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RESEARCH ARTICLE

Locally Artinian Supplemented Modules

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In this paper, we introduce notions of RLA-local modules and locally artinian supplemented modules which are proper generalizations as notions of strongly local modules and ss-supplemented modules, respectively and we study some properties of these module. In particular, we give a characterization of semi perfect rings and left perfect rings.

KEYWORDS:
locally artinian modules, supplement submodule, locally artinian suplemented modules

1 | INTRODUCTION

A submodule N of M will show that $N \subseteq M$. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M and is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow. M is called *locally artinian* if every finitely generated submodule of M is artinian[1, 31]. A submodule K of M is called a *supplement* of N in M if $M = N + K$ and $N \cap K \ll K$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule K of M has *ample supplements* in M if every submodule T of M such that $M = K + T$ contains a supplement of K in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M[1]. In[2], Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M in place of the class of all simple submodules of M, that is, $Soc_s(M) = \sum\{ N \ll M \mid N \text{ is simple } \}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

In this paper, we study notions of RLA-local and locally artinian supplemented modules thank to following notions:

In[3], a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule K of M is called a *ss-supplement* of N in M if $M = N + K$, $N \cap K \subseteq Soc_s(K)$. The module M is called a *ss-supplemented* if every submodule of M has a ss-supplement in M. A submodule K of M has *ample ss-supplements* in M if every submodule T of M such that $M = K + T$ contains a ss-supplement of K in M. The module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. This class of modules was first studied by Kaynar et.al. in[3].

By examining the ss-supplemented modules previously defined in this study we defined and exemplified the concept of RLA-local supplemented modules, which is a more general concept than ss-supplemented modules, and gave its basic properties.

The goal of this paper is to show that, examples were given by defining RLA-local and locally artinian supplemented modules, and locally artinian supplemented modules were characterized on left artinian rings by giving some basic properties of locally artinian supplemeneted modules.

Throughout this paper, R will always denote an associative ring with identity element and modules will be left unital. $Rad(R)$ will denote the Jacobson radical of the ring R. We will use the notation $N \ll M$ to stress that N is small submodule of M. We refer to[4],[3] and[1] for any undefined notion arising in the tex.

2 | RESULTS

In this section, we investigate some properties of RLA-local modules and locally artinian supplemented modules. We mainly study the relation between the notion of these modules and some other notions. In particular, we give a characterization of semi perfect rings and left perfect rings

Definition 1. We call a local module M *RLA-local module* if $Rad(M)$ is a locally artinian submodule of M. If a ring R is the RLA-local module as the left R-module, then we call R a *RLA-local ring*.

Since semisimple modules are locally artinian, we have the following implications hold on modules:

$$\text{strongly local} \Rightarrow \text{RLA-local} \Rightarrow \text{local}$$

The following example shows that the above inclusions are proper. Note that every local artinian module is a RLA-module.

Example 1. (1) Consider finitely generated $\mathbb{Z}$-module $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is local artinian, it is a RLA-local module. On the other hand, by[3, Example 18], $\mathbb{Z}_8$ is not a strongly local module.

(2) Given the Dedekind domain $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \ and \ p \nmid b\}$, where $p \in \mathbb{Z}$ is a prime integer. Therefore, the ring $\mathbb{Z}_{(p)}$ is local which is not RLA-local.

Proposition 1. If M is a RLA-local module, then every factor module of M is RLA-local.

Proof. Assume $N \subset M$. It is clear that $\frac{M}{N}$ is local as a homomorphic image of the local module M. Since local modules are good, it follows from[1, 23.3 (a)] that $Rad(\frac{M}{N}) = \pi(Rad(M))$, where $\pi : M \longrightarrow \frac{M}{N}$ is the canonical homomorphism. Therefore, $Rad(\frac{M}{N}) = \frac{Rad(M)}{N}$ is locally artinian by[1, 31.2 (1) (i)]. Hence $\frac{M}{N}$ is a RLA-local module. □

Definition 2. Let M be a module. M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement V in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule L of M such that $M = U + L$ contains a locally artinian supplement L' of U in M.

We begin by giving some counterexamples seperating locally artinian supplemented modules, ss-supplemented modules, artinian modules and supplemented modules. Note that artinian modules are supplemented, and over a left artinian ring every left module is locally artinian.

Example 2. For a prime integer $p \in \mathbb{Z}$, put $M =_{\mathbb{Z}} \mathbb{Z}_{p^{\infty}}$. Then M is artinian and so it is locally artinian supplemented. However, M is not ss-supplemented according to[3, Example 17].

Example 3. Let R be a left artinian ring and M be the left R-module $R^{(\mathbb{N})}$. Then M is a locally artinian supplemented module which is not artinian.

Example 4. Let R be a local Dedekind domain with quotient field K. Therefore $_RK$ is a hollow module and so it is supplemented. Since $Soc(_RK) = 0$, $_RK$ has no semi artinian submodules. It means that $_RK$ is not locally artinian supplemented.

Under given Examples, we clearly have the following implication on modules:

$$\begin{tikzcd}
ss-supplemented \arrow[dr] & & artinian \arrow[dl] \\
& locally\ artinian\ supplemented \arrow[d] & \\
& supplemented &
\end{tikzcd}$$

Lemma 1. Let M be a supplemented module and $Rad(M)$ be a locally artinian submodule of M. Then M is locally artinian supplemented.

Proof. Let K be an arbitrary submodule of M. Since M is supplemented, we can write $M = K + L$ and $K \cap L \ll L$ for some submodule $K \subseteq M$. Then $K \cap L \subseteq Rad(M)$ because $K \cap L \ll M$. By the hypothesis and[1, 31.2 (1)(i)], we obtain that $K \cap L$ is a locally artinian submodule of M. Therefore M is locally artinian supplemented. □

Theorem 5. Let M be a module with $Rad(M) \ll M$. Then the following statements are equivalent:

(1) M is locally artinian supplemented;

(2) M is supplemented and $Rad(M)$ has a locally artinian supplement in M;

(3) M is supplemented and $Rad(M)$ is locally artinian.

Proof. (1) ⇒ (2) Since M is locally artinian supplemented, M is supplemented and it is obvious that $Rad(M)$ has a locally artinian supplement in M.

(2) ⇒ (3) Since $Rad(M) \ll M$, M is a locally artinian supplement of $Rad(M)$ in M. So, $M = Rad(M) + M$, $Rad(M) = Rad(M) \cap M \ll M$ and $Rad(M)$ is a locally artinian module.

(3) ⇒ (1) Clear from Lemma 1. □

Let $f : P \longrightarrow M$ be an epimorphism. f is called a *cover* if $Ker(f) \ll P$, and a cover f is called a *projective cover* if P is a projective module. A ring R is called *(semi) perfect* if every (finitely generated) left R-module has a projective cover ([1]). It is known in[1, 42.6] that R is semi perfect if and only if ${}_RR$ is supplemented. Using this fact along with the above Theorem we obtain the following:

Corollary 1. Let R be a ring. Then ${}_RR$ is locally artinian supplemented if and only if it is a semi perfect ring and $Rad(R)$ is locally artinian.

Theorem 6. Every RLA-local module is amply locally artinian supplemented.

Proof. Let $M = U + V$. Since M is local, it is amply supplemented and so there exists a submodule V' of V such that $M = U + V'$ and $U \cap V' \ll V'$. Therefore $U \cap V' \subseteq Rad(V') \subseteq Rad(M)$. It follows the hypothesis that $U \cap V'$ is locally artinian. Hence M is amply locally artinian supplemented □

Recall from[1, 31.2 (ii)] that every submodule of a locally artinian module is locally artinian.

Proposition 2. Let M be a module and U be a maximal submodule of M. A submodule V of M is a locally artinian supplement of U if and only if $M = U + V$ and V is a RLA-local module.

Proof. (⇒) Let V be a locally artinian supplement of U in M. So we can write $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is locally artinian. Since U is a maximal submodule of M and V is supplement of U, V is local module by[1, 3, 41.1. (3)]. It follows that $Rad(V) = U \cap V$. So V is a RLA-local module.

(⇐) Since V is a RLA-local module, V is local and $Rad(V)$ is a locally artinian module. Since V is local and U is maximal in M, $U \cap V \subseteq Rad(V)$. It means that $U \cap V$ is locally artinian and $U \cap V \ll V$. Therefore $M = U + V$, $U \cap V \ll V$ and $Rad(V) = U \cap V$ is a locally artinian module, as required. □

To prove that the finite sum of locally artinian supplemented modules is locally artinian supplemented, we use the following standard lemma (see,[1, 41.2]).

Lemma 2. Let M be a module and U, V be submodules of M with U locally artinian supplemented. If $U + V$ has a locally artinian supplement in M, U also has a locally artinian supplement in M.

Proof. Let M be locally artinian supplement of $U + V$ in M and L be locally-artinian supplement of $(K + V) \cap U \subseteq U$. Then $M = U+V+K$, $(U+V)\cap K \ll K$ and $(U+V)\cap K$ is locally artinian. $U = [(K+V)\cap U]+L$ $(K+V)\cap L = [(K+V)\cap U]\cap L \ll L$ and $(K+V)\cap L$ is a locally artinian module. So $M = U+V+K = [(K+V)\cap U]+L+(V+K) = V+(K+L)$. Since $K\cap(U+V)$ and $L\cap(K+V) \ll L$, then we have $V\cap(K+L) \subseteq [K\cap(V+L)]+[L\cap(K+V)] \subseteq [K\cap(U+V)]+[L\cap(K+V)] \ll K+L$, as required. □

Proposition 3. Let U, V be any submodules of a module M such that $M = U+V$. If U and V are locally artinian supplemented, then M is locally artinian supplemented.

Proof. Let K be any submodule of M. The trivial submodule 0 is a locally artinian supplement of $M = U + V + K$ in M. Since U is locally artinian supplemented, $V + K$ has a locally artinian supplement in M by Lemma 2. Again applying Lemma 2, we also have that K has a locally artinian supplement in M. This shows that M is locally-artinian supplemented. □

Using this fact we obtain the following corollary.

Corollary 2. Every finite sum of locally artinian supplemented modules is locally artinian supplemented.

Proposition 4. If a module M is (amply) locally artinian supplemented, then every factor module of M is (amply) locally artinian supplemented.

Proof. Let M be a locally artinian supplemented module and $\frac{M}{N}$ be a factor module of M. By the assumption, for any submodule U of M which contains N, there exists a submodule V of M such that $M = U+V$, $U \cap V \ll V$ and $U \cap V$ is locally artinian. Let $\pi : M \longrightarrow \frac{M}{N}$ the canonical projection. Then we have that $\frac{M}{N} = \frac{U}{N} + \frac{V+N}{N}$ and $\frac{U}{N} \cap \frac{V+N}{N} = \frac{(U\cap V)+N}{N} = \pi(U \cap V) \ll \pi(V) = \frac{V+N}{N}$ by[1, 19.3 (4)]. Since $U \cap V$ is locally artinian, $\pi(U \cap V) = \frac{U}{N} \cap \frac{V+N}{N}$ is locally artinian. That is $\frac{V+N}{N}$ is a locally artinian supplement of $\frac{U}{N}$ in $\frac{M}{N}$, as required. □

Proposition 5. Let M be a module. If every submodule of M is locally artinian supplemented, then M is amply locally artinian supplemented.

Proof. Let K and L be two submodules of M such that $M = K + L$. Since L is locally-artinian supplemented, there exists a submodule L' of L such that $L = (K \cap L) + L'$ and $K \cap L' \ll L'$ is locally artinian. Note that $M = K + L = K + (K \cap L) + L' = K + L'$. It means that K has ample locally artinian supplements in M. Hence M is amply locally artinian supplemented. □

Lemma 3. Let M be amply locally artinian supplemented module and V be a supplement submodule in M. Then V is amply locally artinian supplemented.

Proof. Let V be a supplement of a submodule U of M. Let X and Y be submodules of V such that $V = X + Y$. Then $M = (U + X) + Y$. Since M is amply locally artinian supplemented, $U + X$ has a locally artinian supplement $Y' \subseteq Y$ in M. It follows that $X + Y' \subseteq V$. By the minimality of V, we have $V = X + Y'$. In addition, $X \cap Y' \subseteq (U + X) \cap Y' \ll Y'$, that is, $X \cap Y' \ll Y'$. Since $(U + X) \cap Y'$ is locally artinian, $X \cap Y'$ is also locally artinian by[5, 8.1.5]. It means that Y' is a locally artinian supplement of X in V. Finally, V is amply locally artinian supplemented. □

Proposition 6. Let M be a module. Then, M is amply locally artinian supplemented if and only if every submodule U of M is of the form $U = K + L$, where K is locally artinian supplemented and $L \ll M$ is locally artinian module.

Proof. Let U be a submodule of M. Since M is locally artinian supplemented, U has a locally artinian supplement V in M. Then $M = U + V$. By the assumption, there exists a submodule K of U such that K is a locally artinian supplement of V in M. Put $L = U \cap V$. Since V is a locally artinian supplement of U in M, $U = U \cap M = U \cap (K + V) = K + (U \cap V) = K + L$ by Modular Law. Note that K is locally artinian supplemented by Lemma 3. Since $L \ll V$, we obtain that $L \ll M$. So the proof is complete. □

Proposition 7. Let M be a π-projective and locally artinian supplemented module. Then M is amply locally artinian supplemented.

Proof. Let U and V be submodules of M such that $M = U + V$. Since M is π-projective, there exists an endomorphism φ of M such that $\varphi(M) \subseteq U$ and $(1 - \varphi)(M) \subseteq V$. Note that $(1 - \varphi)(U) \subseteq U$. Let K be a locally artinian supplement of U in M. Then $M = \varphi(M) + (1 - \varphi)(M) = \varphi(M) + (1 - \varphi)(U + K) \subseteq U + (1 - \varphi)(K)$, so that $M = U + (1 - \varphi)(K)$. Note that $(1 - \varphi)(K)$ is a submodule of v. Let $y \in U \cap (1 - \varphi)(K)$. Then, $y \in V$ and $y = (1 - \varphi)(x) = x - \varphi(x)$ for some $x \in K$. Then $x = y + \varphi(x) \in U$ so that $y = (1 - \varphi)(x) \in (1 - \varphi)(U \cap K)$. Since $U \cap (1 - \varphi)(K) \subseteq (1 - \varphi)(U \cap K)$, inverse inclusion can be shown by similar method $U \cap (1 - \varphi)(K) = (1 - \varphi)(U \cap K)$ by[1, 19.3.(4)]. Since $U \cap K$ is locally artinian, $(1 - \varphi)(U \cap K)$ is locally artinian. Since $M = U + (1 - \varphi)(K)$, $U \cap (1 - \varphi)(K) \ll (1 - \varphi)(K)$ and $U \cap (1 - \varphi)(K) = (1 - \varphi)(K)$, M is amply locally artinian supplemented. □

Since every projective module is π-projective, we can obtain the following result.

Corollary 3. Every projective locally artinian supplemented module is amply locally artinian supplemented.

Now, we shall characterize the rings over which all modules are (amply) locally artinian supplemented.

Lemma 4. Let M be a projective module. Then M is locally artinian supplemented if and only if it is supplemented and $Rad(M)$ is locally-artinian.

Proof. Suppose that M is a projective supplemented module. Therefore we have $Rad(M) \ll M$ by[1, 42.5]. Then the proof is obvious from Theorem 5. □

Theorem 7. The following statements are equivalent for a ring R.

(1) R is a left perfect ring and $Rad(R)$ is locally artinian;

(2) Every free left R-module is (amply) locally artinian supplemented;

(3) every left R-module is (amply) locally artinian supplemented;

Proof. (1) ⇒ (2) Let $F = R^{(I)}$ for some index set I. By[1, 43.9], F is supplemented. It follows from[1, 31.2 (2)] that $Rad(F) = Rad(R^{(I)}) = Rad(R)^{(I)}$ is locally artinian. Hence, by Theorem 5, F is locally artinian supplemented.

(2) ⇒ (3) Since every module is a homomorphic image of a free left module, the proof follows from Proposition 4.

(3) ⇒ (1) By Theorem 5 and[1, 43.9]. □

References

1. Wisbauer Robert. *Foundations of module and ring theory* Algebra, Logic and Applications, vol. 3: . Gordon and Breach Science Publishers, Philadelphia, PA; german ed.1991. A handbook for study and research.

2. Zhou D.X., Zhang X.R.. Small-Essential Submodules and Morita Duality. *Southeast Asian Bulletin of Mathematics.* 2011;35:pp.1051-1062.

3. Kaynar E., Çalışıcı H., Türkmen E.. SS-supplemented modules. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat..* 2020;69(1):pp.473-485.

4. Clark John, Lomp Christian, Vanaja Narayanaswami, Wisbauer Robert. *Lifting modules*. Frontiers in MathematicsBirkhäuser Verlag, Basel; 2006. Supplements and projectivity in module theory.

5. Kasch F.. *Modules and rings* London Mathematical Society Monographs, vol. 17: . Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York; 1982. Translated from the German and with a preface by D. A. R. Wallace.

6. Sharpe D. W., Vámos P.. *Injective modules*. Cambridge University Press, London-New York; 1972. Cambridge Tracts in Mathematics and Mathematical Physics, No. 62.

7. Zöschinger H.. Moduln, die in jeder Erweiterung ein Komplement haben. *Mathematica Scandinavica.* 1974;35:pp.267-287.