \breve{y}, 0)$

  where

$$\breve{x} = \frac{bd_2}{a - d_2}$$

which is positive if $a > d_2$ and $\check{y}$ is a root of the following equation;

$$a_1\check{y}^2 + a_2\check{y} + a_3 = 0 \tag{7}$$

where; $a_1 = caf, a_2 = gfb+gf\check{x}+ca+d_1bf+d_1f\check{x}$ and $a_3 = (g+d_1-r)\check{x}+(g+d_1-r)b$. this point is exists provided that $g + d_1 > r$.

- The prey-global warming equilibrium point $c_3 = (\hat{x}, 0, \hat{w})$ where

$$\hat{w} = \frac{\alpha_1}{d_3 - \alpha_3}\hat{x}$$

and $\hat{x}$ is the solution of the following equation

$$r - g\hat{x} - \frac{x_0\alpha_1}{d_3 - \alpha_3}\hat{x}e^{-k_1\hat{x}} - d_1 = 0 \tag{8}$$

which is exists with $d_3 > \alpha_3$

- The coexists equilibrium point $c^* = (X^*, Y^*, W^*)$ where

$$w^* = \frac{\alpha_1 x + \alpha_2 y}{d_3 - \alpha_3} \tag{9}$$

and while $(x*, y*)$ denotes the positive intersection of the following two isoclines:

$$f_1(x,y) = \frac{rx}{1+fy} - gx^2 - \frac{caxy}{b+x} - x_0\frac{\alpha_1 x + \alpha_2 y}{d_3 - \alpha_3}e^{-k_1 x} - d_1 x = 0 \tag{10a}$$

$$f_2(x,y) = \frac{axy}{b+x} - y_0\frac{\alpha_1 x + \alpha_2 y}{d_3 - \alpha_3}e^{-k_2 y} - d_2 y = 0 \tag{10b}$$

The above two isoclines are clearly become $y \to 0$ (and similarly if we chose $x \to 0$)

$$f_1(x,0) = r - gx - x_0\frac{\alpha_1}{d_3 - \alpha_3}e^{-k_1 x} - d_1 = 0 \tag{11a}$$

$$f_2(x,0) = 0 \tag{11b}$$

So, assuming that this equation has a positive root on the $x - axis$ marked by $\zeta > 0$, the above two isoclines have a unique intersection point in the interior of the positive quadrant of the x, y-plane given by (x^*, y^*) if the following adequate conditions are met.

$$f_1(\zeta, 0) = 0 \tag{12a}$$

$$f_2(x, 0) = 0 \tag{12b}$$

$$\frac{dy}{dx} = -\frac{\partial f_1(x,y)/\partial x}{\partial f_1(x,y)/\partial y} < 0 \tag{12c}$$

$$\frac{dy}{dx} = -\frac{\partial f_2(x,y)/\partial x}{\partial f_2(x,y)/\partial y} > 0 \tag{12d}$$

Clearly, these conditions are adequate to ensure the existence of a unique point c^* for the system 4.

3.5 The Local Stability of Steady State Points

Below is a description of the possible stable states points.

Theorem 4. *The steady State point c_0 is conditionally stable provided the conditions $d_1 > r$, $d_3 > \alpha_3$ and the discriminant $D_0(B_0)$ of a polynomial $B_1(\lambda)$ is positive.*

Proof. At c_0, the Jacobian matrix of system 4 is given by

$$J(c_0) = \begin{pmatrix} r - d_1 & 0 & -x_0 \\ 0 & -d_2 & -y_0 \\ \alpha_1 & \alpha_2 & \alpha_3 - d_3 \end{pmatrix}$$

has the follwoing charecterstic equation:

$$\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0$$

where; $q_1 = (d_1 - r) + (d_3 - \alpha_3) + d_2, q_2 = (r - d_1)(\alpha_3 - d_3) + d_2(d_3 - \alpha_3) + \alpha_2 y_0 - x_0\alpha_1$ and $q_3 = d_2[(r - d_1)(\alpha_3 - d_3) + (d_1 - r)] + \alpha_2 y_0(d_1 - r) + x_0\alpha_1 d_1$. So, acorrding to the terms of $q_1 > 0$,$q_3 > 0$ and $q_1 q_2 > q_3$ its clear that those are positive provided that $d_1 > r$ and $d_3 > \alpha_3$. Now, let,

$$B_0(\lambda) = \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3$$

and let,

$$D_0(B_0) = 18q_1q_2q_33 + (q_1q_2)^2 - 4q_3q_1^3 - 27q_3^2.$$

Hence, So, by the conditions of the Matignon's conditions [39] the condition $q_1q_2 > q_3$ is hold with $d_1 > r$ and $d_3 > \alpha_3$, $D_0(B_0)$ is positive. Therefore, $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}, i = 1, 2, 3.$ and, the point c_0 is conditionally stable. □

Theorem 5. *The steady state point c_1 is is conditionally stable provided that; $d_2 > \frac{a\breve{x}}{b+\breve{x}}$, $d_3 > \alpha_3$ and the discriminant $D_1(B_1)$ of a polynomial $B_1(\lambda)$ is positive.*

Proof. At c_1, the Jacobian matrix of system 4 is given by

$$J(c_1) = [a_{ij}] = \begin{pmatrix} -r + d_1 & -\breve{x}\left(rf + \frac{ca}{b+\breve{x}}\right) & -x_0 e^{-k_1\breve{x}} \\ 0 & \frac{a\breve{x}}{b+\breve{x}} - d_2 & -y_0 \\ \alpha_1 & \alpha_2 & \alpha_3 - d_3 \end{pmatrix}$$

which is has the characteristic equation;

$$(a_{11} - \lambda)\left[\lambda^2 - (a_{22} + a_{33})\lambda + a_{11}a_{33} - a_{23}a_{32}\right] + a_{31}\left[a_{12}a_{23} - a_{13}(a_{22} - \lambda)\right] = 0$$

and by forward computations, we get

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + \lambda = 0$$

where; $p_1 = -[a_{11} + a_{22} + a_{33}], p_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{13}a_{31}$ and $p_3 = -a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}a_{12}a_{23}a_{31} - a13a_{22}a_{31}$.the signs of entries as following a_{12}, a_{13} and a_{23} are negative. on other hand, a_{31} and a_{32} are positive.The reader can conclude that the entries a_{11}, a_{22} and a_{33} provided that $d_2 > \frac{a\breve{x}}{b+\breve{x}}$ and $d_3 > \alpha_3$. So, p_1 and p_3 are positive signs and $p_1p_2 > p_3$.

Now, assume that

$$B_1(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$$

and let,

$$D_1(B_1) = 18p_1p_2p_33 + (p_1p_2)^2 - 4p_3p_1^3 - 27p_3^2.$$

be the discriminant of a polynomial $B_1(\lambda)$.

Therefore, according to the routh-howartz creteria and the conditions of the Matignon's conditions [39] the roots of charecterstic equation has negative real parts with $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$ provided that $d_2 > \frac{a\breve{x}}{b+\breve{x}}$, $d_3 > \alpha_3$ and $D_1(B_1) > 0$. So, by the conditions of the Matignon's conditions [39], the point c_1 is conditionally stable. □

Theorem 6. *The steady state point c_2 is conditionally stable provided that $2g\breve{x} > r, d_2 > \frac{a\breve{x}}{b+\breve{x}}, d_3 > \alpha_3$ and $D_2(A) > 0$. where $D_2(A)$ be the discriminant of a polynomial $A(\lambda)$.*

Proof. At c_2, the Jacobian matrix of system 4 is given by

$$J(c_2) = [J_{ij}]_{i,j=1,2,3} =$$
$$\begin{pmatrix} \frac{r}{1+f\breve{y}} - 2g\breve{x} - \frac{cab\breve{y}}{(b+\breve{x})^2} - d_1 & -\breve{x}[\frac{rf}{(1+f\breve{y})} + \frac{ca}{b+\breve{x}}] & -x_0e^{-k_1\breve{x}} \\ \frac{ab\breve{y}}{(b+\breve{x})^2} & \frac{a\breve{x}}{b+\breve{X}} - d_2 & -y_0e^{-k_2\breve{y}} \\ \alpha_1 & \alpha_2 & \alpha_3 - d_3 \end{pmatrix}$$

which is has the characteristic equation;

$$\lambda^3 + \breve{A}_1\lambda^2 + \breve{A}_2\lambda + \breve{A}_3 = 0$$

where;
$\breve{A}_1 = -(j_{11} + j_{22} + j_{33})$
$\breve{A}_2 = -j_{11}(j_{22} + j_{33}) + j_{11}j_{33} - j_{23}j_{32} - j_{12}j_{21} - j_{13}j_{31}$
$\breve{A}_3 = -j_{11}[j_{22}j_{33} - j_{23}j_{32}] + j_{12}[j_{21}j_{33} - j_{23}j_{31}] - j_{13}[j_{21}j_{32} - j_{31}j_{22}]$

When we examine the elements of Jacobian matrix $J(c_2)$, we can see that the signs of elements j_{12}, j_{13} and j_{23} are strictly negative, while the signs of elements j_{21}, j_{31} and j_{32} are strictly positive. on other hand the signs of elements j_{11}, j_{22} and j_{33} are have negative signs provided that $2g\check{x} > r, d_2 > \frac{a\check{x}}{b+\check{x}}, andd_3 > \alpha_3$ respectively. The reader, can easily say that the same conditions gives that all the terms $\check{A}_1, \check{A}_3$ and $\check{A}_1\check{A}_2 > \check{A}_3$ are positive signs. Therefore, the conditions of the Routh-Hurwitz criterion are holds.now, let

$$A(\lambda) = \lambda^3 + \check{A}_1\lambda^2 + \check{A}_2\lambda + \check{A}_3$$

and let,

$$D_2(A) = 18\check{A}_1\check{A}_2\check{A}_3 + (\check{A}_1\check{A}_2)^2 - 4\check{A}_3\check{A}_1^{\ 3} - 27\check{A}_3^{\ 2}.$$

be the discriminant of a polynomial $A(\lambda)$.

Therefore, according to the routh-howartz creteria and the conditions of the Matignon's conditions [39] the roots of charecterstic equation has negative real parts with $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$ provided that $2g\check{x} > r, d_2 > \frac{a\check{x}}{b+\check{x}}, d_3 > \alpha_3$ and $D_2(A) > 0$. So, by the conditions of the Matignon's conditions [39], the point c_2 is conditionally stable.

□

Theorem 7. *The steady state point c_3 is conditionally stable provided that $2g\hat{x} + d_1 > r + x_0\hat{w}k_1r^{-k_1\hat{x}}$, $d_2 > \frac{a\hat{x}}{b+\hat{x}} + y_0k_2\hat{w}$ and $D_3(B) > 0$ where $D_3(B)$ be the discriminant of a polynomial $B(\lambda)$.*

Proof. At c_3, the Jacobian matrix of system 4 is given by

$$J(c_3) = [a_{ij}] \begin{pmatrix} r - 2g\hat{x} + x_0\hat{w}k_1e^{-k_1\hat{x}} & -\hat{x}[\frac{rf}{(1+f\hat{y})^2} + \frac{ca}{b+\hat{x}}] & -x_0e^{-k_1\hat{x}} \\ 0 & \frac{a\hat{x}}{b+\hat{x}} + y_0k_2\hat{w} - d_2 & -y_0 \\ \alpha_1 & \alpha_2 & \alpha_3 - d_3 \end{pmatrix}$$

which is has the characteristic equation;

$$\lambda^3 + \hat{A}_1\lambda^2 + \hat{A}_2\lambda + \hat{A}_3 = 0$$

where; $\hat{A}_1 = -[a_{11} + a_{22} + a_{33}]; \hat{A}_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{23}a_{32} - a_{31}a_{13}$ and $\hat{A}_3 = -[a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}]$.Hence,if $2g\hat{x} + d_1 > r + x_0\hat{w}k_1r^{-k_1\hat{x}}$ and $d_2 > \frac{a\hat{x}}{b+\hat{x}} + y_0k_2\hat{w}$ then the conditions of routh-howartize creteria are holds.now, let

$$B(\lambda) = \lambda^3 + \hat{A}_1\lambda^2 + \hat{A}_2\lambda + \hat{A}_3$$

and let,

$$D_3(B) = 18\hat{A}_1\hat{A}_2\hat{A}_3 + (\hat{A}_1\hat{A}_2)^2 - 4\hat{A}_3\hat{A}_1^{\ 3} - 27\hat{A}_3^{\ 2}.$$

be the discriminant of a polynomial $B(\lambda)$.

Therefore, according to the routh-howartz creteria and the conditions of the Matignon's conditions [39] the roots of charecterstic equation has negative real parts with $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$ provided that $2g\hat{x} + d_1 > r + x_0\hat{w}k_1r^{-k_1\hat{x}}$ and $d_2 > \frac{a\hat{x}}{b+\hat{x}} + y_0k_2\hat{w}$. So, by the conditions of the Matignon's conditions [39], the point c_3 is conditionally stable.

□

Theorem 8. *The steady state point c^* is conditionally stable provided that $A_1^*.A_2^* > A_3^*$ and the condition $D_3(B) > 0$.*

Proof. At c_3, the Jacobian matrix of system 4 is given by

$$J(c_3) = [c_{ij}] \begin{pmatrix} \frac{r}{1+fy^*} - 2gx^* - \frac{caby^*}{(b+x^*)^2} + x_0w^*k_1e^{-k_1x^*} - d_1 & -x^*[\frac{rf}{(1+fy^*)^2} + \frac{ca}{b+x^*}] & -x_0e^{-k_1x^*} \\ \frac{aby^*}{(b+x^*)^*} & \frac{ax^*}{b+x^*} + y_0k_2w^* - d_2 & -y_0e^{-k_2y^*} \\ \alpha_1 & \alpha_2 & \alpha_3 - d_3 \end{pmatrix}$$

which is has the characteristic equation;

$$\lambda^3 + A_1^*\lambda^2 + A_2^*\lambda + A_3^* = 0$$

where; $A_1^* = -[c_{11} + c_{22} + c_{33}]$; $A_2^* = c_{11}c_{22} + c_{11}c_{33} + c_{22}c_{33} + c_{23}c_{32} + c_{21}c_{12} - c_{31}c_{13}$ and $A_3^* = -[c_{11}c_{22}c_{33} - c_{11}c_{23}c_{32} + c_{21}c_{12}c_{33} - c_{21}c_{13}c_{32} + c_{31}c_{12}c_{23} - c_{31}c_{13}c_{22}]$.Hence,if $2gx^* + d_1 > r + x_0w^*k_1e^{-k_1x^*}$ and $d_2 > \frac{ax^*}{b+x^*} + y_0k_2w^*$ then the conditions of routh-howartize creteria $A_1^* > 0, A_3^* > 0$ and $A_1^*.A_2^* > A_3^*$ are holds.now, let

$$B(\lambda) = \lambda^3 + A_1^*\lambda^2 + A_2^*\lambda + A_3^*$$

and let,

$$D_3(B) = 18\hat{A}_1\hat{A}_2\hat{A}_3 + (\hat{A}_1\hat{A}_2)^2 - 4\hat{A}_3\hat{A}_1{}^3 - 27\hat{A}_3{}^2.$$

be the discriminant of a polynomial $B(\lambda)$.

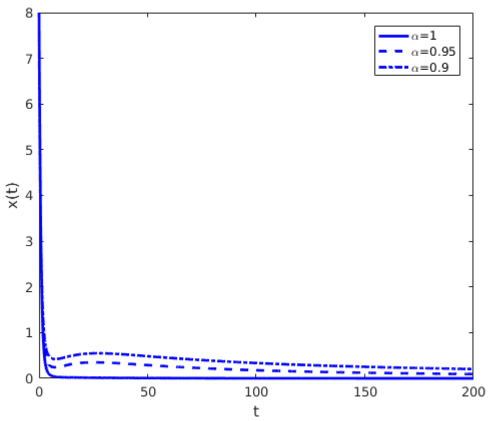
Therefore, the conditions of routh-howartz creteria $A_1^*.A_2^* > A_3^*$ and the condition $D_3(B) > 0$ gives the roots of charecterstic equation has negative real parts with $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$ provided that $2gx^* + d_1 > r + x_0w^*k_1e^{-k_1x^*}$ and $d_2 > \frac{ax^*}{b+x^*} + y_0k_2w^*$. So, by the conditions of the Matignon's conditions [39], the point c^* is conditionally stable.

□

4 Numerical simulation

In this part, we give some numerical simulations to support the obtained results in the theoritical part. We take in all presented figures $\alpha = 1.0$, 0.95 and 0.9,to show the effect of the fractional-order derivatives on the evolution of the considered model.

Let us make some remarks on the obtained figures. For the parameter values $r = 10$, $f = 0$, $g = 0.06$, $c = 1$, $a = 2$, $b = 10$, $x_0 = 0.0001$, $k_1 = 0.02$, $d_1 = 0.33$, $y_0 = 0.3$, $k_2 = 0.1$, $d_2 = 0.3$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\alpha_3 = 0.02$ and $d_3 = 0.4$ and initial values $(x(0), y(0), w(0)) = (8, 5, 4)$, it is clear in the Fig. 1 and Fig. 2 that the free equilibrium c_0 is locally asymptotically stable. In fact, the used values verify the existence condition of this equilibrium ($d_1 > r$ and $d_3 > \alpha_3$). From the Fig.3 and Fig.4, we observe that the prey equilibrium point c_1 is locally asymptotically stable for the same parameter values given in Fig.1 except that $r = 0.4$, $a = 0.45$ and $d_2 = 0.5$. One can see that $d_2 > a$ and $r > d_1$. It is shown in Fig. 5 and Fig.6 that the prey-global warming equilibrium point c_3 is locally asymptotically stable for the same parameter values given in Fig. 1 except that $r = 0.5$, $a = 0.45$ and $d_2 = 0.6$. Forthermore, the stability behavior of the coexists equilibrium point c^* for the same parameter values given in Fig. 1 except that $r = 1.93$, $a = 1$ and $d_2 = 0.5$ is shown in the Fig.7 and Fig.8. Finally, from the Fig.9 and Fig.10, we notice that when we increase the value of fear parameter f (keeping other parameters unchanged), the system dynamics changes, which explain the fear effect on the system dynamics.



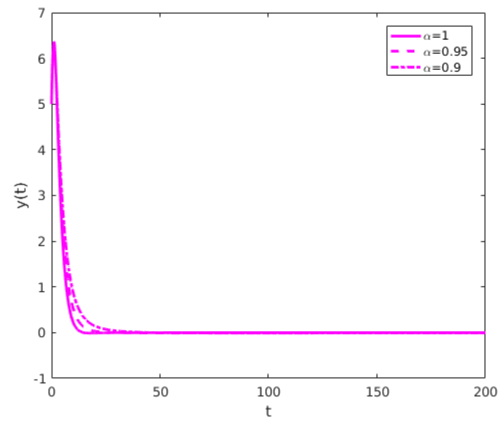


Figure 1: Stability behavior of the free equilibrium point $c_0 = (0, 0, 0)$ for the parameter values $r = 10$, $f = 0$, $g = 0.06$, $c = 1$, $a = 2$, $b = 10$, $x_0 = 0.0001$, $k_1 = 0.02$, $d_1 = 0.33$, $y_0 = 0.3$, $k_2 = 0.1$, $d_2 = 0.3$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\alpha_3 = 0.02$ and $d_3 = 0.4$ and initial values $(x(0), y(0), w(0)) = (8, 5, 6)$.

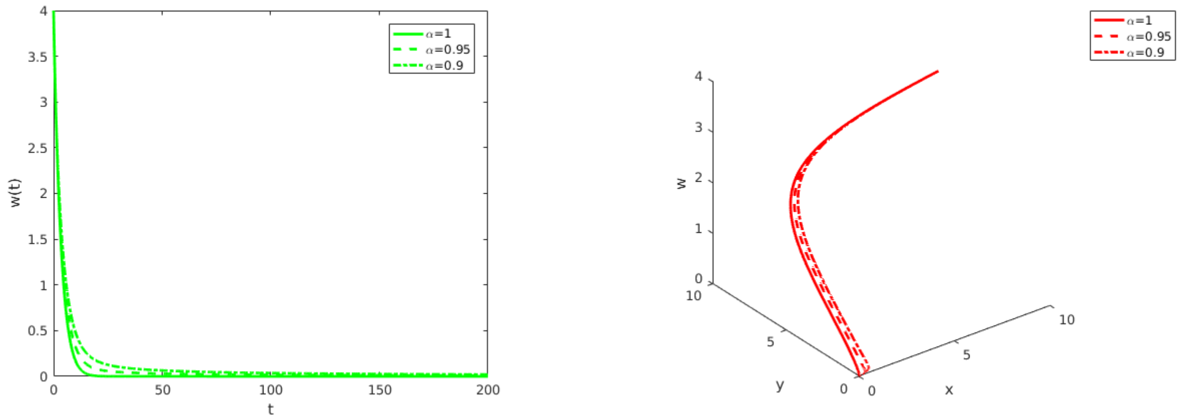


Figure 2: Dynamics of w with respect to time t and phase portrait of the system 4 for the same parameter values given in Fig. 1.

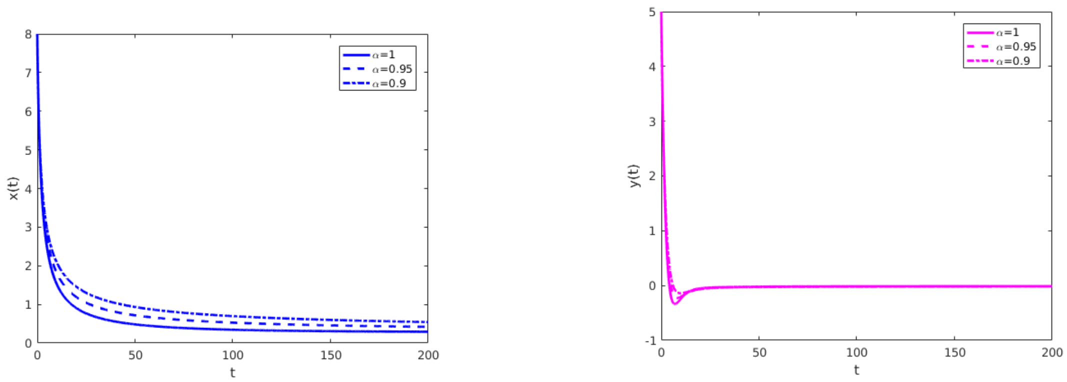


Figure 3: Stability behavior of the prey equilibrium point c_1 for the same parameter values given in Fig. 1 except that $r = 0.4$, $a = 0.45$ and $d_2 = 0.5$ and initial values $(x(0), y(0), w(0)) = (8, 5, 6)$. One can see that $d_2 > a$ and $r > d_1$.

References

[1] Settele, J., Scholes, R., Betts, R. A., Bunn, S., Leadley, P., Nepstad, D., ... & Winter, M. (2015). Terrestrial and inland water systems. In Climate change 2014 impacts, adaptation and vulnerability: Part A: Global and sectoral aspects (pp. 271-360). Cambridge University Press.

[2] Pryor, S. C., Scavia, D., Downer, C., Gaden, M., Iverson, L., Nordstrom, R., ... & Robertson, G. P. (2014). Midwest. Climate change impacts in the United States: The third national climate assessment. In: Melillo, JM; Richmond, TC; Yohe, GW, eds. National Climate Assessment Report. Washington, DC: US Global Change Research

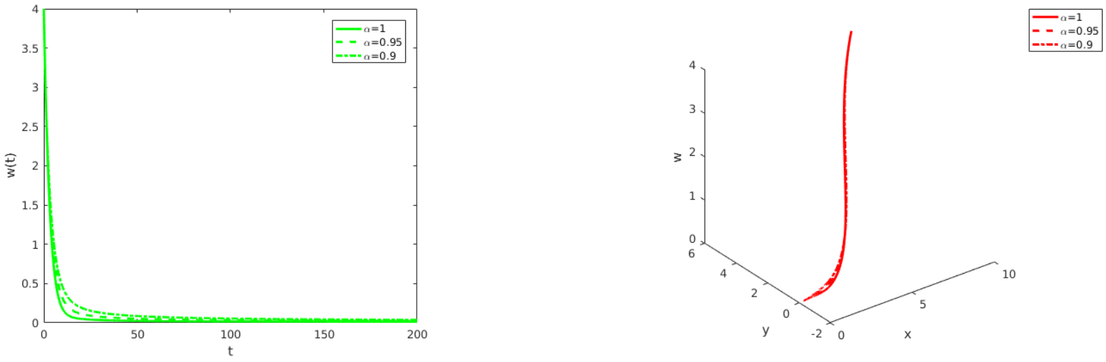


Figure 4: Evolution of w with respect to time t and phase portrait of the system 4 for the same parameter values given in Fig. 3.

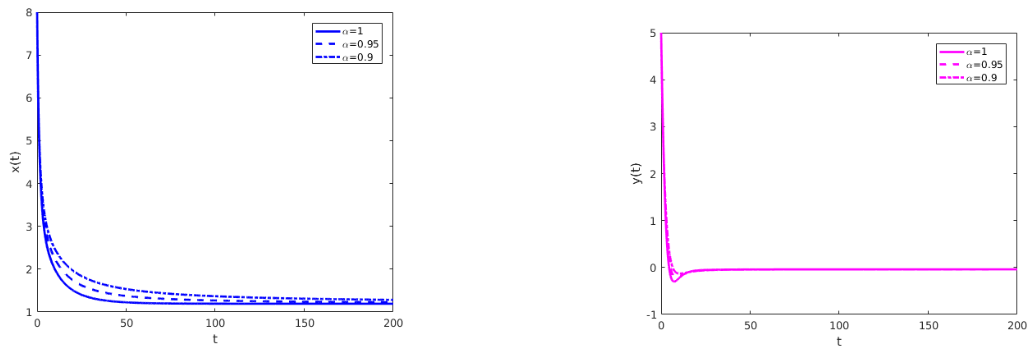


Figure 5: Stability behavior of the prey-global warming equilibrium point c_3 for the same parameter values given in Fig. 1 except that $r = 0.5$, $a = 0.45$ and $d_2 = 0.6$.

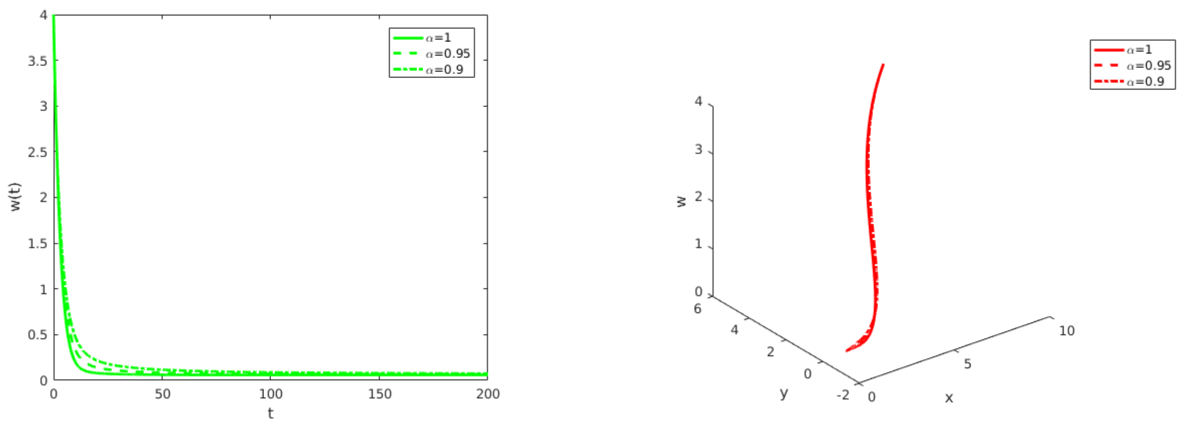


Figure 6: Dynamics of w and phase portrait of the system 4 for the same parameter values given in Fig. 5.

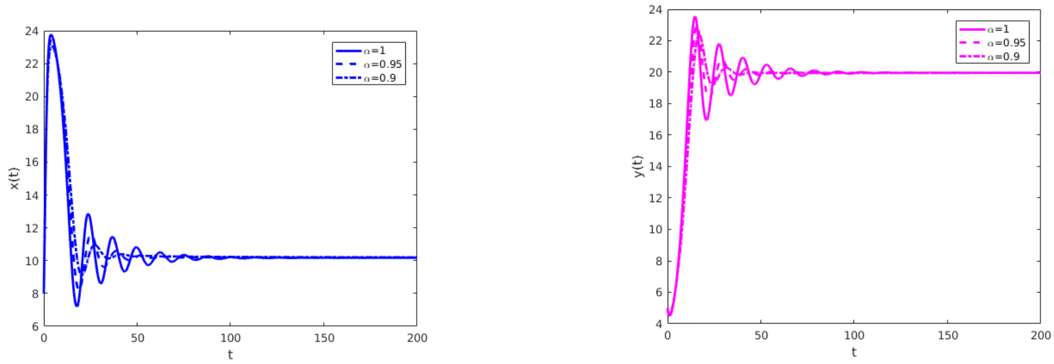


Figure 7: Stability behavior of the coexists equilibrium point c^* for the same parameter values given in Fig. 1 except that $r = 1.93$, $a = 1$ and $d_2 = 0.5$.

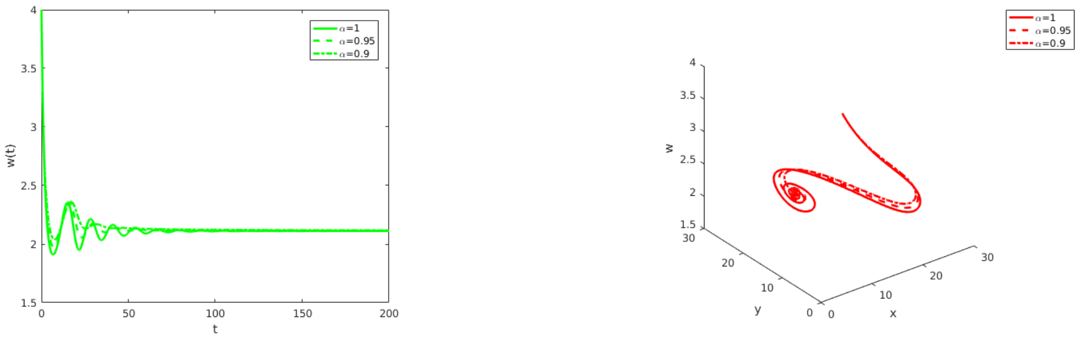


Figure 8: Evolution of w and phase portrait of the system 4 for the same parameter values given in Fig. 7.

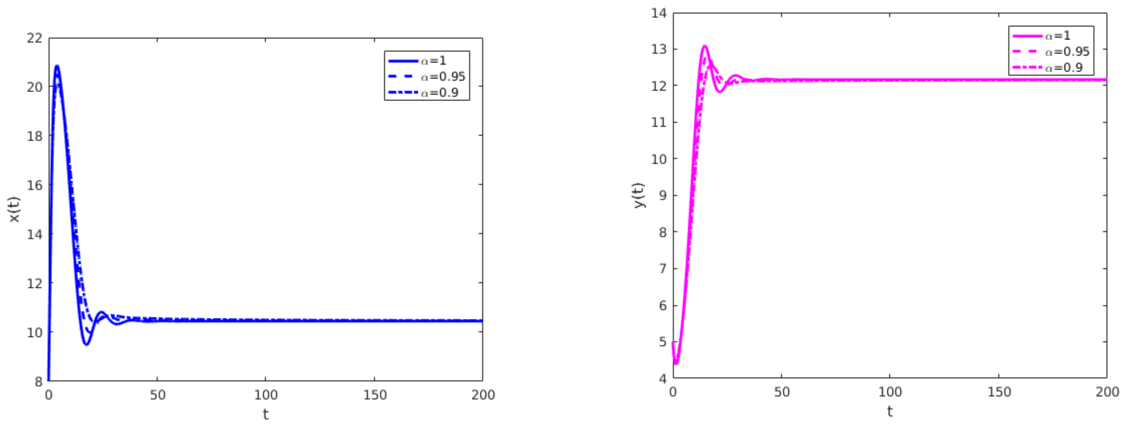


Figure 9: Fear effect on x and y dynamics for the same parameter values given in Fig. 7 except that $f = 0.02$.

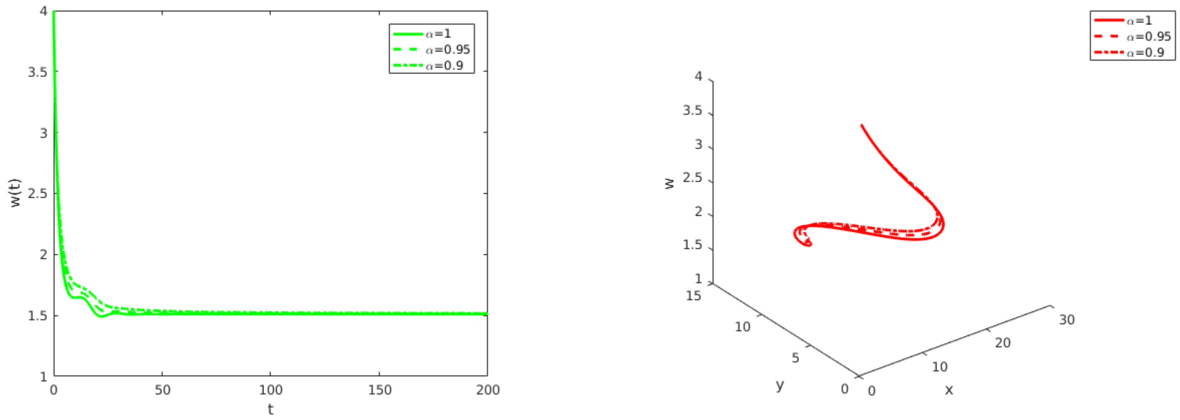


Figure 10: fear effect on w dynamics and phase portrait of the system 4 for the same parameter values given in Fig. 1.

Program: 418-440., 418-440.

[3] Adam, J. C., Hamlet, A. F., & Lettenmaier, D. P. (2009). Implications of global climate change for snowmelt hydrology in the twenty-first century. Hydrological Processes: An International Journal, 23(7), 962-972.

[4] Melillo, J. M., Richmond, T. T., & Yohe, G. (2014). Climate change impacts in the United States. Third national climate assessment, 52.

[5] Karl, T. R., Melillo, J. M., Peterson, T. C., & Hassol, S. J. (Eds.). (2009). Global climate change impacts in the United States. Cambridge University Press.

[6] Pryor, S. C., Scavia, D., Downer, C., Gaden, M., Iverson, L., Nordstrom, R., ... & Robertson, G. P. (2014). Midwest. Climate change impacts in the United States: The third national climate assessment. In: Melillo, JM; Richmond, TC; Yohe, GW, eds. National Climate Assessment Report. Washington, DC: US Global Change Research Program: 418-440., 418-440.

[7] National Research Council, Committee on Climate Change, US Transportation, Transportation Research Board, Division on Earth, & Life Studies. (2008). Potential impacts of climate change on US transportation: Special report 290 (Vol. 290). Transportation Research Board.

[8] Leong, J., Marra, J., Finucane, M., Giambelluca, T., Merrifield, M., Miller, S. E., ... & Wang, B. (2014). Hawai 'i and US Affiliated Pacific Islands. Climate Change Impacts in the United States: The Third National Climate Assessment. Climate Change Impacts in the United States: The Third National Climate Assessment, 537-556.

[9] Saha, S., & Samanta, G. P. (2019). Analysis of a predator–prey model with herd behavior and disease in prey incorporating prey refuge. International Journal of Biomathematics, 12(01), 1950007.

[10] Mondal, S., & Samanta, G. P. (2019). Dynamics of an additional food provided predator–prey system with prey refuge dependent on both species and constant harvest in predator. Physica A: Statistical Mechanics and its Applications, 534, 122301.

[11] Baker, C. M., Diele, F., Lacitignola, D., Marangi, C., & Martiradonna, A. (2019). Optimal control of invasive species through a dynamical systems approach. Nonlinear Analysis: Real World Applications, 49, 45-70.

[12] Gilpin, M. E. (1972). Enriched predator-prey systems: theoretical stability. Science, 177(4052), 902-904.

[13] Kuznetsov, Y. A., & Rinaldi, S. (1996). Remarks on food chain dynamics. Mathematical biosciences, 134(1), 1-33.

[14] Kot, M. (2001). Elements of mathematical ecology. Cambridge University Press.

[15] Murray, J. D. (1993). Models for interacting populations. In Mathematical Biology (pp. 79-118). Springer, New York, NY.

[16] Alves, M. T., & Hilker, F. M. (2017). Hunting cooperation and Allee effects in predators. Journal of theoretical biology, 419, 13-22.

[17] Mondal, S., Maiti, A., & Samanta, G. P. (2018). Effects of fear and additional food in a delayed predator–prey model. Biophysical Reviews and Letters, 13(04), 157-177.

[18] Thirthar, A. A., Majeed, S. J., Alqudah, M. A., Panja, P., & Abdeljawad, T. (2022). Fear effect in a predator-prey model with additional food, prey refuge and harvesting on super predator. Chaos, Solitons & Fractals, 159, 112091.

[19] Zhang, H., Cai, Y., Fu, S., & Wang, W. (2019). Impact of the fear effect in a prey-predator model incorporating a prey refuge. Applied Mathematics and Computation, 356, 328-337.

[20] Mukherjee, D. (2020). Study of fear mechanism in predator-prey system in the presence of competitor for the prey. Ecological Genetics and Genomics, 15, 100052.

[21] Barman, D., Roy, J., Alrabaiah, H., Panja, P., Mondal, S. P., & Alam, S. (2021). Impact of predator incited fear and prey refuge in a fractional order prey predator model. Chaos, Solitons & Fractals, 142, 110420.

[22] Yousef, A., Thirthar, A. A., Alaoui, A. L., Panja, P., & Abdeljawad, T. (2022). The hunting cooperation of a predator under two prey's competition and fear-effect in the prey-predator fractional-order model. AIMS Mathematics, 7(4), 5463-5479.

[23] Yousef, A., & Yousef, F. B. (2019). Bifurcation and stability analysis of a system of fractional-order differential equations for a plant–herbivore model with Allee effect. Mathematics, 7(5), 454.

[24] Öztürk, I., & Özköse, F. (2020). Stability analysis of fractional order mathematical model of tumor-immune system interaction. Chaos, Solitons & Fractals, 133, 109614.

[25] Li, H. L., Zhang, L., Hu, C., Jiang, Y. L., & Teng, Z. (2017). Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing, 54(1), 435-449.

[26] Choi, S. K., Kang, B., & Koo, N. (2014, January). Stability for Caputo fractional differential systems. In Abstract and Applied Analysis (Vol. 2014). Hindawi.

[27] Wei, Z., Li, Q., & Che, J. (2010). Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative. Journal of Mathematical Analysis and Applications, 367(1), 260-272.

[28] Odibat, Z. M., & Momani, S. (2008). An algorithm for the numerical solution of differential equations of fractional order. Journal of Applied Mathematics & Informatics, 26(1_2), 15-27.

[29] Ghanbari, B., Kumar, S., & Kumar, R. (2020). A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. Chaos, Solitons & Fractals, 133, 109619.

[30] Goufo, E. F. D., Kumar, S., & Mugisha, S. B. (2020). Similarities in a fifth-order evolution equation with and with no singular kernel. Chaos, Solitons & Fractals, 130, 109467.

[31] Kumar, S., Kumar, R., Agarwal, R. P., & Samet, B. (2020). A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods. Mathematical Methods in the Applied Sciences, 43(8), 5564-5578.

[32] S Kumar, R Kumar, C Cattani, B Samet, Chaotic behaviour of fractional predator-prey dynamical system, Chaos, Solitons & Fractals 135, 109811 (2020).

[33] Podlubny, I. (1999) Fractional Differential Equations, Academic Press, California, USA.

[34] Kilbas, A. A., Marichev, O. I., & Samko, S. G. (1993). Fractional integrals and derivatives (theory and applications).

[35] Moustafa, M., Mohd, M. H., Ismail, A. I., & Abdullah, F. A. (2018). Dynamical analysis of a fractional-order Rosenzweig–MacArthur model incorporating a prey refuge. Chaos, Solitons & Fractals, 109, 1-13.

[36] Zhou, X., Shi, X., & Cui, J. (2017). Stability and backward bifurcation on a cholera epidemic model with saturated recovery rate. Mathematical Methods in the Applied Sciences, 40(4), 1288-1306.

[37] Choi, S. K., Kang, B., & Koo, N. (2014, January). Stability for Caputo fractional differential systems. In Abstract and Applied Analysis (Vol. 2014). Hindawi.

[38] Hong-L., Long Z., Cheng H., Yao-L.J. and Zhidong T., Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, J. Appl. Math. Comput, DOI 10.1007/s12190-016-1017-8; 26 may 2016.

[39] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). elsevier.